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STABLE SELF-SIMILAR BLOW-UP FOR A FAMILY OF NONLOCAL TRANSPORT EQUATIONS

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We consider a family of nonlocal problems that model the effects of transport and vortex stretching in the incompressible Euler equations. Using modulation techniques, we establish *stable* self-similar blow-up near a family of known self-similar blow-up solutions.

1. Introduction

The dynamics of solutions to the 3-dimensional incompressible Euler equations is guided by many effects which are still not properly understood. Among these effects are

- nonlocality,
- transport,
- vortex stretching.

Nonlocality is physically clear: in an ideal fluid, any disturbance in one location is immediately felt everywhere. Transport refers to the fact that while vortices produce a velocity field, they are also carried by that velocity field to different locations in space. Vortex stretching is the process by which vortices are enhanced due to variations in the velocity gradient in the direction of the vortex. This is succinctly captured in the 3-dimensional Euler system as follows:

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega = \omega \cdot \nabla u, \\ u = \nabla \times (-\Delta)^{-1}\omega. \end{cases} \tag{1-1}$$

Nonlocality is described by the relation $u = \nabla \times (-\Delta)^{-1}\omega$ (called the Biot–Savart law), while the time-evolution of the vorticity ω is determined by the transport term $u \cdot \nabla\omega$ and the vortex-stretching term $\omega \cdot \nabla u$. Notice that the incompressible Euler equation is a system of three equations and that each equation contains seven terms all coupled together through the nonlocal Biot–Savart law. Many authors have written about the different effects of each term, observed through numerical simulations [Hou and Lei 2009], the construction of special solutions [Stuart 1988; Elgindi and Jeong 2020b], and the analysis of model problems [Constantin et al. 1985; De Gregorio 1996; Constantin 2000; Okamoto et al. 2008]. Since a finite-time singularity in the Euler equation can only happen if the magnitude of the vorticity ω becomes unbounded, many have highlighted the vortex stretching as *the* source of a possible singularity.

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Wanting to understand better the qualitative nature of the vortex stretching term, Constantin, Lax and Majda [Constantin et al. 1985] introduced the following model set on $\mathbb{R} \times [0, T_*)$:

$$\begin{cases} \partial_t \omega = -2H(\omega)\omega, \\ H(\omega)(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \omega(y)/(x-y)\, dy. \end{cases} \tag{1-2}$$

From the 3-dimensional Euler equation, one just drops the transport term, makes the system into a single equation, and approximates $\nabla u = \nabla \nabla \times (-\Delta)^{-1}\omega$ by a zeroth-order operator. In one dimension, the natural choice is the Hilbert transform H. With these simplifications, they solve explicitly the equation and prove that solutions can become singular in finite time. This allows one to speculate that singularity formation is possible even in the 3-dimensional Euler equation. There is at least one major problem, however, with this model: while the transport term does not change the magnitude of the vorticity, it can counteract the growing effects of the vortex stretching term. This can be seen easily in the simple model

$$\partial_t \omega + \lambda(t) \sin(x)\, \partial_x \omega = \lambda(t)\omega,$$

set on $[-\pi, \pi]$ with periodic boundary conditions and with $\lambda(t)$ a time-dependent constant. It is not difficult to show that if ω_0 is C^1 and vanishes at 0 and π, then solutions to this equation are *uniformly* bounded in time independent of the size of $\lambda(t)$. In particular, transport can act to "deplete" the growth effects of the vortex stretching term in this simple model. Thus, while the transport term cannot cause a singularity, it can stop it from happening.

After [Constantin et al. 1985], De Gregorio [1996] introduced a model that takes into account both vortex stretching and transport:

$$\begin{cases} \partial_t \omega + 2u\, \partial_x \omega = 2\partial_x u \omega, \\ u = -\Lambda^{-1}\omega(x) = -\int_0^x H\omega(y)\, dy. \end{cases} \tag{1-3}$$

Based on numerical simulations, De Gregorio conjectured by that the addition of the transport term should lead to global regularity. Strong evidence for this conjecture was given in [Jia et al. 2019] and global regularity for special kind of data was given in [Lei et al. 2020]. Inspired by this conjecture, Okamoto, Sakajo, and Wunsch [Okamoto et al. 2008] introduced a new model where they weight the transport term with a parameter a:

$$\begin{cases} \partial_t \omega + au\, \partial_x \omega = 2\partial_x u \omega, \\ u = -\Lambda^{-1}\omega(x) = -\int_0^x H\omega(y)\, dy. \end{cases} \tag{1-4}$$

The purpose of this model was to understand the effects of the modeled vortex stretching and transport terms. Hence, when $a = 2$ we get the De Gregorio model and when $a = 0$ we get the model of Constantin, Lax and Majda. Similarly to [Constantin et al. 1985], Córdoba, Córdoba and Fontelos [Córdoba et al. 2005] introduced a 1-dimensional model to mimic the 2-dimensional quasigeostrophic equation:

$$\begin{cases} \partial_t \omega + 2H(\omega)\omega(x,t) = -2\Lambda^{-1}\omega \partial_x \omega \in \mathbb{R} \times [0, T_*), \\ H(\omega)(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \omega(y)/(x-y)\, dy, \end{cases} \tag{1-5}$$

which corresponds to $a = -2$ in the generalized model (1-4).

Recently, Elgindi and Jeong [2020a] proved the existence of a smooth self-similar profile for a small by using a local continuation argument. The goal here is to prove the stability of those profiles for all a small enough. The proof is based on the modulation technique which has been developed by Merle, Raphael, Martel, Zaag and others. This technique has been very efficient to describe the formation of singularities for many problems like the nonlinear wave equation [Merle and Zaag 2015], the nonlinear heat equation [Merle and Zaag 1997], reaction diffusion systems [Ghoul et al. 2018a; 2018b], the nonlinear Schrödinger equation [Merle and Raphael 2005; Kenig and Merle 2006], the GKDV equation [Martel et al. 2014], and many others. Note that for (1-4) comparing to all the previous models cited above there exists a group of scaling transformations of dimension larger than 2 that leaves the equation invariant. Here this degeneracy is a real difficulty since one does not know in advance which scaling law the flow will select. We remark that similar results to Theorems 1 and 2 were recently established by Chen, Hou, and Huang [Chen et al. 2019]. They were also able to find a singularity for the De Gregorio ($a=2$) model on the whole line using a very interesting argument with computer assistance.

