The asymptotically sharp geometric rigidity interpolation estimate in thin bi-Lipschitz domains

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Abstract

This work is part of a program of development of asymptotically sharp geometric rigidity estimates for thin domains. A thin domain in three dimensional Euclidean space is roughly a small neighborhood of regular enough two dimensional compact surface. We prove an asymptotically sharp geometric rigidity interpolation inequality for thin domains with little regularity. In contrast to that celebrated Friesecke-James-Müller rigidity estimate [*Comm. Pure Appl. Math.,* 55(11):1461-1506, 2002] for plates, our estimate holds for any proper rotations $\boldsymbol{R} \in SO(3)$. Namely, the estimate bounds the L^p distance of the gradient of any $\boldsymbol{y} \in W^{1,p}(\Omega, \mathbb{R}^3)$ field from any constant proper rotation $\boldsymbol{R} \in SO(3)$, in terms of the average L^p distance (nonlinear strain) of the gradient $\nabla \boldsymbol{y}$ from the rotation group $SO(3)$, and the average L^p distance of the field itself from the set of rigid motions corresponding to the rotation $\boldsymbol{R}$. There are several remarkable facts about the estimate: 1. The constants in the estimate are sharp in terms of the domain thickness scaling for any thin domains with the required regularity. 2. In the special cases when the domain has positive or negative Gaussian curvature, the inequality reduces the problem of estimating the gradient $\nabla \boldsymbol{y}$ in terms of the prototypical nonlinear strain $\int_\Omega \text{dist}^p(\nabla \boldsymbol{y}(x), SO(3))dx$ to the easier problem of estimating only the vector field $\boldsymbol{y}$ in terms of the nonlinear strain without any loss in the constant scalings as the Ansätze suggest. The later will be a geometric rigidity Korn-Poincaré type estimate. This passage is major progress in the thin domain rigidity problem. 3. For the borderline energy scaling (bending-to-stretching), the estimate implies improved strong compactness on the vector fields for free. Finally, this being said, our new interpolation inequality reduces the problem of proving "any" geometric one well rigidity problem in thin domains to estimating the vector field itself instead of the gradient, thus reducing the complexity of the problem.

1 Introduction

Let $S \subset \mathbb{R}^3$ be a connected and compact surface that has a normal a.e. (one may assume for instance that S is Lipschitz). Given a small parameter $h > 0$, recall that a shell S^h of thickness h is the $h/2$ neighborhood of S in the normal direction, i.e., $S^h = \{x + t\boldsymbol{n}(x) : x \in S, \ t \in (-h/2, h/2)\}$, where for any point $x \in S$, the vector $\boldsymbol{n}(x)$ is the unit normal to S at x. Thin spatial domains are roughly shells with non-constant thickness. Namely, assume

the functions $g_1^h(x), g_2^h(x) \colon S \to (0, \infty)$ are of order h Lipschitz functions, i.e., they fulfill the uniform conditions

$$h \leq g_1^h(x), g_2^h(x) \leq c_1 h, \quad \text{and} \quad |\nabla g_1^h(x)| + |\nabla g_2^h(x)| \leq c_2 h, \quad \text{for a.e.} \quad x \in S, \tag{1.1}$$

for some fixed constants $c_1, c_2 > 0$. Then the set Ω given by

$$\Omega = \{x + t\boldsymbol{n}(x) \ : \ x \in S, \ t \in (-g_1^h(x), g_2^h(x))\}, \tag{1.2}$$

will be a thin domain if the normal $\boldsymbol{n}(x)$ exists for all $x \in S$. If $\boldsymbol{n}(x)$ exists only a.e. in S, then instead of Ω we can consider the interior of the closure $\bar{\Omega}^\circ$, which will have no holes for any $x \in S^\circ$. This being said we can assume without loss of generality that S has a normal at every point $x \in S$ and that g_1^h and g_2^h are differentiable at every $x \in S$. The problem of determination of the geometric rigidity (which will be defined below) of thin domains is a central task in nonlinear elasticity. The geometric rigidity of a thin domain can be defined through the celebrated geometric rigidity estimate of Friesecke, James and Müller [2], which reads as follows: *Assume $\Omega \subset \mathbb{R}^3$ is open bounded connected and Lipschitz. Then there exists a constant $C = C(\Omega)$ depending only on Ω, such that for every vector field $\boldsymbol{y} \in H^1(\Omega)$, there exists a constant proper rotation $\boldsymbol{R} \in SO(3)$ such that*

$$\|\nabla \boldsymbol{y} - \boldsymbol{R}\|_{L^2(\Omega)}^2 \leq C \int_\Omega \operatorname{dist}^2(\nabla \boldsymbol{y}(x), SO(3)) dx. \tag{1.3}$$

