



Ground state solutions of the complex Gross Pitaevskii equation associated to exciton-polariton Bose-Einstein condensates



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ABSTRACT

We investigate the existence of ground state solutions of a Gross-Pitaevskii equation modeling the dynamics of pumped Bose Einstein condensates (BEC). The main interest in such BEC comes from its important nature as macroscopic quantum system, constituting an excellent alternative to the classical condensates which are hard to realize because of the very low temperature required. Nevertheless, the Gross Pitaevskii equation governing the new condensates presents some mathematical challenges due to the presence of the pumping and damping terms. Following a self-contained approach, we prove the existence of ground state solutions of this equation under suitable assumptions: This is equivalent to say that condensation occurs in these situations. We also solve the Cauchy problem of the nonlinear Schrödinger equation and prove some corresponding laws.

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R É S U M É

Nous étudions l'existence de solutions d'état fondamental d'une équation du type Gross-Pitaevskii modélisant la dynamique des condensateurs de Bose Einstein pompés (BEC). Le principal intérêt de l'étude d'un tel BEC vient de sa particularité importante en tant que système quantique macroscopique constituant une excellente alternative aux condensateurs classiques difficiles à réaliser, en raison de la très basse température. Néanmoins, l'équation de Gross-Pitaevskii régissant cette nouvelle génération de condensateurs présente des défis mathématiques dus notamment à la présence des termes de pompage et d'amortissement. Suivant une approche autonome, nous prouvons l'existence de solutions à l'état fondamental de cette équation sous des hypothèses appropriées : Cela revient à dire que de la condensation se produit dans ces situations. Nous résolvons également le problème de Cauchy

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de l'équation non linéaire de Schrödinger et nous prouvons quelques lois de conservation.

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1. Introduction

The first realization of condensation has been obtained experimentally in a system consisting of about half million alkali atoms cooled down to nano-Kelvin temperature. Thus, a considerable obstacle in the study of (BEC) is the very low temperature required to create the condensate. Completely aware that it is extremely important to explore what kind of condensates can undergo condensation at higher temperatures, huge efforts have been undertaken by scientists to overcome this difficulty right after the first experimental realization of the first (BEC) in 1995. During the last years, a new kind of condensates has attracted the attention of many scientists. Very recently, it turned out that an excellent candidate is a system of exciton-polaritons, which are bosonic quasiparticles that exist inside semiconductor micro-cavities, consisting of a superposition of an exciton and a cavity photon. Above a threshold density, the polaritons macroscopically occupy the same quantum state, forming a condensate. The temperatures that are usually used to form exciton-polariton BECs are around $T=10\text{K}$, far higher than the nano-Kelvin temperatures required for atomic BECs. They are immensely promising in terms of new quantum technologies since quantum effects can appear on a macroscopic level, unlike most systems where quantum effects are rather easily destroyed by temperature and decoherence. As Boson particles are composed of quantum well excitons and optical cavity photons, microcavity exciton-polaritons possess unique intrinsic features: reminiscent excitonic nature leads to important interaction dynamics among exciton-polaritons. Polariton-polariton repulsive interactions are indeed crucial to stimulate scattering processes in order to relax into the ground state Bose-Einstein condensates (BECs). Since the temperature of condensation is inversely proportional to the mass of the particles, the exciton-polariton systems afford relatively high temperatures of condensation. The first drawback of these new condensates is their very short lifetime (approximately 1 ps), inherited also from their photonic component, so that polariton thermalization could be problematic. In fact the polariton gas can become fully thermalized, as a result of strong polariton-polariton interaction caused by their excitonic component. The second important inconvenient comes from the fact that the excitons disappear with the recombination of the electron-hole pairs through emission of photons. One way to overcome these problems is to introduce a polariton reservoir: polaritons are “cooled” and “pumped” from this reservoir into the condensate. At the same time, a low density level is kept in order to reduce the interactions between polaritons. Different mathematical models have been suggested for this new condensate. In this paper we consider the one proposed in [10], called complex Gross-Pitaevski equation. For a more detailed account of these aspects, see [13] and references therein.

In [13], the authors addressed the nature of radially symmetric standing wave-type solutions of the following nonlinear Gross-Pitaevskii equation:

$$i \frac{\partial \psi}{\partial t} = (-\Delta + V(x) + |\psi|^2)\psi + i(\sigma(x) - \alpha|\psi|^2)\psi, \quad (\text{GPPD})$$

where $\psi = \psi(x, t)$ is a complex-valued function defined on $\mathbb{R}^2 \times \mathbb{R}$, Δ is the Laplace operator on \mathbb{R}^2 , $V(x) = |x|^2$ is the harmonic potential, $\sigma(x) \geq 0$ and $\alpha \geq 0$.

To achieve their goals, they have developed a numerical collocation method but they did not provide any theoretical justification of their claims. The main objective of this paper is to rigorously prove the existence of ground state solutions of the Gross-Pitaevskii equation under study. We believe that this is a challenging and immensely important scientific question. The principle challenge comes from the fact that all classical methods do not seem to be applicable to discuss the existence of stationary solutions to (GPPD). This is

essentially due to the simultaneous presence of the dissipation and pumping terms in the equation. Let us note that the establishment of ground state solutions avoids costly and very difficult experiments in the “classical” BEC. To achieve this goal, let us first introduce some important quantities associated to (GPPD).

Recall that the mass \mathcal{M} , the Hamiltonian \mathcal{H} , the action \mathcal{S}_μ ($\mu > 0$) and the functional \mathcal{K} associated to the equation (GPPD) are given by:

$$\mathcal{M}(u) := \|u\|_{L^2}^2, \quad (1.1)$$

$$\mathcal{H}(u) := \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|xu\|_{L^2}^2) + \frac{1}{4}\|u\|_{L^4}^4 := \mathcal{H}_0(u) + \frac{1}{4}\|u\|_{L^4}^4 \quad (1.2)$$

$$\mathcal{S}_\mu(u) := -\frac{\mu}{2}\mathcal{M}(u) + \mathcal{H}(u), \quad (1.3)$$

$$\mathcal{K}(u) := \int_{\mathbb{R}^2} (\sigma(x) - \alpha|u(x)|^2)|u(x)|^2 dx, \quad (1.4)$$

respectively. Observe that

$$\frac{d}{dt}\mathcal{M}(\psi(t)) = 2\mathcal{K}(\psi(t)). \quad (1.5)$$

Identity (1.5) shows that, at least formally, the mass and the energy are pumped into the system through the term $i\sigma\psi$ involving the parameter σ and they are nonlinearly damped by the term $-i\alpha|\psi|^2\psi$ involving the parameter α . Contrarily to the complex Ginzburg-Landau equation (when a dissipative term of the form $i\Delta\psi$ is added to the RHS of (GPDP)), one cannot obtain time-uniform estimates of the solution in the energy space. The complex Gross-Pitaevski equation reflects the non-equilibrium dynamics described above by adding pumping and decaying terms to the GP equation.

Before going any further, we recall a few results about the linear equation without dissipation and pumping. The equation then reads

$$i\frac{\partial\phi}{\partial t} = (-\Delta + V(x))\phi.$$

We define the energy space $\Sigma := H^1(\mathbb{R}^2) \cap \{u : xu \in L^2\}$, endowed with the L^2 -scalar product $(u, v)_2 := \int_{\mathbb{R}^2} u(x)\bar{v}(x) dx$, by

$$(u, v)_\Sigma = (\nabla u, \nabla v)_2 + (xu, xv)_2 + (u, v)_2 : \|u\|_\Sigma^2 = \|\nabla u\|_2^2 + \left\| (1 + |\cdot|^2)^{\frac{1}{2}} u \right\|_2^2.$$

Also, define the dual space Σ^* of Σ as follows. For any $v \in \Sigma^*$, there exists a unique $u \in \Sigma$ such that $H_0 u = v$ with the norm on Σ^* given by

$$\|H_0 u\|_{\Sigma^*} = \|v\|_{\Sigma^*} := \|u\|_\Sigma.$$

Recall that $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^2)$. It is well known that the unbounded operator $H_0 := -\Delta + V$ defined on

$$D(H_0) := \{u \in \Sigma : H_0(u) \in L^2(\mathbb{R}^2)\}$$

is self-adjoint. Moreover, the lowest eigenvalue of H_0 denoted by $\omega_1 = 2$ is simple with eigenfunction $\varphi_1(x) = \frac{1}{\sqrt{\pi}}e^{-|x|^2/2}$. Notice that (φ_1, ω_1) can be constructed variationally as

$$\omega_1 = \min_{\|u\|_{L^2}=1} \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + |x|^2 |u|^2 dx := \min_{\|u\|_{L^2}=1} \mathcal{H}_0.$$

In particular, for any $u \in D(H_0)$, we have

$$2\|u\|_{L^2}^2 \leq \|xu\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.$$

For more details, we refer for example to [9].

When the chemical potential is complex $\mu = \mu_r + i\mu_i$, solitary wave solution $\psi(x, t) = Q(x)e^{-it\mu} = Q(x)e^{t\mu_i}e^{-it\mu_r}$ would grow exponentially fast as $|t| \rightarrow \infty$ which can be bad for the analysis as well as for numerics and experiments. Assuming that $\mu = \mu_r$, yields the following stationary problem for Q :

$$\mu Q = (-\Delta + V(x) + |Q|^2)Q + i(\sigma(x) - \alpha|Q|^2)Q, \quad Q \in \Sigma \setminus \{0\}. \quad (\mu\text{-SP})$$

Multiplying $(\mu\text{-SP})$ by \bar{Q} and integrating gives the following identity.

$$\mu \mathcal{M}(Q) = 2\mathcal{H}(Q) + 1/2\|Q\|_{L^4}^4 + i\mathcal{K}(Q).$$

The condition for the chemical potential μ of being real is then equivalent to the fact that Q is a zero of \mathcal{K} .