Main theorem. We introduce first the weighted space

$$L^2_\phi(\mathbb{R}) = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \int_{\mathbb{R}} |f|^2 \phi < \infty \right\},$$

equipped with the norm and inner product

$$\|f\|^2_{L^2_\phi} = (f\phi, f)_{L^2},$$

where $\phi = (1+y^2)^2/y^4$. From now on we focus on (1-4). In [Elgindi and Jeong 2020a], the authors show the existence of self-similar solutions of (1-4) of the form

$$\omega(x,t) = \frac{1}{T-t} F_a\left(\frac{x}{(T-t)^{1+\gamma(a)}}\right), \tag{1-6}$$

where F_a solves

$$F_a + \big((1+\gamma(a))y - a\Lambda^{-1}F_a\big)F_a' + 2HF_aF_a = 0, \tag{1-7}$$

and $\gamma(a) = a(-2+\ln 4) + O(a^2)$. When $a=0$, the profile F_0 has the form

$$F_0(y) = \frac{y}{1+y^2}, \qquad HF_0(y) = -\frac{1}{y^2+1},$$

while for $|a|$ small

$$|\gamma(a)| + |F_a - F_0|_{H^3} \le C|a|,$$

with $C>0$ a universal constant. In fact, we also have the expansions

$$F_a(y) = \begin{cases} y + O(y^3), & y \le 1, \\ C_1|y|^{-\frac{1}{1+\gamma(a)}} + O(|y|^{-\frac{2}{1+\gamma(a)}}), & y \ge 1, \end{cases} \tag{1-8}$$

$$HF_a(y) = \begin{cases} -(2+\gamma(a))/(2-a) + O(y^2), & y \le 1, \\ C_2|y|^{-\frac{1}{1+\gamma(a)}} + O(|y|^{-\frac{2}{1+\gamma(a)}}), & y \ge 1. \end{cases} \tag{1-9}$$

The main result of this work is the dynamic stability of these blow-up profiles. In particular, this allows us to construct compactly supported solutions with local self-similar blow-up and cusp formation in finite time (a phenomenon numerically conjectured to occur in the case $a = -2$).

To prove the stability of the profiles F_a we rescale (1-4). A natural change of variables to do here is

$$z = \frac{x}{\lambda^{1+\gamma(a)}}, \quad \frac{ds}{dt} = \frac{1}{\lambda}, \quad \omega(x,t) = \frac{1}{\lambda} v\left(\frac{x}{\lambda^{1+\gamma(a)}}, s\right). \tag{1-10}$$

Hence, in these new variables we get the following equation on v:

$$\begin{cases} v_s - (\lambda_s/\lambda)(v + (1+\gamma(a))z v_z) + 2H(v)v = a\Lambda^{-1} v v_z \in \mathbb{R} \times [0,\infty), \\ H(v)(y) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} v(x)/(z-x)\,dx. \end{cases} \tag{1-11}$$

Note that this change of variables leaves the $C^{\alpha(a)}$ norm of the velocity $u = -\Lambda^{-1}\omega$ unchanged, with $\alpha(a) = 1 - 1/(1+\gamma(a))$. This indicates that the velocity u will form a C^α cusp. Note that (1-7) is invariant under the scaling

$$F_{a,\mu}(z) := F_a(\mu z).$$

We will make an abuse of notation by denoting $F_{a,\mu}$ by F_a. Actually, this will induce an instability as one can see on the spectrum of the linearized operator around the profile F_a in (1-8). To fix this instability, we will allow μ to depend on time and fix it through an orthogonality condition. Hence, we introduce

$$v(z) = w(\mu z), \quad y = \mu z,$$

where w solves

$$w_s + \frac{\mu_s}{\mu} y w_y - \frac{\lambda_s}{\lambda}(w + (1+\gamma(a)) y w_y) + 2H(w)w = a\Lambda^{-1} w w_y, \tag{1-12}$$

and

$$\mathscr{S}_a(w) = w + (1+\gamma(a)) y \partial_y w.$$

Now we linearize around F_a by setting

$$w = F_a + q,$$

where q solves

$$q_s + \frac{\mu_s}{\mu} y (F_a' + q_y) - \left(\frac{\lambda_s}{\lambda} + 1\right) \mathscr{S}_a(F_a + q) = \mathscr{M}_a q - 2Hqq + a\Lambda^{-1} q q_y, \tag{1-13}$$

where

$$\mathscr{M}_a q = -(2HF_a + 1)q - 2HqF_a - ((1+\gamma(a))y - a\Lambda^{-1}F_a) q_y + a\Lambda^{-1} q F_a'. \tag{1-14}$$

Theorem 1. *Let a be small enough and $\tilde{\alpha}(a) = 1 - 1/(1+\gamma(a))$. Then there exists an open set of odd initial data of the form $\omega(t{=}0) = F_a + q_0$, with $\partial_x q_0(0) = Hq_0(0) = 0$ and where*

$$C_2 \int_{\mathbb{R}} |q_0(y)|^2 \phi(y)\,dy + C_1 \int_{\mathbb{R}} |y \partial_y q_0|^2 \phi(y)\,dy < \epsilon, \tag{1-15}$$

such that there exist $T(q_0) > 0$, $\mu_*(q_0)$, and $C(q_0) > 0$ with ω satisfying

$$\omega(x,t) = \frac{1}{\lambda(t)} \left(F_a\left(\frac{x\mu(t)}{\lambda^{1+\gamma(a)}}\right) + q\left(\frac{x\mu(t)}{\lambda^{1+\gamma(a)}}, t\right)\right),$$

with $\lambda(t)/(T-t) \to C(q_0)$ *and* $\mu(t) \to \mu^*$ *as* $t \to T$ *and*

$$C_1 \int_{\mathbb{R}} |y\,\partial_y q(y,t)|^2 \phi(y)\,dy + C_2 \int_{\mathbb{R}} |q(y,t)|^2 \phi(y)\,dy \lesssim_\delta (T-t)^{1-\delta}$$

for any $\delta > 0$. *When* $a < 0$ (*so that* $\gamma(a) > 0$) *we have*

$$\sup_{t\in[0,T]} \|u(t)\|_{C^{\tilde{\alpha}(a)}} < \infty, \quad \textit{with } u(t,x) = -\Lambda^{-1}\omega.$$

Remark 1.1. The last statement of the theorem on the C^α regularity of the velocity field when $a < 0$ is related to the $C^{1/2}$ conjecture made in [Kiselev 2010; Silvestre and Vicol 2016].

Remark 1.2. The assumptions of the theorem do not require that q_0 be differentiable everywhere (just that $q \in H^1$ and that q vanishes to high order near 0). Note also that the assumptions that $\partial_x q_0(0) = Hq_0(0) = 0$ can be trivially removed using Lemma 3.1 since F_a could be replaced by a slightly rescaled version of F_a to make the perturbation and its Hilbert transform vanish to second order at 0.

Remark 1.3. Note that the open set of initial data contains a slowly decaying solution, and also compactly supported solutions. Indeed, one can impose that $q_0 \sim -F_a$ at infinity.

Our next theorem is in the same spirit as Theorem 1. The difference is that it applies to the nonsmooth self-similar solutions constructed in [Elgindi and Jeong 2020a] and, thus, applies even for a large. Indeed, in that work, the authors constructed a family of self-similar solutions to (1-4) which are smooth functions of the variable $X = |x|^\alpha$ with speed $1 + \tilde{\gamma}_\alpha(a)$ whenever $|a\alpha|$ is small enough. Denoting these solutions by F_a^α, we have the following stability theorem.