It is known that for thin domains Ω, the constant C in (1.3) blows up as h goes to zero [2,3,4,5,6,7,8,9,10,11,12,13,21,22,23], and it typically has the asymptotic form $C = ch^\alpha$ for some $c > 0$ and $\alpha < 0$. The geometric rigidity of Ω is then the exponent α in $C = ch^\alpha$ provided it exists as $h \to 0$; the bigger α is, the more rigid the domain is. A large amount of evidence in the literature suggests the yet unproven fact that α should depend only on the domain mid-surface S, while the constant c will depends also on the constants c_1 and c_2 in (1.1). In particular, for the case when S has positive or negative Gaussian curvature the author has proven in [10] that $\alpha = 1$ or $\alpha = 4/3$ respectively for the around identity linearized analogue of (1.3). Let us provide somewhat more details on the later. If one rotates the field $\boldsymbol{u}$ by the rotation $\boldsymbol{R}$ in (1.3), then the resulting estimate shows that one can assume without loss of generality that $\boldsymbol{R} = \boldsymbol{I}$. Recall that the field $\boldsymbol{y} - \boldsymbol{x}$ is the displacement in nonlinear elasticity, thus for small enough deformations, one has the equality $\nabla \boldsymbol{y} = \boldsymbol{I} + \nabla \boldsymbol{u}$, where the quantity $\nabla \boldsymbol{u}$ is small. Thus we have the linearization

$$\operatorname{dist}(\nabla \boldsymbol{u}(x) + \boldsymbol{I}, SO(3)) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) = 2e(\boldsymbol{u}),$$

where $e(\boldsymbol{u})$ is the linear strain in the linear elasticity theory. Hence, upon this linearization one arrives at a new inequality, called Korn's first inequality

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}^2 \leq C \|e(\boldsymbol{u})\|_{L^2(\Omega)}^2. \tag{1.4}$$

The inequality (1.4) has been introduced by Korn [21,22] to prove coerciveness of the linear elastic energy, see also [1,16,19,20] for more details. Certainly (1.4) follows from (1.3) with the

same constant C by taking $\boldsymbol{y} = \boldsymbol{x} + \epsilon \boldsymbol{u}$ and letting ϵ go to zero. The problem of determining the rigidity of a given thin domain (the asymptotics of C in terms of the thickness parameter h in (1.3)) has been solved for plates[1] by Friesecke, James and Müller [2], yielding the result $C = ch^{-2}$. That is when $\Omega = \omega \times (-h/2, h/2)$ for some connected compact Lipschitz set $\omega \subset \mathbb{R}^2$. Also, it has been shown in [4] that in fact the estimate (1.3) holds with $\alpha = -2$ for any shells S^h, i.e., there exist constant $C, h_0 > 0$ depending only on S such that

$$\|\nabla \boldsymbol{y} - \boldsymbol{R}\|_{L^2(S^h)}^2 \leq \frac{C}{h^2} \int_{S^h} \text{dist}^2(\nabla \boldsymbol{y}(x), SO(3)) dx. \tag{1.5}$$

for all $h \in (0, h_0)$. Among others, very significant applications of the inequality (1.3) are that it allows one to derive shell theories from three dimensional elasticity for certain scaling regimes of the elastic energy in terms of the thickness h, [2,3,4,14,15], as well as it allows one to calculate the critical buckling load in shell buckling problems [7,8]. In both problems knowing the the asymptotics of the optimal constant in (1.3) is crucial. In this work we are concerned with studying (1.3) for thin domains $\Omega \subset \mathbb{R}^3$. We prove an interpolation version of (1.3) which reduces the problem to the estimation of the deviation of the vector field $\boldsymbol{u}$ itself (not the gradient) from the group of rigid motions. Despite the fact that the proof of the inequality is surprisingly elementary and easy, this is apparently a significant reduction of the complexity of the problem, taking into account the fact that in the case of uniformly positive or negative Gaussian curvature thin domains (this refers to the Gaussian curvature of the mid-surface S), no loss of the constant in terms of the asymptotics in h is expected as the lower bounds and the Ansätze in [10] suggest. We provide a more detailed observation on this in the next section after the formulation of the main result.

2 Main result

For the mid-surface S to have a normal a.e. and the main result to hold, we will impose some mild regularity condition on S. While in the literature a usual assumption on the mid-surface S would be C^2, we will assume that S is compact connected and bi-Lipschitz, i.e., it has a finite atlas with bi-Lipschitz patches. This will imply the following geometric condition: *There exist $\sigma, \delta \in (0, 1)$ such that for every $r \in (0, \delta)$ and every $x \in S$ one has*

$$\frac{\mathcal{H}^2(B_r(x) \cap S)}{\mathcal{H}^2(B_{2r}(x) \cap S)} \leq \sigma, \tag{2.1}$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure (surface measure in this case). Condition (2.1) roughly means that there can be no infinitesimal local concentrations of the surface S.