It is important to emphasize that due to the presence of the dissipation and pumping mechanisms, we find it hard to apply the standard variational or PDE methods to construct soliton-type solutions of (GPPD) (i.e. a solution Q of $(\mu\text{-SP})$). In this paper, our idea to construct a solution of $(\mu\text{-SP})$ with real chemical potential μ goes along a perturbative way by introducing a small parameter factor in the dissipation and pumping term. More precisely, for all $\varepsilon > 0$, consider

$$i \frac{\partial \psi}{\partial t} = (-\Delta + V(x) + |\psi|^2)\psi + i\varepsilon(\sigma(x) - \alpha|\psi|^2)\psi, \quad (\text{GPPD}_\varepsilon)$$

and its corresponding stationary equation

$$\mu Q = (-\Delta + V(x) + |Q|^2)Q + i\varepsilon(\sigma(x) - \alpha|Q|^2)Q \quad Q \in \Sigma \setminus \{0\}. \quad (\mu\text{-SP}_\varepsilon)$$

The object is to construct a solution $(Q_\varepsilon, \mu_\varepsilon)$ in the form

$$Q_\varepsilon = Q_\varepsilon^a + \psi_\varepsilon, \quad \text{and} \quad \mu_\varepsilon = \mu_\varepsilon^a + \mu_\varepsilon,$$

where the approximate solution $(Q_\varepsilon^a, \mu_\varepsilon^a)$ will be given explicitly, and $(\psi_\varepsilon, \mu_\varepsilon)$ is the error term that needs to be found. To define $(Q_\varepsilon^a, \mu_\varepsilon^a)$, we need to introduce some notation and state a few preliminary useful results. The first Theorem of this paper reads as follows:

Theorem 1.1. *Let $\sigma(x) \geq 0$ be a continuous and bounded nontrivial function. There exist $\alpha_0 \gg 1$ and a positive ε_0 small such that, for any $0 < \varepsilon < \varepsilon_0$ and $\alpha > \alpha_0$, the complex Gross-Pitaevskii equation $(\text{GPPD}_\varepsilon)$ has a solitary wave solution $\psi^\varepsilon(x, t) = e^{it\mu_\varepsilon} Q^\varepsilon(x)$ with $(Q^\varepsilon, \mu_\varepsilon) \in \Sigma \times (2, \infty)$, with $Q^\varepsilon \neq 0$, solving $(\mu\text{-SP}_\varepsilon)$.*

Remark 1.1. It would be very desirable to extend the branch of standing wave solutions we constructed for ε small to all values of ε . Unfortunately, so far we were not able to do so given the non-equilibrium structure of the model.

Our second result concerns the Cauchy problem associated to (GPPD) . We have.

Theorem 1.2. Assume $\alpha \geq 0$, and $\sigma \in L^\infty(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$. For any $\psi_0 \in L^2(\mathbb{R}^2)$, there exists a unique global solution $\psi \in C([0, \infty), L^2(\mathbb{R}^2)) \cap L^4_{loc}([0, \infty), L^4(\mathbb{R}^2))$ of (GPPD) with $\psi(x, 0) = \psi_0(x)$. Moreover, for any $T > 0$, we have

$$\int_0^T \int_{\mathbb{R}^2} |\psi(x, t)|^4 dx dt \lesssim e^{\|\sigma\|_{L^\infty} T} \|\psi_0\|_{L^2}^4.$$

The paper is organized as follows: In the next section, some preliminary results are proven. This will prepare the field to the establishment of ground state solutions. In section 3, we will present our self-contained proof built up to prove the existence of ground state solutions. The last section of this paper is dedicated to the Cauchy problem. We show the existence and uniqueness of solutions for a large class of damping and pumping terms. We also discuss the non-conservation of some important functionals associated to the Schrödinger equation.

2. Preliminaries

Here we focus on the problem without pumping and decay of the energy, that is when $\varepsilon = 0$. We start by recalling a few known facts about the space Σ , for which the proof can for example be found in Kavian-Weissler [9].

Lemma 2.1. The Hilbert space Σ is compactly embedded in $L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$.

Throughout this paper, we suppose that $\sigma \geq 0$ is nontrivial continuous and is in $L^\infty(\mathbb{R}^2)$ function.

Lemma 2.2. For any $M > 0$, there exists a unique $v_M \in \Sigma$ solving the following constrained variational problem:

$$(V_M) : \quad \mu_M = \inf \left\{ \mathcal{H}(u) : \int u^2 = M \right\};$$

In addition, the minimizer v_M is non-negative, radial and radially decreasing satisfying

$$|\partial^\beta v_M(x)| \lesssim \exp(-|x|^2), \quad x \in \mathbb{R}^2,$$

for all multi-index β with $|\beta| \leq 2$.

Proof. It is sufficient to show the existence of a minimizer of (V_M) . The uniqueness of the minimizer follows directly from the strict convexity of the functional \mathcal{H} , and the decay was proven in Proposition 6.9 in [9].

Now let us fix $M > 0$, let (v_n) be a minimizing sequence of (V_M) , i.e., $\lim_{n \rightarrow \infty} \mathcal{H}(v_n) = \mu_M$ and $\int v_n^2 = M$. Then

$$\mathcal{H}(v_n) \geq \frac{1}{2} \|\nabla v_n\|_2^2 + \frac{1}{2} \|x v_n\|_2^2.$$

Therefore, we can find $K_M > 0$ such that

$$\|\nabla v_n\|_2^2 + \|x v_n\|_2^2 \leq K_M.$$

This implies that

$$\|v_n\|_\Sigma^2 \leq M + K_M. \quad (2.1)$$

Consequently, there exists $u \in \Sigma$ such that

$$v_n \rightharpoonup u \quad \text{in } \Sigma.$$

This implies, thanks to Lemma 2.1, that $v_n \rightarrow u$ in $L^2(\mathbb{R}^2)$ and $L^4(\mathbb{R}^2)$. Thus, we certainly have that $\int u^2 = M$ implying that u is non-trivial, and by the lower semi-continuity, we can write:

$$\mathcal{H}(u) \leq \liminf_n \mathcal{H}(v_n) = \mu_M.$$

Therefore, $\mathcal{H}(u) = \mu_M$. On the other hand, let u be the unique minimizer of (V_M) , then u is a non-negative function in Σ since

$$\mathcal{H}(|u|) \leq \mathcal{H}(u), \quad \text{and} \quad M(|u|) = M(u).$$

Furthermore, by rearrangement inequalities [6,7], we have:

$$\begin{aligned} \int |u|^2 &= \int (|u|^*)^2 \\ \int |u|^4 &= \int (|u|^*)^4 \\ \int |x|^2 |u|^2 &\geq \int |x|^2 (|u|^*)^2 \\ \int |\nabla |u||^2 &\geq \int |\nabla (|u|^*)|^2. \end{aligned}$$

Combining these identities, it follows that

$$\mathcal{H}(|u|^*) \leq \mathcal{H}(|u|). \quad \square$$

The next Lemma, addresses the regularity of the Hamiltonian \mathcal{H} , as well as the map $M \rightarrow \mu_M$.

Lemma 2.3. *The Hamiltonian \mathcal{H} is in $C^1(\Sigma, \mathbb{R})$. Moreover, for all $u \in \Sigma$ we have*

$$\|\mathcal{H}'(u)\|_{\Sigma^{-1}} \leq C\{\|u\|_{\Sigma} + \|u\|_{\Sigma}^3\} \quad \text{for all } u \in \Sigma, \quad (\text{i})$$

and the function

$$M \rightarrow \mu_M = \mathcal{H}(v_M), \quad \text{is continuous on } (0, \infty). \quad (\text{ii})$$

Proof. The proof of (i) follows from standard arguments. For example, we refer to reference [8], and we just prove (ii).

Fix $M > 0$. Let $M_n \subset (0, \infty)$ be a sequence of positive real numbers such that $M_n \rightarrow M$. We will first prove that

$$\limsup_n \mu_{M_n} \leq \mu_M. \quad (2.2)$$

Let (v_n) be a sequence such that $\int v_n^2 = M$ and $\mathcal{H}(v_n) \rightarrow \mu_M$. By (2.1), we can find $L > 0$ such that

$$\|v_n\|_{\Sigma}^2 \leq L.$$

Now let $w_n = \frac{M_n}{M}v_n$, then $\int w_n^2 = M_n$ and

$$\|v_n - w_n\|_\Sigma = |1 - \frac{M_n}{M}| \|v_n\|_\Sigma \leq |1 - \frac{M_n}{M}| L$$

for any $n \in \mathbb{N}$.

Therefore, we can find n_0 such that

$$\|v_n - w_n\|_\Sigma \leq L + 1$$

for any $n \geq n_0$.

It follows from (i) that there exists a constant $K(L)$ such that $\|\mathcal{H}'(u)\|_{\Sigma^{-1}} \leq K(L)$ for all $u \in \Sigma$ such that $\|u\|_\Sigma \leq 2L + 1$.

Thus, for all $n \geq n_0$,

$$\begin{aligned} |\mathcal{H}(w_n) - \mathcal{H}(v_n)| &= \left| \int_0^1 \frac{d}{dt} \mathcal{H}(tw_n + (1-t)v_n) dt \right| \\ &\leq \sup_{\|u\|_\Sigma \leq 2L+1} \|\mathcal{H}'(u)\|_{\Sigma^{-1}} \|v_n - w_n\|_\Sigma \\ &\leq K(L)L |1 - \frac{M_n}{M}|. \end{aligned}$$

Consequently, $\mu_{M_n} \leq \mathcal{H}(w_n) \leq \mathcal{H}(v_n) + K(L)L |1 - \frac{M_n}{M}|$.

