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RESOLUTION ANALYSIS OF INVERTING THE GENERALIZED RADON TRANSFORM FROM DISCRETE DATA IN $\mathbb{R}^3$*

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Abstract. A number of practically important imaging problems involve inverting the generalized Radon transform (GRT) $\mathcal{R}$ of a function f in $\mathbb{R}^3$. On the other hand, not much is known about the spatial resolution of the reconstruction from discretized data. In this paper we study how accurately and with what resolution the *singularities* of f are reconstructed. The GRT integrates over a fairly general family of surfaces $\mathcal{S}_y$ in $\mathbb{R}^3$. Here y is the parameter in the data space, which runs over an open set $\mathcal{V} \subset \mathbb{R}^3$. Assume that the data $g(y) = (\mathcal{R}f)(y)$ are known on a regular grid y_j with step-sizes $O(\epsilon)$ along each axis, and suppose $\mathcal{S} = \text{singsupp}(f)$ is a piecewise smooth surface. Let f_ϵ denote the result of reconstruction from the discrete data. We obtain explicitly the leading singular behavior of f_ϵ in an $O(\epsilon)$-neighborhood of a generic point $x_0 \in \mathcal{S}$, where f has a jump discontinuity. We also prove that under some generic conditions on $\mathcal{S}$ (which include, e.g., a restriction on the order of tangency of $\mathcal{S}_y$ and $\mathcal{S}$), the singularities of f do not lead to nonlocal artifacts. For both computations, a connection with the uniform distribution theory turns out to be important. Finally, we present a numerical experiment, which demonstrates a good match between the theoretically predicted behavior and actual reconstruction.

Key words. generalized Radon transform, discrete data, resolution analysis, uniform distribution

AMS subject classifications. 44A12, 65R10, 94A08

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1. Introduction.

A large number of practically important imaging problems involve inversion of the generalized Radon transform (GRT), i.e., recovering an unknown function f from its integrals over a family of surfaces. The reconstruction may involve finding f itself, or finding f modulo smoother terms. Most of the time, the surfaces are not planes. Below is a list of some of the most common integral transforms with some of the most prominent examples of their use.

1. Integration over spheres. Applications include ultrasound imaging and, in particular, SONAR (see [17] and references therein), as well as thermoacoustic and photoacoustic tomography [14, 23].
2. Integration over ellipses. This transform arises in linearized seismic imaging with a common offset between the sources and receivers [7].
3. Integration over cones arises in Compton camera imaging. Applications are single-scattering optical tomography, Compton camera medical imaging, and homeland security (see [21] for a recent review).

In all of the above cases, one collects a discrete data set and reconstructs f using a numerical algorithm. Frequently, reconstruction is achieved by applying a linear inversion formula (as opposed to a nonlinear reconstruction algorithm based on fidelity functional minimization). In all of the above examples it is of fundamental importance to know the resolution of the method as a function of (1) the data sampling rate and (2) specific implementation of the inversion formula that is used. Despite the

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significance of this problem, not much is known about the resolution of reconstruction from discrete data. The main reason for this is that the classical sampling theory, which addresses such problems, can be applied only to the classical Radon transform (CRT) and only under restricted conditions [16]. The known results are quite scarce, and they are of a semiqualitative nature (see, e.g., pp. 784–786 in [6]). Very recently, a more flexible approach to sampling based on semiclassical analysis was proposed in [20]. Let A be a Fourier integral operator (FIO). The idea of [20] is to determine how the data Af should be sampled to allow for accurate interpolation of its values on a lattice provided that f is semiclassically bandlimited. If the sampling condition is violated, then reconstruction from the discrete values of Af (i.e., applying a parametrix A^{-1} to the interpolated Af) leads to aliasing artifacts, which are also analyzed in [20].

An alternative approach to the analysis of resolution was proposed recently in [11, 12]. The idea is to investigate how accurately and with what resolution the *singularities* of f are reconstructed. For some of the above problems there is no exact inversion formula, and inversion modulo smoother terms is the most one can hope for. In such cases, spatial resolution of the recovery of singularities is all one needs. Note that in this paper both f and $g = \mathcal{R}f$ are assumed to have singularities in the sense of a conventional, classical wavefront set (see, e.g., [8]). In contrast, the main assumption in [20] is that f and, consequently, the data Af have only semiclassical singularities (see, e.g., [25]). It is possible to apply the approach of [20] to the analysis of classical singularities, but this would require summing a series over "folded" frequencies in the Fourier domain, which is complicated.

In [11, 12] the author considers the inversion of the CRT of f in $\mathbb{R}^2$ and $\mathbb{R}^3$. The parametrization of the data is standard, i.e., in terms of the affine and angular variables. Suppose the step-sizes along the angular and affine variables are $O(\epsilon)$. Let f_ϵ denote the result of reconstruction from the discrete data. The author picks a point x_0, where f has a jump singularity, and obtains explicitly the leading singular behavior of f_ϵ in an $O(\epsilon)$-neighborhood of x_0 as $\epsilon \to 0$. The obtained behavior, which we call *edge response*, provides the desired resolution of the reconstruction algorithm. It is shown also that convex parts of the singular support of f do not create nonlocal artifacts. The case when f changes during the scan (so-called dynamic CT) is considered in the two-dimensional setting as well [11].

In this paper we generalize the approach of [11, 12]. The reconstruction problem is now formulated in terms of the GRT $\mathcal{R}$, which integrates f over a fairly general family of surfaces $\mathcal{S}_y$ in $\mathbb{R}^3$. Here $\text{supp}(f) \subset \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^3$ is an open set, and y is the parameter in the data space. For the problem to be well-determined, we assume that y runs over an open set $\mathcal{V} \subset \mathbb{R}^3$. As is seen, our setting is fairly general and covers all the problems mentioned above. The GRT in this paper is very close to that considered by Beylkin in [2], only the parametrization of the surfaces $\mathcal{S}_y$ is slightly different. This gives us more flexibility to connect our results with practical applications, where GRTs arise.

Assume that the data $g = \mathcal{R}f$ are known on a regular grid y_j with step-sizes $O(\epsilon)$ along each axis. Suppose $\mathcal{S} = \text{singsupp}(f)$ is a piecewise smooth surface. Similarly to [11, 12], we obtain explicitly the leading singular behavior of f_ϵ in an $O(\epsilon)$-neighborhood of a generic point $x_0 \in \mathcal{S}$, where f has a jump discontinuity. We also prove that under some generic conditions on $\mathcal{S}$ (which include, e.g., a restriction on the order of tangency of $\mathcal{S}_y$ and $\mathcal{S}$), the singularities of f do not lead to nonlocal artifacts. For both computations, a connection with the uniform distribution theory [15] turns out to be important. It is possible that violation of the imposed conditions leads to artifacts. Analysis of such artifacts and analysis of more general surfaces $\mathcal{S}$ will be the subject of future research.

The reconstruction formula $g \to \check{f}$, which contains a suitably adapted adjoint $\mathcal{R}^*$, is one specific example of an FIO. Here $\check{f}$ is such that $\check{f} - f$ is smoother than f. Thus, the reconstruction algorithm can be viewed as an application of an FIO to discrete data $(\mathcal{R}f)(y_j)$. A number of methods for computing the action of FIOs on discrete data have been proposed; see, e.g., [4, 5, 1, 24] and references therein. To the best of the author's knowledge, here we propose the first method to compute the *resolution* of the reconstruction obtained by applying an FIO to discrete data that comes from an image with classical singularities (jump discontinuities). Extension of the method to more general FIOs and more general singularities will also be the subject of future work. Some results along this direction are in [13].

The paper is organized as follows. In section 2 we define the GRT $\mathcal{R}$ via an incidence relation $\mathcal{C} \subset \mathcal{U} \times \mathcal{V}$, list the properties of the function $\Phi(x, y)$ that defines the incidence relation, define generic points, and specify the continuous and discrete inversion formulas that are used in the analysis. The main result is formulated in section 3; see Theorem 3.1. In this theorem we give a simple and explicit expression for the edge response of a reconstruction from $(\mathcal{R}f)(y_j)$. Instead of some cumulative characteristic of resolution represented by a single number (e.g., a common measure is full width at half maximum, or FWHM for short), we compute the entire edge response curve. Hence, this result constitutes a comprehensive answer to the question of how accurately and with what resolution the jumps of f can be reconstructed from discrete GRT data. Section 3 also contains the beginning of the proof of Theorem 3.1. The entire proof spans sections 3–7. In section 3 we obtain the behavior of g near its singular support, which generalizes one of the results of [18, 19] from the CRT to the GRT. The behavior of the interpolated data g_ϵ near singsupp(g) is obtained in section 4. The contribution of the leading singular term to the edge response at a generic point $x_0 \in \mathcal{S}$ is computed in section 5. In section 6 we show that lower order terms do not contribute to the edge reponse. In section 7 we prove that, under some assumptions, remote singularities do not contribute to the edge response either. For convenience of the reader, in section 8 we introduce a GRT that integrates over a family of spheres tangent to a plane and obtain explicitly some of the key quantities that arise in our derivation. The example is continued in section 9, where we work out the details of the inversion formula for this transform. We also present the results of a numerical experiment, which show a good match between the theoretically predicted behavior and actual reconstruction. Finally, Appendix A contains the statement of the Morse lemma and a short explanation of how it is used in our work.