THEOREM **2.1.** *Let the surface $S \subset \mathbb{R}^3$ be compact connected and bi-Lipschitz, and let $h > 0$ be a small parameter. Assume the family of functions $g_1^h, g_2^h \colon S \to (0, \infty)$ fulfills the uniform conditions*

$$h \leq g_1^h(x), g_2^h(x) \leq c_1 h, \quad \text{and} \quad |\nabla g_1^h(x)| + |\nabla g_2^h(x)| \leq c_2 h, \quad \text{for all} \quad x \in S, \tag{2.2}$$

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[1]Or for shells that have a flat part

for some constants $c_1, c_2 > 0$, and denote the thin domains

$$\Omega^h = \{x + t\boldsymbol{n}(x) \ : \ x \in S, \ t \in (-g_1^h(x), g_2^h(x))\}.$$

Let $1 < p < \infty$ and let $\|\cdot\|_p$ denote the $L^p(\Omega^h)$ norm. Then there exists a constants $C, h_0 > 0$, depending only on S and the constants $c_1, c_2 > 0$ such that for any vector field $\boldsymbol{y} \in W^{1,p}(\Omega^h)$, any proper rotation $\boldsymbol{R} \in SO(3)$, and any constant vector $\boldsymbol{b} \in \mathbb{R}^3$ one has the estimate

$$\|\nabla \boldsymbol{y} - \boldsymbol{R}\|_p^2 \leq C \left(\frac{\|\boldsymbol{y} - \boldsymbol{R}x - \boldsymbol{b}\|_p \|\text{dist}(\nabla \boldsymbol{y}, SO(3))\|_p}{h} + \|\boldsymbol{y} - \boldsymbol{R}x - \boldsymbol{b}\|_p^2 + \|\text{dist}(\nabla \boldsymbol{y}, SO(3))\|_p^2 \right),$$

$$(2.3)$$

for all $h \in (0, h_0)$. Moreover, if in addition S has a C^2 piece that admits a local chart in terms of the principal coordinates, then the exponent of h in the inequality (2.2) is optimal for Ω^h, i.e., there exists a deformation $\boldsymbol{y} \in W^{1,p}(\Omega^h, \mathbb{R}^3)$ realizing the asymptotics of h in (2.2).

Some remarks are in order.

1. Note first that for any given displacement $\boldsymbol{u} \in W^{1,p}(\Omega^h, \mathbb{R}^3)$, taking the sequence of deformations $\boldsymbol{y}^\epsilon = \boldsymbol{x} + \epsilon \boldsymbol{u}$ and then letting ϵ go to zero we derive from (2.3) the estimate

$$\|\nabla \boldsymbol{u}\|_p^2 \leq C \left(\frac{\|\boldsymbol{u}\|_p \|e(\boldsymbol{u})\|_p}{h} + \|\boldsymbol{u}\|_p^2 + \|e(\boldsymbol{u})\|_p^2 \right), \qquad (2.4)$$

   where $\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the linear strain. The estimate (2.4) is the linear version of (2.3). A stronger version of (2.4) has been proven in [11,12] for the L^2 norm and in [13] for every $1 < p < \infty$, where in place of the product $\|\boldsymbol{u}\|_2 \|e(\boldsymbol{u})\|_2$ one has $\|\boldsymbol{n} \cdot \boldsymbol{u}\|_2 \|e(\boldsymbol{u})\|_2$, i.e., only the out-of-plane component of the field enters the estimate.

2. Next denote the Gaussian curvature of S by K. Tovstik and Smirnov have constructed an Ansatz [24] that realizes the asymptotics $\alpha = -4/3$ in the constant $C = ch^\alpha$ in (1.3) in the case $K < 0$. Also the author has constructed an Ansatz [10] that gives the asymptotics $\alpha = -1$ in (1.3) in the case $K > 0$. Furthermore, it has been proven in [10] that if zero boundary condition is imposed on the vector field $\boldsymbol{u}$ on the thin face of the thin domain Ω^h, then in the linear version (1.4) one indeed has $\alpha = -4/3$ and $\alpha = -1$ in the cases $K < 0$ and $K < 0$ respectively. Also, it has been shown [10] that

$$\|\boldsymbol{u}\|_{L^2(\Omega^h)}^2 \leq Ch^\beta \|e(\boldsymbol{u})\|_{L^2(\Omega)}^2,$$

   where $\beta = -1/3$ for $K < 0$ and $\beta = 0$ for $K > 0$. This implies that the estimate (2.3) indeed does not suffer an asymptotic loss of constants at least in the cases $K >< 0$, and thus from this point on, the interpolation estimate (2.3) may be utilized for the purpose of proving asymptotically optimal rigidity estimates on gradient fields.

