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Brief paper

# A generalization of Bellman's equation with application to path planning, obstacle avoidance and invariant set estimation ☆

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ABSTRACT

The standard Dynamic Programming (DP) formulation can be used to solve Multi-Stage Optimization Problems (MSOP's) with additively separable objective functions. In this paper we consider a larger class of MSOP's with monotonically backward separable objective functions; additively separable functions being a special case of monotonically backward separable functions. We propose a necessary and sufficient condition, utilizing a generalization of Bellman's equation, for a solution of a MSOP, with a monotonically backward separable cost function, to be optimal. Moreover, we show that this proposed condition can be used to efficiently compute optimal solutions for two important MSOP's; the optimal path for Dubin's car with obstacle avoidance, and the maximal invariant set for discrete time systems.



## 1. Introduction

Throughout Engineering, Economics, and Mathematics many problems can be formulated as Multi-Stage Optimization Problems (MSOP's):

$$\min\left\{J(u(0),\dots,u(T-1),x(0),\dots,x(T))\right\}$$
$$x(0)=x_0,\ \ x(t+1)=f(x(t),u(t),t) \text{ for } t=0,\dots,T-1$$
$$x(t)\in X_t\subset\mathbb{R}^n,\ \ u(t)\in U\subset\mathbb{R}^m \text{ for } t=0,\dots,T.$$

Such problems consist of (1) a cost function $J:\mathbb{R}^{m\times T}\times\mathbb{R}^{n\times(T+1)}\to\mathbb{R}$, (2) an underlying discrete-time dynamical system governed by the plant equation $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{N}\to\mathbb{R}^n$, (3) a state space $X_t\subset\mathbb{R}^n$, (4) an admissible input space $U\subset\mathbb{R}^m$, and (5) a terminal time $T>0$. Examples of such optimization problems include: optimal battery scheduling to minimize consumer electricity bills (Jones & Peet, 2017); energy-optimal speed planning for road vehicles (Zeng & Wang, 2018); optimal maintenance of manufacturing systems (Liu, Dong, Lv, & Ye, 2019); etc.

MSOP's are members of the class of constrained nonlinear optimization problems. Such optimization problems can be solved using nonlinear solvers such as SNOPT (Gill, Murray, & Saunders, 2005) over small time horizons. However, the most commonly used class of methods for solving MSOP's is Dynamic Programming (DP) (Bertsekas, 1995). DP methods exploit the structure of MSOP's to decompose the optimization problem into lower dimensional sub-problems that can be solved recursively to give the solution to the original higher dimensional MSOP. Typically, DP is used to solve problems with cost functions of the form $J(\mathbf{u},\mathbf{x})=\sum_{t=0}^{T-1}c_t(x(t),u(t))+c_T(x(T))$. These functions (Definition 2) are called additively separable functions, as they can be additively separated into sub-functions, each of which only depend on a single time-stage, $t\in\{0,\dots,T\}$. In the additively separable case it was shown in Bellman (1966) that if we can find a function $F$ that satisfies Bellman's Equation,

$$F(x,T)=c_T(x)\quad\text{for all } x\in X_T$$
$$F(x,t)=\inf_{u\in\Gamma_{x,t}}\left\{c_t(x,u)+F(f(x,u,t),t+1)\right\}$$
$$\text{for all } x\in X_t,\ t\in\{0,\dots,T-1\},$$

where $\Gamma_{x,t}:=\{u\in U: f(x,u,t)\in X_t\}$, then a necessary and sufficient condition for a feasible input and state sequence, $\mathbf{u}=(u(0),\dots,u(T-1))$ and $\mathbf{x}=(x(0),\dots,x(T))$, to be optimal is

$$u(t)\in\arg\inf_{u\in\Gamma_{x(t),t}}\left\{c_t(x(t),u)+F(f(x(t),u,t),t+1)\right\}$$
$$\text{for all } t\in\{0,\dots,T-1\}.$$

We consider MSOP's with cost functions of the more general form $J(\mathbf{u},\mathbf{x})=\phi_0(x(0),u(0),\phi_1(x(1),u(1),\dots\phi_T(x(T))\dots))$, where maps $\phi_t:X_t\times U\times\mathbb{R}\to\mathbb{R}$ are monotonic in their third argument

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for $t = 0, \dots, T-1$. Such functions are called monotonically backward separable, defined in Definition 3, and shown to contain the class of additively separable functions in Lemma 4. For MSOP's with monotonically backward separable cost functions we show in Theorem 9 that if we can find a function $V$ that satisfies

$$V(x,T) = \phi_T(x) \quad \text{for all } x \in X_T \tag{1}$$
$$V(x,t) = \inf_{u \in \Gamma_{x,t}} \left\{ \phi_t(x, u, V(f(x,u,t), t+1)) \right\} \quad \text{for all } x \in X_t, t \in \{0, \dots, T-1\},$$

where $\Gamma_{x,t} := \{u \in U : f(x,u,t) \in X_t\}$, then a necessary and sufficient for a feasible input and state sequence, $\mathbf{u} = (u(0), \dots, u(T-1))$ and $\mathbf{x} = (x(0), \dots, x(T))$, to be optimal is

$$u(t) \in \arg \inf_{u \in \Gamma_{x(t),t}} \left\{ \phi_t\left(x(t), u, V(f(x(t),u,t), t+1)\right) \right\} \quad \text{for all } t \in \{0, \dots, T-1\}.$$

Eq. (1) can be thought of as a generalization of Bellman's Equation; as it is shown in Corollary 10 that in the special case when the cost function is additively separable Eq. (1) reduces to Bellman's Equation. We therefore refer to Eq. (1) as the Generalized Bellman's Equation (GBE). Through several examples we show a solution, $V$, to the GBE can be obtained numerically by recursively solving the GBE backwards in time for each element of $X_t$, the same way Bellman's Equation is solved, thereby extending traditional DP methods to solve a larger class of MSOP's with non-additively separable cost functions. Moreover, in Section 4 it is shown how Approximate Dynamic Programming (ADP) methods can be modified to solve the GBE.

By recursively solving the GBE (1) it is possible to synthesize optimal input sequences for many important practical problems. In this paper we consider two such problems; path planning with obstacle avoidance and maximal invariant sets. First, we define the path planning problem as the search for a sequence of inputs that drives a dynamical system to a target set in minimum time while avoiding obstacles defined by subsets of the state–space. In Section 5 we show that such problems can be formulated as an MSOP with monotonically backward separable objective, of form $J(\mathbf{u}, \mathbf{x}) = \min\{\inf\{t \in [0,T] : x(t) \in S\}, T\}$, implying that the solution to the path planning problem can be found using the solution to the GBE. Similarly, in Section 6 we show that computation of maximal invariant sets can be formulated as an MSOP with monotonically backward separable objective of form $J(\mathbf{u}, \mathbf{x}) = \max\{\max_{0 \le k \le T-1}\{c_k(u(k), x(k))\}, c_T(x(T))\}$.

Path planning with obstacle avoidance has been extensively studied (see surveys Dreyfus, 1969; Gallo & Pallottino, 1988) and has many applications; including UAV surveillance (Xie, Jin, & Garcia Carrillo, 2019). In Rippel, Bar-Gill, and Shimkin (2005) the path planning problem is separated into two separate problems: the "geometric problem", in which the shortest curve, $\tilde{x}(t)$, between the initial set and target set is calculated, and the "tracking problem", in which a controller, $u(t)$, is synthesized so that $\sum_{t=0}^{T} \|x(t) - \tilde{x}(t)\|_2^2$ is minimized, where $x(t+1) = f(x(t), u(t), t)$ and $\|\cdot\|_2$ is the Euclidean norm. Separating the path planning problem allows for the use of efficient algorithms such as $A^*$-search or tangent graphs (Liu & Arimoto, 1992) to solve the "geometric problem" and LQR control to solve the "tracking problem", however, there is no guaranteed that this method will produce the true solution to the original path planning problem. The same approach is used in Cowlagi and Tsiotras (2011), where it is shown through numerical examples that a controller closer to optimality can be derived when the state space is augmented with historic trajectory information. Our approach of using the GBE to solve the path planning does not separate the problem into the "geometric or "tracking" problem and thus does not require any state augmentation. For systems described in continuous time (rather than the discrete systems considered in this paper) with obstacles that satisfy certain boundary curvature assumptions, assumptions not made in this paper, it has been shown in Savkin and Hoy (2013) that a path planning sliding mode controller can be efficiently computed. Furthermore, this sliding mode controller can be used for effective path planning in unknown environments, a case not considered in this paper.