2. Preliminary construction. Let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^3$ be two open connected sets, where $\mathcal{U}$ is the image domain, and $\mathcal{V}$ is the data domain. Each $y \in \mathcal{V}$ determines a smooth surface $\mathcal{S}_y \subset \mathcal{U}$. Let $\mathcal{C}$ be an incidence relation $\mathcal{C} \in \mathcal{U} \times \mathcal{V}$, which is defined in terms of a smooth function $\Phi(x, y) \in C^\infty(\mathcal{U} \times \mathcal{V})$:

$$\mathcal{C} := \{(x, y) \in \mathcal{U} \times \mathcal{V} : \Phi(x, y) = 0\}. \tag{2.1}$$

Another way to state (2.1) is that $x \in \mathcal{S}_y$ if and only if $\Phi(x, y) = 0$. Define the submanifold:

$$\mathcal{T}_x := \{y \in \mathcal{V} : \Phi(x, y) = 0\},\ x \in \mathcal{U}. \tag{2.2}$$

Thus, $\mathcal{T}_x$ is the collection of all $y \in \mathcal{V}$ such that $\mathcal{S}_y$ contains x. The main assumptions about Φ are as follows ("DF" stands for defining function):

DF1. Φ is real-valued and nondegenerate, i.e.,

$$\Phi'_x(x,y) \neq 0,\ \Phi'_y(x,y) \neq 0,\ (x,y) \in \mathcal{C}. \tag{2.3}$$

DF2. For each $x \in \mathcal{U}$, the map $\mathcal{T}_x \to S^2$ defined by $y \to \pm\Phi'_x(x,y)/|\Phi'_x(x,y)|$, $y \in \mathcal{T}_x$, is surjective.

DF3. For each $y \in \mathcal{V}$, the vectors $\Phi'_y(x,y)$ and $\Phi'_y(z,y)$ are not parallel whenever $x, z \in \mathcal{S}_y$, $x \neq z$.

DF4. The mixed Hessian of Φ is nondegenerate,

$$\det\left(\frac{\partial^2\Phi(x,y)}{\partial x_i \partial y_j}\right) \neq 0,\ (x,y) \in \mathcal{C}, \tag{2.4}$$

where $\partial/\partial x_i$, $i = 1,2$, and $\partial/\partial y_j$, $j = 1,2$, are basis vectors in the tangent spaces to the submanifolds $\mathcal{S}_y$ and $\mathcal{T}_x$ at x and y, respectively.

By DF1, $\mathcal{S}_y$ and $\mathcal{T}_x$ are immersed submanifolds. Condition DF2 means that the tomographic data are complete, i.e., any singularity is visible. Condition DF3 says that there are no conjugate points. Condition DF4 is a local version of the Bolker condition. Conditions DF3 and DF4 imply the (global) Bolker condition. Conditions DF1–DF4 are analogous to Conditions (I)–(IV) in [2]. Conditions DF1 and DF4 combined are equivalent to the condition (cf. equation (4.23) in [22, p. 335]) that at every point $(x,y) \in \mathcal{C}$,

$$\det\begin{pmatrix} \Phi''_{xy} & (\Phi'_x)^T \\ \Phi'_y & 0 \end{pmatrix} \neq 0. \tag{2.5}$$

Indeed, let W denote the matrix in (2.5). Pick any $(x,y) \in \mathcal{C}$. By DF1, we can augment $x_i, i = 1,2$, and $y_j, j = 1,2$ (cf. condition DF4), to local coordinates in $\mathcal{U}$ and $\mathcal{V}$, respectively, so that $\Phi'_x = (1,0,\dots,0)$ and $\Phi'_y = (1,0,\dots,0)$. Subtracting the appropriate multiples of the bottom row $(\Phi'_y, 0)$ and the right column $(\Phi'_x, 0)^T$ from the rest of W transforms W to the matrix

$$\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0}^T & \tilde{W} & 0 \\ 1 & \mathbf{0} & 0 \end{pmatrix},\ \mathbf{0} := (0,0), \tag{2.6}$$

where $\tilde{W}$ is the 2×2 matrix in (2.4). Hence conditions DF1 and DF4 are equivalent to (2.5).

In the paper we consider functions, which can be represented as a finite sum

$$f(x) = \sum_j \chi_{D_j} f_j(x), \tag{2.7}$$

where χ_{D_j} is the characteristic function of the domain $D_j \subset \mathcal{U}$. For each j,

1. D_j is compact,
2. D_j is bounded by finitely many smooth surfaces, i.e., $\partial D_j \subset \cup_{l=1}^{L_j} \mathcal{S}_{jl}$, and
3. f_j is C^∞ in a domain containing the closure of D_j.

Denote $\mathcal{S} := \cup_j \partial D_j$. By construction, singsupp$(f) \subset \mathcal{S}$. Similarly to (2.2), let

$$\mathcal{T}_{\mathcal{S}} := \{y \in \mathcal{V} : \mathcal{S}_y \text{ is tangent to } \mathcal{S}\}. \tag{2.8}$$

Generally, $\mathcal{S}$ is not necessarily smooth, so the notion of tangency in (2.8) needs to be suitably generalized (see [18]). In this paper, we consider only the cases when $\mathcal{S}_y$ is tangent to $\mathcal{S}$ at points where $\mathcal{S}$ is smooth. The GRT of f is given by

$$g(y) = (\mathcal{R}f)(y) := \int_{\mathcal{S}_y} b(x,y)f(x)dx,\ y \in \mathcal{V}, \tag{2.9}$$

where the weight b is smooth (i.e., C^∞) and nonvanishing, and dx is the area element on $\mathcal{S}_y$. The discrete data are given by

$$g(\epsilon j),\ j \in r + \mathbb{Z}^3, \tag{2.10}$$

for some $r \in \mathbb{R}^3$.

Even though (2.10) assumes that the step-size along each data axis equals ϵ, this is a nonrestrictive assumption. Indeed, consider a smooth diffeomorphism ψ: $\mathcal{V} \to \tilde{\mathcal{V}}$ for some open $\tilde{\mathcal{V}} \subset \mathbb{R}^2$, so that ψ maps an irregular grid covering $\mathcal{V}$ into a regular, square grid covering $\tilde{\mathcal{V}}$. Introducing a new defining function $\tilde{\Phi}(x,\tilde{y}) := \Phi(x,\psi^{-1}(\tilde{y}))$, we can transform any smoothly sampled data set into the one with a square grid. Clearly, if Φ satisfies DF1–DF4, then $\tilde{\Phi}$ satisfies DF1–DF4 as well.

Conditions DF1–DF4 imply that (cf. [2] and [22, sections VIII.5 and VIII.6]) the following:

1. $\mathcal{R}$ is an FIO with phase function $\lambda\Phi(x,y)$.
2. The corresponding canonical relation is

$$C := \{((x,\lambda\Phi'_x(x,y)),(y,-\lambda\Phi'_y(x,y))) :\ \Phi(x,y)=0, \lambda \in \mathbb{R}\setminus 0, x \in \mathcal{U}, y \in \mathcal{V}\}, \tag{2.11}$$

   which is a local canonical graph.
3. Any suitably modified adjoint of $\mathcal{R}$, denoted $\mathcal{R}^*$, is also an FIO, whose canonical relation C^* is obtained from (2.11) by switching the $(x,\xi) \in T^*\mathcal{U}$ and $(y,\eta) \in T^*\mathcal{V}$ variables.
4. The composition $\mathcal{R}^*(\dots)\mathcal{R}$, where the dots denote a cut-off combined with a suitable differential operator, is a pseudodifferential operator (ΨDO), i.e., $C^* \circ C$ is a subset of the diagonal in $T^*\mathcal{U}$.

Given a point $x \in \mathcal{S}$ where $\mathcal{S}$ is smooth, find $y = y(x)$ (which is locally smooth under our assumptions; see assertion 1 of Lemma 3.4 below) such that $\mathcal{S}_y$ is tangent to $\mathcal{S}$ at x. Denote

$$N(x) := \mathrm{II}_{\mathcal{S}_y}(x) - \mathrm{II}_{\mathcal{S}}(x), \tag{2.12}$$

where $\mathrm{II}_{\mathcal{S}}(x)$ is the matrix of the second fundamental form of $\mathcal{S}$ at $x \in \mathcal{S}$ written in an orthonormal basis of $T_x\mathcal{S}$.