3. In the derivation of shell theories from nonlinear elasticity by $\Gamma-$convergence, one usually assumes that the elastic energy density has a quadratic growth at the group of proper rotations (which is the elastic energy well) [2,3,14,15,23], i.e.,

$$W(\nabla \boldsymbol{y}) \geq C \cdot \text{dist}^2(\nabla \boldsymbol{y}, SO(3))$$

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for some fixed constant $C > 0$ and all vector fields $\boldsymbol{y} \in W^{1,2}(\Omega, \mathbb{R}^3)$. Then one considers different scaling regimes of the elastic energy $\int_\Omega^h W(\nabla \boldsymbol{y}(x)) dx$ in terms of the thickness h. A critical energy scaling is $E_{el} \sim h^3$, which corresponds to bending. It is then worth mentioning that while Friesecke-James-Müller estimate (1.3) for $\alpha = -2$ implies weak relative compactness in L^2 for the sequence of the rescaled (in the normal variable) gradients[2] $\nabla \boldsymbol{y}_h$ and strong compactness in L^2 for the sequence of rescaled fields $\boldsymbol{y}_h$ (as $h \to 0$), our estimate (2.3) then will clearly yield strong compactness of the rescaled gradients $\nabla \boldsymbol{y}_h$ immediately.

3   Proof of the main result

Proof of Theorem 2.1. We divide the proof into several steps for the convenience of the reader.

1. We first somewhat simplify the estimate (2.3). First of all a translation by a fixed vector $\boldsymbol{y} = \boldsymbol{v} + \boldsymbol{b}$ does not change the gradient, thus we can assume without loss of generality that $\boldsymbol{b} = 0$. Next, denoting $\boldsymbol{v} = \boldsymbol{R}\boldsymbol{w}$, the left hand side of (2.3) will become $\|\boldsymbol{w} - \boldsymbol{I}\|_p^2$, and the right hand side of (2.3) will remain the same expression written out for $\boldsymbol{w}$ in place of $\boldsymbol{v}$. This being said we can assume without loss of generality that $\boldsymbol{R} = \boldsymbol{I}$ and $\boldsymbol{b} = 0$ in (2.3). Finally, making a change of variables $\boldsymbol{y} = \boldsymbol{w} + \boldsymbol{x}$ will transform the new form of (2.3) to the estimate

$$\|\nabla \boldsymbol{w}\|_p^2 \leq C \left(\frac{\|\boldsymbol{w}\|_p \|\mathrm{dist}(\nabla \boldsymbol{w} + \boldsymbol{I}, SO(3))\|_p}{h} + \|\boldsymbol{w}\|_p^2 + \|\mathrm{dist}(\nabla \boldsymbol{w} + \boldsymbol{I}, SO(3))\|_p^2 \right), \quad (3.1)$$

to be now proven.

2. In the second step we prove the following statement: *Under the conditions of Theorem 2.1, the estimate (3.1) holds if and inly if one has for any field $\boldsymbol{v} \in W^{1,p}(\Omega^h, \mathbb{R}^3)$ the estimate*

$$\|\nabla \boldsymbol{v}\|_p^2 \leq C_1 \left(\frac{\|\boldsymbol{v}\|_p^2}{h^t} + \frac{\|\mathrm{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_p^2}{h^{2-t}} \right), \quad (3.2)$$

for any $t \in [0,2]$. Here $C_1 > 0$, and C and C_1 in (3.1) and (3.2) are of the same order, namely, $1/2 \leq \frac{C_1}{C} \leq 2$.

Evidently as $h > 0$ is small, we have $h^t, h^{2-t} \leq 1$ for $t \in [0,2]$ and (3.1) implies (3.2) by the arithmetic-geometric mean inequality with $C_1 = \frac{3}{2}C$. Assume now (3.2) holds. Given the fixed vector field $\boldsymbol{w} \in W^{1,p}(\Omega^h, \mathbb{R}^3)$, if

$$\frac{\|\boldsymbol{w}\|_p^2}{h^{t_0}} = \frac{\|\mathrm{dist}(\nabla \boldsymbol{w} + \boldsymbol{I}, SO(3))\|_p^2}{h^{2-t_0}} \quad \text{for some} \quad t_0 \in [0,2],$$

then we choose $t = t_0$ in (3.2) and get the estimate

$$\|\nabla \boldsymbol{w}\|_p^2 \leq 2C_1 \frac{\|\boldsymbol{w}\|_p \|\mathrm{dist}(\nabla \boldsymbol{w} + \boldsymbol{I}, SO(3))\|_p}{h}.$$

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[2]However there are tools to deduce strong convergence of the gradients