Then $\limsup \mu_{M_n} \leq \lim \mathcal{H}(v_n) = \mu_M$ and then

$$\limsup \mu_{M_n} \leq \mu_M. \quad (2.3)$$

Now let us prove that if $M_n \rightarrow M$, then

$$\mu_M \leq \liminf \mu_{M_n}. \quad (2.4)$$

For all $n \in \mathbb{N}$, there exists (v_n) a sequence of functions in Σ such that $\int v_n^2 = M_n$ and

$$\mu_{M_n} \leq \mathcal{H}(v_n) \leq \mu_{M_n} + \frac{1}{n}.$$

Combining the proof of (2.1) and (2.4), we can find $K > 0$ such that $\|v_n\|_\Sigma \leq K$ for all $n \in \mathbb{N}$. Setting $w_n = \frac{M}{M_n}v_n$, we have that $\int w_n^2 = M$ and

$$\|v_n - w_n\|_\Sigma \leq K |1 - \frac{M}{M_n}|.$$

Thus, following the proof of (2.4), we certainly get:

$$|\mathcal{H}(w_n) - \mathcal{H}(v_n)| \leq L(K)K |1 - \frac{M}{M_n}|.$$

Consequently, we have:

$$\mu_{M_n} \geq \mathcal{H}(v_n) - \frac{1}{n} \geq \mathcal{H}(w_n) - L(K)K |1 - \frac{M}{M_n}| - \frac{1}{n},$$

yielding $\liminf \mu_{M_n} \geq \mu_M$ as desired. \square

Proposition 2.4. *Let $M > 0$, and $(M_n) \subset (0, \infty)$ be a sequence of positive real numbers such that $M_n \rightarrow M$. Denote by v_{M_n} the unique minimizer of (V_{M_n}) , and v_M the unique minimizer of (V_M) . Then*

$$\mathcal{K}(v_{M_n}) \rightarrow \mathcal{K}(v_M),$$

and

$$\mathcal{H}(v_{M_n}) \rightarrow \mathcal{H}(v_M).$$

Proof. We will first prove that there exists $\bar{u} \in \Sigma$ such that v_{M_n} converges weakly in Σ to \bar{u} ($v_{M_n} \rightharpoonup \bar{u}$ in Σ). First obviously $\|v_{M_n}\|_2^2 \leq A$. Now noticing that

$$\mu_{M_n} = \frac{1}{2} \|\nabla v_{M_n}\|_2^2 + \frac{1}{2} \|x v_{M_n}\|_2^2 + \frac{1}{4} \|v_{M_n}\|_4^2,$$

one has

$$\mu_{M_n} \geq \frac{1}{2} \|\nabla v_{M_n}\|_2^2 + \frac{1}{2} \|x v_{M_n}\|_2^2.$$

Therefore, using (2.4), there exists a constant $B > 0$ such that

$$\|v_{M_n}\|_{\Sigma} \leq B.$$

Thus, (up to a subsequence), there exists $\bar{u} \in \Sigma$ such that

$$v_{M_n} \rightharpoonup \bar{u} \quad \text{in } \Sigma.$$

Now using Lemma 2.1, we have that

$$v_{M_n} \rightarrow \bar{u} \quad \text{in } L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2).$$

In particular, $\int \bar{u}^2 = M$. Thus,

$$\mu_M \leq \mathcal{H}(\bar{u}) \leq \liminf \mathcal{H}(v_{M_n}) = \liminf \mu_{M_n}$$

and then $\mathcal{H}(\bar{u}) = \mu_M$. This shows that \bar{u} is the unique minimizer of (V_M) . To end the proof, we need to show that

$$\int \sigma(x) v_{M_n}^2(x) \rightarrow \int \sigma(x) v_M^2(x) \tag{2.5}$$

and

$$\int v_{M_n}^4(x) \rightarrow \int v_M^4(x). \tag{2.6}$$

To prove (2.5), it is sufficient to notice that $\sigma \in L^\infty(\mathbb{R}^2)$ and $v_n \rightarrow v$ in $L^2(\mathbb{R}^2)$, while (2.6) follows from the fact that $v_n \rightarrow v$ in $L^4(\mathbb{R}^2)$. \square

3. Ground state solutions

Always in the case $\varepsilon = 0$, and within the class of minimizers v_M we have just constructed, we would like to intersect it with the co-dimension one manifold characterized by the zeros of the functional \mathcal{K} . Before doing so, let us first fix our assumptions on the decay and pumping parameters.

First we deal with case $\varepsilon = 0$ i.e. the standard nonlinear Schrödinger equation in the absence of both the pumping and dissipation. Equation $(\mu\text{-SP}_\varepsilon)$ then becomes

$$\mu Q = (-\Delta + V(x) + |Q|^2)Q, \quad Q \in \Sigma \setminus \{0\}. \quad (\mu\text{-SP}_0)$$

The first preliminary result is the first iteration. We have the following result:

Proposition 3.1. *There exists a non-negative radial function $Q_0 \in \Sigma$ and $\mu_0 > 2$ solving $(\mu\text{-SP}_0)$. Moreover, Q_0 satisfies*

$$\mathcal{K}(Q_0) = 0.$$

Remark 3.1. (Q_0, μ_0) will be the first approximate solution in the iteration process to construct the full solution $(Q_\varepsilon, \mu_\varepsilon)$ of $(\mu\text{-SP}_\varepsilon)$.

Proof of Proposition 3.1. It is sufficient to prove that the functional \mathcal{K} changes sign when the mass of the ground state v_M given by Lemma 2.2 varies. Then the conclusion will follow using Lemma 2.3. We divide the proof into several steps. STEP 1: POSITIVITY OF $\mathcal{K}(Q_M)$ FOR SMALL MASSES: Now, because of the positivity of $Q = Q_M$, first observe that for any nontrivial non-negative continuous function σ , we have $\int_{\mathbb{R}^2} \sigma |Q|^2 dx > 0$. Moreover, on the one hand, by the Gagliardo-Nirenberg inequality, there is a constant $C_* > 0$ such that for any $u \in H^1$, we have

$$\|u\|_{L^4}^4 \leq C_* \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2.$$

On the other hand, multiplying $(\mu\text{-SP}_0)$ by \bar{Q} and integrating shows that any solution Q of $(\mu\text{-SP}_0)$ satisfies

$$\mu \|Q\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2 + \|xQ\|_{L^2}^2 + \|Q\|_{L^4}^4.$$

Thus, if $\|Q\|_{L^2}^2 = M$ we have

$$\|Q\|_{L^4}^4 \lesssim M^2 \mu_M.$$

This shows that when $M \leq 1$, we have $\mu_M \lesssim 1$ and thus $\mathcal{K}(Q) \geq \int_{\mathbb{R}^2} \sigma |Q|^2 dx - CM^2$, for some positive constant C . Now since $\sigma \geq 0$ is a nontrivial continuous function, there exists a nontrivial open set $\mathcal{O} \subset \mathbb{R}^2$ and a positive constant $c_0 > 0$ such that $\sigma(x) \geq c_0$, for all $x \in \mathcal{O}$. We have $\int_{\mathbb{R}^2} \sigma |Q|^2 dx \geq c_0 \int_{\mathcal{O}} |Q|^2 dx \geq c_1 M$, for some small positive constant c_1 . This implies that $\mathcal{K}(Q_M) \geq 0$ as $M \rightarrow 0$. (Remember we simply denoted Q_M by Q .)

STEP 2: NEGATIVITY OF $\mathcal{K}(Q_M)$ FOR LAREG MASSES: More precisely, first we will prove that

$$\mathcal{H}(Q) \lesssim M^{\frac{3}{2}}, \quad \text{as } M \rightarrow \infty. \quad (3.1)$$

If we let $\mathcal{H}_{\text{int}}(Q) := \frac{1}{2}(\|xQ\|_{L^2}^2 + \frac{1}{2}\|Q\|_{L^4}^4)$, then clearly

$$\mathcal{H}_{\text{int}}(Q) \leq \mathcal{H}(Q).$$

Now, we will explicitly calculate the minimizer

$$\nu_M := \inf_{\|u\|_{L^2}^2 = M} \mathcal{H}_{\text{int}}(u), \quad u \in \Sigma_{\text{int}},$$

where $\Sigma_{\text{int}} = \{u \in L^2(\mathbb{R}^2), u \in L^4(\mathbb{R}^2) : \int |x|^2 u^2 < \infty\}$ with the norm

$$\|u\|_{\Sigma_2^4} = \|u\|_2 + \|u\|_4 + \|x|u\|_2.$$

Let (u_n) be a minimizing sequence of ν_M that is

$$\|u_n\|_{L^2}^2 = M, \quad \text{and} \quad \frac{1}{2}(\|xu_n\|_{L^2}^2 + \frac{1}{2}\|u_n\|_{L^4}^4) \rightarrow \nu_M. \quad (3.2)$$

From the above bounds, let us just denote by u (instead of u_M), an L^2 -weak limit of (u_n) . Denote by $f_n := u_n^2$. First we show that $\|f\|_{L^1(\mathbb{R}^2)} = M$. Up to an extraction, we may assume that a subsequence of (f_n) (also denoted by (f_n)) converges weakly to f in the sense of distributions; that is for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ (smooth and compactly supported function), we have

$$\int_{\mathbb{R}^2} \varphi f_n \, dx \rightarrow \int_{\mathbb{R}^2} \varphi f \, dx.$$

To show strong convergence in L^1 , we observe that (see for example [5])

$$\limsup_n \|f_n - f\|_{L^1} \leq C(\{f_n, n = 1, 2, \dots\}),$$

where, for any subset $\mathcal{A} \subset L^1(\mathbb{R}^2)$, the function $C(\mathcal{A})$ introduced by H. P. Rosenthal [12] is given by

$$C(\mathcal{A}) = \inf_{\varepsilon} \sup_{|A| < \varepsilon} \sup_n \int_A f_n \, dx.$$