The GBE can also be used in the application of computing the Finite Time Horizon Maximal Invariant Set (FTHMIS), defined as the largest set of initial conditions for a discrete time process such that there exists a feasible input sequence for which the state of the system never violates a time-varying constraint. Knowledge of this set can be used to design controllers that ensure the system never violates given safety constraints. We show that FTHMIS's are equivalent to the sublevel set of solutions to the GBE. To the best of the authors knowledge the problem of computing FTHMIS's has not previously been addressed in the literature. However, a proposed methodology for computing maximal invariant sets over infinite time horizons can be found in Esterhuizen, Aschenbruck, and Streif (2019), Wang, Jungers, and Ong (2019) and Xue and Zhan (2018). Similar continuous-time formulations of this problem can be found in Jones and Peet (2019, 2019b).

Substantial work on generalizations of Bellman's Equation for both infinite and finite time MSOP's can be found in Bertsekas (2018). Our work differs from Bertsekas (2018) as rather than attempting to generalize the "Bellman's operator", as Bertsekas (2018) does, we consider a wider class of cost functions associated with MSOP's, introducing monotonically backward separable cost functions, leading to a derivation of the GBE (1). Unlike in Bertsekas (2018), we formalize the link between the cost function of an MSOP and the GBE (1). Other examples in the literature of MSOP's with non-additively separable cost functions can be found in the pioneering work of Li (1990) and Li and Haimes (1990a, 1990b, 1991). Li considered MSOP's with $k$-separable cost functions; functions of the form $J(\mathbf{u}, \mathbf{x}) = H(J_1(\mathbf{u}, \mathbf{x}), \dots, J_k(\mathbf{u}, \mathbf{x}))$, where $H : \mathbb{R}^k \to \mathbb{R}$ is strictly increasing and differentiable, and each of the functions, $J_i$, are differentiable monotonically backward separable functions. Li showed that for problems in this class of MSOP, an equivalent multi-objective optimization problem with k-separable cost functions can be constructed. The multi-objective optimization problem can then be analytically solved, using methods relying of the differentiability of the cost function, to find the optimal input sequence for the MSOP. We do not assume, as in Li, that the cost function is differentiable or $k$-separable and our solution does not require the solution of a multi-objective optimization problem.

In related work, coherent risk measures, from Ruszczyński (2010), Shapiro and Ugurlu (2016) and Shapiro (2009), result in MSOP's with non-additively separable cost functions of the form $J(\mathbf{u}, \mathbf{x}) = c_0(x(0), u(0)) + \rho_1(c_1(x(1), u(1)) + \rho_2(c_2(x(2), u(2)) + \dots + \rho_T(c_T(x(T)))\dots))$. Such MSOP's are solved recursively using a modified Bellman's Equation. Coherent risk measure functions are a special case of monotonically backward separable functions; in this case our GBE reduces to the previously proposed modified Bellman's equation.

## 2. Multi-stage optimization problems with backward separable cost functions

In this section we will introduce a class of general Multi-Stage Optimization Problems (MSOP's). We show this class contains problems that classical DP theory is able to solve; MSOP's with additively separable cost functions (Eq. (3)). We then propose a more general class of cost functions called monotonically backward separable functions (Eq. (4)) that contain the class of additively separable functions. Using this framework we are then able to derive necessary and sufficient conditions for an input sequence to solve an MSOP with monotonically backward separable cost function. Such conditions are shown to reduce to the classical conditions proposed by Bellman (1966) in the special case when the cost function is additively separable.

**Definition 1.** For a given initial condition $x_0 \in \mathbb{R}^n$, for every tuple of the form $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, where $J : \mathbb{R}^{m\times T} \times \mathbb{R}^{n\times(T+1)} \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n$, $X_t \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$, and $T \in \mathbb{N}$, we associate a MSOP of the following form

$$(\mathbf{u}^*, \mathbf{x}^*) \in \arg\min_{\mathbf{u},\mathbf{x}} J(\mathbf{u}, \mathbf{x}) \text{ subject to:} \tag{2}$$
$$x(t+1) = f(x(t), u(t), t) \text{ for } t = 0, \dots, T-1$$
$$x(0) = x_0,\ \ x(t) \in X_t \subset \mathbb{R}^n \text{ for } t = 0, \dots, T$$
$$u(t) \in U \subset \mathbb{R}^m \text{ for } t = 0, \dots, T-1$$
$$\mathbf{u} = (u(0), \dots, u(T-1)) \text{ and } \mathbf{x} = (x(0), \dots, x(T))$$

Classical DP theory is concerned with the special case when the cost function, $J : \mathbb{R}^{m\times T} \times \mathbb{R}^{n\times(T+1)} \to \mathbb{R}$, has an additively separable structure defined as follows.

**Definition 2.** The function $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is said to be **additively separable** if there exist functions, $c_T(x) : X_T \to \mathbb{R}$, and $c_t : X_t \times U \to \mathbb{R}$ for $t = 0, \dots, T-1$ such that,

$$J(\mathbf{u}, \mathbf{x}) = \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)), \tag{3}$$

where $\mathbf{u} = (u(0), \dots, u(T-1))$ and $\mathbf{x} = (x(0), \dots, x(T))$.

We consider the class of "monotonic backward separable" cost functions defined next. The definition of this class of functions uses the image set of a function. Specifically, for $f : X \to Y$ we denote the image set of the function as $Image\{f\} := \{y \in Y :$ there exists $x \in X$ such that $f(x) = y\}$.

**Definition 3.** The function $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$, where $U \subset \mathbb{R}^m$ and $X_t \subset \mathbb{R}^n$ is said to be **monotonically backward separable** if there exist representation maps, $\phi_T : X_T \to \mathbb{R}$, and $\phi_t : X_t \times U \times Image\{\phi_{t+1}\} \to \mathbb{R}$ for $t = 0, \dots, T-1$ such that the following holds:

(1) The function $J$ can be expressed as the composition of representation maps, $\{\phi_t\}_{t=0}^T$, ordered backwards in time. That is $J$ satisfies

$$J(\mathbf{u}, \mathbf{x}) = \phi_0(x(0), u(0), \phi_1(x(1), u(1), \dots \phi_T(x(T)) \dots)), \tag{4}$$

where $\mathbf{u} = (u(0), \dots, u(T-1))$ and $\mathbf{x} = (x(0), \dots, x(T))$.

(2) Each representation map, $\phi_t$, is monotonic in its third argument. That is if $z, w \in Image\{\phi_{t+1}\}$ are such that $z \ge w$ then

$$\phi_t(x, u, z) \ge \phi_t(x, u, w) \text{ for all } (x, u) \in X_t \times U \tag{5}$$

Moreover if $J$ also satisfies the following properties than we say $J$ is **naturally monotonically backward separable**:

(1) Each representation map, $\phi_t$, is upper semi-continuous in its third argument. That is for any $t \in \{0, \dots, T-1\}$, $x \in X_t$, $u \in U$ and any monotonically decreasing sequence $\{z_n\}_{n\in\mathbb{N}} \subset Image\{\phi_{t+1}\}$, such that $z_{n+1} \le z_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \phi_t(x, u, z_n) = \phi_t(x, u, \lim_{n\to\infty} z_n). \tag{6}$$

(2) Each representation map, $\phi_t$, satisfies the following boundedness property. For any $t \in \{0, \dots, T-1\}$ and $(x, u, z) \in X_t \times U \times Image\{\phi_{t+1}\}$ we have $|\phi_t(x, u, z)| < \infty$ and for all $x \in X_T$ we have $|\phi_T(x)| < \infty$; that is for each $t \in \{0, \dots, T\}$ there exists $R > 0$ such that

$$Image\{\phi_t\} \subset \{x \in \mathbb{R} : |x| < R\}. \tag{7}$$

We show in Section 3 that monotonically backward separable functions share a deep connection with Bellman's Principle of Optimality (Definition 11). However, we also consider naturally monotonically backward separable functions as the added semi-continuity and boundedness properties are used in the derivation of necessary and sufficient conditions for an input sequence to solve an MSOP with naturally monotonically backward separable cost function (Theorem 9).

We next show the class of MSOP's with monotonically backward separable cost functions includes the class of MSOP's with additively separable cost functions as a special case.