If
$$\frac{\|\boldsymbol{w}\|_p^2}{h^t} < \frac{\|\mathrm{dist}(\nabla\boldsymbol{w}+\boldsymbol{I}, SO(3))\|_p^2}{h^{2-t}} \quad \text{for all} \quad t \in [0,2],$$
then (3.2) implies
$$\|\nabla\boldsymbol{w}\|_p^2 \le 2C_1 \mathrm{dist}(\nabla\boldsymbol{w}+\boldsymbol{I}, SO(3))\|_p^2,$$
which in tern yields (3.1) with $C = 2C_1$. The case
$$\frac{\|\boldsymbol{w}\|_p^2}{h^t} < \frac{\|\mathrm{dist}(\nabla\boldsymbol{w}+\boldsymbol{I}, SO(3))\|_p^2}{h^{2-t}} \quad \text{for all} \quad t \in [0,2]$$
is analogous.

3. Now we focus our attention to the simplified estimate (3.2) with no product terms. We will first prove it on the shell $S^h = \{x + t\boldsymbol{n}(x) : x \in S, \ t \in (-h/2, h/2)\}$ with thickness h around S and then pass to the thin domain Ω^h employing a localization argument originated in [18] and successfully employed in the derivation of the estimate (1.5), as well as in [11] to pass from shells to thin domains. Fix $\gamma \in [0,1]$ and divide the shell S^h into small compact shells with in-plane size of order h^γ. Denoting $m = [1/h^\gamma]$, we have $N = O(m^2)$ shells $S_1^h, S_2^h, \ldots, S_N^h$, with thickness h and in-plane size roughly h^γ. It is known that the estimate (1.5) holds in L^p for all $1 < p < \infty$, thus if we scale each S_i^h by h^γ to get a new shell $\frac{1}{h^\gamma}S_i^h$ with in-plane size of order one and thickness $h^{1-\gamma}$, and apply the estimate (1.5) to the fields $h^\gamma(\boldsymbol{v}+\boldsymbol{x})$, we obtain for the vector field $\boldsymbol{v}+\boldsymbol{x}$ the estimate

$$\|\nabla\boldsymbol{v}+\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(S_i^h)} \le \frac{C}{h^{1-\gamma}}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I}, SO(3))\|_{L^p(S_i^h)}, \quad i = 1, 2, \ldots, N, \qquad (3.3)$$

for some local rotations $\boldsymbol{R}_i \in SO(3)$ and some uniform constant $C > 0$ that depends only on S. Consequently we obtain from (3.3) the bound

$$\|\nabla\boldsymbol{v}\|_{L^p(S_i^h)} \le \|\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(S_i^h)} + \frac{C}{h^{1-\gamma}}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I}, SO(3))\|_{L^p(S_i^h)}, \quad i = 1, 2, \ldots, N. \quad (3.4)$$

For the average fields $\boldsymbol{b}_i = \fint_{S_i^h}(\boldsymbol{v}(x) + (\boldsymbol{I}-\boldsymbol{R}_i)x)dx$ we have by the Poincaré inequality and (3.3) that

$$\|\boldsymbol{v}+(\boldsymbol{I}-\boldsymbol{R}_i)x - \boldsymbol{b}_i\|_{L^p(S_i^h)} \le Ch^\gamma\|\nabla\boldsymbol{v}+\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(S_i^h)} \qquad (3.5)$$
$$\le Ch^{2\gamma-1}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I}, SO(3))\|_{L^p(S_i^h)},$$

which gives the bound

$$\|(\boldsymbol{I}-\boldsymbol{R}_i)x - \boldsymbol{b}_i\|_{L^p(S_i^h)} \le \|\boldsymbol{v}\|_{L^p(S_i^h)} + Ch^{2\gamma-1}\|\mathrm{dist}(\nabla\boldsymbol{v}+\boldsymbol{I}, SO(3))\|_{L^p(S_i^h)}, \quad i = 1, 2, \ldots, N. \qquad (3.6)$$

Next we claim that

$$\|(\boldsymbol{I}-\boldsymbol{R}_i)x - \boldsymbol{b}_i\|_{L^p(S_i^h)} \ge Ch^\gamma\|\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(S_i^h)}, \qquad (3.7)$$

for some $C > 0$ uniformly in $i = 1, 2, \ldots, N$. Indeed, on one hand we have the obvious equality

$$\|\boldsymbol{I}-\boldsymbol{R}_i\|_{L^p(S_i^h)}^p = |\boldsymbol{I}-\boldsymbol{R}_i|^p |S_i^h|, \quad i = 1, 2, \ldots, N. \qquad (3.8)$$