Using Hölder inequality and the above bounds (3.2), we have for any $R > 0$

$$\begin{aligned} \int_A f_n \, dx &\leq \sqrt{|A|} \sqrt{\int_A f_n^2 \, dx} + \frac{1}{R^2} \int_{A \cap \{|x| > R\}} |x|^2 f_n \, dx \\ &\lesssim \sqrt{\varepsilon} + \frac{1}{R^2}, \end{aligned}$$

which clearly shows that $C(\{f_n, n = 1, 2, \dots\}) = 0$, and thus $\|u_n - u\|_{L^2} \rightarrow 0$ and $\|f\|_{L^1(\mathbb{R}^2)} = \|u\|_{L^2(\mathbb{R}^2)}^2 = M$, as desired. Moreover, by the lower semi-continuity of the norms, we have

$$\frac{1}{2}(\|xu\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^4}^4) = \frac{1}{2}(\| |x|^2 f \|_{L^1} + \frac{1}{2}\|f^2\|_{L^2}^2) \leq \liminf_n \frac{1}{2}(\|xu_n\|_{L^2}^2 + \frac{1}{2}\|u_n\|_{L^4}^4) \leq \nu_M.$$

If the estimate were strict that would contradict the minimality of $\nu = \nu_M$. The convergence $u_n \rightarrow u$ is therefore strong in L^2 , and at the minimizer $u = u_M$, we have

$$|x|^2 u + u^3 = \nu u, \quad u^2 = (\nu - |x|^2)_+$$

yielding

$$M = \|u_M\|_{L^2}^2 = \int_{\mathbb{R}^2} (\nu - V)_+ dx = \int_{\{|x|^2 < \nu\}} (\nu - |x|^2)_+ dx = \frac{\pi}{2} \nu^2,$$

and, as $M \rightarrow \infty$,

$$\|u_M\|_{L^4}^4 = \int_{\mathbb{R}^2} (\nu - |x|^2)_+ |u|^2 dx = \int_{\mathbb{R}^2} (\nu - |x|^2)_+^2 dx \leq \frac{\pi}{3} \nu^3 \sim M^{\frac{3}{2}}.$$

Now we mollify v_M in order to get an upper bound for ν_M . Set

$$\tilde{u}_M := ((\nu - |x|^2)_+^2 + 1)^{\frac{1}{4}} - 1, \quad w_M := \sqrt{M} \frac{\tilde{u}_M}{\|\tilde{u}_M\|_{L^2}}.$$

Calculating $\|\tilde{u}_M\|_{L^2}^2$ shows that

$$\|\tilde{u}_M\|_{L^2}^2 = \int_0^\mu \left((s^2 + 1)^{\frac{1}{4}} - 1 \right)^2 ds \sim \mu^2 = M \quad \text{as } M \rightarrow \infty. \quad (3.3)$$

Moreover, similar calculation enables us to see that

$$\|\nabla \tilde{u}_M\|_{L^2}^2 \lesssim \nu^3 \quad \text{and} \quad \|x|\tilde{u}_M\|_{L^2}^2 \lesssim \nu^3. \quad (3.4)$$

In summary, in virtue of (3.3) and (3.4), we have

$$\|w_M\|_{L^2}^2 = M \quad \text{and} \quad \|x|w_M\|_{L^2}^2 \lesssim M^{\frac{3}{2}}, \quad (3.5)$$

which implies, thanks to the fact that $\mathcal{H}(Q_M) \leq \mathcal{H}(w_M)$,

$$\|Q_M\|_{L^2}^2 = M, \quad \text{and} \quad \|xQ_M\|_{L^2}^2 \lesssim M^{\frac{3}{2}}, \quad \text{as } M \rightarrow \infty.$$

The above estimates automatically imply the following key estimate

$$M^{\frac{3}{2}} \lesssim \|Q_M\|_{L^4}^4. \quad (3.6)$$

STEP 3: PROOF OF THE KEY ESTIMATE (3.6): By contradiction, assume (3.6) does not hold, then there would exist a sequence $M_n \rightarrow \infty$, and $(u_n)_n$ satisfying

$$\|u_n\|_{L^2}^2 = M_n \quad \text{and} \quad \|x|u_n\|_{L^2}^2 \lesssim M_n^{\frac{3}{2}}$$

and

$$\|u_n\|_{L^4}^4 \leq \frac{M_n^{\frac{3}{2}}}{n}.$$

On the other hand, for all $R > 0$ and $n \in \mathbb{N}$

$$\begin{aligned} \|u_n\|_{L^2}^2 &\lesssim \frac{M_n^{\frac{3}{2}}}{R^2} + R \|u_n\|_{L^4}^2 \\ &\lesssim \frac{M_n^{\frac{3}{2}}}{R^2} + R \frac{M_n^{\frac{3}{4}}}{n^{\frac{1}{2}}}. \end{aligned}$$

Now choosing $R = M_n^{\frac{1}{4}} n^{\frac{1}{8}}$, gives the bound

$$1 \lesssim \frac{1}{n^{\frac{1}{4}}}$$

leading to a contradiction by taking $n \rightarrow \infty$. Clearly, (3.6) shows that $\mathcal{K}(Q_M)$ becomes negative as $M \rightarrow \infty$ which finishes the proof. \square

Notice that to construct a nonlinear solution to $(\mu\text{-SP}_0)$, one can use several techniques. Variationally like in [1], for any given amount of mass $M > 0$, we have shown that a radial positive solution (u_M, μ_M) to $(\mu\text{-SP}_0)$ can be constructed through the following minimizing problem

$$\mu_M = \mathcal{H}(u_M) := \min_{\|u\|_{L^2}^2 = M} \mathcal{H}(u).$$

Moreover, this family of solutions is included in the branch of solutions constructed using bifurcation arguments pioneered by Rabinowitz, and Crandall-Rabinowitz [4]. Indeed, (u, μ) is a solution to $(\mu\text{-SP}_0)$ if and only if $(I - \mu K)u = \mathcal{N}(u)$, where $K = A^{-1}B$, $\mathcal{N} = A^{-1}G'(u)$, and the operators A , B and G are defined by

$$\begin{aligned} A : \Sigma &\rightarrow \Sigma^*, \quad \text{for any } u, v \in \Sigma; \quad \langle Au, v \rangle := (\nabla u, \nabla v)_2 + (xu, xv)_2, \\ B : \Sigma &\rightarrow \Sigma^*, \quad \text{for any } u, v \in \Sigma; \quad \langle Bu, v \rangle := (u, v)_2, \end{aligned}$$

and

$$G : \Sigma \rightarrow \mathbb{R}, \quad \text{for any } u \in \Sigma; \quad G(u) = -\frac{1}{4}\|u\|_{L^4}^4.$$

Indeed, the following proposition shows that a branch of solutions of $(\mu\text{-SP}_0)$ emerging from the linear solution (φ_1, ω_1) can be constructed. The proof of the proposition is included in the proof of the spectral assumption given in the Appendix. (See section 5.)

Proposition 3.2. *There exists a unique solution $u = u(\eta) \in \Sigma$, $\mu(\eta) > 2$ of $(\mu\text{-SP}_0)$ parametrized by $\eta > 0$; $0 < \eta < \eta_0$ for some $\eta_0 > 0$, such that*

$$u(\eta) = \sqrt{\eta}(a(\eta)\varphi_1 + z(\eta)),$$

with $z \in \Sigma$, $z(0) = 0$ and $(z(\eta), \varphi_1)_2 = 0$.

For the solution (Q_0, μ_0) to $(\mu\text{-SP}_0)$ satisfying $\mathcal{K}(Q_0) = 0$ given by Proposition 3.1, denote by

$$L_- := -\Delta + V + Q_0^2 - \mu_0,$$

and

$$L_+ := -\Delta + V + 3Q_0^2 - \mu_0.$$

The second preliminary result concerns the operators L_{\pm} . We have the following important property of L_{\pm} .

Proposition 3.3. *Let $\langle Q_0 \rangle^{\perp}$ be the subspace of Σ consisting of all functions L^2 -orthogonal to Q_0 . Then we have*

$$\ker(L_-) = \{Q_0\}, \quad \text{and} \quad L_- : \langle Q_0 \rangle^\perp \rightarrow \langle Q_0 \rangle^\perp \quad \text{is bijective.}$$

Moreover, there exists $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$,

$$L_+ : \Sigma \rightarrow \Sigma^* \quad \text{is bijective.}$$

The property of L_+ comes from the breakdown of the spatial translation symmetry due to the presence of the potential. We refer to the Appendix (Section 5) for the proof of Proposition 3.3.