**Lemma 4.** *Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is an additively separable function (Definition 2), with associated cost functions $\{c_t\}_{t=0}^T$. Then $J$ is a monotonically backward separable function (Definition 3). Moreover, if the functions $\{c_t\}_{t=0}^T$ are bounded over $X_t \times U$ then $J$ is naturally monotonically backward separable function.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

Further examples of monotonically backward separable functions, including instances where the representation maps are non-differentiable, are given in Section 2.3.

### *2.1. Exchanging the order of composition and infimum for monotonically backward separable functions*

As we will show in Lemma 5, monotonically backward separable functions have the special property that the order of an infimum and composition of representation maps can be interchanged.

Before stating Lemma 5 we introduce notation for the set of feasible controls. Given a tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$ for $x \in X_t$ and $s \in [0, T-1]$ we denote

$$\Gamma_{x,s} := \{u \in U : f(x, u, s) \in X_{s+1}\}.$$

$$\text{Moreover we say,} \quad (u(s), \dots, u(T-1)) \in \Gamma_{x_0,[s,T-1]} \tag{8}$$

if $u(t) \in \Gamma_{x(t),t}$ for all $t \in \{s, \dots, T-1\}$, where $x(s) = x_0$ and $x(k+1) = f(x(k), u(k), k)$ for $k \in \{s, \dots, T-1\}$.

**Lemma 5.** *Consider an MSOP of Form (2) associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$. Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is a naturally monotonically backward separable function (Definition 3) with representation maps $\{\phi_t\}_{t=0}^T$ and $\Gamma_{x,t} \ne \emptyset$ for all $(x, t) \in X_t \times \{0, \dots, T-1\}$. Then for $k \in \{0, \dots, T-1\}$ and any $x \in X_k$ we have*

$$\inf_{u(k)\in\Gamma_{x,k}} \Bigg\{ \phi_k \Bigg( x(k), u(k), \inf_{(u(k+1),\dots,u(T-1))\in\Gamma_{x(k+1),[k+1,T-1]}} \Bigg\{ \phi_{k+1}( x(k+1), u(k+1), \phi_{k+2}(x(k+2), u(k+2), \dots \phi_T(x(T))\dots) \Bigg\} \Bigg) \Bigg\}$$
$$= \inf_{(u(k),\dots,u(T-1))\in\Gamma_{x,[k,T-1]}} \Bigg\{ \phi_k(x(k), u(k), \phi_{k+1}(x(k+1), u(k+1), \dots \phi_T(x(T))\dots) \Bigg\}, \tag{9}$$

*where $x(t+1) = f(x(t), u(t), t)$ for $t \in \{k, \dots, T-1\}$ and $x(k) = x$.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

### 2.2. Main result: A generalization of Bellman's equation

When $J$ is additively separable, the MSOP, given in Eq. (2), associated with the tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, can be solved recursively using Bellman's Equation (Bellman, 1966). In this section we show that a similar approach can be used to solve MSOP's with naturally monotonically backward separable cost functions.

We next define conditions under which a function, $V$, is said to be a *value function* for an associated MSOP.

**Definition 6.** Consider a monotonically backward separable function $J : \mathbb{R}^{m\times T} \times \mathbb{R}^{n\times(T+1)} \to \mathbb{R}$ with representation functions $\{\phi_t\}_{0\le t\le T}$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$, and $T \in \mathbb{N}$. We say the function $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is a **value function** of the MSOP associated with the tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$ if for all $x \in X_T$

$$V(x, T) = \phi_T(x), \tag{10}$$

and for all $x \in X_t$ and $t \in [0, T-1]$

$$V(x, t) = \inf_{u(t)\in\Gamma_{x,t},\ldots,u(T-1)\in\Gamma_{x(T-1),T-1}} \Big\{ \phi_t(x(t), u(t), \phi_{t+1}(x(t+1), u(t+1), \ldots \phi_T(x(T))\ldots)) \Big\}, \tag{11}$$

where $x(t) = x$ and $x(k+1) = f(x(k), u(k), k)$ for $k \in \{t, \ldots, T-1\}$.

We note that the value function has the special property that $V(x_0, 0) = J^*$, where $J^*$ is the minimum value of the cost function of the MSOP (2). In the special case when $J$ is an additively separable function the value function defined in this way reduces to the optimal cost-to-go function.

**Proposition 7** (*Generalized Bellman's Equation (GBE)*). *Consider an MSOP of Form (2) associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$. Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is a naturally monotonically backward separable function (Definition 3) with representation maps $\{\phi_t\}_{t=0}^T$ and $\Gamma_{x,t} \neq \emptyset$ for all $(x, t) \in X_t \times \{0, \ldots, T-1\}$. Then if $F : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ satisfies*

$$F(x, T) = \phi_T(x) \text{ for all } x \in X_T, \quad \text{and} \tag{12}$$
$$F(x, t) = \inf_{u\in\Gamma_{x,t}} \Big\{ \phi_t(x, u, F(f(x, u, t), t+1)) \Big\} \quad \text{for all } x \in X_t,\ t \in \{0, \ldots, T-1\},$$

*then $F$ is a value function (Definition 6) of the MSOP associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

We next propose sufficient conditions showing an input sequence is optimal if it recursively minimizes the right hand side of the GBE (12). Later in Theorem 9 we propose necessary and sufficient conditions involving the GBE (12).

**Proposition 8** (*Sufficient Conditions*). *Consider an MSOP of Form (2) associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$. Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is a naturally monotonically backward separable function (Definition 3) with representation maps $\{\phi_t\}_{t=0}^T$, $\Gamma_{x,t} \neq \emptyset$ for all $(x, t) \in X_t \times \{0, \ldots, T-1\}$, $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ satisfies the GBE (12), and the state sequence $\mathbf{x}^* = (x^*(0), \ldots, x^*(T))$ and input sequence $\mathbf{u}^* = (u^*(0), \ldots, u^*(T-1))$ satisfy*

$$u^*(k) \in \arg \inf_{u\in\Gamma_{x^*(k),k}} \Big\{ \phi_t(x^*(k), u, V(f(x^*(k), u, k), k+1)) \Big\} \quad \text{for } k \in \{0, \ldots, T-1\}. \tag{13}$$

$$x^*(0) = x_0, \quad x^*(k+1) = f(x^*(k), u^*(k), k) \quad \text{for } k \in \{0, \ldots, T-1\}. \tag{14}$$

*Then $(\mathbf{u}^*, \mathbf{x}^*)$ solve the MSOP given in Eq. (2), associated with the tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

Consider an MSOP associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, where $J$ is naturally monotonically backward separable (Definition 3). As we will show next, if the representation maps $\{\phi_t\}_{t=0}^T$, associated with $J$ are strictly monotonic (Eq. (15)) then Eqs. (13) and (14) of Proposition 8 become sufficient and necessary for optimality. In Section 2.3 we will give several examples of naturally monotonically backward functions with associated strictly monotonic representation maps.

**Theorem 9** (*Necessary and Sufficient Conditions*). *Consider an MSOP of Form (2) associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$. Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is a naturally monotonically backward separable function (Definition 3) with representation maps $\{\phi_t\}_{t=0}^T$, and $\Gamma_{x,t} \neq \emptyset$ for all $(x, t) \in X_t \times \{0, \ldots, T-1\}$. Furthermore, suppose the representation maps are strictly monotonic in their third argument. That is if $z, w \in Image\{\phi_{t+1}\}$ are such that $z > w$ then*

$$\phi_t(x, u, z) > \phi_t(x, u, w) \text{ for all } (x, u) \in X_t \times U. \tag{15}$$

*Then $(\mathbf{u}^*, \mathbf{x}^*)$ solve the MSOP if and only if $(\mathbf{u}^*, \mathbf{x}^*)$ satisfy Eqs. (13) and (14).*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

In the next corollary we show that when the cost function, $J$, is additively separable, the GBE (12) reduces to Bellman's Eq. (16); thus showing Bellman's Equation is an implication of the GBE. Therefore we have generalized the necessary and sufficient conditions for optimality encapsulated in Bellman's Equation to the GBE. The GBE provides optimality conditions for a larger class of MSOP's with monotonically backward separable cost functions; that no longer need be additively separable.