Theorem 2. *Define*

$$\phi_*(Y) = \frac{(1+Y)^4}{Y^4}.$$

There exists $c_0 > 0$ *so that if* $a \in \mathbb{R}$ *and* $\alpha > 0$ *satisfy* $(|a|+1)\alpha < c_0$ *and* $\tilde{\beta}_\alpha(a) = 1 - 1/(1+\tilde{\gamma}_\alpha(a))$, *then there exists an open set of odd initial data of the form* $\omega(t{=}0) = F_a^\alpha + q_0$, *with* $\partial_X q_0(0) = Hq_0(0) = 0$ *and where*

$$C_2 \int_{\mathbb{R}} |q_0(Y)|^2 \phi_*(Y)\,dY + C_1 \int_{\mathbb{R}} |Y\,\partial_Y q_0|^2 \phi_*(Y)\,dY < \epsilon, \tag{1-16}$$

such that there exist $T(q_0) > 0$, $\mu_*(q_0)$, *and* $C(q_0) > 0$ *with* ω *satisfying*

$$\omega(x,t) = \frac{1}{\lambda(t)}\left(F_a\left(\frac{X\mu(t)}{\lambda^{1+\tilde{\gamma}_\alpha(a)}}\right) + q\left(\frac{X\mu(t)}{\lambda^{1+\tilde{\gamma}_\alpha(a)}}, t\right)\right),$$

with $\lambda(t)/(T-t) \to C(q_0)$ *and* $\mu(t) \to \mu^*$ *as* $t \to T$ *and*

$$C_1 \int_{\mathbb{R}} |Y\,\partial_Y q(Y,t)|^2 \phi_*(Y)\,dy + C_2 \int_{\mathbb{R}} |q(Y,t)|^2 \phi_*(Y)\,dY \lesssim_\delta (T-t)^{1-\delta}$$

for any $\delta > 0$ *and*

$$\sup_{t\in[0,T]} \|u(t)\|_{C^{\tilde{\beta}_\alpha(a)}} < \infty, \quad \textit{with } u(t,x) = -\Lambda^{-1}\omega.$$

Remark 1.4. Note that in Theorem 2 we allow the parameter a to be anything in $\mathbb{R}$ but we pay the price on the regularity, since we need to pick α so that $|a\alpha|$ is small enough. Note also that $\tilde{\beta}_\alpha(a) \to 1$ as $\alpha \to 0$ for any fixed a. This means that as $\alpha \to 0$, the blow-up becomes more and more mild.

Remark 1.5. The proof of Theorem 2 is sketched in Section 6; the only main difference between the proofs of Theorems 1 and 2 is the coercivity of the linearized operator and an extra change of variables in the proof of Theorem 2.

Organization of the paper. In Section 2 we establish coercivity estimates for the linearized operator $\mathscr{M}_a$ under the assumption that the perturbation q vanishes to high order at 0 along with its Hilbert transform. This is the core of the argument. In Section 3, we modulate the free parameters λ and μ to propagate the vanishing condition on q. Then we prove long-time decay estimates on q (in self-similar variables) in Section 4, which show that the perturbation q becomes small relative to the self-similar profile as we approach the blow-up time. We establish Theorem 2 in Section 6.

2. Coercivity

Proposition 2.1. *There exists a universal constant $C > 0$ so that if a is small enough and if f is odd, $f'(0) = Hf(0) = 0$ and*

$$\int_{\mathbb{R}} |f|^2 \phi(y)\, dy < +\infty,$$
$$\int_{\mathbb{R}} f \mathscr{M}_a f \phi(y)\, dy \le -\left(\tfrac{1}{2} - C|a|\right) \int_{\mathbb{R}} f(y)^2 \phi(y)\, dy. \tag{2-1}$$

The proof of this lemma requires a weighted identity for the Hilbert transform, which we show in Lemma A.1.

Proof. We write

$$\begin{aligned}\mathscr{M}_a f &= -(2HF_a + 1) f - 2Hf F_a - \big((1+\gamma(a))y - a\Lambda^{-1}F_a\big) f_y + a\Lambda^{-1} f F_a' \\ &= -(2HF_0 + 1) f - 2Hf F_0 - y f_y + a\overline{\mathscr{M}}_a f.\end{aligned}$$

Observe that if a is small enough, there exists a universal constant $C > 0$ (independent of a) so that we have the estimate

$$\left|\int_{\mathbb{R}} f \overline{\mathscr{M}}_a f \phi(y)\, dy\right| \le C \int (|Hf|^2 + |f|^2)\phi(y)\, dy \le C \int |f|^2 \phi(y)\, dy,$$

using Lemma A.1. This follows from the observation

$$\left|\frac{1}{\phi(y)}\, \partial_y(\phi(y)\Lambda^{-1}F_a)\right|_{L^\infty} + |F_a{}'|_{L^\infty} + \frac{|\gamma(a)|}{|a|} + \frac{1}{|a|}|F_a - F_0|_{L^\infty} + \frac{1}{|a|}|HF_a - HF_0|_{L^\infty} \le C.$$

The only one which is not a direct consequence of the expansion given in [Elgindi and Jeong 2020a] is the first one, which we see can be estimated by

$$\left|\frac{1}{\phi(y)}\, \partial_y \phi(y)\Lambda^{-1}F_a\right|_{L^\infty} + |HF_a|_{L^\infty} \le \left|\frac{1}{y}\Lambda^{-1}F_a\right|_{L^\infty} + |HF_a|_{L^\infty} \le C|HF_a|_{L^\infty} \le C.$$

Thus, we must consider only the quantity

$$\int ((-2HF_0-1)f-2HfF_0-yf_y)f\phi\,dy.$$

First let us observe

$$\int HffF_0\phi=0.$$

Indeed,

$$\begin{aligned}\int HffF_0\phi=\int fHf\frac{y^2+1}{y^3}=\int\frac{fHf}{y}+\int\frac{fHf}{y^3}&=H(fHf)(0)+\tfrac{1}{2}H(\partial_{yy}(fHf))(0)\\ &=\tfrac{1}{2}(H(f)^2(0)-f(0)^2)+\tfrac{1}{4}\partial_{yy}(H(f)^2-f^2)(0)=0,\end{aligned}$$

by the assumptions[1] on f. This leaves us with

$$\int(-2HF_0-1)f^2\phi(y)-yff_y\phi\,dy=\int\left(-2HF_0-1+\frac{1}{2}\frac{\partial_y(y\phi)}{w}\right)|f(y)|^2\phi(y)\,dy.$$

Next observe that

$$-2HF_0-1+\frac{1}{2}\frac{\partial_y\phi}{\phi}=\frac{2}{1+y^2}-1+\frac{y^2-3}{2(y^2+1)}=-\frac{1}{2}.$$

This completes the proof. □

3. Modulation equation and derivation of the law

Since our coercivity estimate from the previous section relies on $\partial_y q(0)=H(q)(0)=0$, we will use that we have the "free" parameters μ and λ to fix these conditions. To find precisely how to do this, we will just differentiate (1-13) with respect to y and apply the Hilbert transform to (1-13) and evaluate both at $y=0$. We will prove now by using the implicit function theorem that there exists a unique decomposition to the solution w of (1-12). Indeed, in the following lemma we fix μ and λ such that $q_y(s,0)=Hq(0)=0$.