We have

$$L_+(Q_0) = 2Q_0^3. \quad (3.7)$$

Since $\mathcal{K}(Q_0) = (Q_0, (\sigma - \alpha|Q_0|^2)Q_0)_2 = 0$, then thanks to Proposition 3.3, one can uniquely define Q_{1i} by

$$L_-Q_{1i} := (\alpha|Q_0|^2 - \sigma)Q_0.$$

Observe that given the smoothness and the decay of Q_0 , we have $Q_{1i} \in \text{Dom} L_-$. Moreover, we have

$$L_+^{-1} : L^2 \rightarrow \text{Dom}(L_+) \quad \text{is bounded, and} \quad L_+(Q_{1i}) = \alpha Q_0^3 + 2Q_0^2 Q_{1i} - \sigma Q_0. \quad (3.8)$$

Now, define Q_{2r} and Q_{3i} by

$$L_+Q_{2r} = \mu_2 Q_0 + (\sigma - \alpha|Q_0|^2)Q_{1i} - Q_0 Q_{1i}^2, \quad (3.9)$$

and

$$L_-Q_{3i} = (2Q_{2r}Q_0 - Q_{1i}^2)Q_{1i} + \mu_2 Q_{1i} + ((2 + \alpha)Q_0^2 - \sigma)Q_{2r} + Q_{1i}^2 Q_0. \quad (3.10)$$

The bijectivity of L_+ enables us to determine Q_{2r} , and again the regularity of Q_0 shows that $Q_{2r} \in \text{Dom} L_+$. Thus it only remains to determine the coefficient μ_2 , and Q_{3i} . They are determined by the orthogonality condition

$$(L_-Q_{3i}, Q_0)_2 = 0.$$

Indeed, substituting Q_{2r} (given by inverting (3.9)) into (3.10) gives

$$L_-Q_{3i} = \mu_2 [Q_{1i} + ((2 + \alpha)Q_0^2 - \sigma + 2Q_0 Q_{1i})L_+^{-1}Q_0] + Q_{1i}^2 Q_0 - Q_{1i}^3 \quad (3.11)$$

$$+ ((2 + \alpha)Q_0^2 - \sigma + 2Q_0 Q_{1i})L_+^{-1}((\sigma - Q_0^2)Q_{1i} - Q_0 Q_{1i}^2). \quad (3.12)$$

Now since $(Q_0, Q_{1i})_2 = 0$, then clearly

$$(L_+^{-1}((2 + \alpha)Q_0^2 - \sigma + 2Q_0 Q_{1i}), Q_0) = \|Q_0\|_{L^2}^2 \neq 0,$$

which insures that μ_2 is uniquely determined in terms of Q_0 , Q_{1i} which were already defined. Then Q_{3i} follows by inverting L_- using the orthogonality $(Q_{3i}, Q_0)_2 = 0$. Now, set

$$Q_\varepsilon^a := Q_0 + i\varepsilon Q_{1i} + \varepsilon^2 Q_{2r} + i\varepsilon^3 Q_{3i}, \quad \text{and} \quad \mu_\varepsilon^a = \mu_0 + \varepsilon^2 \mu_2. \quad (3.13)$$

The main result of this section is the following.

Theorem 3.1. For σ non-negative, non-trivial and bounded function and $\alpha > \alpha_0$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, equation (GPPD $_\varepsilon$) has a solution $(Q_\varepsilon, \mu_\varepsilon) \in \Sigma \times (2, \infty)$ that can be decomposed as

$$(Q_\varepsilon, \mu_\varepsilon) = (Q_\varepsilon^a + \psi_\varepsilon, \mu_\varepsilon^a + \kappa_\varepsilon), \quad (3.14)$$

with $\psi_\varepsilon = \psi_{\varepsilon,r} + i\psi_{\varepsilon,i}$ satisfying

$$|\kappa_\varepsilon| + \|\psi_{\varepsilon,r}\|_\Sigma \lesssim \varepsilon^4 \quad (3.15)$$

$$\|\psi_{\varepsilon,i}\|_\Sigma \lesssim \varepsilon^5. \quad (3.16)$$

Proof of Theorem 3.1. First, we write an equation for $(Q_\varepsilon, \mu_\varepsilon)$ being a solution of (μ -SP $_\varepsilon$). We start by further decomposing $Q_\varepsilon^a = Q_{\varepsilon,r}^a + iQ_{\varepsilon,i}^a$ and observe that

$$|Q_\varepsilon|^2 = |Q_{\varepsilon,r}^a|^2 + |Q_{\varepsilon,i}^a|^2 + 2Q_{\varepsilon,r}^a\psi_{\varepsilon,r} + 2Q_{\varepsilon,i}^a\psi_{\varepsilon,i} + |\psi_{\varepsilon,r}|^2 + |\psi_{\varepsilon,i}|^2.$$

Substituting this in equation (μ -SP $_\varepsilon$) and splitting the real and imaginary parts, we obtain

$$\begin{aligned} (\mu_\varepsilon^a + \kappa_\varepsilon)(Q_{\varepsilon,r}^a + \psi_{\varepsilon,r}) &= (-\Delta + V + |Q_\varepsilon|^2)(Q_{\varepsilon,r}^a + \psi_{\varepsilon,r}) \\ &\quad - \varepsilon(\sigma - \alpha|Q_\varepsilon|^2)(Q_{\varepsilon,i}^a + \psi_{\varepsilon,i}), \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} (\mu_\varepsilon^a + \kappa_\varepsilon)(Q_{\varepsilon,i}^a + \psi_{\varepsilon,i}) &= (-\Delta + V + |Q_\varepsilon|^2)(Q_{\varepsilon,i}^a + \psi_{\varepsilon,i}) \\ &\quad + \varepsilon(\sigma - \alpha|Q_\varepsilon|^2)(Q_{\varepsilon,r}^a + \psi_{\varepsilon,r}), \end{aligned} \quad (3.18)$$

respectively. The identity coming from the real part can be rewritten in the following way.

$$\begin{aligned} L_+\psi_{\varepsilon,r} &= \mu_\varepsilon^a Q_{\varepsilon,r}^a - (-\Delta + V + |Q_\varepsilon^a|^2)\psi_{\varepsilon,r} + \varepsilon(\sigma - \alpha|Q_\varepsilon^a|^2)Q_{\varepsilon,i}^a \\ &\quad + \kappa_\varepsilon Q_{\varepsilon,r}^a + \varepsilon^2 \mu_2 \psi_{\varepsilon,r} - 2Q_{\varepsilon,i}^a Q_{\varepsilon,r}^a \psi_{\varepsilon,r} + \varepsilon(\sigma - \alpha|Q_\varepsilon^a|^2)\psi_{\varepsilon,i}^a \\ &\quad - 2|Q_{\varepsilon,r}^a|^2 \psi_{\varepsilon,i}^a + \kappa_\varepsilon \psi_{\varepsilon,r} + \psi_{\varepsilon,r}(2Q_{\varepsilon,r}^a \psi_{\varepsilon,r} + 2Q_{\varepsilon,i}^a \psi_{\varepsilon,i} + \psi_{\varepsilon,r}^2 + \psi_{\varepsilon,i}^2) \\ &\quad - \varepsilon \psi_{\varepsilon,r}(2Q_{\varepsilon,r}^a \psi_{\varepsilon,r} + 2Q_{\varepsilon,i}^a \psi_{\varepsilon,i} + \psi_{\varepsilon,r}^2 + \psi_{\varepsilon,i}^2) \\ &:= \kappa_\varepsilon Q_0 + \varepsilon^4 g_1 + F_\varepsilon(\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_\varepsilon) \end{aligned}$$

where g_1 is given by

$$g_1 := \mu_2 Q_{2r} - Q_{1i}^2 Q_{2r} - (Q_{2r}^2 + 2Q_{3i} Q_{1i})Q_0 + (\sigma - \alpha Q_0^2)Q_{3i} - (2Q_0 Q_{2r} + Q_{1i}^2)Q_{1i}$$

and F_ε can be explicitly computed. In particular it satisfies

$$\|F_\varepsilon(\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_\varepsilon)\|_\Sigma \lesssim \varepsilon^6.$$

The identity coming from the imaginary part can be rewritten in the following way.

$$\begin{aligned} L_-\psi_{\varepsilon,i} &= \mu_\varepsilon^a Q_{\varepsilon,i}^a - (-\Delta + V + |Q_\varepsilon^a|^2)Q_{\varepsilon,i}^a - \varepsilon(\sigma - \alpha|Q_\varepsilon^a|^2)Q_{\varepsilon,r}^a \\ &\quad + \kappa_\varepsilon Q_{\varepsilon,i}^a + \varepsilon^2 \mu_2 \psi_{\varepsilon,i} - 2Q_{\varepsilon,i}^a(Q_{\varepsilon,r}^a \psi_{\varepsilon,r} + Q_{\varepsilon,i}^a \psi_{\varepsilon,i}) - \varepsilon(\sigma - \alpha|Q_0|^2)\psi_{\varepsilon,r} \\ &\quad + 2\varepsilon Q_{\varepsilon,r}^a(Q_{\varepsilon,r}^a \psi_{\varepsilon,r} + Q_{\varepsilon,i}^a \psi_{\varepsilon,i}) \end{aligned}$$

$$\begin{aligned}
& -2\psi_{\varepsilon,i}(Q_{\varepsilon,r}^a\psi_{\varepsilon,r} + Q_{\varepsilon,i}^a\psi_{\varepsilon,i}) + 2\varepsilon\psi_{\varepsilon,r}(Q_{\varepsilon,r}^a\psi_{\varepsilon,r} + Q_{\varepsilon,i}^a\psi_{\varepsilon,i}) \\
& + \varepsilon Q_{\varepsilon,r}^a(\psi_{\varepsilon,r}^2 + \psi_{\varepsilon,i}^2) - \psi_{\varepsilon,i}(\psi_{\varepsilon,r}^2 + \psi_{\varepsilon,i}^2) + \varepsilon\psi_{\varepsilon,r}(\psi_{\varepsilon,r}^2 + \psi_{\varepsilon,i}^2) + \kappa_{\varepsilon}\psi_{\varepsilon,i} \\
& := \varepsilon(\kappa_{\varepsilon}Q_{1i} + ((2+\alpha)Q_0^2 - \sigma - 2Q_0Q_{1i})\psi_{\varepsilon,r}) \\
& + \varepsilon^5\varphi_2 + G_{\varepsilon}(Q_{\varepsilon,r}, Q_{\varepsilon,i}, \kappa_{\varepsilon}),
\end{aligned}$$

where φ_2 is given by

$$\varphi_2 := -(2Q_0Q_{2r} + Q_{1i}^2)Q_{3i} - (Q_{2r}^2 + 2Q_{1i}Q_{3i})Q_{1i} + (2Q_0Q_{2r} + Q_{1i}^2)Q_{2r} + (Q_{2r}^2 + 2Q_{1i}Q_{3i})Q_0$$

and G_{ε} can be explicitly computed. In particular it satisfies

$$\|G_{\varepsilon}(\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_{\varepsilon})\|_{\Sigma} \lesssim \varepsilon^7.$$