For any $x \in \mathcal{U}$, introduce the set (see Figure 1)

$$\Gamma_x := \{y \in \mathcal{V} :\ x \in \mathcal{S}_y,\ \mathcal{S}_y \text{ is tangent to } \mathcal{S} \text{ at some } z,\ z \neq x\}. \tag{2.13}$$

Definition 2.1. *A pair $(x_0,y_0) \in \mathcal{C}$ is* globally generic *if whenever $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at some $z \neq x_0$ the following conditions hold:*

GG1. *$\mathcal{S}$ is smooth at z, and $N(z)$ is either positive or negative definite.*

GG2. *Let $\dot{\Gamma}_{x_0}$ be a nonvanishing at any point tangent vector field along Γ_{x_0}. There exists an open set $\mathcal{V}_1$, $y_0 \in \mathcal{V}_1 \subset \mathcal{V}$, such that for each $m \in \mathbb{Z}^3$, $|m| > 0$, and all $\delta > 0$ sufficiently small,*

1. *the set $\{y \in \Gamma_{x_0} \cap \mathcal{V}_1 :\ |m \cdot \dot{\Gamma}_{x_0}(y)| \le \delta\}$ is contained in a finite number of segments of Γ_{x_0} (this number may depend on m and δ), and*
2. *the sum of the lengths of these segments goes to zero as $\delta \to 0$.*

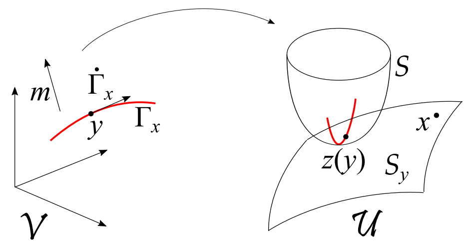

FIG. 1. *Illustration of the curve Γ_x, which is shown in red on the left. The red curve on the right shows the points of tangency $z(y)$ of $\mathcal{S}$ and $\mathcal{S}_y$.*

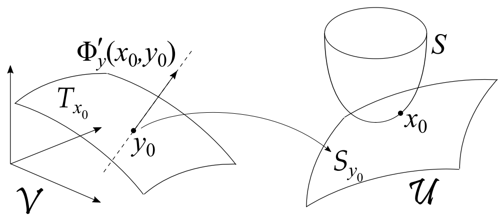

FIG. 2. *Illustration of the geometric setup in the definition of a locally generic pair $(x_0, y_0) \in \mathcal{C}$.*

As is shown in section 7, conditions GG1 and DF3 imply that Γ_{x_0} is a smooth curve, so condition GG2 makes sense.

An example when condition GG2 is violated is when Γ_{x_0} contains a straight line segment and $m \cdot \dot{\Gamma}_{x_0}(y) \equiv 0$ on this segment for some $m \in \mathbb{Z}^3$, $|m| > 0$.

DEFINITION 2.2. *A pair $(x_0, y_0) \in \mathcal{C}$, $x_0 \in \mathcal{S}$, is* locally generic *if, whenever $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at x_0 (see Figure 2), the following conditions hold:*
LG1. *$\mathcal{S}$ is smooth at x_0, and $N(x_0)$ is either positive or negative definite.*
LG2. *There is no $\lambda \neq 0$ such that $\lambda \Phi'_y(x_0, y_0) \in \mathbb{Z}^3$.*

DEFINITION 2.3. *A pair $(x_0, y_0) \in \mathcal{C}$ is* generic *if it is both locally and globally generic.*

Let φ be an interpolating kernel (IK), i.e., $\varphi(0) = 1$ and $\varphi(j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$. Suppose also that φ satisfies the following assumptions:
IK1. φ is exact up to the order 2, i.e.,

$$\sum_{j\in\mathbb{Z}^3} j^m \varphi(u - j) = u^m, \quad m \in (0 \cup \mathbb{N})^3,\ |m| \leq 2,\ u \in \mathbb{R}^3. \tag{2.14}$$

IK2. φ is compactly supported.
IK3. All partial derivatives of φ up to order 2 are continuous.
IK4. All partial derivatives of φ of order 3 are piecewise continuous and bounded.
IK5. φ is normalized, i.e., $\int \varphi(y)dy = 1$.
The interpolated version of g can be written in the form

$$g_\epsilon(y) := \sum_{j\in r+\mathbb{Z}^3} g(\epsilon j)\varphi\left(\frac{y - \epsilon j}{\epsilon}\right). \tag{2.15}$$

First, we derive a microlocal inversion formula for the GRT $\mathcal{R}$, which reconstructs exactly the leading singularities of f. Pick any $(x_0, y_0) \in \mathcal{C}$. Let α_0 be a unit vector normal to $\mathcal{S}_{y_0}$ at x_0. For $(x, \alpha) \in \mathcal{U} \times S^2$ close to (x_0, α_0) and for t, $|t| \ll 1$, find the local solution $y = Y(\alpha, t; x)$ such that $x + t\alpha \in \mathcal{S}_y$ and α is normal to $\mathcal{S}_y$ at $x + t\alpha$. By construction, $y_0 = Y(\alpha_0, t = 0; x_0)$. Here we use the assumption that the data are complete, i.e., such a solution exists. It is shown below (see (4.2)) that the map $(\alpha, t) \to y = Y(\alpha, t; x)$ is a local diffeomorphism that depends smoothly on x.

Let $\mathcal{V}_1$ be a small neighborhood of y_0. Pick any $\chi \in C_0^\infty(\mathcal{V}_1)$ such that $\chi \equiv 1$ near y_0. The inversion formula with continuous data is given by

$$f_\chi(x) = -\frac{1}{4\pi^2} \int_{S_+^2} \frac{\chi(Y(\alpha, 0; x))}{b(x, Y(\alpha, 0; x))} \left(\frac{\partial}{\partial t}\right)^2 g(Y(\alpha, t; x))\bigg|_{t=0} d\alpha. \tag{2.16}$$

This inversion formula emulates the CRT inversion formula by backprojecting a second order derivative of the GRT. The affine variable t is computed relative to x as opposed to the origin, as is the case with the CRT. Hence the GRT analogue of the usual term $\alpha \cdot x$ is missing from (2.16), because it is absorbed by the function Y. Due to the symmetry $g(Y(\alpha, t; x)) = g(Y(-\alpha, -t; x))$, in (2.16) we integrate over half of the unit sphere S_+^2.

Using consequences 1–4 of conditions DF1–DF4 (see above (2.12)), it is easy to show that the map $f \to f_\chi$ is a ΨDO of degree zero with principal symbol 1 microlocally near (x_0, α_0) (see, e.g., [2, 10]). Thus, the singularities of f and f_χ are the same to leading order (e.g., in the scale of Sobolev spaces) microlocally near (x_0, α_0). An inversion formula that recovers all the singularities of f can be obtained by combining (2.16) with a microlocal partition of unity. In the case of discrete data, we use the same inversion formula (2.16), but replace g with g_ϵ. The corresponding reconstruction is denoted $f_{\chi\epsilon}$.

3. Statement of main result. Beginning of proof.

3.1. Statement of main result.

Pick $(x_0, y_0) \in \mathcal{C}$ such that $\mathcal{S}$ is smooth at $x_0 \in \mathcal{S}$, $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at x_0 (see Figure 3), and $N(x_0)$ is either positive definite or negative definite. Fix some orthonormal basis in the common plane tangent to $\mathcal{S}$ and $\mathcal{S}_{y_0}$ at x_0. Let α_0 be the unit vector normal to $\mathcal{S}$ at x_0. For convenience of calculations, we sometime use the Cartesian coordinates $(x_1, x^\perp)$ determined by

$$x = x_1\alpha_0 + x^\perp,\ x^\perp \in \alpha_0^\perp, \tag{3.1}$$

where $\alpha_0^\perp$ is the plane through x_0 and normal to α_0. This plane is tangent to both $\mathcal{S}$ and $\mathcal{S}_{y_0}$ at x_0 (see Figure 3). The direction of α_0 is chosen so that $N(x_0)$ is *negative definite*. The side of $\mathcal{S}$ where α_0 points is called *interior*. The other side of $\mathcal{S}$ is called *exterior*. If necessary, multiply Φ by (-1) so that $\Phi'_x(x_0, y_0)/|\Phi'_x(x_0, y_0)| = -\alpha_0$.