Lemma 3.1 (modulation). *For $q\in L^1_{\mathrm{loc}}$ for which $Hq(0)$ and $q'(0)$ exist and*

$$|Hq(0)|+|q'(0)|\le\tfrac{1}{2},$$

there exists a unique pair $(\mu,\lambda)\in(0,\infty)^2$ so that

$$\tilde{q}:=F_a(y)+q-\widetilde{F}_{a,\mu,\lambda},$$

with

$$\widetilde{F}_{a,\mu,\lambda}(y)=\frac{1}{\lambda}F_a\left(\frac{y\mu}{\lambda^{1+\gamma(a)}}\right),$$

satisfies

$$\tilde{q}_y(0)=H\tilde{q}(0)=0.$$

[1]Note that, strictly speaking, $fH(f)$ is not twice differentiable but the equality $\int HffF_0\phi=0$ can be made rigorous by applying the smoothing procedure in Lemma A.2.

In fact,

$$\lambda = 1 - H(q)(0)\frac{(2+\gamma(a))}{2-a} \quad \textit{and} \quad \mu = (1+q'(0))\lambda^{2+\gamma(a)}.$$

Proof of Lemma 3.1. We want to find μ, λ so that

$$\tilde{q} := F_a + q - \widetilde{F}_{a\mu,\lambda}$$

satisfies $\tilde{q}_y(0) = H\tilde{q}(0) = 0$. Observe that

$$(\widetilde{F}_{a\mu,\lambda})'(0) = \frac{\mu}{\lambda^{2+\gamma(a)}} F_a'(0), \quad H(\widetilde{F}_{a\mu,\lambda})(0) = \frac{1}{\lambda} H F_a(0),$$

while

$$F_a'(0) = 1, \quad H F_a(0) = -\frac{2+\gamma(a)}{2-a}. \qquad \square$$

Let w_0, $y\partial_y w_0$ be in $L^2_\phi(\mathbb{R})$ with a small enough norm and let w be its corresponding solution. Consequently, thanks to Lemma 3.1 the solution admits a unique decomposition on some time interval $[s_0, s^*)$:

$$w(y,s) = \tilde{F}_{a,\mu,\lambda} + q, \tag{3-1}$$

where

$$q_y(s,0) = Hq(s,0) = 0.$$

The bootstrap regime. We will define first in which sense the solution is initial close to the self-similar profile.

Definition 3.2 (initial closeness). Let $\delta > 0$ small enough, $s_0 \gg 1$, and $w_0 \in H^1_\phi$. We say that w_0 is initially close to the blow-up profile if there exists $\lambda_0 > 0$ and $\mu_0 > 0$ such that the following properties are satisfied. In the variables (y, s) one has

$$w_0(y) = F + q_0, \tag{3-2}$$

and the remainder and the parameters satisfy

(i) (initial values of the modulation parameters)

$$\tfrac{1}{2}e^{\frac{s_0}{2}} < \lambda_0 < 2e^{\frac{s_0}{2}}, \quad \tfrac{1}{2} < \mu_0 < 2, \tag{3-3}$$

(ii) (initial smallness)

$$\|y\partial_y q_0\|^2_{L^2_\phi} + \frac{1}{\delta}\|q_0\|^2_{L^2_\phi} < e^{-\frac{s_0}{8}}. \tag{3-4}$$

We are going to prove that solutions initially close to the self-similar profile in the sense of Definition 3.2 will stay close to this self-similar profile in the following sense.

Definition 3.3 (trapped solutions). Let $K \gg 1$. We say that a solution w is trapped on $[s_0, s^*]$ if it satisfies the properties of Definition 3.2 at time s_0, and if it can be decomposed as

$$w = F + q(y,s)$$

for all $s \in [s_0, s^*]$ with

(i) (values of the modulation parameters)

$$\frac{1}{K}e^{-\frac{s}{8}} < \lambda(s) < Ke^{-\frac{s}{8}}, \quad \frac{1}{K} < \mu(s) < K, \tag{3-5}$$

(ii) (smallness of the remainder)

$$\|y\partial_y q\|_{L^2_\phi}^2 + \frac{1}{\delta}\|q\|_{L^2_\phi}^2 < Ke^{-\frac{s}{8}}, \tag{3-6}$$

Proposition 3.4. *There exist universal constants $K, s_0^* \gg 1$ such that the following holds for any $s_0 \geq s_0^*$. All solutions w initially close to the self-similar profile in the sense of Definition 3.2 are trapped on $[s_0, +\infty)$ in the sense of Definition 3.3.*

Define for $\delta > 0$ small enough

$$\mathscr{E}(s) = \|y\partial_y q\|_{L^2_\phi}^2 + \frac{1}{\delta}\|q\|_{L^2_\phi}^2. \tag{3-7}$$

The proof of the proposition will be done later by using energy estimates. Before this we will derive the "law" that μ and λ will satisfy.

Indeed, we will prove the following:

Proposition 3.5. *To ensure that*

$$q_y(s,0) = Hq(0) = 0,$$

it suffices to impose that μ and λ satisfy the following:

$$\frac{\mu_s}{\mu} = (2+\gamma(a))\left(\frac{\lambda_s}{\lambda}+1\right), \tag{3-8}$$

$$\left(\frac{\lambda_s}{\lambda}+1\right)\frac{2+\gamma(a)}{2-a} = a\big(H(\Lambda^{-1}F_a q_y)(0,s) + H(\Lambda^{-1}qF_a')(0,s) + H(\Lambda^{-1}qq_y)(0,s)\big), \tag{3-9}$$

$$\left|\frac{\lambda_s}{\lambda}+1\right| \leq C|a|\sqrt{\mathscr{E}}. \tag{3-10}$$

Proof. Dividing (1-13) by y and evaluating at $y=0$ and using that q is odd we get

$$\partial_s(q_y(0,s)) + \frac{\mu_s}{\mu}(F_a'(0)+q_y(0)) - \left(\frac{\lambda_s}{\lambda}+1\right)\partial_y((1+(1+\gamma(a))y\,\partial_y)(F_a+q))\big|_{y=0}$$
$$= \partial_y \mathscr{M}_a q\big|_{y=0} - 2\partial_y(Hqq)\big|_{y=0} + a\,\partial_y(\Lambda^{-1}qq_y)\big|_{y=0}.$$