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To estimate the other term we interpret it geometrically. Each local rotation $\boldsymbol{R}_i$ rotates around a unit vector $\boldsymbol{n}_i \in \mathbb{R}^3$, which means that the operator $\boldsymbol{R}_i x - \boldsymbol{b}_i \colon \mathbb{R}^3 \to \mathbb{R}^3$ projects onto the plane π_i orthogonal to $\boldsymbol{n}_i$, then rotates by $\boldsymbol{R}_i$ inside π_i, and then translates by the vector $-\boldsymbol{b}_i$. Assume the plane π_i is applied at the tip of the vector $\boldsymbol{b}_i$. Note that as each shell S_i^h has in-plane size of order h^γ, then invoking condition (2.1) (first uncsaling each piece by h^γ) we have that for some fixed $\tau \in (0,1)$ (uniform in i and independent on h), at most half of S_i^h (in terms of the measure) gets projected into the disc D_i, which is centered at $\boldsymbol{b}_i$, lies in π_i and has radius τh^γ. As in two dimensions one always has $|(\boldsymbol{I} - \boldsymbol{R})x| = |\boldsymbol{I} - \boldsymbol{R}||x|$ for any rotation $\boldsymbol{R} \in SO(2)$ and any vector $x \in \mathbb{R}^2$, then taking into account the above observation, we can write the obvious estimate

$$\|(\boldsymbol{I} - \boldsymbol{R}_i)x - \boldsymbol{b}_i\|_{L^p(S_i^h)}^p \geq \frac{(\tau h^\gamma)^p |\boldsymbol{I} - \boldsymbol{R}_i|^p |S_i^h|}{2}, \quad i = 1, 2, \ldots, N. \tag{3.9}$$

The estimate (3.7) immediately follows from (3.8) and (3.9). Finally putting together (3.4), (3.6) and (3.7) and summing up the obtained estimates in i, we discover the bound

$$\|\nabla \boldsymbol{v}\|_{L^p(S^h)} \leq C \left(\frac{1}{h^\gamma} \|\boldsymbol{v}\|_{L^p(S^h)} + \frac{1}{h^{1-\gamma}} \|\mathrm{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(S^h)} \right), \tag{3.10}$$

which is equivalent to (3.2) and finishes the Ansatz-free lower bound part of the Theorem for shells S^h. Now we pass from S^h to the thin domain Ω^h by a localization argument. Fix any $h > 0$ small enough divide the mid-surface S into N small parts that are of order h in two in-plane orthogonal directions, where N is roughly $1/h^2$. We can do the division so that all the small pieces of surface have uniformly bounded Lipschitz constants for all small enough $h > 0$. Then denote the h neighborhood in the normal direction of each small piece of surface by s_i^h and the part bounded between the functions g_1^h and g_2^h in the normal direction by ω_i^h for $i = 1, 2, \ldots, N$. For any $i = 1, 2, \ldots, N$, the domains s_i^h and ω_i^h have uniformly bounded Lipschitz constants, thus we have by the estimate (1.3)

$$\|\nabla \boldsymbol{v} + \boldsymbol{I} - \boldsymbol{R}_i^1\|_{L^p(s_i^h)} \leq C \|\mathrm{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(s_i^h)}, \tag{3.11}$$

$$\|\nabla \boldsymbol{v} + \boldsymbol{I} - \boldsymbol{R}_i^2\|_{L^p(\omega_i^h)} \leq C \|\mathrm{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)},$$

for some uniform constant $C > 0$ and constant rotations $\boldsymbol{R}_i^1$ and $\boldsymbol{R}_i^2$. As (2.2) suggests s_i^h is a subset of ω_i^h thus we have by the triangle inequality and from (3.11) that

$$\|\boldsymbol{R}_i^1 - \boldsymbol{R}_i^2\|_{L^p(s_i^h)} \leq C \|\mathrm{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)}. \tag{3.12}$$

Recalling the fact that s_i^h and ω_i^h have uniformly bounded Lipschitz constants, the condition (2.2) implies that they have uniformly comparable volumes too, thus we get from (3.11),

(3.12) and the triangle inequality:

$$\|\nabla \boldsymbol{v}\|_{L^p(\omega_i^h)} \leq \|\nabla \boldsymbol{v} + \boldsymbol{I} - \boldsymbol{R}_i^2\|_{L^p(\omega_i^h)} + \|\boldsymbol{I} - \boldsymbol{R}_i^2\|_{L^p(\omega_i^h)}$$
$$\leq C\|\text{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)} + C\|\boldsymbol{I} - \boldsymbol{R}_i^2\|_{L^p(s_i^h)}$$
$$\leq C\|\text{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)} + C\|\boldsymbol{I} - \boldsymbol{R}_i^1\|_{L^p(s_i^h)} + \|\boldsymbol{R}_i^1 - \boldsymbol{R}_i^2\|_{L^p(s_i^h)}$$
$$\leq C\|\text{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)} + C\|\nabla \boldsymbol{v} + \boldsymbol{I} - \boldsymbol{R}_i^1\|_{L^p(s_i^h)} + C\|\nabla \boldsymbol{v}\|_{L^p(s_i^h)}$$
$$\leq C\|\text{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\omega_i^h)} + C\|\nabla \boldsymbol{v}\|_{L^p(s_i^h)}.$$