**Corollary 10** (*Bellman's Equation*). *Consider an MSOP of Form (2) associated with $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$. Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is an additively separable function (Definition 2), with associated cost functions $\{c_t\}_{t=0}^T$ that are bounded over $X_t \times U$. Then if $F : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ satisfies*

$$F(x, T) = c_T(x) \quad \text{for all } x \in X_T, \tag{16}$$
$$F(x, t) = \inf_{u\in\Gamma_{x,t}} \Big\{ c_t(x, u) + F(f(x, u, t), t+1) \Big\} \quad \text{for all } x \in X_t,\ t \in \{0, \ldots, T-1\},$$

*then $F$ is a value function for the MSOP associated with the tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$.*

*Moreover, if $\Gamma_{x,t} \neq \emptyset$ for all $(x, t) \in X_t \times \{0, \ldots, T\}$ then $\mathbf{x}^* = (x^*(0), \ldots, x^*(T))$ and $\mathbf{u}^* = (u^*(0), \ldots, u^*(T-1))$ solve the MSOP if and only if the following is satisfied*

$$u^*(k) \in \arg \inf_{u\in\Gamma_{x^*(k),k}} \{c_k(x^*(k), u) + F(f(x^*(k), u, k), k+1)\}, \tag{17}$$

$$x^*(0) = x_0, \quad x^*(k+1) = f(x^*(k), u^*(k), k) \quad \text{for } k \in \{0, \ldots, T-1\}. \tag{18}$$

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

### 2.3. Examples: Backward separable functions

In Section 2.2, we have shown that MSOP's with cost functions that are naturally monotonically backward separable (Definition 3) can be solved efficiently using the GBE (12). We now give examples of non-additively separable, yet monotonically backward separable functions, which may be of significant interest. This is not a complete list of all monotonically backward separable functions. Currently little is known about size and structure of the set of all monotonically backward separable functions.

The first function we consider is the point-wise maximum function. This function occurs in MSOP's when demand charges are present (Jones & Peet, 2018) and in maximal invariant set estimation (Xue & Zhan, 2018).

**Example 1** (*Point Wise Maximum Function*). Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is of the form

$$J(\mathbf{u}, \mathbf{x}) = \max \left\{ \max_{0 \le k \le T-1} \{c_k(x(k), u(k))\}, c_T(x(T)) \right\},$$

where $\mathbf{u} = (u(0), \dots, u(T-1))$, $\mathbf{x} = (x(0), \dots, x(T))$, $U \subseteq \mathbb{R}^m$, $X_t \subseteq \mathbb{R}^n$, $c_k : X_k \times U \to \mathbb{R}$ and $c_T : X_T \to \mathbb{R}$. Then $J$ is a monotonically backward separable function. Moreover, if $\{c_t\}_{t=0}^T$ are bounded functions, then $J$ is naturally monotonically backward separable.

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

In the next example we consider multiplicative costs. A special case of this cost function, of the form $J(\mathbf{u}, \mathbf{x}) = \mathbb{E}_{\mathbf{w}}[\exp(\sum_{t=0}^{T-1} c_t(x(t), u(t), w(t)) + c_T(x(T), w(t)))] := \int \exp(\sum_{t=0}^{T-1} c_t(x(t), u(t), w(t)) + c_T(x(T), w(t)))p(\mathbf{w})d\mathbf{w}$, where $p(\mathbf{w})$ is the probability density function of $\mathbf{w} = (w(0), \dots, w(T))$, has previously appeared (Glover & Doyle, 1988; Jacobson, 1973).

**Example 2** (*Multiplicative Function*). Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is of the form

$$J(\mathbf{u}, \mathbf{x}) = \mathbb{E}_{\mathbf{w}}[c_T(x(T), w(T))\Pi_{t=0}^{T-1} c_t(x(t), u(t), w(t))]$$
$$:= \int_{I_0 \times .. I_T} c_T(x(T), w(T))\Pi_{t=0}^{T-1} c_t(x(t), u(t), w(t))$$
$$p_T(x(T), w(T))\Pi_{t=0}^{T-1} p_t(x(t), u(t), w(t))dw(0)...dw(T),$$

where $\mathbf{u} = (u(0), \dots, u(T-1))$, $\mathbf{x} = (x(0), \dots, x(T))$, $\mathbf{w} = (w(0), \dots, w(T))$, $U \subset \mathbb{R}^m$ and $X_t \subset \mathbb{R}^n$, $I_t \subset \mathbb{R}^k$, $c_t : X_t \times U \times I_t \to \mathbb{R}^+$ for $0 \le t \le T-1$, $c_T : X_T \times I_T \to \mathbb{R}$, and $p_t : X_t \times U \times I_t \to \mathbb{R}^+$, $p_T : X_T \times I_T \to \mathbb{R}$ satisfy $\int_{I_t} p_t(x, u, w)dw = 1$ and $\int_{I_T} p_T(x, w)dw = 1$ for $0 \le t \le T-1$ and any $(x, u) \in X_t \times U$. Then $J$ is a monotonically backward separable function. Moreover, if $\{c_t\}_{t=0}^T$ and $\{p_t\}_{t=0}^T$ are bounded functions, and sets $\{I_t\}_{t=0}^T$ have finite measure, then $J$ is naturally monotonically backward separable. Furthermore, if $\int_{I_i} p_i(x, u, w)c_i(x, u, w)dw \neq 0$ for all $(x, u, i) \in X_i \times U \times \{0, \dots, T-1\}$ then the associated representation maps are strictly monotonic (Eq. (15)).

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

In the next example we consider a function that can be interpreted as the expectation of cumulative stochastically stopped additive costs, where at each time stage, $t \in \{0, \dots, T-1\}$, a cost $c_t(x(t), u(t))$ is added and there is an independent probability, $p_t(x(t), u(t)) \in [0, 1]$, of stopping and incurring no further future costs. For a state and input trajectory, $(\mathbf{u}, \mathbf{x}) \in U^T \times \Pi_{t=0}^T X_t$, let us denote the stopping time by $T(\mathbf{u}, \mathbf{x})$; it then follows the distribution of this random variable is given as

$$\mathbb{P}(T(\mathbf{u}, \mathbf{x}) = T) = p_T(x(T))\Pi_{i=1}^{T-1}(1 - p_i(x(i), u(i))),$$
$$\text{and for all } t \in \mathbb{N}, \tag{19}$$
$$\mathbb{P}(T(\mathbf{u}, \mathbf{x}) = t) = p_t(x(t), u(t))\Pi_{i=1}^{t-1}(1 - p_i(x(i), u(i))),$$

where we slightly abuse notation to write $\Pi_{i=1}^{-1}(1-p_i(x(i), u(i))) = 1$ so $\mathbb{P}(T(\mathbf{u}, \mathbf{x}) = 0) = p_0(x(0), u(0))$. The stopped additive function is then given as

$$J(\mathbf{u}, \mathbf{x}) = \mathbb{E}_{T(\mathbf{u},\mathbf{x})} \Bigg[ \sum_{t=0}^{\min\{T(\mathbf{u},\mathbf{x}), T-1\}} c_t(x(t), u(t)) + \mathbb{1}_{\{(\mathbf{u},\mathbf{x}) \in U^T \times \Pi_{t=0}^T X_t : T(\mathbf{u},\mathbf{x}) = T\}}(\mathbf{u}, \mathbf{x}) c_T(x(T)) \Bigg]. \tag{20}$$

To show the function in Eq. (20) is monotonically backward separable we will assume the probability of the stopping time occurring inside the finite time horizon $\{0, \dots, T\}$ is one; this gives us the following "law of total probability" equation $\sum_{t=0}^T \mathbb{P}(T(\mathbf{u}, \mathbf{x}) = t) = 1$ for all $(\mathbf{u}, \mathbf{x}) \in U^T \times \Pi_{t=0}^T X_t$, which can be rewritten in terms of its probability density functions as,

$$\sum_{t=0}^{T-1} p_t(x(t), u(t))\Pi_{i=1}^{t-1}(1 - p_i(x(i), u(i))) + p_T(x(T))\Pi_{i=1}^{T-1}(1 - p_i(x(i), u(i))) \equiv 1. \tag{21}$$

Note, if $p_T(x(T)) \equiv 1$ then it can be trivially shown that Eq. (21) holds for any functions $p_i : X_i \times U \to [0, 1]$.