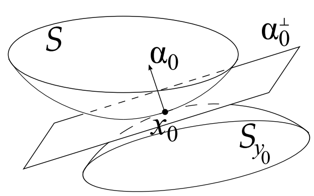

FIG. 3. *Illustration of the surface $\mathcal{S}_{y_0}$, which is tangent to $\mathcal{S}$ at x_0, and the associated plane $\alpha_0^\perp$.*

Consider the point

$$x_\epsilon := x_0 + \epsilon\tilde{x},\ \tilde{x} \in \tilde{\mathcal{U}}, \tag{3.2}$$

where $\tilde{\mathcal{U}}$ is a bounded set. Denote

$$\begin{gathered} f_\chi(x_{0\pm}) := \lim_{\epsilon\to 0^+} f_\chi(x_0 \pm \epsilon\alpha_0),\ f_0 := \lim_{\epsilon\to 0^+}(f(x_0+\epsilon\alpha_0) - f(x_0-\epsilon\alpha_0)), \\ \nu := \frac{|\Phi_x'|}{|\Phi_y'|},\ \beta_0 = \frac{\Phi_y'}{|\Phi_y'|}. \end{gathered} \tag{3.3}$$

Here and in what follows the convention is that if the arguments of Φ and its derivatives are omitted, then they are evaluated at x_0, y_0.

We also introduce local y-coordinates with the origin at y_0:

$$y = (y_1, y^\perp) = y_1\beta_0 + y^\perp. \tag{3.4}$$

Thus, equation $y_1 = 0$ determines the plane tangent to the submanifold $\mathcal{T}_{x_0}$ at y_0. We frequently denote this plane $\beta_0^\perp$.

Finally, we use the extension of the CRT to all of $\mathbb{R}^3 \setminus 0$ according to

$$\hat{f}(u,s) := \int f(x)\delta(u\cdot x - s)dx,\ u \in \mathbb{R}^3 \setminus 0, \tag{3.5}$$

for a sufficiently smooth and rapidly decreasing f.

The main result of the paper is the following.

THEOREM 3.1. *Pick a generic pair $(x_0, y_0) \in \mathcal{C}$ such that $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at $x_0 \in \mathcal{S}$. Then*

$$\lim_{\epsilon\to 0} f_{\chi\epsilon}(x_\epsilon) = f_\chi(x_{0^+}) - f_0 \int_{\nu h}^{\infty} \hat{\varphi}(\beta_0, s)ds, \tag{3.6}$$

where $h = \tilde{x}\cdot\alpha_0$, and $\hat{\varphi}$ is the CRT of φ.

By IK5, $\int_{\mathbb{R}} \hat{\varphi}(\beta_0, s)ds = 1$. The inversion formula (2.16) reconstructs jumps of f accurately, so $f_\chi(x_{0^+}) - f_\chi(x_{0^-}) = f_0$ and (3.6) can be written as follows:

$$\lim_{\epsilon\to 0} f_{\chi\epsilon}(x_\epsilon) = f_\chi(x_{0^-}) + f_0 \int_{-\infty}^{\nu h} \hat{\varphi}(\beta_0, s)ds. \tag{3.7}$$

Theorem 3.1 implies that for small $\epsilon > 0$, the reconstructed image near a generic jump discontinuity of f can be approximated by the right-hand side of (3.6) (or (3.7)). Based on this formula one can perform various analyses of the accuracy of reconstruction, e.g., compute conventional measures of resolution like FWHM, investigate how resolution and accuracy depend on location and orientation of the edge, and many others. The integral on the right-hand side of (3.6), which represents the *unit edge response*, is explicitly computable using fairly simple formulas. Indeed, only three quantities are required for this: ν and β_0, which are given in (3.3), and the CRT of the interpolation kernel φ. See sections 8 and 9, where the edge response is computed for a GRT that integrates over spheres.

Formulas (3.6), (3.7) can be understood intuitively as follows. Let $\theta(t) = 0$, $t \le 0$, and $\theta(t) = 1$, $t > 1$, be the Heaviside step function. As $\epsilon \to 0$, the reconstruction points x_ϵ are confined to a shrinking neighborhood of $x_0 \in \mathcal{S}$. In this neighborhood,

$\mathcal{S}$ is close to the tangent plane $(x-x_0)\cdot\alpha_0=0$, and the approximation $f(x_\epsilon)\approx f(x_{0-})+f_0\theta((x_\epsilon-x_0)\cdot\alpha_0)$ is increasingly accurate as $\epsilon\to 0$. Moreover, because (2.16) preserves values of jumps, we can view the function $\theta((x_\epsilon-x_0)\cdot\alpha_0)$ as the unit edge response of the reconstruction from continuous data. Clearly,

$$\int_{-\infty}^{\nu h}\hat\varphi(\beta_0,s)ds=\int_{\mathbb{R}}\theta\left(h-\frac{|\Phi_y'|}{|\Phi_x'|}s\right)\hat\varphi(\beta_0,s)ds. \tag{3.8}$$

Thus the unit edge response of the reconstruction from discrete data equals to the convolution of the ideal, continuous data edge response with a smoothing kernel (see also [13] for related results). Since $(x_0,\alpha_0)\in WF(f)$, analyzing the reconstruction of f at x_0 probes the edge response along the direction α_0. The canonical relation C^* of the FIO in (2.16) maps (y_0,β_0) to $(x_0,\lambda\alpha_0)$ for some $\lambda\neq 0$, so it is clear that a vanishingly small conical neighborhood of (y_0,β_0) should be contributing to the discrete reconstruction at (x_0,α_0) as $\epsilon\to 0$. This is indeed confirmed by our derivation; see (4.3), (5.20), and (6.3) below. As $g(y)$ is a conormal distribution and $(y_0,\beta_0)\in WF(g)$, in a small neighborhood of y_0, $g(y)$ can be approximated by a function $g_0((y-y_0)\cdot\beta_0)$, which is constant along planes perpendicular to β_0. The interpolation kernel φ is applied to the data, so it is natural that the CRT of φ along β_0 appears in (3.8), and the convolution is with respect to the affine variable s along β_0. The ratio $|\Phi_y'|/|\Phi_x'|$ is a microlocal (i.e., depending on location and orientation) "conversion factor," which converts distances in the data domain $\mathcal{V}$ (e.g., s and the sampling rate ϵ) to distances in the image domain $\mathcal{U}$.

By linearity, the proof of Theorem 3.1 can be split into two parts: local and global. The local part is formulated as follows.

THEOREM 3.2. *Pick a locally generic pair $(x_0,y_0)\in\mathcal{C}$ such that $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at $x_0\in\mathcal{S}$. Suppose* supp(f) *is contained in a sufficiently small neighborhood of x_0. Then*

$$\lim_{\epsilon\to 0}f_{\chi\epsilon}(x_\epsilon)=f_\chi(x_{0^+})-f_0\int_{\nu h}^{\infty}\hat\varphi(\beta_0,s)ds, \tag{3.9}$$

where $h=\tilde x\cdot\alpha_0$, and $\hat\varphi$ is the CRT of φ.

Theorems 3.2 and 7.1 (see section 7) state that the edge response of the reconstruction near x_0 is determined only by the behavior of f near x_0.

Since $\mathcal{R}$ is an FIO with canonical relation (2.11), $g=\mathcal{R}f$ is singular only when $\mathcal{S}_y$ is tangent to $\mathcal{S}$. Therefore, we are interested in the behavior of g in a neighborhood of y_0.

Remark 3.3. Strictly speaking, one has to distinguish between the original coordinates that describe points $x\in\mathcal{U}$, $y\in\mathcal{V}$ and those in (3.1), (3.4), respectively. For example, one should write $x=\hat x_1\alpha_0+\hat x^\perp$ instead of $x=x_1\alpha_0+x^\perp$. Such notation would emphasize that $\hat x_1$ is not the first component of x in the original coordinates, i.e., $\hat x_1\neq x_1$. Similarly, a derivative like $\partial\Phi(x,y)/\partial x_1$, if written in full, becomes $\partial\Phi(x(\hat x_1,\hat x^\perp),y(\hat y_1,\hat y^\perp))/\partial\hat x_1$. To avoid burdensome notation, whenever the coordinates $(\hat x_1,\hat x^\perp)$ and $(\hat y_1,\hat y^\perp)$ are used, we will stick with the simplified notation and assume that the above convention holds.

3.2. Behavior of g near its singular support. Because $\mathcal{S}$ is smooth in a neighborhood of $x_0\in\mathcal{S}$, there is a smooth local diffeomorphism $x\to(z,p)$ so that

$$x=z+pn(z),\ z\in\mathcal{S},\ n(z)\text{ is normal to }\mathcal{S}\text{ at }z,\ |n(z)|\equiv 1. \tag{3.10}$$

The normal $n(z)$ is chosen so that $N(z)$ is negative definite. Thus $n(x_0) = \alpha_0$. Clearly, we can extend the function $n(z)$, $z \in \mathcal{S}$, to $n(x)$ defined in a neighborhood of x_0 by the formula $n(z + pn(z)) := n(z)$. With a slight abuse of notation, the extended function will also be denoted $n(\cdot)$.