By inspection, using that $F_a'(0)=1$ we see that

$$\partial_s(q_y(0,s)) + \frac{\mu_s}{\mu} - (2+\gamma(a))\left(\frac{\lambda_s}{\lambda}+1\right) = AHq(0,s) + Bq_y(0,s)$$

for some s-dependent numbers A, B depending only on q, F_a, μ and λ. Before finding the second law, we note the following simple fact for decaying functions f with $f_y \in L^1$:

$$H(y\,\partial_y f)(0) = \int f_y = 0.$$

Now we apply H to (1-13) and evaluating at $y=0$ we will get the second law

$$\partial_s(Hq(0,s)) - \left(\frac{\lambda_s}{\lambda}+1\right)(HF_a(0)+Hq(0,s)) = -H(q)(0)+2H(q)(0,s)H(F_a)(0)+aH(\Lambda^{-1}F_a q_y)(0,s) \\ +aH(\Lambda^{-1}qF_a')(0,s)-Hq(0,s)^2+aH(\Lambda^{-1}(q)q_y)(0,s).$$

In particular, if

$$\left(\frac{\lambda_s}{\lambda}+1\right)\frac{2+\gamma(a)}{2-a} = a\big(H(\Lambda^{-1}F_a q_y)(0,s)+H(\Lambda^{-1}qF_a')(0,s)\big)+H(\Lambda^{-1}qq_y)(0,s),$$

$$\frac{\mu_s}{\mu} = (2+\gamma(a))\left(\frac{\lambda_s}{\lambda}+1\right),$$

and if $Hq(0,0)=q_y(0,0)=0$ we get

$$Hq(0,s)=q_y(0,s)=0$$

for all $s \geq 0$. In addition, we get that

$$\left|\frac{\lambda_s}{\lambda}+1\right| \lesssim \int_0^\infty \frac{|a|(|\Lambda^{-1}F_a q_y|+|\Lambda^{-1}qF_a'|)+|\Lambda^{-1}qq_y|}{y}\,dy. \tag{3-11}$$

By using that

$$\left|\frac{\Lambda^{-1}q}{y}\right|_{L^\infty} \lesssim |H(q)|_{L^\infty} \lesssim |q|_{H^1} \lesssim \sqrt{\mathscr{E}(s)},$$

we deduce that

$$\left|\frac{\lambda_s}{\lambda}+1\right| \leq |a|C\sqrt{\mathscr{E}(s)}. \qquad \square$$

4. Energy estimates

The goal of this section is to establish energy estimates for q in a suitable space. Let us first define our energy

$$\mathscr{E}(q) = \frac{1}{\delta}\int |q|^2\phi(y)\,dy + \int |y\,\partial_y q|^2\phi(y)\,dy,$$

where δ will be chosen to be small enough. We will prove that if a is small enough, then

$$\frac{d}{ds}\mathscr{E}(q) \leq -\tfrac{1}{4}\mathscr{E}(q)+C\mathscr{E}(q)^{3/2} \tag{4-1}$$

for a universal constant $C>0$.

Observe that from our choice of μ and λ in Proposition 3.5, we have the following estimate (again assuming that a is small enough):

$$\left|\frac{\lambda_s}{\lambda}+1\right| \leq aC(\sqrt{\mathscr{E}(q)}+\mathscr{E}(q))$$

for some universal constant $C>0$. Next, we use

$$\frac{\mu_s}{\mu} = (2+\gamma(a))\left(\frac{\lambda_s}{\lambda}+1\right)$$

in (1-13) to deduce

$$q_s + \left(\frac{\lambda_s}{\lambda} + 1\right)(F_a - yF_a' + q - yq_y) = \mathscr{M}_a q - 2Hqq + a\Lambda^{-1}qq_y. \tag{4-2}$$

Taking the (weighted) inner product of (4-2) with q we get

$$\frac{1}{2}\frac{d}{ds}\|q\|^2_{L^2_\phi} \le (\mathscr{M}_a q, q) + aC(\mathscr{E}(q) + \mathscr{E}(q)^{\frac{3}{2}}) - 2(H(q)q, q)_{L^2_\phi} + a(\Lambda^{-1}qq_y, q)_{L^2_\phi},$$

where we have used that

$$\left|\frac{\partial_y(y\phi(y))}{\phi(y)}\right|_{L^\infty} + |F_a - yF_a'|_{L^2_\phi} \le C$$

for some universal constant C independent of a. Now, we have

$$|H(q)|_{L^\infty} \le C|q|_{H^1} \le C\mathscr{E}(q).$$

Furthermore,

$$(\Lambda^{-1}qq_y, q\phi) = -\tfrac{1}{2}(q^2\,\partial_y(\phi\Lambda^{-1}q)) = -\tfrac{1}{2}(q^2\Lambda^{-1}q, \partial_y\phi) - \tfrac{1}{2}(q^2, H(q)\phi).$$

Now observe that

$$\left|\frac{\Lambda^{-1}q}{y}\right|_{L^\infty} \le C|H(q)|_{L^\infty} \quad \text{and} \quad \left|\frac{y\,\partial_y\phi}{\phi}\right| \le C.$$

Thus,

$$\frac{1}{2}\frac{d}{ds}\|q\|^2_{L^2_\phi} \le -\left(\frac{1}{2} - C|a|\right)|q|^2_{L^2_\phi} + |a|C\mathscr{E}(q) + C\mathscr{E}(q)^{\frac{3}{2}}. \tag{4-3}$$

Now we establish the first derivative estimate. First we apply $y\,\partial_y$ to (4-2) and get

$$\partial_s(yq_y) + \left(\frac{\lambda_s}{\lambda} + 1\right)(-y^2F_a'' - y^2q_{yy}) = \mathscr{M}_a(yq_y) + [\mathscr{M}_a, y\,\partial_y]q - 2y\,\partial_y(Hqq) + ay\,\partial_y(\Lambda^{-1}qq_y).$$

Now we multiply by $y\phi q_y$ and integrate to get

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|y\,\partial_y q\|^2_{L^2_\phi} \le (\mathscr{M}_a(yq_y), yq_y)_{L^2_\phi} + ([\mathscr{M}_a, y\,\partial_y]q, yq_y)_{L^2_\phi} - 2(y\,\partial_y(Hqq), yq_y)_{L^2_\phi}\\ + (ay\,\partial_y(\Lambda^{-1}qq_y), yq_y)_{L^2_\phi} + |a|C(\mathscr{E}(q) + \mathscr{E}(q)^{\frac{3}{2}}).\end{aligned}$$