Assuming Eq. (21) holds and using the law of total expectation, conditioning on the probability of each stopping time, it follows

$$\begin{aligned} J(\mathbf{u}, \mathbf{x}) &= \mathbb{E}_{T(\mathbf{u},\mathbf{x})} \Bigg[ \sum_{t=0}^{\min\{T(\mathbf{u},\mathbf{x}), T-1\}} c_t(x(t), u(t)) + \mathbb{1}_{\{(\mathbf{u},\mathbf{x}) \in U^T \times \Pi_{t=0}^T X_t : T(\mathbf{u},\mathbf{x}) = T\}}(\mathbf{u}, \mathbf{x}) c_T(x(T)) \Bigg] \\ &= \sum_{t=0}^{T-1} \left( \sum_{s=0}^{t} c_s(x(s), u(s)) \right) \mathbb{P}(T(\mathbf{u}, \mathbf{x}) = t) + \left( \sum_{s=0}^{T} c_s(x(s), u(s)) + c_T(x(T)) \right) \mathbb{P}(T(\mathbf{u}, \mathbf{x}) = T) \\ &= \sum_{t=0}^{T-1} \left( \sum_{s=0}^{t} c_s(x(s), u(s)) \right) p_t(x(t), u(t))\Pi_{i=0}^{t-1}(1 - p_i(x(i), u(i))) \\ &\quad + \left( \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)) \right) \times p_T(x(T))\Pi_{i=0}^{T-1}(1 - p_i(x(i), u(i))). \end{aligned} \tag{22}$$

We next state and prove that the $J$ given in Eq. (22) is monotonically backward separable.

**Example 3** (*Stochastically Stopped Additive Cost*). Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is of the form

$$\begin{aligned} J(\mathbf{u}, \mathbf{x}) = & \sum_{t=1}^{T-1} \left( \sum_{s=0}^{t} c_s(x(s), u(s)) \right) p_t(x(t), u(t))\Pi_{i=0}^{t-1}(1 - p_i(x(i), u(i))) \\ & + \left( \sum_{t=0}^{T-1} c_t(x(t), u(t)) + c_T(x(T)) \right) \times p_T(x(T))\Pi_{i=0}^{T-1}(1 - p_i(x(i), u(i))), \end{aligned} \tag{23}$$

where $p_k : X_k \times U \to [0, 1]$ and $p_T : X_T \to [0, 1]$ satisfy Eq. (21), $\mathbf{u} = (u(0), \dots, u(T-1))$, $\mathbf{x} = (x(0), \dots, x(T))$, $U \subset \mathbb{R}^m$ and $X_t \subset \mathbb{R}^n$, $c_k : X_k \times U \to \mathbb{R}$ and $c_T : X_T \to \mathbb{R}$. Then $J$ is a monotonically backward separable function. Moreover, if $\{c_t\}_{t=0}^T$ are bounded functions, then $J$ is naturally monotonically backward separable. Furthermore, if $p_i(x, u) \neq 1$ for all $(x, u, i) \in X_i \times U \times \{0, \dots, T-1\}$

then the associated representation maps are strictly monotonic (Eq. (15)).

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

In the next example we introduce a function representing the number of time-steps a trajectory spends outside some target set. Later, in Section 5, we will use this function as the cost function for path planning problems.

**Example 4** (*Minimum Time Set Entry Function*). Suppose $J : U^T \times \Pi_{t=0}^T X_t \to \mathbb{R}$ is of the form

$$J(\mathbf{u},\mathbf{x}) = \min\left\{\inf\left\{t \in [0,T] : x(t) \in S\right\}, T\right\}, \tag{24}$$

where $\mathbf{u} = (u(0),\dots,u(T-1))$, $u(t) \in \mathbb{R}^m$, $\mathbf{x} = (x(0),\dots,x(T))$, $x(t) \in \mathbb{R}^n$, $U \subset \mathbb{R}^m$ and $X_t \subset \mathbb{R}^n$, and $S \subset \mathbb{R}^n$. If the set $\{t \in [0,T] : x(t) \in S\}$ is empty, we define the infimum to be infinity. Then $J$ is a naturally monotonically backward separable function.

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

## 3. The principle of optimality: A necessary condition for monotonic backward separability

Given a function, $J : \mathbb{R}^{m\times T} \times \mathbb{R}^{n\times(T+1)} \to \mathbb{R}$, there is no obvious way to determine whether $J$ is monotonically backward separable. Instead, in this section we will recall a necessary condition proposed by Bellman (1966), called the Principle of Optimality (Definition 11), that we show all MSOP's with monotonically backward separable cost functions satisfy (Proposition 12). Before recalling the definition of the Principle of Optimality let us consider a family of MSOP's, associated with the tuples $\{J_{t_0}, f, \{X_t\}_{t_0\le t\le T}, U, T\}_{t_0=0}^T$, each initialized at $(x_0,t_0) \in \mathbb{R}^n \times \{0,\dots,T\}$, and of the form:

$$\begin{aligned}
&(\mathbf{u}^*,\mathbf{x}^*) \in \arg\min_{\mathbf{u},\mathbf{x}} J_{t_0}(\mathbf{u},\mathbf{x}) \text{ subject to:} \qquad (25)\\
&x(t+1) = f(x(t),u(t),t) \text{ for } t = t_0,\dots,T-1\\
&x(t_0) = x_0,\;\; x(t) \in X_t \subset \mathbb{R}^n \text{ for } t = t_0,\dots,T\\
&u(t) \in U \subset \mathbb{R}^m \text{ for } t = t_0,\dots,T-1\\
&\mathbf{u} = (u(t_0),\dots,u(T-1)) \text{ and } \mathbf{x} = (x(t_0),\dots,x(T))
\end{aligned}$$

**Definition 11.** We say the family of MSOP's of Form (25) satisfies **the Principle of Optimality** at $x_0 \in X_0$ if the following holds. For any $t$ with $0 \le t < T$, if $\mathbf{u} = (u(0),\dots,u(T-1))$ and $\mathbf{x} = (x(0),\dots,x(T))$ solve the MSOP initialized at $(x_0,0)$ then $\mathbf{v} = (u(t),\dots,u(T-1))$ and $\mathbf{h} = (x(t),\dots,x(T))$ solve the MSOP initialized at $(x(t),t)$.

**Proposition 12.** *Consider a family of MSOP's of Form (25) associated with $\{J_t, f, \{X_t\}_{t\le s\le T}, U, T\}_{t=0}^T$. Suppose the MSOP's initialized at $(x_0,0)$ has a unique solution and $J_t : U^{T-t} \times \Pi_{s=t}^T X_s \to \mathbb{R}$ is monotonically backward separable (Definition 3). Then the family of MSOP's of Form (25) associated with $\{J_t, f, \{X_t\}_{t\le s\le T}, U, T\}_{t=0}^T$ satisfies the Principle of Optimality at $x_0 \in X_0$.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

Proposition 12 shows the Principle of Optimality (Definition 11) is a necessary condition that all families of MSOP's with unique solutions and monotonically backward separable cost functions must satisfy. We now conjecture a necessary and sufficient condition. The following notation is used in this conjecture. Given $J_t$, $\{X_t\}_{0\le t\le T}$ and $U$ let us denote the set $\mathscr{F}$, where $(f,x_0) \in \mathscr{F}$ if $x_0 \in X_0$ and the MSOP associated with $\{J_0, f, \{X_t\}_{0\le t\le T}, U, T\}$ initialized at $(x_0,0)$ has a unique solution.

**Conjecture 13.** *Consider $\{X_t\}_{0\le t\le T} \subset \mathbb{R}^{n\times T}$, $U \subset \mathbb{R}^m$ and $J_t : U^{T-t} \times \Pi_{s=t}^T X_s \to \mathbb{R}$. Then, for any $(f,x_0) \in \mathscr{F}$ the family of MSOP's associated with $\{J_t, f, \{X_t\}_{t\le s\le T}, U, T\}_{t=0}^T$ satisfy the Principle of Optimality at $x_0 \in X_0$ if and only if $J_t$ is monotonically backward separable.*

Regardless of whether Conjecture 13 is true, Proposition 12 is useful. Proposition 12 provides a way of proving a function $J_t : U^{T-t} \times \Pi_{s=t}^T X_s \to \mathbb{R}$ is not monotonically backward separable. Rather than showing $J_t$ does not satisfy Definition 3 for every family of representation maps $\{\phi_s\}_{s=t}^T$, for which there are an uncountably many, we find any $f$ for which the family of MSOP's associated with $\{J_t, f, \{X_s\}_{t\le s\le T}, U, T\}_{t=0}^T$ has a unique solution for some initialization $(x_0,0)$ and does not satisfy the Principle of Optimality. Then Proposition 12 shows $J_t$ is not monotonically backward separable. We demonstrate this proof strategy in the following lemma.