Using (3.10), define $\Psi(z + pn(z)) := p$. Then $\Psi(x) = 0$ is the equation of $\mathcal{S}$ near x_0, and Ψ is smooth. By construction, $\Psi(x) > 0$ on the interior side of $\mathcal{S}$.

Consider the system of equations

$$\Phi'_x(z + pn(z), y) - \lambda n(z) = 0,\ \Phi(z + pn(z), y) = 0,\ \Psi(z) = 0. \tag{3.11}$$

If we set $p = 0$ and solve (3.11) for y, we find a local patch of the submanifold $\mathcal{T}_\mathcal{S}$ near y_0. Recall that $y \in \mathcal{T}_\mathcal{S}$ implies that $\mathcal{S}_y$ is tangent to $\mathcal{S}$. We also need to solve these equations for z, p, and λ in terms of y.

Lemma 3.4. *Pick $(x_0, y_0) \in \mathcal{C}$ such that (1) $\mathcal{S}$ is smooth at x_0, (2) $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at $x_0 \in \mathcal{S}$, and (3) $N(x_0)$ is negative definite. Then*

1. *$\mathcal{T}_\mathcal{S} \subset \mathcal{V}$ is a smooth submanifold of codimension 1 near y_0, and the vector $\Phi'_y(x_0, y_0)$ is normal to $\mathcal{T}_\mathcal{S}$ at y_0;*
2. *the solutions $z = Z(y)$, $p = P(y)$, and $\lambda = \Lambda(y)$ to (3.11) depend smoothly on y in a neighborhood of y_0, and*

$$P'_y(y_0) = \frac{1}{|\Phi'_x|}\Phi'_y \neq 0; \tag{3.12}$$

3. *equations (3.11) determine a smooth function $y = \overline{Y}(z, p)$, $(z, p) \in \mathcal{S} \times \mathbb{R}$, in a neighborhood of $(x_0, 0)$.*

Proof. Differentiate (3.11) with respect to z, p, λ, and y, and set $z = x_0$, $p = 0$, $y = y_0$ to obtain the 5×8 matrix

$$\begin{bmatrix} \Phi''_{xx} - \lambda n'_x & \Phi''_{xx}\alpha_0 & -\alpha_0 & \Phi''_{xy} \\ \Phi'_x & \Phi'_x \cdot \alpha_0 & 0 & \Phi'_y \\ \Psi'_x & 0 & 0 & 0 \end{bmatrix}. \tag{3.13}$$

Here n'_x is the derivative of the function $n(x)$ extended to a neighborhood of x_0 as described following (3.10). Since $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at $x_0 \in \mathcal{S}$, we have $\Phi'_x \parallel \alpha_0$, so $\Phi'_x \cdot \alpha_0 = \lambda = -|\Phi'_x| \neq 0$. This also gives the value of λ to be used in (3.13). By (3.3), $\Phi'_y = |\Phi'_y|\beta_0$. Also,

$$\Psi'_x(z + pn(z)) \equiv n(z), \tag{3.14}$$

which implies that $\Psi'_x(x_0) = \alpha_0$. Combining these observations, and dividing the second from the bottom row by $-|\Phi'_x|$, and the bottom row – by $|\Phi'_x|$, the matrix in (3.13) transforms to

$$\begin{bmatrix} \Phi''_{xx} + |\Phi'_x| n'_x & \Phi''_{xx}\alpha_0 & -\alpha_0^T & \Phi''_{xy} \\ \alpha_0 & 1 & 0 & (-1/\nu)\beta_0 \\ \alpha_0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.15}$$

where α_0 and β_0 are row vectors.

To prove (3.14) note that, by construction, $\Psi(x(z, p)) \equiv p$. Here $(z, p) \to x$ is the local diffeomorphism defined in (3.10). Suppose points $z \in \mathcal{S}$ near x_0 are parametrized by some two-dimensional parameter $u = (u_1, u_2)$. Differentiating with respect to u

and p gives $\Psi'_x(x)x'_u \equiv 0$ (because p and z are independent) and $\Psi'_x(x)x'_p \equiv 1$, where $x'_p = n(z)$. By construction, $z(u) \in \mathcal{S}$ and $|n(u)| \equiv 1$ (where we replaced $n(z(u))$ with $n(u)$ with a slight abuse of notation). Hence $x'_{u_j} = z'_{u_j} + pn'_{u_j}$, $j = 1, 2$. Since z'_{u_j} is tangent to $\mathcal{S}$ at z and $n'_{u_j} \cdot n \equiv 0$, we get $n(z) \cdot x'_{u_j} \equiv 0$, and (3.14) immediately follows.

To prove the first part of the first assertion of the lemma we need to show that z, λ, and y_1 are smooth functions of $y^\perp$. Remove the columns corresponding to the derivatives with respect to p (because $p = 0$ is fixed) and $y^\perp$ to obtain a 5×5 submatrix. By applying elementary row and column operations, it is clear that this submatrix is full-rank if and only if the following matrix has rank two:

$$\Phi''_{x^\perp x^\perp} + |\Phi'_x|(n^\perp)'_{x^\perp}. \tag{3.16}$$

Here $\Phi''_{x^\perp x^\perp} : \alpha_0^\perp \to \alpha_0^\perp$ is the appropriate submatrix of Φ''_{xx} in the coordinates (3.1), and $n^\perp$ is the projection of n onto $\alpha_0^\perp$. It is easy to see that

$$\Phi''_{x^\perp x^\perp} + |\Phi'_x|(n^\perp)'_{x^\perp} = |\Phi'_x|N(x_0). \tag{3.17}$$

The desired assertion follows from condition LG1 (see Definition 2.2).

Next, set $p = 0$ in (3.11) and assume that z, y_1 are functions of $y^\perp$. Differentiating the last two equations in (3.11) with respect to $y^\perp$ and using that $\Phi'_y \cdot \beta_0 = |\Phi'_y| \neq 0$ gives $\Phi'_x z'_{y^\perp} = 0$ and $\partial y_1/\partial y^\perp = 0$. This proves the second part of the first assertion.

The first part of the second assertion follows by retaining the columns corresponding to the derivatives with respect to z, p, and λ. As before, the resulting 5×5 submatrix is full-rank because the matrix in (3.16) has rank two. The second part of the second assertion follows by considering z, p, and λ as functions of y, differentiating the last two equations in (3.11) with respect to y, and using that $\Phi'_x, \Phi'_y \neq 0$.

The function $z = Z(y)$ leads to a shorter proof of the first assertion by noting that $\Phi(Z(y), y) = 0$ is also a local equation of $\mathcal{T}_\mathcal{S}$ near y_0 and, by construction, $d_y\Phi(Z(y), y)|_{y=y_0} = d_y\Phi(x_0, y)|_{y=y_0}$.

The third assertion follows immediately from data completeness and the Bolker condition (conditions DF2 and DF4, respectively). ∎

We need the following lemma, which generalizes one of the results of Ramm and Zaslavsky [18, 19] from the CRT to the GRT.

LEMMA 3.5. *Pick $(x_0, y_0) \in \mathcal{C}$ such that (1) $\mathcal{S}$ is smooth at x_0, (2) $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at $x_0 \in \mathcal{S}$, and (3) $N(x_0)$ is negative definite. Suppose* supp(f) *is contained in a small neighborhood of x_0. For any $z \in \mathcal{S}$ and p in small neighborhoods of x_0 and 0, respectively, one has*

$$g(\overline{Y}(z,p)) = p_+G(z,p) + G_1(z,p) \text{ and } G(z,0) = f_0(z)b(z,\overline{Y}(z,0))\frac{2\pi}{\sqrt{detN(z)}} \tag{3.18}$$

for some smooth $G(z,p)$, $G_1(z,p)$.

Proof. Recall that $\overline{Y}(z,p)$ is the smooth function of $(z,p) \in \mathcal{S} \times \mathbb{R}$ defined by the conditions that $z + pn(z) \in \mathcal{S}_y$ and $n(z)$ be normal to $\mathcal{S}_y$ at the point $z + pn(z)$; see assertion 3 of Lemma 3.4. By assumption, $N(x_0)$ is negative definite in coordinates (3.1). By linearity, we may assume that $f \equiv 0$ on the exterior side of $\mathcal{S}$. In particular, $f(x_0 - \epsilon\alpha_0) \equiv 0$, $\epsilon > 0$, in (3.3). In this case we have to prove (3.18) with $G_1 \equiv 0$. By construction,

$$g(y) = \int_{\mathcal{S}_y} f(x)b(x,y)\theta\left(\Psi(x)\right)dx, \tag{3.19}$$

where θ is the unit step function (Heaviside function). Consider the system

$$\Psi'_x(x) - \mu\Phi'_x(x,y) = 0,\ \ \Phi(x,y) = 0, \tag{3.20}$$

which we solve for x and μ in terms of y. Equations (3.20) determine the stationary point $x_*(y)$ of $\Psi(x)$ on the surface $\mathcal{S}_y$ (the parameter μ, which corresponds to $1/\lambda$ in (3.11), is the Lagrange multiplier). From (3.11) and the property of Ψ'_x, the solution $x_*(y)$, $\mu(y)$ to (3.20) can be obtained from the solution $z = Z(y), p = P(y), \lambda = \Lambda(y)$ to (3.11): $x_*(y) = Z(y) + P(y)n(Z(y))$, $\mu(y) = 1/\Lambda(y)$. By assertion 2 of Lemma 3.4, $x_*(y)$ is smooth near y_0.