Note that yq_y is an odd function and $H(yq_y)(0) = 0 = (yq_y)_y(0)$. Thus,

$$(\mathscr{M}_a(yq_y), yq_y)_{L^2_\phi} \le -\left(\tfrac{1}{2} - aC\right)\|yq_y\|^2_{L^2_\phi},$$

using Proposition 2.1. It is also easy to see that, as before,

$$|(y\,\partial_y(Hqq), yq_y)_{L^2_\phi}| + |(y\,\partial_y(\Lambda^{-1}qq_y), yq_y)_{L^2_\phi}| \le C\mathscr{E}(q)^{3/2}$$

once we observe that $yH(\partial_y q) = H(y\,\partial_y q)$ and remember that H is an isometry on odd functions in L^2_ϕ whose first derivative and Hilbert transform vanish at 0. Thus it remains to estimate the commutator term

$$([\mathscr{M}_a, y\,\partial_y]q, yq_y)_{L^2_\phi}.$$

Note that y and ∂_y both commute with the Hilbert transform (when the argument of the Hilbert transform is an odd function). Let us first recall the form of $\mathscr{M}_a$:

$$\mathscr{M}_a q = -(2HF_a+1)q - 2HqF_a - \big((1+\gamma(a))y - a\Lambda^{-1}F_a\big)q_y + a\Lambda^{-1}qF_a'.$$

In particular,

$$[\mathscr{M}_a, y\,\partial_y]q = \sum_{i=1}^{6} I_i,$$

where

$$I_1 = -2qy\,\partial_y HF_a, \quad I_2 = -2Hqy\,\partial_y F_a, \quad I_3 = a\,q_y y\,\partial_y \Lambda^{-1}F_a,$$
$$I_4 = -a\Lambda^{-1}F_a\,\partial_y q, \quad I_5 = a\Lambda^{-1}qyF_a'', \quad I_6 = aF_a'[\Lambda^{-1}, y]\,\partial_y q.$$

Thus we see readily

$$|(I_1+I_2, yq_y)_{L^2_2}| \le C|q|_{L^2_\phi}|yq_y|_{L^2_\phi}(|y\,\partial_y F_a|_{L^\infty} + |y\,\partial_y HF_a|_{L^\infty}) \le C|q|_{L^2_\phi}|yq_y|_{L^2_\phi}.$$

Moreover,

$$|(I_3+I_4+I_5, yq_y)_{L^2_\phi}| \le |a|\mathscr{E}(q).$$

Now note that

$$\begin{aligned}[\Lambda^{-1}, y]\,\partial_y q &= y\int_0^y \partial_z H(q)(z)\,dz - \int_0^y H(z\,\partial_z q)\,dz = yH(q) - \int_0^y H(z\,\partial_z q) \\ &= yH(q) - H(yq) + \int_0^y H(q) = \int_0^y H(q).\end{aligned} \tag{4-4}$$

Thus,

$$|(I_6, yq_y)_{L^2_\phi}| = |(aF_a'\Lambda^{-1}q, yq_y)_{L^2_\phi}| = |a|\left|\Big(yF_a'\frac{1}{y}\Lambda^{-1}q, yq_y\Big)_{L^2_\phi}\right| \le |a|\,|yF_a'|_{L^2_\phi}\mathscr{E}(q) \le |a|C\mathscr{E}(q).$$

Therefore, we get

$$\frac{1}{2}\frac{d}{ds}\|y\,\partial_y q\|^2_{L^2_\phi} \le -\Big(\frac{1}{2} - C|a|\Big)|yq_y|^2_{L^2_\phi} + C|q|_{L^2_\phi}|yq_y|_{L^2_\phi} + C|a|\mathscr{E}(q) + C\mathscr{E}(q)^{3/2}, \tag{4-5}$$

with C a universal constant independent of a (when a is small enough). Now we first choose a so small that we have

$$\frac{d}{ds}\|y\,\partial_y q\|^2_{L^2_\phi} \le -\tfrac{1}{2}|yq_y|^2_{L^2_\phi} + C|q|^2_{L^2_\phi} + C|a|\mathscr{E}(q) + C\mathscr{E}(q)^{3/2},$$
$$\frac{d}{ds}\|q\|^2_{L^2_\phi} \le -\tfrac{1}{2}|q|^2_{L^2_\phi} + |a|C\mathscr{E}(q) + C\mathscr{E}(q)^{\frac{3}{2}}.$$

Next, we recall that

$$\mathscr{E}(q) = \frac{1}{\delta}\|q\|^2_{L^2_\phi} + \|y\,\partial_y q\|^2_{L^2_\phi}.$$

Thus we take δ so that $1/\delta > 10C$ and use again that a is small to see that

$$\frac{d}{ds}\mathscr{E}(q) \le -\tfrac{1}{4}\mathscr{E}(q) + C\mathscr{E}(q)^{3/2}.$$

Now we prove the closure of the bootstrap.

5. Proof of Proposition 3.4

By using (4-1) and the bootstrap assumptions, we deduce that

$$\frac{d}{ds}(e^{\frac{s}{4}}\mathscr{E}(q)) \leq CA_1^3 e^{-\frac{s}{8}}. \tag{5-1}$$

Hence, by integrating the previous inequality between s_0 and s we deduce that

$$\mathscr{E}(s) \leq \mathscr{E}(s_0)e^{-\frac{(s-s_0)}{4}} + \frac{CA_1^3}{8}e^{-\frac{s}{4}}(e^{-\frac{s_0}{8}} - e^{-\frac{s}{8}}). \tag{5-2}$$

Also, we have from (3-8) that

$$\frac{\lambda_s}{\lambda}+1 = O(e^{-\frac{s}{8}}), \quad \frac{\mu_s}{\mu} = O(e^{-\frac{s}{8}}). \tag{5-3}$$

Hence, we can easily deduce that

$$\lambda(s) = Ce^{-s+O(e^{-\frac{s}{8}})}, \quad \mu(s) = O(e^{e^{-\frac{s}{8}}}). \tag{5-4}$$

Let an initial datum satisfy (3-4) at time s_0. Let $\tilde{s}$ be the supremum of times when the solution is trapped on $[s_0, \tilde{s}]$. Suppose that $\tilde{s} < +\infty$. Hence, from Definition 3.3 and a continuity argument, one of the inequalities (3-5) or (3-6) must be an equality at time $\tilde{s}$. This contradicts (5-2), and (5-4) for K large enough, which implies $\tilde{s} = +\infty$ and concludes the proof of Proposition 3.4.

6. Proof of Theorem 2

The proof of Theorem 2 is very similar to the proof of Theorem 1 so we content ourselves with only giving a sketch. We discuss the two key elements which are different: a change of variables and the coercivity for the linearized operator. All the nonlinear estimates are almost identical.