**Lemma 14.** *The function $J_t : \mathbb{R}^{m\times(T-t)} \times \mathbb{R}^{n\times(T+1-t)} \to \mathbb{R}$, defined as*

$$J_t(\mathbf{u},\mathbf{x}) := \sum_{s=t}^{T-1} c_s(u(s)) + \max_{t\le s\le T} d(x(s)), \tag{26}$$

*is not monotonically backward separable (Definition 3) for all functions $c_k : \mathbb{R}^m \to \mathbb{R}$ and $d_k : \mathbb{R}^n \to \mathbb{R}$.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

**Remark 15.** The function given in Eq. (26) can clearly be expressed as the addition of two monotonically backward separable functions, $J_1(\mathbf{u},\mathbf{x}) = \sum_{s=t}^{T-1} c_s(u(s))$ (Lemma 4) and $J_2(\mathbf{u},\mathbf{x}) = \max_{t\le s\le T} d(x(s))$ (Example 1). Therefore, Lemma 14 shows that the property of monotonically backward separability is not preserved under addition.

## 4. Comparison with state augmentation methods

We proposed an alternative method for solving MSOP's with non-additively separable costs in Jones and Peet (2018); where cost functions are forward separable:

$$\begin{aligned}
J(\mathbf{u},\mathbf{x}) = \psi_T(x(T), \psi_{T-1}(x(T-1), u(T-1), \psi_{T-2}(\dots,\\
\psi_1(x(1), u(1), \psi_0(x(0), u(0)))),\dots,)),
\end{aligned} \tag{27}$$

where $\psi_0 : X_0 \times U \to \mathbb{R}^k$, $\psi_t : X_t \times U \times Image\{\psi_{t-1}\} \to \mathbb{R}^k$ for $t \in \{1,\dots,T-1\}$, and $\psi_T : X_T \times Image\{\psi_{T-1}\} \to \mathbb{R}$.

It was shown that for $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, where $J$ is of the Form (27), an equivalent MSOP with additively separable cost function, $\{\tilde J, \tilde f, \{\tilde X_t\}_{0\le t\le T}, U, T\}$, can be constructed, where $\tilde J(\mathbf{u},\mathbf{x}) = \psi_T(x(T))$, $\tilde f([x_1,x_2]^T, u, t) = [f(x_1,u,t), \psi_t(x_1,u,x_2)]^T$, and $\tilde X_t = X_t \times Image\{\psi_t\}$. The augmented MSOP, $\{\tilde J, \tilde f, \{\tilde X_t\}_{0\le t\le T}, U, T\}$, can then be solved using the classical Bellman Eq. (16). This state augmentation method is particularly useful when solving MSOP's with cost functions that are not monotonically backward separable, for instance the function in Eq. (26). However, the augmented MSOP has a larger state space dimension. Therefore, in the case when the cost function is both forward separable, of Form (27), and monotonically backward separable, of Form (4), it is computationally more efficient to solve the GBE (12) rather than augmenting and solving Bellman's

Eq. (16). We now demonstrate this in the following numerical example.

Consider the MSOP

$$\min_{\mathbf{u},\mathbf{x}} \sqrt{x(0)+u(0)+\sqrt{....\sqrt{x(T-1)+u(T-1)+\sqrt{x(T)}}}}$$

subject to: (28)

$$x(t+1)=\begin{cases}2 \text{ if } u=0.5\\ 1 \text{ if } u=1\end{cases} \quad \text{for } t=0,\dots,T,$$

$$x(0)=2,\ x(t)\in\{1,2\} \text{ for } t=0,\dots,T,$$

$$u(t)\in\{0.5,1\} \text{ for } t=0,\dots,T-1.$$

The cost function in the above MSOP is naturally monotonically backward separable and can be written in the Form (4) with representation maps

$$\phi_T(x)=\sqrt{x},\ \ \phi_t(x,u,z)=\sqrt{x+u+z} \text{ for } t\in\{0,\dots,T-1\}. \quad (29)$$

Moreover the cost function is also forward separable and can be written in the Form (27) with representation maps

$$\psi_0(x,u)=[x,u]^T,\quad \psi_t(x,u,z)=[z,x,u]^T, \quad (30)$$

$$\psi_T(x,z)=\sqrt{z_1+z_2+\sqrt{....\sqrt{z_{2T-1}+z_{2T}+\sqrt{x}}}}.$$

We solved the MSOP in Eq. (28) using both the GBE and the state augmentation method, plotting the computation time results in Fig. 1. The green points represent the computation time required to construct the value function by solving the GBE (12) with representation maps given in Eq. (29), and then to synthesize the optimal input sequence using Eq. (13). The red points represent the computation time required to construct the value function by solving Bellman's Eq. (16) for the state augmented MSOP and then to construct the optimal input sequence. The green points increase linearly as a function of the terminal time, $T\in\mathbb{N}$, of order $\mathcal{O}(T)$, whereas the red points increase exponentially with respect to $T$, of order $\mathcal{O}(2^T)$ (due to the fact that using representation maps, given in Eq. (30), results in an augmented state space of size $2^T$). Moreover, Fig. 1 also includes blue dots representing computation times required to solve the GBE approximately, as discussed in the next section.

### 4.1. Approximate dynamic programming using the GBE

Rather than solving the MSOP (28) exactly using the GBE, as we did in the previous section, we now use an Approximate Dynamic Programming (ADP)/Reinforcement Learning (RL) algorithm to heuristically solve the MSOP and numerically show these algorithms can result in lower computational times when compared to methods that solve the GBE exactly. This demonstrates that MSOP's with monotonically backward separable cost functions can be heuristically solved using the same methods developed in the ADP literature with the aid of the methodology developed in this paper.

Typically ADP methods use parametric function fitting (neural networks, linear combinations of basis functions, decision tree's, etc.) to approximate the value function from data. The approximated value function is then used to synthesize a suboptimal input sequence. To see how this works, suppose an ADP algorithm constructs some approximate value function, denoted $\tilde{V}(x,t)$, then an approximate optimal input sequence, $\tilde{\mathbf{u}}=(\tilde{u}(0),\dots,\tilde{u}(T))$, can be constructed by solving

$$\tilde{u}(k)\in\arg\inf_{u\in\Gamma_{\tilde{x}(k),k}}\left\{\phi_t(\tilde{x}(k),u,\tilde{V}(f(\tilde{x}(k),u,k),k+1))\right\} \quad \text{for } k\in\{0,\dots,T-1\}.$$

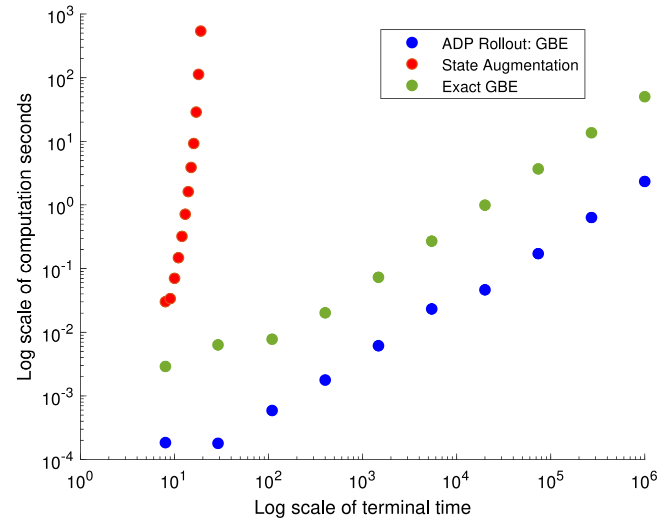


**Fig. 1.** Log log graph showing computation time for solving MSOP (28) using state augmentation (red points), via exactly solving GBE (green points), and via approximately solving the GBE using the rollout (blue points) algorithm versus the terminal time of the problem.

$$\tilde{x}(0)=x_0,\quad \tilde{x}(k+1)=f(\tilde{x}(k),\tilde{u}(k),k) \quad \text{for } k\in\{0,\dots,T-1\}. \quad (31)$$

One way to obtain an approximate value function, $\tilde{V}$, is to use the rollout algorithm found in the textbook (Bertsekas, 1995). This algorithm supposes a base policy is known $\mu_{base}:\mathbb{R}^n\times\mathbb{N}\to U$ and approximates the value function as follows

$$\begin{aligned}\tilde{V}(x,t)=&\phi_t(x(t),u(t),\phi_{t+1}(x(t+1),u(t+1),\dots\phi_T(x(T))\dots)),\\ &\text{where } x(t)=x \text{ and for all } s\in\{t,\dots,T-1\},\\ &x(s+1)=f(x(s),u(s),t),\ \ u(s)=\mu_{base}(x(s),s).\end{aligned}$$

Using the base policy $\mu_{base}(x,t)=\begin{cases}1 \text{ if } t/4\in\mathbb{N}\\ 0.5 \text{ otherwise}\end{cases}$ we used the rollout algorithm to solve the MSOP (28) for terminal times $T=8$ to $10^6$. Computation times are plotted as the blue points in Fig. 1 showing better performance than solving the GBE exactly or using state augmentation.