As is easily checked, when the matrix $\Psi''_{xx}(x_0) - \mu(y_0)\Phi''_{xx}(x_0,y_0)$ is viewed as a quadratic form on $\alpha_0^\perp$, it coincides with $N(x_0)$. The latter is negative definite, hence $x_*(y)$ is the local maximum of $\Psi(x)$ on $\mathcal{S}_y$. By the Morse lemma (see Appendix A), find local coordinates ω on $\mathcal{S}_y$, which depend smoothly on y, such that $\omega(x_*(y)) = 0$ and $\Psi(x) = \Psi(x_*(y)) - |\omega|^2$, $x = x(\omega; y) \in \mathcal{S}_y$. Since f, b, and $x(\omega; y)$ are all smooth, we get from (3.19)

$$g(y) = \int \theta\left(\Psi(x_*(y)) - |\omega|^2\right) F(\omega, y) d\omega \tag{3.21}$$

for some smooth F. Expand F in the Taylor series around $\omega = 0$ and integrate in spherical coordinates $\omega = r\Theta$. Integration with respect to Θ removes all the odd powers of r, i.e., only the even powers of r remain. By construction,

$$\Psi(x_*(y)) = \Psi(z + pn(z)) = p\ \text{ if }\ y = \overline{Y}(z,p), \tag{3.22}$$

and the first statement in (3.18) follows.

To prove the second statement, we use the local coordinates (3.1). In these coordinates, the local equation of $\mathcal{S}_y$ becomes

$$x_1 = x_1(x^\perp, y) = \phi(y) + a(y)\cdot x^\perp + \frac{A(y)x^\perp\cdot x^\perp}{2} + O(|x^\perp|^3) \tag{3.23}$$

for some smooth ϕ, a, and A. By construction, $y = \overline{Y}(x_0,p)$ implies that $x_*(y) = x_0 + p\alpha_0$. Substitute such a pair $(x_*(y), y)$ into (3.20) and use (3.23) to conclude that $\phi(y) \equiv p$, $a(y) \equiv 0$, $y = \overline{Y}(x_0,p)$. Let $x_1 = Q(x^\perp)$ be the equation of $\mathcal{S}$ in the coordinates (3.1). By construction, $Q'(0) = 0$. Clearly, $A(y_0)$ and $Q''(0)$ are the matrices of the second fundamental form of $\mathcal{S}_{y_0}$ and $\mathcal{S}$, respectively, at x_0 in the coordinates (3.1). From (3.19),

$$\begin{aligned} g(y) &= \int f(x)b(x,y)\theta\left(\psi(x^\perp,y)\right)\sqrt{1+|Q'(x^\perp)|^2}\,dx^\perp,\\ \psi(x^\perp,y) &:= x_1(x^\perp,y) - Q(x^\perp),\ x = (x_1(x^\perp,y), x^\perp). \end{aligned} \tag{3.24}$$

Setting $y = \overline{Y}(x_0,p)$ in (3.24), integrating by diagonalizing $N(x_0) = \psi''_{x^\perp x^\perp}(0,y_0)$ and changing variables, and taking the limit as $p \to 0$ (hence $y \to y_0$), we find

$$G(x_0, 0) = f(x_0)b(x_0,y_0)\frac{2\pi}{\sqrt{\det N(x_0)}}. \tag{3.25}$$

By noting that instead of $(x_0,y_0) \in \mathcal{C}$ one can take any $(z,y) \in \mathcal{C}$, where $z \in \mathcal{S}$ is sufficiently close to x_0 (and $y = \overline{Y}(z,0)$), we finish the proof. ∎

It follows from the results obtained in this section that the properties of the matrix function $N(x)$ are very important for our analysis. The nondegeneracy of $N(x)$ ensures that $\mathcal{T}_{\mathcal{S}} \subset \mathcal{V}$ is a smooth submanifold of co-dimension 1 near x_0. See the proof of assertion 1 of Lemma 3.4. Also, the specific behavior of $g(y)$ near its singular support (recall that $\text{singsupp}(g) \subset \mathcal{T}_{\mathcal{S}}$) established in Lemma 3.5 holds only when $N(x_0) < 0$ (i.e., $N(x_0)$ is negative definite). The case $N(x_0) > 0$ can be obtained from the case $N(x_0) < 0$ by a change of variables (cf. the paragraph following (3.1)).

4. Local behavior of interpolated data. Similarly to (3.11), consider the equations

$$\Phi'_x(x + t\alpha, y) - \lambda\alpha = 0,\ \Phi(x + t\alpha, y) = 0, \tag{4.1}$$

which we solve to find $y = Y(\alpha, t; x)$ assuming (α, t, x) is in a neighborhood of $(\alpha_0, 0, x_0)$. Differentiating (4.1) with respect to λ and y and setting $x = x_0$, $t = 0$, $\alpha = \alpha_0$, $\lambda = -|\Phi'_x|$, we obtain similarly to (3.13) a matrix, which is nondegenerate. As opposed to (3.13), the key reason why it is nondegenerate is the Bolker condition. Hence $Y(\alpha, t; x)$ is a smooth function of α, t, and x. Next we substitute $y = Y(\alpha, t; x)$ into (4.1) and obtain a few useful properties of Y. The first one is that $\partial Y_1/\partial\alpha^\perp = 0$. Indeed, differentiate the second equation in (4.1) with respect to $\alpha^\perp$ and set $x = x_0$, $t = 0$, $\alpha = \alpha_0$ to obtain $\Phi'_y \partial Y/\partial\alpha^\perp = 0$. The desired assertion follows from (3.4). In a similar fashion, we have

$$\det(\partial Y^\perp/\partial\alpha^\perp) \neq 0,\ \partial Y_1/\partial t = |\Phi'_x|/|\Phi'_y| \neq 0,\ \det(\partial Y/\partial(\alpha^\perp, t)) \neq 0. \tag{4.2}$$

The first result is obtained by differentiating the first equation in (4.1) with respect to $\alpha^\perp$ and using the Bolker condition and that $\partial Y_1/\partial\alpha^\perp = 0$. The second result is obtained by differentiating the second equation in (4.1) with respect to t and using the properties of the selected coordinates (3.1), (3.4). The last result is an obvious consequence of the first two and that $\partial Y_1/\partial\alpha^\perp = 0$.

In view of (4.2), given any small $\omega > 0$, we can find a sufficiently small open set $\mathcal{V}_1$, $y_0 \in \mathcal{V}_1 \subset \mathcal{V}$, such that $Y(\alpha, t; x_\epsilon) \in \mathcal{V}_1$ implies $|\alpha^\perp| < \omega$ for all x_ϵ provided that ϵ is sufficiently small. This relationship between $\mathcal{V}_1$ and ω is assumed in what follows. The results in (4.2) imply also that the inversion formula (2.16) requires finitely many data points $y_j = \epsilon j$ (cf. (2.10)) for each fixed $x \in \mathcal{U}$ and $\epsilon > 0$.

Since $\text{supp}(\chi) \subset \mathcal{V}_1$, the domain of integration in (2.16) can be split into two sets:

$$\Omega_1 := \{\alpha \in S^2_+ :\ |\alpha^\perp| < A\epsilon^{1/2}\},\ \Omega_2 := \{\alpha \in S^2_+ :\ A\epsilon^{1/2} < |\alpha^\perp| < \omega\} \tag{4.3}$$

for some small (but fixed) $\omega > 0$. Here $A > 0$ is a large parameter. Let $f^{(j)}_{\chi\epsilon}$ denote the result of integrating in (2.16) (with g replaced by g_ϵ) over Ω_j, $j = 1, 2$. The behavior of $f^{(1)}_{\chi\epsilon}$ is investigated first. This is done in section 5. In the remainder of this section we lay the groundwork for that investigation by deriving the behavior of $Y(\alpha, t; x_\epsilon)$ and $g_\epsilon(y)$ in a neighborhood of $(\alpha_0, 0, x_0)$ and y_0, respectively.