We make the following change of variables in (1-4). First, since we are only looking at odd solutions, we consider the spatial domain to be $[0, \infty)$. For some $0 < \alpha < 1$ we define

$$X = x^\alpha$$

and set

$$\omega(x,t) = \Omega(X,t) \quad \text{and} \quad u(x,t) = xU(X,t).$$

Then the evolution equation in (1-4) becomes

$$\partial_t\Omega + a\alpha UX\,\partial_X\Omega = 2U\Omega + 2\alpha X\,\partial_X U\Omega.$$

Now let us study the relation between U and Ω:

$$\begin{aligned}\partial_x(xU(X,t)) &= \partial_x u(x,t) = -H(\omega)(x,t) = -\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{\omega(y,t)}{x-y}\,dy.\\ &= -\frac{1}{\pi}PV\int_0^\infty \frac{y\omega(y,t)}{x^2-y^2} = -\frac{1}{\pi}PV\int_0^\infty\frac{y\Omega(Y,t)}{x^2-y^2}\,dy = -\frac{1}{\pi}\int_0^\infty\frac{Y^{1/\alpha}\Omega(Y,t)}{X^{2/\alpha}-Y^{2/\alpha}}\,d(Y^{1/\alpha})\\ &= -\frac{1}{\pi\alpha}\int_0^\infty\frac{Y^{2/\alpha-1}}{X^{2/\alpha}-Y^{2/\alpha}}\Omega(Y,t)\,dY := -\mathscr{H}_\alpha(\Omega),\end{aligned}$$

using the oddness of ω. In particular,

$$U+\alpha X\,\partial_X U=-\mathscr{H}_\alpha(\Omega).$$

Therefore,

$$\partial_X(X^{1/\alpha}U)=-\frac{1}{\alpha}X^{\frac{1}{\alpha}-1}\mathscr{H}_\alpha(\Omega).$$

Now define

$$\mathscr{L}_\alpha(f)=\frac{1}{\alpha}X^{-\frac{1}{\alpha}}\int_0^X Y^{\frac{1}{\alpha}-1}f(Y)\,dY.$$

Thus, (1-4) becomes

$$\begin{aligned}\partial_t\Omega+a\alpha UX\,\partial_X\Omega&=-2\Omega\mathscr{H}_\alpha(\Omega),\\ U&=-\mathscr{L}_\alpha\mathscr{H}_\alpha(\Omega),\\ \mathscr{H}_\alpha(\Omega)&=\frac{1}{\pi\alpha}\int_0^\infty\frac{Y^{2/\alpha}}{X^{2/\alpha}-Y^{2/\alpha}}\frac{\Omega(Y,t)}{Y}\,dY.\end{aligned}\tag{6-1}$$

Now, as shown in [Elgindi and Jeong 2020a], when $a=0$ for each α, we have the explicit self-similar profiles

$$\Omega(X,t)=\frac{1}{1-t}F_0^{(\alpha)}\left(\frac{X}{1-t}\right),$$

where

$$F_0^{(\alpha)}(Y)=-\frac{\sin\left(\frac{\pi}{2}\alpha\right)Y}{1+2\cos\left(\frac{\pi}{2}\alpha\right)Y+Y^2},\qquad \mathscr{H}_\alpha(F_0^{(\alpha)})(Y)=\frac{1+\cos\left(\frac{\pi}{2}\alpha\right)Y}{1+2\cos\left(\frac{\pi}{2}\alpha\right)Y+Y^2}.$$

In particular,

$$\left|\sin\left(\tfrac{\pi}{2}\alpha\right)\widetilde{F}_0-F_0^{(\alpha)}\right|_{H^3_Y}\le C\alpha^2,$$

where

$$F_0(Y)=\frac{Y}{(1+Y)^2}$$

as in the expression below (1-7). For the analysis we also need that

$$|\mathscr{H}_\alpha(F_0^{(\alpha)})-\tilde{H}F_0|_{H^3_Y}\le C\alpha,\tag{6-2}$$

where

$$\tilde{H}F_0(Y)=-\frac{1}{1+Y}.$$

We also need that

$$\alpha\|\mathscr{H}_\alpha\|_{H^1\to H^1}+\|\mathscr{L}_\alpha\|_{H^1\to H^1}\le C,$$

where C is independent of α.

Linearized operator. Following the proof of Theorem 1, we mainly need to establish coercivity properties of the main linearized operator. We thus content ourselves with establishing the analogue of Proposition 2.1. We note that linearizing around F_0^α leads to

$$\mathscr{M}_a^\alpha q=\mathscr{M}_0 q+P_a^\alpha q,$$

where

$$\mathscr{M}_0^\alpha q=q+Y\,\partial_Y q+2\mathscr{H}_\alpha(F_0^{(\alpha)})q+2F_0^{(\alpha)}\mathscr{H}_\alpha(q)$$

and $P_a^\alpha(q)$ satisfies

$$|(P_a^\alpha(q), q)_{\mathscr{E}}| \le C(\alpha|a| + \alpha)\mathscr{E}(q),$$

exactly as in the proof of Proposition 2.1.

Now let us introduce the weight

$$\phi_*(Y) := \frac{(Y+1)^4}{Y^4},$$

which was used in [Elgindi 2019]. Then, recalling (6-2), we have

$$(q + Y\,\partial_Y q + 2\mathscr{H}_\alpha(F_0^{(\alpha)})q, q\phi_*) \ge \bigl(\tfrac{1}{2} - C\alpha\bigr)|q\sqrt{\phi_*}|_{L^2}^2.$$

It remains to study $(\mathscr{H}_\alpha(q), qF_0^{(\alpha)}\phi_*)_{L^2}$.

Claim. $$(\mathscr{H}_\alpha(q), qF_0^{(\alpha)}\phi_*)_{L^2} \ge C\alpha|q\sqrt{\phi_*}|_{L^2}^2.$$

Once the claim is established, the L^2 coercivity once α is small follows and the rest of the proof of Theorem 2 is similar to that of Theorem 1. We have

$$\begin{aligned}(\mathscr{H}_\alpha(q), qF_0^{(\alpha)}\phi_*)_{L^2} &= \int_0^\infty\int_0^\infty \frac{Y^{2/\alpha}}{X^{2/\alpha} - Y^{2/\alpha}}\frac{(X+1)^2}{X^2}\frac{q(X)q(Y)}{XY}\,dX\,dY\\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{Y^{2/\alpha+2}(X+1)^2 - X^{2/\alpha+2}(Y+1)^2}{X^{2/\alpha} - Y^{2/\alpha}}\frac{q(X)q(Y)}{X^3Y^3}\,dX\,dY.\end{aligned}$$

All we have done in the second equality is symmetrize the kernel. Now let us study the symmetrized kernel

$$K_\alpha(X,Y) := \frac{Y^{2/\alpha+2}(X+1)^2 - X^{2/\alpha+2}(Y+1)^2}{X^{2/\alpha} - Y^{2/\alpha}}.$$

Observe that

$$K_\alpha(X,Y) = \frac{(Y/X)^{1/\alpha}Y^2(X+1)^2 - (X/Y)^{1/\alpha}X^2(Y+1)^2}{(X/Y)^{1/\alpha} - (Y/X)^{1/\alpha}}.$$