## 5. Application: Path planning and obstacle avoidance

In this section we design a full state feedback controller (Markov Policy) for a discrete time dynamical system with the objective of reaching a target set in minimum time while avoiding moving obstacles.

### 5.1. MSOP's for path planning

We say the MSOP, associated with tuple $\{J,f,\{X_t\}_{0\le t\le T},U,T\}$, defines a Path Planning DP problem if

- $J(\mathbf{u},\mathbf{x})=\min\left\{\inf\left\{t\in[0,T]:x(t)\in S\right\},T\right\}$.
- $S=\{x\in\mathbb{R}^n:g(x)<0\}$, where $g:\mathbb{R}^n\to\mathbb{R}$.
- $X_t=\mathbb{R}^n/(\cup_{i=1}^N O_{t,i})$, where $O_{t,i}=\{x\in\mathbb{R}^n:h_{t,i}(x)<0\}$ and $h_{t,i}:\mathbb{R}^n\to\mathbb{R}$.
- There exists a feasible solution, $(\mathbf{u},\mathbf{x})$, to the MSOP (2) associated with the tuple $\{J,f,\{X_t\}_{0\le t\le T},U,T\}$ such that $x(k)\in S$ for some $k\in\{0,\dots,T\}$.

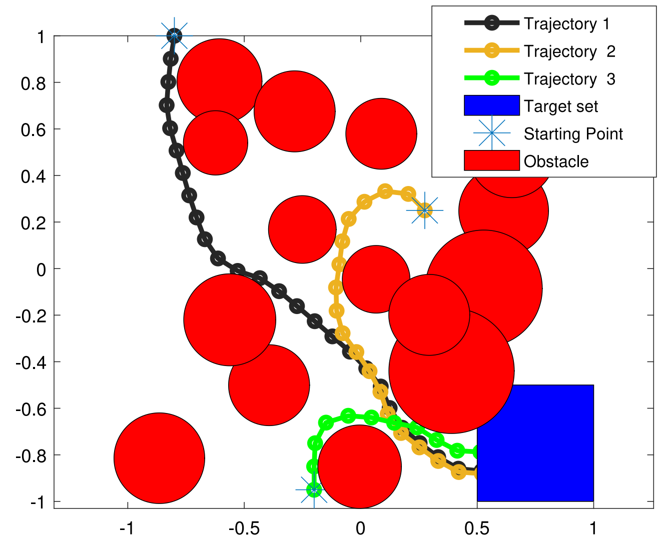


**Fig. 2.** Graph showing approximate optimal trajectories, shown as the gold, black and green curves, with dynamics given in Eq. (32) and the goal of reaching the target set, shown as the blue square, while avoiding obstacles, shown as red circles.

Clearly, solving the MSOP (2) associated with a path planning problem tuple, $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, is equivalent to finding the input sequence that drives a discrete time system, governed by the vector field $f$, to a target $S$ in minimum time while avoiding the moving obstacles, represented as sets $O_{t,i} \subset \mathbb{R}^n$. Moreover, as shown in Example 4, the cost function $J$ is a naturally forward separable function (Definition 3).

### 5.2. Path planning for Dubin's car

We now solve the path planning problem with dynamics as defined in Maidens, Barrau, Bonnabel, and Arcak (2018); also known as the Dubin's car dynamics.

$$f(x,u,t) = \left[x_1 + v\cos(x_3), x_2 + v\sin(x_3), x_3 + \frac{v}{L}\tan(u)\right]^T, \quad (32)$$

where $(x_1, x_2) \in \mathbb{R}^2$ is the position of the car, $x_3 \in \mathbb{R}$ denotes the angle the car is pointing, $u \in \mathbb{R}$ is the steering angle input, $v \in \mathbb{R}$ is the fixed speed of the car, and $L$ is a parameter that determines the turning radius of the car.

We solve the path planning problem using a discretization scheme, similar to Jones and Peet (2018); such discretization schemes are known to be parallelizable (Maidens, Packard, & Arcak, 2016). The target set, obstacles, state space, and input constraint sets are given by

$$S = \{(x_1,x_2) \in \mathbb{R}^2 : -0.25 < x_1 - 0.75 < 0.25, \quad -0.25 < x_2 + 0.75 < 0.25\}$$

$$O_{t,i} = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1 - X_i)^2 + (x_2 - Y_i)^2 - R_i^2 < 0\} \text{ for } i \in \{1,\dots,15\} \text{ and } t \in \{0,\dots,T\}$$

$$X_t = [-1,1]^2 \times \mathbb{R} \text{ for } t \in \{0,\dots,T\}, \quad U = [-1,1],$$

where $X, Y, R \in \mathbb{R}^{15}$ are randomly generated vectors. The parameters of the system are set to $v = 0.1$ and $L = 1/6$.

Fig. 2 shows three approximately optimal state sequences starting from different initial conditions. These state sequences are found by numerically solving the GBE (12), where $\{\phi_t\}_{t=0}^T$ are as in Example 4. To numerically solve the GBE (12) the state space, $X_t \subset \mathbb{R}^3$, is discretized as a $60 \times 60 \times 60$-grid between $[-1,1]^2 \times [0, 2\pi]$ and the input space, $U \subset \mathbb{R}$, is discretized as 100 grid points within $[-1, 1]$. The first state sequence was chosen to have initial condition $[-0.8, 1, -0.55\pi]^T \in \mathbb{R}^3$ (the furthest of the three trajectories from the target) and took 25 steps to reach its goal. The second state sequence was chosen to have initial condition $[0.275, 0.25, 0.75\pi]^T \in \mathbb{R}^3$; in this case as $x_3(0) = 0.75\pi$ Dunbin's car initially is directed towards the top left corner. The input sequence successfully turns the car downwards between two obstacles and into the target set, taking 18 steps. The third trajectory was chosen to have initial condition $[-0.2, 0.95, 0.5\pi]^T \in \mathbb{R}^3$-starting very closely to an obstacle facing upwards. This trajectory had to use the full turning radius of the car to navigate around the obstacle towards the target set and took 10 steps.

### 5.3. Path planning in 3D

We now solve a three dimensional path planning problem with dynamics given by

$$f(x,u,t) = [x_1 + u_1, x_2 + u_2, x_3 + u_3]^T. \quad (33)$$

The target set, obstacles, state space and input constraint set were respectively are given by

$$S = \{(x_1,x_2,x_3) \in \mathbb{R}^2 : -0.25 < x_1 - 0.75 < 0.25, \quad -0.25 < x_2 + 0.75 < 0.25, -0.25 < x_2 + 0.75 < 0.25\}$$

$$O_{t,i} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1 - A_i - \alpha_i t)^2 + (x_2 - B_i - \beta_i t)^2 + (x_2 - C_i - \gamma_i t)^2 - R_i^2 < 0\} \text{ for } i \in \{1,\dots,35\}, t \in \{0,\dots,T\}$$

$$X_t - [-1,1]^3 \text{ for } t \in \{0,\dots,T\}, \qquad U = [-0.05, 0.05]^3,$$

where $A, B, C, \alpha, \beta, \gamma, R \in \mathbb{R}^{35}$ are randomly generated vectors. Note, when $\alpha, \beta, \gamma$ are non-zero the centre of the spherical obstacles moves with time. For presentation purposes in this paper we consider stationary obstacles, selecting $\alpha = \beta = \gamma = 0$, however, a downloadable .gif file showing the numerical solution for moving obstacles can be found at Jones and Peet (2020b).

This path planning problem can be numerically solved by computing the solution to the GBE (12) using $\{\phi_t\}_{t=0}^T$ as given in Example 4. To numerically solve the GBE (12) we discretized the state and input space, $X_t \subset \mathbb{R}$ and $U \subset \mathbb{R}^3$, as a $40 \times 40 \times 40$ uniform grid on $[-1,1]^3$ and a $5 \times 5 \times 5$ uniform grid on $[-0.05, 0.05]^3$ respectively. Fig. 3 shows four optimal state sequences, shown as green lines, starting from various initial conditions. All trajectories successfully avoid the obstacles, represented as red spheres, and reach the target set, shown as a blue cube.