4.1. Local behavior of $\boldsymbol{Y(\alpha, t; x_\epsilon)}$. The first step is to obtain the leading term behavior of the function $Y(\alpha, t; x_\epsilon)$ for $t = O(\epsilon)$ and $\alpha \in \Omega_1$, i.e., for $|\alpha^\perp| = O(\epsilon^{1/2})$. In this section we continue using the coordinates (3.1).

Expanding $y = Y(\alpha, t; x_\epsilon)$ in the Taylor series around $x = x_0$, $t = 0$, $\alpha = \alpha_0$ and using that $|x_\epsilon - x_0| = O(\epsilon)$, $t = O(\epsilon)$, $|\alpha^\perp| = O(\epsilon^{1/2})$, and $\partial Y_1/\partial\alpha^\perp = 0$ gives

$$y_1 = O(\epsilon),\ |y^\perp| = O(\epsilon^{1/2}). \tag{4.4}$$

For x in an $O(\epsilon)$ neighborhood of the origin (i.e., x_0) and for y in an $O(\epsilon^{1/2})$ neighborhood of the origin (i.e., y_0) we have

$$\begin{aligned}\Phi(x,y) &= \Phi'_x\cdot x + \Phi'_y\cdot y + \frac{\Phi''_{yy}y\cdot y}{2} + O(\epsilon^{3/2}),\\ \Phi'_x(x,y) &= \Phi'_x + \Phi''_{xy}y + O(\epsilon).\end{aligned} \tag{4.5}$$

To find $y = Y(\alpha,t;x_\epsilon)$, substitute $x = x_\epsilon$ into (4.5) and solve

$$\begin{aligned}&\Phi'_x\cdot(\epsilon\tilde{x}+t\alpha) + \Phi'_y\cdot y + \frac{\Phi''_{yy}y\cdot y}{2} = O(\epsilon^{3/2}),\\ &\Phi'_x + \Phi''_{xy}y = \lambda\left(1,\alpha^\perp\right) \pmod{O(\epsilon)}.\end{aligned} \tag{4.6}$$

Recall that $h = \tilde{x}\cdot\alpha_0$. Switching to the coordinates (3.1), (3.4), using (4.4), and keeping only the terms of order $O(\epsilon)$ in the first equation in (4.6) gives

$$-(\epsilon h + t)|\Phi'_x| + |\Phi'_y|y_1 + \frac{\Phi''_{y^\perp y^\perp}y^\perp\cdot y^\perp}{2} = O(\epsilon^{3/2}). \tag{4.7}$$

Projecting the second equation in (4.6) onto α_0 and onto $\alpha_0^\perp$ implies

$$\lambda = -|\Phi'_x| + O(\epsilon^{1/2}),\ \Phi''_{x^\perp y^\perp}y^\perp = \lambda\alpha^\perp + O(\epsilon), \tag{4.8}$$

leading to

$$Y^\perp(\alpha,t;x_\epsilon) = -|\Phi'_x|(\Phi''_{x^\perp y^\perp})^{-1}\alpha^\perp + O(\epsilon). \tag{4.9}$$

By the Bolker condition (2.4), $\Phi''_{x^\perp y^\perp}$ is nondegenerate. Substitution into (4.7) now yields

$$\begin{aligned}&Y_1(\alpha,t;x_\epsilon) = \nu\left(\epsilon h + t - \frac{M\alpha^\perp\cdot\alpha^\perp}{2}\right) + O(\epsilon^{3/2}),\\ &M := |\Phi'_x|(\Phi''_{x^\perp y^\perp})^{-T}\Phi''_{y^\perp y^\perp}(\Phi''_{x^\perp y^\perp})^{-1},\ M:\alpha_0^\perp \to \alpha_0^\perp.\end{aligned} \tag{4.10}$$

Recall that ν is defined in (3.3).

4.2. The leading local behavior of $g(Y(\alpha,t;x_\epsilon))$. We plan to substitute $y = Y(\alpha,t=0;x_\epsilon)$ into (2.15). Hence the second step is to find the leading behavior of $g(y)$ when $|y - Y(\alpha,0;x_\epsilon)| = O(\epsilon)$ and $|\alpha^\perp| = O(\epsilon^{1/2})$. This is done by finding the asymptotics of $z = Z(y)$ and $p = P(y)$, which are determined by solving (3.11).

Recall that the local equation of $\mathcal{S}$ in the coordinates (3.1) and the interior unit normal are given by

$$\begin{aligned}z_1 = Q(z^\perp) &= \frac{Q''z^\perp\cdot z^\perp}{2} + O(|z^\perp|^3),\\ n(z) &= (1+O(|z^\perp|^2), -Q''z^\perp + O(|z^\perp|^2)),\ Q'' := Q''(0).\end{aligned} \tag{4.11}$$

By (4.9), (4.10), $|y_1| = O(\epsilon)$, $|y^\perp| = O(\epsilon^{1/2})$. From (3.12), $\partial p/\partial y^\perp = 0$ at $y = y_0$, so this implies $|p| = O(\epsilon)$, $x_1 = O(\epsilon)$, and $|z^\perp|, |x^\perp| = O(\epsilon^{1/2})$, where $x = z + pn(z)$. Equation (4.11) leads to

$$\begin{aligned}x &= \left(\frac{Q''z^\perp\cdot z^\perp}{2} + O(\epsilon^{3/2}), z^\perp\right) + p(1+O(\epsilon^{1/2}), -Q''z^\perp + O(\epsilon))\\ &= \left(p + \frac{Q''z^\perp\cdot z^\perp}{2} + O(\epsilon^{3/2}), z^\perp + O(\epsilon^{3/2})\right).\end{aligned} \tag{4.12}$$

From (4.12), $x^\perp = z^\perp + O(\epsilon^{3/2})$.

Given that now $|x^\perp| = O(\epsilon^{1/2})$, the expansion in the first line in (4.5) should include additional terms. The second equation in (3.11) becomes

$$-|\Phi'_x|x_1+|\Phi'_y|y_1+\frac{\Phi''_{x^\perp x^\perp}x^\perp\cdot x^\perp}{2}+\Phi''_{x^\perp y^\perp}y^\perp\cdot x^\perp+\frac{\Phi''_{y^\perp y^\perp}y^\perp\cdot y^\perp}{2}=O(\epsilon^{3/2}). \tag{4.13}$$

Solving for x_1 we find

$$x_1=\frac{1}{\nu}y_1+\frac{1}{|\Phi'_x|}\left\{\frac{\Phi''_{x^\perp x^\perp}x^\perp\cdot x^\perp}{2}+\Phi''_{x^\perp y^\perp}y^\perp\cdot x^\perp+\frac{\Phi''_{y^\perp y^\perp}y^\perp\cdot y^\perp}{2}\right\}+O(\epsilon^{3/2}). \tag{4.14}$$

Using (4.9) and that $|y-Y(\alpha,0;x_\epsilon)|=O(\epsilon)$, we have

$$y^\perp=-|\Phi'_x|(\Phi''_{x^\perp y^\perp})^{-1}\alpha^\perp+O(\epsilon), \tag{4.15}$$

and the unit normal vector to $\mathcal{S}_Y$ is thus

$$\left(1+O(\epsilon^{1/2}),\alpha^\perp-\frac{\Phi''_{x^\perp x^\perp}x^\perp}{|\Phi'_x|}+O(\epsilon)\right). \tag{4.16}$$

The big-O terms in (4.16) follow by noticing that differentiation with respect to x_1 in (4.13) converts $O(\epsilon^{3/2})$ into $O(\epsilon^{1/2})$, and differentiation with respect to $x^\perp$ converts $O(\epsilon^{3/2})$ into $O(\epsilon)$. This follows by noticing that (a) the term $O(\epsilon^{3/2})$ on the right in (4.13) represents a smooth function (which is the difference between $\Phi(x,y)$ and the left-hand side of (4.13)), and (b) the Taylor expansion of this function starts with the terms of order $O(\epsilon^{3/2})$ (e.g., $x_1x_j^\perp$, $x_1y_j^\perp$, $y_1y_j^\perp$, $y_1y_j^\perp$, ...).