Consequently summing up in $1, 2, \ldots, N$ we arrive at

$$\|\nabla \boldsymbol{v}\|_{L^p(\Omega^h)} \leq C\|\text{dist}(\nabla \boldsymbol{v} + \boldsymbol{I}, SO(3))\|_{L^p(\Omega^h)} + \|\nabla \boldsymbol{v}\|_{L^p(S^h)}, \tag{3.13}$$

coupling which with (3.10) we discover (3.2), i.e., the estimate for the thin domain Ω^h. This completes the Ansatz-free lower bound part of the proof.

Recall that in the case when S has patch that is C^2, then an Ansatz realizing the asymptotics of h in Korn's first inequality for shells with positive Gaussian curvature has been constructed in [10]. It turns out that the same Ansatz also works for (2.3). For the sake of completeness and convenience of the reader we recall the Ansatz construction here. Assume S is C^2 and has a single patch given by the parametrization $\boldsymbol{r} = \boldsymbol{r}(\theta, z)$ in the principal coordinates θ and z. Then, introducing the normal coordinate t, we obtain the set of orthogonal curvilinear coordinates (t, θ, z) on the patch given by

$$\boldsymbol{R}(t, \theta, z) = \boldsymbol{r}(z, \theta) + t\boldsymbol{n}(z, \theta),$$

where $\boldsymbol{n}$ is the *outward* unit normal. We choose a part $S_0 \subset S$ of the patch that is given by $S_0 = \{(\theta, z) \ : \ \theta \in [0, \delta], \ z \in [0, \delta]\}$. Denote next

$$A_z = \left|\frac{\partial \boldsymbol{r}}{\partial z}\right|, \qquad A_\theta = \left|\frac{\partial \boldsymbol{r}}{\partial \theta}\right|,$$

the two nonzero components of the metric tensor of S_0 and the two principal curvatures by κ_z and κ_θ. The signs of κ_z and κ_θ are chosen such that κ_z and κ_θ are positive for a sphere. Then the gradient of a vector field $\boldsymbol{y} = (y_t, y_\theta, y_z) \in W^{1,p}(S_0^h, \mathbb{R}^3)$ has the form

$$\nabla \boldsymbol{y} = \begin{bmatrix} y_{t,t} & \dfrac{y_{t,\theta} - A_\theta \kappa_\theta y_\theta}{A_\theta(1 + t\kappa_\theta)} & \dfrac{y_{t,z} - A_z \kappa_z y_z}{A_z(1 + t\kappa_z)} \\[3mm] y_{\theta,t} & \dfrac{A_z y_{\theta,\theta} + A_z A_\theta \kappa_\theta y_t + A_{\theta,z} y_z}{A_z A_\theta(1 + t\kappa_\theta)} & \dfrac{A_\theta y_{\theta,z} - A_{z,\theta} y_z}{A_z A_\theta(1 + t\kappa_z)} \\[3mm] y_{z,t} & \dfrac{A_z y_{z,\theta} - A_{\theta,z} y_\theta}{A_z A_\theta(1 + t\kappa_\theta)} & \dfrac{A_\theta y_{z,z} + A_z A_\theta \kappa_z y_t + A_{z,\theta} y_\theta}{A_z A_\theta(1 + t\kappa_z)} \end{bmatrix} \tag{3.14}$$

in the orthonormal basis $\boldsymbol{e}_t, \boldsymbol{e}_\theta, \boldsymbol{e}_z$, where $f, x = \partial_x f$. We choose $\boldsymbol{y} = \boldsymbol{I} + \epsilon \boldsymbol{u}$, where

$$\begin{cases} u_t = W\left(\frac{\theta}{\sqrt{h}}, z\right) \\[2mm] u_\theta = -\dfrac{t \cdot W_{,\theta}\left(\frac{\theta}{\sqrt{h}}, z\right)}{A_\theta \sqrt{h}} \\[3mm] u_z = -\dfrac{t \cdot W_{,z}\left(\frac{\theta}{\sqrt{h}}, z\right)}{A_z}, \end{cases} \tag{3.15}$$

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where w is a smooth function compactly supported on the mid-surface S_0. For the rotation $\boldsymbol{R} = \boldsymbol{I}$ and the vector field $\boldsymbol{b} = 0$, this choice will give equality in (2.3) for every fixed $h > 0$ by choosing $\epsilon > 0$ small enough.

$\square$

Acknowledgements

References

[1] K.O. Friedrichs. On the boundary-value problems of the theory of elasticity and Korn's inequality, *Annals of Math.,* 48 No. 2 (1947), 441-471.