**GPU Implementation** All DP methods involving discretization fall prey to the curse of dimensionality, where the number of points required to sample a space increases exponentially with respect to the dimension of the space. For this reason solving MSOP's in dimensions greater than three can be computationally challenging. Fortunately, our discretization approach to solving the GBE (12), can be parallelized at each time-step. To improve the scalability of the proposed approach, we have therefore constructed in Matlab a GPU accelerated DP algorithm for solving the 3D path planning problem. This code is available for download at Code Ocean (Jones & Peet, 2019a).

## 6. Application: Maximal invariant sets

The Finite Time Horizon Maximal Invariant Set (FTHMIS) is the largest set of initial conditions such that there exists an input sequence that produces a feasible state sequence over a finite time period. Computation of the maximal robust invariant sets over infinite time horizons was considered in Xue and Zhan (2018). Before we define the FTHMIS we introduce some notation.

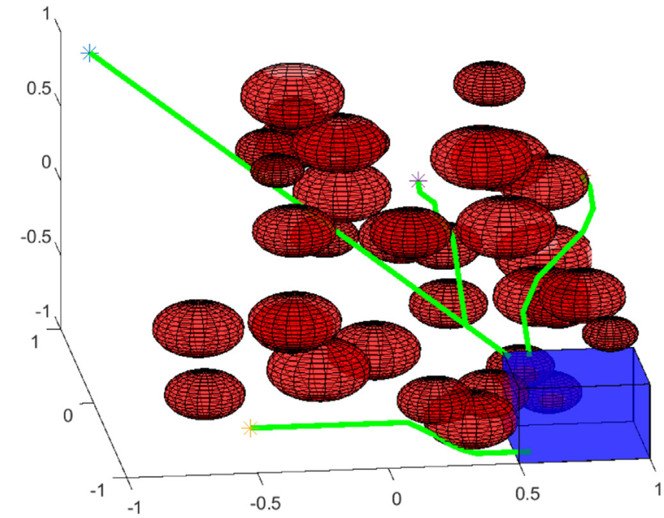

**Fig. 3.** Graph showing approximate optimal trajectories, shown as the green curves, with dynamics given in Eq. (33) and the goal of reaching the target set, shown as the blue cube, while avoiding obstacles, shown as red spheres.

For $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n$ we say the map $\rho_f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{m\times(T-1)} \to \mathbb{R}^n$ is the solution map associated with $f$ if the following holds for all $(x_0, t, \mathbf{u}) \in \mathbb{R}^n \times \{0,\dots,T\} \times \mathbb{R}^{m\times(T-1)}$

$$\rho_f(x_0, t, \mathbf{u}) = x(t),$$

where $\mathbf{u} = (u(0),\dots,u(T-1))$, $x(k+1) = f(x(k), u(k), k)$ for all $k \in \{0,\dots,k-1\}$, and $x(0) = x_0$.

**Definition 16.** For $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}^n$, $X_t \subseteq \mathbb{R}^n$, $U \subset \mathbb{R}^m$, $T \in \mathbb{N}$, and $\mathscr{A}_t \subseteq \mathbb{R}^n$ we define *the Finite Time Horizon Maximal Invariant Set* (FTHMIS), denoted by $\mathscr{R}$, by

$$\mathscr{R} := \{x_0 \in \mathbb{R}^n : \text{there exists } \mathbf{u} \in \Gamma_{x_0,[0,T-1]} \text{ such that } \rho_f(x_0, t, \mathbf{u}) \in \mathscr{A}_t \text{ for all } t \in \{0,\dots,T\}\},$$

where the notation $\Gamma_{x_0,[0,T-1]}$ is as in Eq. (8).

We next show that the sublevel set of the value function associated with a certain MSOP can completely characterize the FTHMIS.

**Theorem 17.** *Consider the sets $\mathscr{A}_t = \{x \in \mathbb{R}^n : g_t(x) < 0\}$, where $g_t : \mathbb{R}^n \to \mathbb{R}$. Suppose $V$ is a value function associated with the MSOP, defined by the tuple $\{J, f, \{X_t\}_{0\le t\le T}, U, T\}$, where $J(\mathbf{u},\mathbf{x}) = \max_{0\le k\le T} g_k(x(k))$. Then*

$$\mathscr{R} = \{x \in \mathbb{R}^n : V(x,0) < 0\}, \tag{34}$$

*where the set $\mathscr{R} \subset \mathbb{R}^n$ is the FTHMIS as in Definition 16.*

**Proof.** Please see extended Arxiv paper (Jones & Peet, 2020a). ■

*6.1. Numerical example: Maximal invariant sets*

Value functions can characterize FTHMIS's, as shown by Theorem 17. We now approximate a FTHMIS by computing a value function using a discretization scheme for solving the GBE (12) using $\{\phi_t\}_{t=0}^T$ as given in Example 1. Let us consider a discrete time switching system, whose Robust Maximal Invariant Set (RMIS) was previously computed in Xue and Zhan (2018):

$$f(x,u,t) = \begin{cases} \begin{bmatrix} x_1 \\ (0.5+u)x_1 - 0.1x_2 \end{bmatrix} & \text{if } 1-(x_1-1)^2 - x_2^2 \le 0 \\ \begin{bmatrix} x_2 \\ 0.2x_1 - (0.1+u)x_2 + x_2^2 \end{bmatrix} & \text{otherwise.} \end{cases} \tag{35}$$

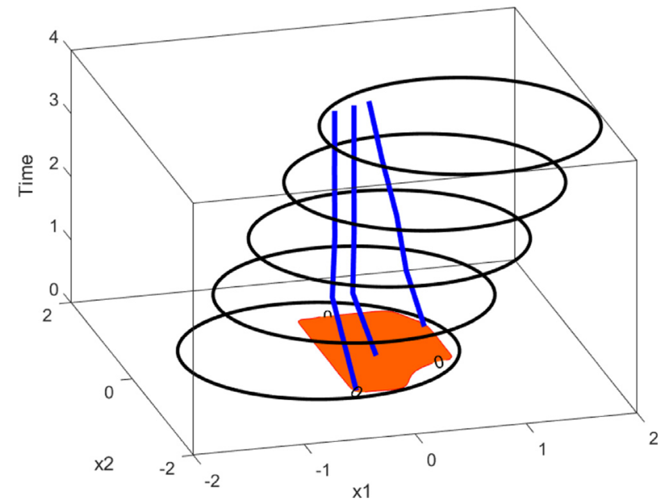


**Fig. 4.** Figure showing an approximation of $L(V,0) := \{x \in \mathbb{R}^n : V(x,0) \le 0\}$, shown in the shaded orange region, where $V$ is the value function of the MSOP associated with Eq. (35). The $z$-axis represents time and the black circular lines represent the boundary of $\mathscr{A}_t$ for $t = 1,2,3,4$. Three sample trajectories, shown in blue, start in $L(V,0)$ and remain in the sets $\mathscr{A}_t$ for the time-steps $t = 1,2,3,4$; giving numerical evidence that $L(V,0)$ is indeed an approximation of the FTHMIS.

We now compute the FTHMIS, denoted by $\mathscr{R}$, associated with

$$\mathscr{A}_t = \{x \in \mathbb{R}^2 : g_t(x) \le 0\} \text{ for all } t \in \{0,\dots,T\},$$

$$g_t(x) = \left(x_1 - \frac{(t-1)}{4}\right)^2 + \left(x_2 - \frac{(t+1)}{4}\right)^2 - 1.5,$$

$$X_t = [-1,1]^2 \text{ for all } t \in \{0,\dots,T\},$$

$$U = \{u \in \mathbb{R} : u^2 - 0.01 \le 0\}, \quad T = 4.$$

Fig. 4 shows the FTHMIS, $\mathscr{R}$, found by using a discretization scheme to solve the GBE (12) for $5 \times 5$ state grid points in $[-1,1]^2$. To represent $\mathscr{R}$ in $\mathbb{R}^2$, once the value function, $V$, is found at each grid point a polynomial function is fitted and its zero-sublevel set, shown as the orange shaded region, approximately gives $\mathscr{R}$.

## 7. Conclusion

For MSOP's with monotonically backward separable cost functions we have derived necessary and sufficient conditions for solutions to be optimal. We have shown that by solving the Generalized Bellman's Equation (GBE) one can derive an optimal input sequence. Furthermore, we have demonstrated the GBE can be numerically solved using a discretization scheme and Approximate Dynamic Programming (ADP) techniques such as Rollout. We have shown our numerical methods can solve current practical problems of interest; such as path planning and the computation of maximal invariant sets.

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