Now we define a map $\Phi_{\varepsilon} : \Sigma \times \Sigma \times (0, \infty) \rightarrow \Sigma \times \Sigma \times (0, \infty)$ by

$$\Phi_{\varepsilon}(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_{\varepsilon}) = (\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_{\varepsilon})$$

where, $(\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_{\varepsilon})$ solves

$$\begin{cases} L_+\psi_{\varepsilon,r} = \kappa_{\varepsilon}Q_0 + \varepsilon^4g_1 + F_{\varepsilon}(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_{\varepsilon}) \\ L_-\psi_{\varepsilon,i} = \varepsilon(\kappa_{\varepsilon}Q_{1i} + ((2+\alpha)Q_0^2 - \sigma - 2Q_0Q_{1i})\psi_{\varepsilon,r}) + \varepsilon^5\varphi_2 + G_{\varepsilon}(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_{\varepsilon}), \\ (L_-\psi_{\varepsilon,i}, Q_0)_2 = 0. \end{cases} \quad (3.19)$$

Now the purpose is to show that there are positive constants C_1, C_2 and C_3 such that the above map is a contraction on the ball

$$B_{\varepsilon} := \{(\psi_{\varepsilon,r}, \psi_{\varepsilon,i}, \kappa_{\varepsilon}) : |\kappa_{\varepsilon}| \leq C_1\varepsilon^4, \|\psi_{\varepsilon,r}\|_{\Sigma} \leq C_2\varepsilon^4, \|\psi_{\varepsilon,i}\|_{\Sigma} \leq C_3\varepsilon^5\},$$

for $\varepsilon > 0$ sufficiently small. The ball B_{ε} is endowed with the norm

$$\max \left\{ \frac{|\kappa_{\varepsilon}|}{C_1\varepsilon^4}, \frac{\|\psi_{\varepsilon,r}\|_{\Sigma}}{C_2\varepsilon^4}, \frac{\|\psi_{\varepsilon,i}\|_{\Sigma}}{C_3\varepsilon^5} \right\}. \quad (3.20)$$

Thanks to the equation on $\psi_{\varepsilon,r}$ and the invertibility of L_+ , we can write

$$\psi_{\varepsilon,r} = \kappa_{\varepsilon}Q'(\mu_0) + \varepsilon^4L_+^{-1}(g_1) + L_+^{-1}(F_{\varepsilon}(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_{\varepsilon})). \quad (3.21)$$

Plugging the above identity in the equation on $\psi_{\varepsilon,i}$, we obtain

$$\begin{aligned}
L_-\psi_{\varepsilon,i} & = \varepsilon\kappa_{\varepsilon}(Q_{1i} + ((2+\alpha)Q_0^2 + 2Q_0Q_{1i} - \sigma)Q'(\mu_0) \\
& + \varepsilon^5((2+\alpha)Q_0^2 + 2Q_0Q_{1i} - \sigma)L_+^{-1}(g_1) + L_+^{-1}(F_{\varepsilon}(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_{\varepsilon})).
\end{aligned}$$

Since,

$$(Q_{1i} + ((2+\alpha)Q_0^2 - 2Q_0Q_{1i} - \sigma)Q'(\mu_0), Q_0)_2 = \|Q_0\|_{L^2}^2$$

then the choice of

$$\kappa_\varepsilon = -\frac{\varepsilon^4}{\|Q_0\|_{L^2}^2} \int_{\mathbb{R}^2} Q_0 g_1 \, dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} Q_0 L_+^{-1} (F_\varepsilon(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_\varepsilon)) \, dx$$

makes

$$(L_- \psi_{\varepsilon,i}, Q_0)_2 = 0,$$

which enables us to invert L_- and thus calculate $\psi_{\varepsilon,i}$:

$$\begin{aligned} \psi_{\varepsilon,i} &= \varepsilon \kappa_\varepsilon L_-^{-1} (Q_{1,i} + ((2 + \alpha)Q_0^2 - 2Q_0 Q_{1,i} - \sigma)Q'(\mu_0)) + L_-^{-1} L_+^{-1} (F_\varepsilon(\tilde{\psi}_{\varepsilon,r}, \tilde{\psi}_{\varepsilon,i}, \tilde{\kappa}_\varepsilon)) \\ &\quad + \varepsilon^5 L_-^{-1} (((2 + \alpha)Q_0^2 - 2Q_0 Q_{1,i} - \sigma)L_+^{-1}(g_1)) \end{aligned} \quad (3.22)$$

Let

$$\begin{aligned} C_1 &:= \frac{2\|g_1\|_{L^2}}{\|Q_0\|_{L^2}^2}, \\ C_2 &= 2(C_1\|Q'(\mu_0)\|_\Sigma + \|L_+^{-1}(g_1)\|_\Sigma), \end{aligned}$$

and

$$\begin{aligned} C_3 &:= 2C_1\|L_-^{-1}(Q_{1,i} + ((2 + \alpha)Q_0^2 - 2Q_0 Q_{1,i} - \sigma)Q'(\mu_0))\|_\Sigma \\ &\quad + 2\|L_-^{-1}(((2 + \alpha)Q_0^2 - 2Q_0 Q_{1,i} - \sigma)L_+^{-1}(g_1))\|_\Sigma \end{aligned}$$

To show that Φ_ε is a contraction, consider $(\tilde{\psi}_{\varepsilon,r}^a, \tilde{\psi}_{\varepsilon,i}^a, \tilde{\kappa}_\varepsilon^a)$ and $(\tilde{\psi}_{\varepsilon,r}^b, \tilde{\psi}_{\varepsilon,i}^b, \tilde{\kappa}_\varepsilon^b)$ in the ball B_ε and denote by $(\psi_{\varepsilon,r}^a, \psi_{\varepsilon,i}^a, \kappa_\varepsilon^a)$ and $(\psi_{\varepsilon,r}^b, \psi_{\varepsilon,i}^b, \kappa_\varepsilon^b)$ their respective images through the map Φ_ε . We have

$$\begin{aligned} \kappa_\varepsilon &:= \kappa_\varepsilon^a - \kappa_\varepsilon^b = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} Q_0 L_+^{-1} [F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^b, \tilde{\psi}_{\varepsilon,i}^b, \tilde{\kappa}_\varepsilon^b) - F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^a, \tilde{\psi}_{\varepsilon,i}^a, \tilde{\kappa}_\varepsilon^a)] \, dx, \\ \psi_{\varepsilon,r} &:= \psi_{\varepsilon,r}^a - \psi_{\varepsilon,r}^b = \kappa_\varepsilon Q'(\mu_0) - L_+^{-1} (F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^b, \tilde{\psi}_{\varepsilon,i}^b, \tilde{\kappa}_\varepsilon^b) - F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^a, \tilde{\psi}_{\varepsilon,i}^a, \tilde{\kappa}_\varepsilon^a)), \end{aligned}$$

and

$$\begin{aligned} \psi_{\varepsilon,i} &:= \psi_{\varepsilon,i}^a - \psi_{\varepsilon,i}^b = \varepsilon \kappa_\varepsilon L_-^{-1} (Q_{1,i} + ((2 + \alpha)Q_0^2 - 2Q_0 Q_{1,i} - \sigma)Q'(\mu_0)) \\ &\quad - L_-^{-1} L_+^{-1} (F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^b, \tilde{\psi}_{\varepsilon,i}^b, \tilde{\kappa}_\varepsilon^b) - F_\varepsilon(\tilde{\psi}_{\varepsilon,r}^a, \tilde{\psi}_{\varepsilon,i}^a, \tilde{\kappa}_\varepsilon^a)). \end{aligned}$$

Estimating κ_ε , $\psi_{\varepsilon,r}$ and $\psi_{\varepsilon,i}$ using the above bounds on F_ε and G_ε yields

$$|\kappa_\varepsilon| \lesssim \varepsilon^5, \quad \|\psi_{\varepsilon,r}\|_\Sigma \lesssim \varepsilon^5, \quad \|\psi_{\varepsilon,i}\|_\Sigma \lesssim \varepsilon^6$$

showing the contraction of the map Φ_ε \square

4. The Cauchy problem

In this section, we study the Cauchy problem:

$$\begin{cases} i\partial_t \psi + \Delta \psi = V(x)\psi + |\psi|^2 \psi + i(\sigma(x) - \alpha|\psi|^2)\psi, & t > 0, x \in \mathbb{R}^2, \\ \psi|_{t=0} &= \psi_0. \end{cases} \quad (4.1)$$

We will first assume that $\sigma \in L^4(\mathbb{R}^2)$, we set $U(t) = e^{-it(-\Delta+V)}$.

Definition 4.1. A pair (p, q) is admissible if $2 \leq q < \infty$ and

$$\frac{2}{p} = \delta(q) := 2\left(\frac{1}{2} - \frac{1}{q}\right).$$

Recall the following Strichartz estimates for the Schrödinger equation with potential are due to [3].

Proposition 4.1. Let $T > 0$.

1. For any admissible pair (p, q) , there exists $C_q(T)$ such that

$$\|U(\cdot)\varphi\|_{L^p([0,T];L^q)} \leq C_q(T)\|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbb{R}^2). \quad (4.2)$$

2. Denote

$$D(F)(t, x) = \int_0^t U(t-s)F(s, x)d\tau.$$

For all admissible pairs (p_1, q_1) and (p_2, q_2) , there exists $C = C_{q_1, q_2}(T)$ such that

$$\|D(F)\|_{L^{p_1}([0,T];L^{q_1})} \leq C\|F\|_{L^{p_2'}([0,T];L^{q_2'})}, \quad (4.3)$$

for all $F \in L^{p_2'}([0, T]; L^{q_2'})$ and $0 \leq \tau \leq T$.