[2] G. Friesecke, R. D. James, and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461-1506, 2002.

[3] G. Friesecke, R. D. James, and S. Müller. A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence. *Arch. Ration. Mech. Anal.*, 180(2):183-236, 2006.

[4] G. Friesecke, R.D. James, M.G. Mora, and S. Müller. Derivation of nonlinear bending theory for shells from three dimensional nonlinear elasticity by Gamma-convergence. *C. R. Acad. Sci. Paris, Sér. I.* 336 (2003), 697–702.

[5] Y. Grabovsky and D. Harutyunyan. Exact scaling exponents in Korn and Korn-type inequalities for cylindrical shells. *SIAM Journal on Mathematical Analysis*, 46(5), pp. 3277-3295, 2014.

[6] Y. Grabovsky and D. Harutyunyan. Korn inequalities for shells with zero Gaussian curvature. *Annales de l'Institute Henri Poincare (C) Non Linear Analysis.* 2018, Vol. 35, Iss. 1, pp. 267–282.

[7] Y. Grabovsky and D. Harutyunyan. Rigurous derivation of the formula for the buckling load in axially compressed circular cylindrical shells. *Journal of Elasticity,* 120(2), pp. 249-276, 2015

[8] Y. Grabovsky and D. Harutyunyan. Scaling instability of the buckling load in axially compressed circular cylindrical shells. *Journal of Nonlinear Science.* Vol. 26, Iss. 1, pp. 83–119, Feb. 2016.

[9] D. Harutyunyan. New asyptotically sharp Korn and Korn-like inequalities in thin domains. *Journal of Elasticity,* 117(1), pp. 95-109, 2014.

[10] D. Harutyunyan. Gaussian curvature as an identifier of shell rigidity. *Archive for Rational Mechanics and Analysis,* Nov. 2017, Vol. 226, Iss. 2, pp. 743–766.

[11] D. Harutyunyan. On the Korn interpolation and second inequalities in thin domains. *SIAM J. Math. Anal,* 50(5), pp. 4964–4982, 2018.

[12] D. Harutyunyan. The Korn interpolation and second inequalities for shells. *Compt. Rend. Math.* Vol. 356, Iss. 5, May 2018, pp. 575–580.

[13] D. Harutyunyan. The sharp L^p Korn interpolation and second inequalities in thin domains. *Preprint,* https://arxiv.org/abs/1809.04439

[14] P. Hornung, M. Lewicka, and R. Pakzad. Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells. *Journal of Elasticity,* Vol. 111, Nr. 1 (2013).

[15] M. Lewicka, M. G. Mora and R. Pakzad. Shell theories arising as low energy $\Gamma-$limit of $3D$ nonlinear elasticity, *Ann. Scoula Norm. Sup. Pisa Cl. Sci.,* (5) Vol. IX (2010), 1-43.

[16] C.O. Horgan. Korn's inequalities and their applications in continuum mechanics, *SIAM Rev.,* 37 (1995), no. 4, 491-511.

[17] R. V. Kohn. New integral estimates for deformations in terms of their nonlinear strain. *Arch. Rat. Mech. Anal.* 78, (1982) 131-172.

[18] R.V. Kohn and M. Vogelius, A new model for thin plates with rapidly varying thickness. II:A convergence proof, Quart. Appl. Math. 43 (1985), 1–22.

[19] V. Kondratiev and O. Oleinik. On Korn's inequalities, *C.R. Acad. Sci. Paris,* 308 Serie I (1989), 483-487.

[20] V. A. Kondratiev and O. A. Oleinik. Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities. *Uspekhi Mat. Nauk 43,* 5(263) (1988), 55-98, 239.

[21] A. Korn. Solution générale du problème d'équilibres dans la théorie de l'élasticité dans le cas où les efforts sont donnés à la surface, *Ann. Fac. Sci. Toulouse,* ser. 2. 10 (1908), 165-269.

[22] A. Korn. Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen, Bull. Int. Cracovie Akademie Umiejet, Classe des Sci. Math. Nat., (1909) 705-724.

[23] S. Müller. *Mathematical problems in thin elastic sheets: scaling limits, packing, crumpling and singularities,* Vector-valued partial differential equations and applications, Vol: 2179 of Lecture Notes in Math, Springer, Cham. pp: 125-193 (2017).

[24] P. E. Tovstik and A. L. Smirnov. *Asymptotic methods in the buckling theory of elastic shells, volume 4 of Series on stability, vibration and control of systems.* World Scientific, 2001.

[25] P. Yao. Optimal exponentials of thickness in Korn's inequalities for parabolic and elliptic shells. *preprint*, https://arxiv.org/abs/1807.11114

[26] P. Yao. Linear strain tensors on hyperbolic surfaces and asymptotic theories for thin shells. *preprint*, https://arxiv.org/abs/1708.07202