Now, it is easy to see that $\lim_{\alpha\to 0^+} K(\alpha, X, Y) = -Y^2(X+1)^2\mathbf{1}_{X<Y} - X^2(Y+1)^2\mathbf{1}_{X\ge Y}$ on $\mathbb{R}^2\setminus\{X{=}Y\}$. Let us try to get some more quantitative information. By symmetry, we may restrict ourselves to the region where $X < Y$. Observe that

$$\begin{aligned}K_\alpha(X,Y) + Y^2(X+1)^2 &= \Bigl(\frac{X}{Y}\Bigr)^{1/\alpha}\frac{Y^2(X+1)^2 - X^2(Y+1)^2}{(X/Y)^{1/\alpha} - (Y/X)^{1/\alpha}} = \Bigl(\frac{X}{Y}\Bigr)^{1/\alpha}\frac{(2XY + X + Y)(Y - X)}{(X/Y)^{1/\alpha} - (Y/X)^{1/\alpha}}\\ &= \frac{(2XY + X + Y)(Y-X)}{1 - (Y/X)^{2/\alpha}} = -X^2(2Y + 1 + \sigma)\frac{\sigma - 1}{\sigma^{2/\alpha} - 1},\end{aligned}$$

with $\sigma = Y/X$. Defining $f(\sigma) = \sigma^{2/\alpha} - 1$, let us note that $f, f', f'' \ge 0$. Therefore,

$$f(\sigma) \ge (\sigma - 1)f'(1) = (\sigma - 1)\Bigl(\frac{2}{\alpha} - 1\Bigr)$$

if $\sigma \ge 1$. Thus,

$$\frac{\sigma - 1}{\sigma^{2/\alpha} - 1} \le \frac{\alpha}{2 - \alpha}.$$

Consequently,

$$|K_\alpha(X,Y)+Y^2(X+1)^2| \le 2YX^2\alpha + 2X^2\alpha + 2XY\alpha$$

if $\alpha \le 1$. Now let us note that

$$\begin{aligned}\int_0^\infty \frac{|q|(X)}{X}\int_X^\infty \frac{|q|(Y)}{Y^3}\,dY\,dX &\le |q\sqrt{\phi_*}|_{L^2}\int_0^\infty \frac{|q|(X)}{|X|^2}\,dX \\ &= \int_0^\infty \frac{|q|(X)(X+1)^2}{X^2}\frac{1}{(X+1)^2}\,dX \le 10|q\sqrt{\phi_*}|^2_{L^2}.\end{aligned}$$

The claim now follows once we show that

$$\int_0^\infty\int_X^\infty Y^2(X+1)^2\frac{q(X)q(Y)}{Y^3X^3}\,dX\,dY \le 0,$$

whenever $q(0)=q'(0)=\int_0^\infty q(Y)/Y\,dY=0$. Indeed,

$$\begin{aligned}\int_0^\infty\int_X^\infty Y^2(X+1)^2\frac{q(X)q(Y)}{Y^3X^3}dXdY &= \int_0^\infty \frac{q(X)}{X}\frac{(X+1)^2}{X^2}\int_X^\infty \frac{q(Y)}{Y}\,dY \\ &= -\frac{1}{2}\int_0^\infty \frac{d}{dX}\biggl(\int_X^\infty \frac{q(Y)}{Y}dY\biggr)^2\frac{(X+1)^2}{X^2}\,dX \\ &= -\frac{1}{2}\int_0^\infty \frac{d}{dX}\biggl(\int_X^\infty \frac{q(Y)}{Y}\,dY\biggr)^2\biggl(1+\frac{2}{X}+\frac{1}{X^2}\biggr)\,dX \\ &= -\frac{1}{2}\int_0^\infty\biggl(\int_X^\infty \frac{q(Y)}{Y}\,dY\biggr)^2\biggl(\frac{2}{X^2}+\frac{2}{X^3}\biggr) \le 0.\end{aligned}$$

Appendix

Weighted identities.

Lemma A.1. *Let $\phi(y)=(1+y^2)^2/y^4$. For all $f\in C_c^\infty(\mathbb{R})$ odd on $\mathbb{R}$ and satisfying $f'(0)=Hf(0)=0$, we have*

$$\int |Hf(y)|^2\phi(y)\,dy = \int |f(y)|^2\phi(y)\,dy.$$

Proof. Note that $w(y)=1+2/y^2+1/y^4$. Thus, it suffices to show that

$$\int \frac{|Hf(y)|^2}{y^k}\,dy = \int \frac{|f(y)|^2}{y^k}\,dy$$

under the condition that f is odd and $f'(0)=Hf(0)=0$ for $k=0,2,4$. Note that $k=0$ is just the isometry property of H. The cases $k=2$ and $k=4$ are similar so we only do the more difficult case of $k=4$. Let us write $f=y^2g$. Observe that, by assumption, we have $0=Hf(0)=\int yg$ and $\int g=0$ since g is odd. Thus,

$$H(y^2g)=y^2H(g).$$

In particular, we have

$$\int \frac{|Hf|^2}{y^4}\,dy = \int |H(g)|^2 = \int |g|^2 = \int \frac{|f|^2}{y^4}. \qquad \square$$

Smoothing procedure. We now give a lemma which allows us to justify some of the computations in the coercivity and energy estimates.

Lemma A.2. *Let q be such that q is odd, $H(q) = q'(0) = 0$ and $\mathscr{E}(q) < \infty$. Then, there exists a sequence $q_n \in \mathscr{S}(\mathbb{R})$ with*

- *q_n is odd and $q_n'(0) = Hq_n(0) = 0$,*
- *$\mathscr{E}(q_n) \leq 2\mathscr{E}(q)$,*
- *$q_n \to q$ uniformly on $\mathbb{R}$.*

Proof. Take

$$q_n^1(y) = \frac{ny^2}{ny^2+1}\phi_n * q, \quad \text{where } \phi_n = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2).$$

Clearly, q_n^1 is odd and $q_n^{1\prime}(0) = 0$. Moreover, $q_n^1 \in \mathscr{S}(\mathbb{R})$ and $q_n^1 \to q$ uniformly on $\mathbb{R}$. It may be, however, that $H(q_n^1)(0) \neq 0$. Now let's define the function

$$\psi(y) = y^3\exp(-y^2).$$

Clearly, $\psi \in \mathscr{S}$ and $\mathscr{E}(\psi) < \infty$. Moreover,

$$H(\psi)(0) \neq 0.$$

Thus we define

$$q_n(y) = q_n^1(y) - \frac{H(q_n^1)(0)}{H(\psi)(0)}\psi(y).$$

Clearly, q_n is odd and $q_n'(0) = Hq_n(0) = 0$. Now let's compute $\mathscr{E}(q_n)$. First, $H(q_n^1)(0) \to 0$ as $n \to \infty$ by the dominated convergence theorem and $\mathscr{E}(\psi) < \infty$. Thus, the second term in the definition of q_n converges uniformly to 0 in $\mathbb{R}$ in the energy norm. It is also easy to see that if n is large enough, $\mathscr{E}(q_n^1) \leq 2\mathscr{E}(q)$. □

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