Proposition 4.2. There exists $\delta > 0$ such that if $\psi_0 \in L^2(\mathbb{R}^2)$ and $T \in [0, 1]$ are such that

$$\|U(\cdot)\psi_0\|_{L^4([0,T]\times\mathbb{R}^2)} \leq \delta \quad \text{and} \quad T^{3/4}\|\sigma\|_{L^4(\mathbb{R}^2)} \leq \frac{1}{8},$$

then (4.1) has a unique solution

$$\psi \in C([0, T]; L^2) \cap L^4([0, T] \times \mathbb{R}^2).$$

Proof of Proposition 4.2. Let

$$X = \{\psi \in C([0, T]; L^2) \cap L^4([0, T] \times \mathbb{R}^2), \quad \|\psi\|_{L^4([0,T]\times\mathbb{R}^2)} \leq 2\delta\}.$$

In view of Duhamel's formula, and for $\psi \in X$, introduce the map

$$\Phi(\psi)(t) := U(t)\psi_0 - i \int_0^t U(t-s)(1-i\alpha)|\psi|^2\psi(s)ds + \int_0^t U(t-s)(\sigma\psi)(s)ds.$$

From Strichartz inequalities

$$\begin{aligned} \|\Phi(\psi)\|_{L^4([0,T]\times\mathbb{R}^2)} &\leq \|U(\cdot)\psi_0\|_{L^4([0,T]\times\mathbb{R}^2)} + C(1+\alpha)\|\psi\|^2\psi\|_{L^{4/3}([0,T]\times\mathbb{R}^2)} \\ &\quad + \|\sigma\psi\|_{L^1([0,T];L^2)} \\ &\leq \delta + C(1+\alpha)\|\psi\|_{L^4([0,T]\times\mathbb{R}^2)}^3 + \|\sigma\|_{L^4}\|\psi\|_{L^1([0,T];L^4)} \\ &\leq \delta + C(1+\alpha)(2\delta)^3 + \|\sigma\|_{L^4}T^{3/4}\|\psi\|_{L^4([0,T]\times\mathbb{R}^2)} \\ &\leq \delta + C(1+\alpha)(2\delta)^3 + 2\delta T^{3/4}\|\sigma\|_{L^4} \\ &\leq \delta + C(1+\alpha)(2\delta)^3 + \frac{\delta}{4}. \end{aligned}$$

By choosing $\delta > 0$ sufficiently small, the right hand side does not exceed 2δ : X is stable under the action of Φ . For the contraction, let $\psi_1, \psi_2 \in X$:

$$\begin{aligned} \|\Phi(\psi_1) - \Phi(\psi_2)\|_{L^4([0,T] \times \mathbb{R}^2)} &\leq C(1 + \alpha) \| |\phi_1|^2 \psi_1 - |\psi_2|^2 \psi_2 \|_{L^{4/3}([0,T] \times \mathbb{R}^2)} \\ &\quad + \|\sigma(\psi_1 - \psi_2)\|_{L^1([0,T]; L^2)} \\ &\leq C(\|\psi_1\|_{L^4([0,T] \times \mathbb{R}^2)}^2 + \|\psi_2\|_{L^4([0,T] \times \mathbb{R}^2)}^2) \|\psi_1 - \psi_2\|_{L^4([0,T] \times \mathbb{R}^2)} \\ &\quad + \|\sigma\|_{L^4 T^{3/4}} \|\psi_1 - \psi_2\|_{L^4([0,T] \times \mathbb{R}^2)} \\ &\leq (C\delta^2 + \frac{1}{8}) \|\psi_1 - \psi_2\|_{L^4([0,T] \times \mathbb{R}^2)}. \end{aligned}$$

Up to decreasing δ again, the factor on the right hand side does not exceed $1/2$ and Φ is a contraction on X . This proves the existence part of the proposition. The uniqueness part readily follows from the remark that if $\psi \in L^4([0, T] \times \mathbb{R}^2)$, then $[0, T]$ can be split finitely many times on intervals where

$$\|\psi\|_{L^4(I_j \times \mathbb{R}^2)} \leq 2\delta,$$

so uniqueness on X can be deduced. \square

Theorem 4.1. *Let $\psi_0 \in L^2(\mathbb{R}^2)$, $\sigma \in L^4(\mathbb{R}^2)$. Then (4.1) has a unique, maximal solution*

$$\psi \in C([0, T_{max}); L^2) \cap L^4_{loc}([0, T_{max}); L^4(\mathbb{R}^2)).$$

Moreover, in $[0, T_{max})$:

$$\frac{d}{dt} \|\psi(t)\|_{L^2}^2 + \alpha \|\psi(t)\|_{L^4}^4 - \int_{\mathbb{R}^2} \sigma(x) |\psi(t, x)|^2 dx = 0. \quad (4.4)$$

It is maximal in the sense that if T_{max} is finite, then

$$\int_0^{T_{max}} \int_{\mathbb{R}^2} |\psi(t, x)|^4 dt dx = \infty.$$

Proof. Since $\psi_0 \in L^2$, the homogeneous Strichartz inequality (2.1) implies $U(\cdot)\psi_0 \in L^4([0, 1] \times \mathbb{R}^2)$, hence

$$\|U(\cdot)\psi_0\|_{L^4([0,T] \times \mathbb{R}^2)} \xrightarrow{T \rightarrow 0} 0.$$

Moreover, Proposition 4.2 yields a local solution satisfying (4.1). For the notion of maximality, we proceed as in [2]. Suppose that $\psi \in C([0, T_{max}); L^2) \cap L^4([0, T_{max}) \times \mathbb{R}^2)$, with T_{max} finite, and that ψ cannot be extended to larger time. Let $t \in [0, T_{max})$ and $s \in [0, T_{max} - t)$. Duhamel's formula implies

$$U(s)\psi(t) = \psi(t+s) + i \int_0^s U(s-s')(1-i\alpha)|\psi|^2\psi(t+s')ds' - \int_0^s U(s-s')(\sigma\psi)(t+s')ds'.$$

In view of the same inequality as in the proof of Proposition 4.2.

$$\begin{aligned} \|U(\cdot)\psi(t)\|_{L^4((t, T_{max}-t) \times \mathbb{R}^2)} &\leq \|\psi\|_{L^4((t, T_{max}) \times \mathbb{R}^2)} + C\|\psi\|_{L^4(t, T_{max}) \times \mathbb{R}^2}^3 \\ &\quad + \frac{1}{8} \|\psi\|_{L^4((t, T_{max}) \times \mathbb{R}^2)}. \end{aligned}$$

The right hand side is less than δ if t is close to T_{max} . Proposition 4.3 shows that ψ can be extended after T_{max} , in contradiction with the definition of T_{max} . \square

Corollary 4.2. *If $\psi_0 \in L^2(\mathbb{R}^2)$ and $\sigma \in L^4 \cap L^\infty(\mathbb{R}^2)$, then $T_{max} = \infty$, and for all $t \geq 0$,*

$$\|\psi(t)\|_{L^2}^2 \leq \|\psi_0\|_{L^2}^2 e^{t\|\sigma\|_{L^\infty}}.$$

Proof. Form (5.1)

$$\frac{d}{dt} \|\psi(t)\|_{L^2}^2 + \alpha \|\psi(t)\|_{L^4}^4 - \|\sigma\|_{L^\infty} \|\psi(t)\|_{L^2}^2 \leq 0,$$

hence

$$\frac{d}{dt} \left(e^{-\|\sigma\|_{L^\infty} t} \|\psi(t)\|_{L^2}^2 \right) + \alpha e^{-\|\sigma\|_{L^\infty} t} \|\psi(t)\|_{L^4}^4 \leq 0,$$

and

$$e^{-\|\sigma\|_{L^\infty} t} \|\psi(t)\|_{L^2}^2 + \alpha \int_0^T e^{-\|\sigma\|_{L^\infty} t} \|\psi(t)\|_{L^4}^4 dt \leq \|\psi_0\|_{L^2}^2.$$

Therefore, for all T finite,

$$\int_0^T \|\psi(t)\|_{L^4}^4 dt \leq \frac{e^{T\|\sigma\|_{L^\infty}}}{\alpha} \|\psi_0\|_{L^2}^2,$$

hence $T_{max} = \infty$ in Theorem 4.1. \square

Corollary 4.3. *If $\psi_0 \in \Sigma$ and $\sigma \in L^4 \cap W^{1,\infty}(\mathbb{R}^2)$, then (4.1) has a unique, global solution ψ , such that*

$$\psi, \nabla \psi, x\psi \in C([0, \infty); L^2(\mathbb{R}^2)) \cap L^4_{loc}([0, \infty); L^4(\mathbb{R}^2)).$$

The analogue of Proposition 4.2 becomes, if we just assume $\sigma \in L^\infty(\mathbb{R}^2)$:

Proposition 4.3. *There exists $\delta > 0$ such that if $\psi_0 \in L^2(\mathbb{R}^2)$ and $T \in (0, 1]$ are such that*

$$\|U(\cdot)\psi_0\|_{L^4([0,T]\times\mathbb{R}^2)} \leq \delta \quad \text{and} \quad T\|\sigma\|_{L^\infty}\|\psi_0\|_{L^2} \leq \frac{\delta}{8},$$

then (4.1) has a unique solution

$$\psi \in C([0, T]; L^2) \cap L^4([0, T] \times \mathbb{R}^2).$$

The proof is similar to the proof of Proposition 4.2, by working in

$$Y = \{\psi \in C([0, T]; L^2) \cap L^4([0, T] \times \mathbb{R}^2), \|\psi\|_{L^4([0,T]\times\mathbb{R}^2)} \leq 2\delta, \|\psi\|_{L^\infty([0,T];L^2)} \leq 2\|\psi_0\|_{L^2}\}.$$

and estimating

$$\|\sigma\psi\|_{L^1([0,T];L^2)} \leq T\|\sigma\|_{L^\infty([0,T];L^2)}.$$

Now, we still have (5.1), hence

$$\|\psi(t)\|_{L^2}^2 \leq \|\psi_0\|_{L^2}^2 e^{t\|\sigma\|_{L^\infty}} \quad \text{and} \quad \int_0^T \|\psi(t)\|_{L^4}^4 dt \leq \frac{e^T \|\sigma\|_{L^\infty}}{\alpha} \|\psi_0\|_{L^2}^2.$$

Therefore, the solution is global again.

5. Appendix: Proof of Proposition 3.3

Proof. We start by proving (i). Consider the minimizing problem

$$\ell_{\mu_0} := \inf\{ \langle L_- v; v \rangle_{\Sigma^*, \Sigma} : u \in \Sigma \quad \text{and} \quad \|v\|_{L^2} = 1 \},$$

and observe that since $L_-(Q_0) = 0$, then $\ell_{\mu_0} \leq 0$.

On the one hand, arguing as in the proof of Proposition 3.1, one can easily show that a minimizer u of the above problem exists. Next, for any test function φ , we have

$$\langle L_-(u + \varepsilon\varphi); (u + \varepsilon\varphi) \rangle_{\Sigma^*, \Sigma} \geq \ell_{\mu_0} (1 + \varepsilon^2 \|\varphi\|_{L^2}^2 + 2\varepsilon \operatorname{Re}(u, \varphi)_2).$$

That is

$$\begin{aligned} & \ell_{\mu_0} + \varepsilon^2 (\|\nabla\varphi\|_{L^2}^2 + \|x\varphi\|_{L^2}^2 + \|Q_0\varphi\|_{L^2}^2 - \mu_0 \|\varphi\|_{L^2}^2) \\ & + 2\varepsilon \operatorname{Re}((\nabla u, \nabla\varphi)_2 + (xu, x\varphi)_2 + (Q_0 u, Q_0 \varphi)_2 - \mu_0 (u, \varphi)_2) \\ & \geq \ell_{\mu_0} (1 + \varepsilon^2 \|\varphi\|_{L^2}^2 + 2\varepsilon \operatorname{Re}(u, \varphi)_2). \end{aligned}$$

Since ε is arbitrary and can have any sign, then we deduce that

$$L_- u = \ell_{\mu_0} u,$$

and therefore u is an eigenvector of L_- corresponding to the first (and simple) eigenvalue ℓ_{μ_0} . On the other hand, it follows from Theorem 11.8 in [11] that the minimizer is unique and up to a phase change, we can take a positive minimizer $\tilde{u} = e^{i\theta} u$ with $\theta \in \mathbb{R}$. Now, to conclude it suffices to show that $\ell_{\mu_0} = 0$. We see from $L_- Q_0 = 0$ that

$$\ell_{\mu_0} (\tilde{u}, Q_0)_2 = \langle L_- \tilde{u}; Q_0 \rangle_{\Sigma^*, \Sigma} = \langle \tilde{u}; L_- Q_0 \rangle_{\Sigma^*, \Sigma} = 0,$$

yielding (given that Q_0 and \tilde{u} are positive) $\ell_{\mu_0} = 0$. This finishes the proof.

Now we prove (ii). The proof of the bijectivity goes through several steps. First we prove Proposition 3.2. Set

$$\begin{aligned} \mu &= 2 + \eta \\ u &= \sqrt{\eta} q, \quad \text{where } 0 < \eta < 1. \end{aligned} \tag{5.1}$$

Then $(\mu_0 - \operatorname{SP}_0)$ reads as

$$(-\Delta + |x|^2 - 2)q = \eta(q - q^3).$$

Now we can decompose $q = a\varphi_1 + \varphi_1^\perp$, where φ_1 is the first simple eigenfunction of the operator $H_0 := -\Delta + |x|^2$, φ_1^\perp denotes an element of the vector space L^2 -orthogonal to φ_1 , and $a(\eta)$ is a scalar. Therefore, we have

$$\begin{aligned} L_2\varphi_1^\perp &:= (-\Delta + |x|^2 - 2)\varphi_1^\perp = \eta \left[a\varphi_1 + \varphi_1^\perp - (a\varphi_1 + \varphi_1^\perp)^3 \right] \\ &= \eta F(a, \varphi_1^\perp). \end{aligned} \quad (5.2)$$

Now let us define the projection Π by: $\Pi(c\varphi_1 + \varphi_1^\perp) = \varphi_1^\perp$. Thus, (5.2) can be rewritten in the following way

$$\varphi_1^\perp = \eta(\Pi L_2)^{-1} \Pi F(a, \varphi_1^\perp) \quad (5.3)$$

and

$$(I - \Pi) F(a, \varphi_1^\perp) = 0. \quad (5.4)$$

We will first solve (5.3) using the implicit function theorem.

First, we notice that for $\eta = 0$, and a_0 satisfying:

$$|\varphi_1|_2^2 - a_0^2 |\varphi_1|_4^4 = 0,$$

we have $\varphi_1^\perp(0, a_0) = 0$ is a solution of (5.3). On the other hand,

$$\frac{d}{d\varphi_1^\perp} |_{(\eta=0, a_0)} \left(\varphi_1^\perp - \eta(\Pi L_2)^{-1} \Pi F(a, \varphi_1^\perp) \right)$$

is invertible for $0 < |\eta, a - a_0| \ll 1$. Thus, using the implicit function theorem, there exists a unique $\varphi_1^\perp = \gamma(\eta, a)$ solving (5.3). Now we are going to solve (5.4) for $a = a(\eta)$.

We have $(F(a, \varphi_1^\perp), \varphi_1)_2 = 0$, which after expansion, leads to

$$a|\varphi_1|_2^2 - a^3 |\varphi_1|_4^4 + O(\eta) = 0,$$

that in particular yields

$$-|\varphi_1|_2^2 - 3a^2 |\varphi_1|_4^4 < 0, \quad (5.5)$$

by an appropriate choice of $|\eta| \ll 1$.

In summary, we can assert that there exists $0 < \eta_0 \ll 1$ such that for all $0 < \eta < \eta_0$, there exists a unique solution $(\varphi_1^\perp, a) = (\varphi_1^\perp, a(\eta))$ solving (5.3)-(5.4), finishing the proof of Proposition 3.2.

Second, we consider the eigenvalue problem of the linearized operator around $u(\eta)$

$$L_+\varphi = (-\Delta + |x|^2 - 3(u(\eta))^2 - \mu(\eta))\varphi = \lambda\varphi, \quad (5.6)$$

and track how the zero eigenvalue of L_+ moves for small values of η . Note that since q (and hence u) decays exponentially fast in space, the operator L_+ has a discrete spectrum with the same asymptotic of the eigenvalues as the Harmonic oscillator. Recalling that $\mu = \eta + 2$ and $u(\eta) = \sqrt{\eta}q$ with $q = a\varphi_1 + \varphi_1^\perp$, equation (5.6) is rewritten as

$$L_\eta\varphi(\eta) = \lambda(\eta)\varphi(\eta)$$

with $\|\varphi(\eta)\|_2^2 = 1$, and $L_\eta := -\Delta + |x|^2 - 3\eta q^2 - \eta - 2$.

When $\eta = 0$, we have $\lambda(0) = 0$, $\varphi(0) = \varphi_1$, and $\lambda(0)$ is a simple isolated eigenvalue of L_0 .

Taking the derivative with respect to η , we obtain at $\eta = 0$:

$$\left(-\Delta + |x|^2 - 2 - 3(a(0)\varphi_1)^2 - 1\right) + L(0)\frac{d\varphi(0)}{d\eta} = \frac{d\lambda}{d\eta}\varphi(0) + \lambda\frac{d\varphi(0)}{d\eta}. \quad (5.7)$$

Multiplying the identity (5.7) by $\varphi(0)$ and taking the L^2 scalar product, we get

$$\left((-\Delta + |x|^2 - 2 - 3(a(0)\varphi_1)^2 - 1)\varphi_1, \varphi_1\right)_2 = -3a^2(0)|\varphi_1|_4^4 - |\varphi_1|_2^2. \quad (5.8)$$

(5.8) is strictly negative by identity (5.5). Therefore, $\frac{d\lambda}{d\eta}\Big|_{\eta=0} < 0$.

Third, we show that for small masses M , the ground state Q_M minimizing V_M given by Lemma 2.2 is indeed equal to the unique $u(\eta)$ given in proposition (3.2) by choosing $M \sim \eta < \eta_0$. More precisely, it is sufficient to show that $\frac{Q_M}{\sqrt{M}} \rightarrow \varphi_1$ and $\mu_M \rightarrow 2$, as $M \rightarrow 0$ where $(-\Delta + |x|^2)\varphi_1 = 2\varphi_1$ and $\|\varphi_1\|_2 = 1$.

To prove the latter assertion, let us first notice that

$$\begin{aligned} 2M &\leq \|\nabla Q_M\|_2^2 + \|xQ_M\|_2^2 + \|Q_M\|_4^4 \\ &\leq 2M + M^2\|\varphi_1\|_4^4. \end{aligned} \quad (5.9)$$

This implies that $\mu_M \rightarrow 2$ as $M \rightarrow 0$.

Additionally, (5.9) implies that $\frac{Q_M}{\sqrt{M}} = v_M$ is bounded in Σ and satisfies

$$-\Delta v_M + |x|^2 v_M + M|v_M|^2 v_M = \mu_M v_M.$$

Taking the weak limit in the latter inequality, we deduce that $v_M \rightarrow \varphi_1$ in Σ as $M \rightarrow 0$. Hence, choosing M small enough; $M \sim \eta < M_0 := \eta_0$ so that $\frac{Q_M}{\sqrt{M}}$ is in a neighborhood of φ_1 in Σ .

In the last part of the proof, we choose

$$\alpha = \frac{\int \sigma(x)Q_M^4(x)dx}{\int Q_M^2(x)dx} > \alpha_0 := \frac{\int \sigma(x)Q_{M_0}^4(x)dx}{\int Q_{M_0}^2(x)dx} \sim \frac{1}{M_0}$$

for $M < M_0$ yielding that Q_M is a zero of the functional \mathcal{K} . \square

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