From (4.11) and (4.16), the two normal vectors are parallel (the first equation in (3.11)) if

$$\alpha^\perp-\frac{\Phi''_{x^\perp x^\perp}z^\perp}{|\Phi'_x|}+O(\epsilon)=-Q''z^\perp+O(\epsilon), \tag{4.17}$$

which implies

$$z^\perp=N_0^{-1}\alpha^\perp+O(\epsilon),\ x^\perp=N_0^{-1}\alpha^\perp+O(\epsilon),\ N_0:=N(x_0). \tag{4.18}$$

Matching the first components in (4.12) and using (4.14), (4.15), (4.18) gives after simple transformations

$$p=\frac{1}{\nu}\left(y_1+\frac{M_1\alpha^\perp\cdot\alpha^\perp}{2}\right)+O(\epsilon^{3/2}),\ M_1:=\nu(M-N_0^{-1}), \tag{4.19}$$

where M is defined in (4.10). Summarizing (4.18) and (4.19) we have

$$\begin{aligned}&Z^\perp(y)=N_0^{-1}\alpha^\perp+O(\epsilon),\ P(y)=\frac{1}{\nu}\left(y_1+\frac{M_1\alpha^\perp\cdot\alpha^\perp}{2}\right)+O(\epsilon^{3/2}),\\ &|y-Y(\alpha,0;x_\epsilon)|=O(\epsilon),\ |\alpha^\perp|=O(\epsilon^{1/2}).\end{aligned} \tag{4.20}$$

Substituting into (3.18) we find

$$g(y)=f(Z(y))b(Z(y),y^*)\frac{2\pi}{\sqrt{\det N(Z(y))}}P_+(y)+O\left(P_+^2(y)\right), \tag{4.21}$$

where $y^* = \overline{Y}(z, p=0)$ whenever $y = \overline{Y}(z,p)$.

From (4.11), (4.18) and (4.9), (4.10) it follows that $|Z(y) - x_0| = O(\epsilon^{1/2})$ and $|y - y_0| = O(\epsilon^{1/2})$ whenever $|y - Y(\alpha, 0; x_\epsilon)| = O(\epsilon)$, hence

$$\frac{f(Z(y))b(Z(y), y^*)}{\sqrt{\det N(Z(y))}} = \frac{f(x_0)b(x_0, y_0)}{\sqrt{\det N(x_0)}} + O(\epsilon^{1/2}). \tag{4.22}$$

4.3. Local behavior of the interpolated data. In this subsection we find the behavior of the interpolated data near $y_0 = 0$. Combining (4.20)–(4.22) we get

$$\begin{aligned} g(\epsilon j) =& \frac{2\pi}{\nu}\left(\frac{f(x_0)b(x_0,y_0)}{\sqrt{\det N_0}} + O(\epsilon^{1/2})\right)\left(\epsilon j_1 + \frac{M_1\alpha^\perp\cdot\alpha^\perp}{2} + O(\epsilon^{3/2})\right)_+ \\ &+ O\left(P_+^2(\epsilon j)\right), \end{aligned} \tag{4.23}$$

where M_1 is defined in (4.19). From (4.10) and (4.20), $|P(\epsilon j)| = O(\epsilon)$. Using (4.9), (4.10), and (4.23) gives

$$\begin{aligned} &g_\epsilon(Y(\alpha,t;x_\epsilon)) \\ &= \frac{2\pi\epsilon}{\nu}\sum_{j\in r+\mathbb{Z}^3}\left(\frac{f(x_0)b(x_0,y_0)}{\sqrt{\det N_0}} + O(\epsilon^{1/2})\right)\left(j_1 + \frac{M_1\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2} + O\left(\epsilon^{1/2}\right)\right)_+ \\ &\quad\times\varphi\left(\nu\left(h + \tilde t - \frac{M\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}\right) - j_1 + O\left(\epsilon^{1/2}\right), -\frac{|\Phi_x'|}{\epsilon^{1/2}}(\Phi''_{x^\perp y^\perp})^{-1}\tilde\alpha^\perp - j^\perp + O(1)\right) \\ &\quad + O(\epsilon^2), \quad \tilde t := t/\epsilon,\ \tilde\alpha^\perp := \alpha^\perp/\epsilon^{1/2}. \end{aligned} \tag{4.24}$$

Since φ is compactly supported, the number of terms in the sum in (4.24) is bounded. Using additionally that $|\tilde\alpha^\perp| \le A < \infty$, the two $O\left(\epsilon^{1/2}\right)$ terms on the second line of (4.24) are uniform in j, and the sum itself is bounded as well. Finally, combining with the fact that $|(a+O(\epsilon^{1/2}))_+ - a_+| = O(\epsilon^{1/2})$ uniformly in $a\in\mathbb{R}$ and using (4.9) and (4.10) gives

$$\begin{aligned} &g_\epsilon(Y(\alpha,t;x_\epsilon)) \\ &= \frac{f(x_0)b(x_0,y_0)2\pi\epsilon/\nu}{\sqrt{\det N_0}}\sum_{j\in r+\mathbb{Z}^3}\left(j_1 + \frac{M_1\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}\right)_+ \\ &\quad\times\varphi\left(\nu\left(h + \tilde t - \frac{M\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}\right) - j_1 + O\left(\epsilon^{1/2}\right), -\frac{|\Phi_x'|}{\epsilon^{1/2}}(\Phi''_{x^\perp y^\perp})^{-1}\tilde\alpha^\perp - j^\perp + O(1)\right) \\ &\quad + O(\epsilon^{3/2}). \end{aligned} \tag{4.25}$$

In view of (4.25), denote

$$\psi(q,u) := \sum_{j\in r+\mathbb{Z}^3}(j\cdot\beta_0 + q)_+\varphi(u - j),\ q\in\mathbb{R},\ u\in\mathbb{R}^3. \tag{4.26}$$

Here we will need a higher order approximation of $Y(\alpha,t;x_\epsilon)$ than the one in (4.9), (4.10):

$$\begin{aligned} Y^\perp/\epsilon &= -|\Phi_x'|(\Phi''_{x^\perp y^\perp})^{-1}\tilde\alpha^\perp + A_1(\tilde\alpha^\perp,\tilde\alpha^\perp) + A_2\tilde t + A_3 h + O(\epsilon^{1/2}), \\ Y_1/\epsilon &= \nu\left(h + \tilde t - \frac{M\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}\right) + O(\epsilon^{1/2}), \end{aligned} \tag{4.27}$$

where A_1 is a bilinear map $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, and $A_2, A_3 \in \mathbb{R}^2$. Moreover, the two $O(\epsilon^{1/2})$ terms in (4.27) depend smoothly on $\tilde{t}$ and $\tilde{\alpha}^\perp$. In particular, differentiation with respect to $\tilde{t}$ does not change the order of these terms as $\epsilon \to 0$. Hence we rewrite (4.25) as follows:

(4.28)
$$\begin{aligned} &g_\epsilon(Y(\alpha,t;x_\epsilon)) = \frac{f_0 b_0 2\pi\epsilon/\nu}{\sqrt{\det N_0}}\psi\left(q(\tilde{\alpha}^\perp), u(\tilde{\alpha}^\perp,\tilde{t})\right) + O(\epsilon^{3/2}),\ q(\tilde{\alpha}^\perp) := \frac{M_1\tilde{\alpha}^\perp\cdot\tilde{\alpha}^\perp}{2},\\ &u(\tilde{\alpha}^\perp,\tilde{t}) := \left(\nu\left(h+\tilde{t}-\frac{M\tilde{\alpha}^\perp\cdot\tilde{\alpha}^\perp}{2}\right)+O\left(\epsilon^{1/2}\right),\right.\\ &\qquad\left. -\frac{|\Phi_x'|}{\epsilon^{1/2}}(\Phi''_{x^\perp y^\perp})^{-1}\tilde{\alpha}^\perp + A_1(\tilde{\alpha}^\perp,\tilde{\alpha}^\perp) + A_2\tilde{t} + A_3 h + O(\epsilon^{1/2})\right). \end{aligned}$$

We also have

$$\begin{aligned} \partial u(\tilde{\alpha}^\perp,\tilde{t})/\partial\tilde{t}\big|_{\tilde{t}=0} &= \nu\left(1+O(\epsilon^{1/2}), A_2/\nu + O(\epsilon^{1/2})\right),\\ \partial^2 u(\tilde{\alpha}^\perp,\tilde{t})/\partial\tilde{t}^2\big|_{\tilde{t}=0} &= O(\epsilon^{1/2}). \end{aligned} \tag{4.29}$$

5. Estimating the term $f^{(1)}_{\chi\epsilon}$. To study $f^{(1)}_{\chi\epsilon}$ we need the following lemma, which follows immediately from (4.26) and the properties IK1–IK3 of φ.

LEMMA 5.1. *Partial derivatives of $\psi(q,u)$ with respect to u up to the order two are continuous. Also, one has*

$$\psi(q,u) = \psi(q + m\cdot\beta_0, u - m)\ \forall m \in \mathbb{Z}^3, \tag{5.1}$$

and, for some $c > 0$,

$$\psi(q,u) = 0 \text{ if } u\cdot\beta_0 + q < -c; \quad \psi(q,u) = u\cdot\beta_0 + q \text{ if } u\cdot\beta_0 + q > c. \tag{5.2}$$

Denote

$$U(q,u) := -\left.\frac{\partial^2}{\partial\tau^2}\psi(q, u + \tau(1, A_2/\nu))\right|_{\tau=0}. \tag{5.3}$$

The following lemma is a direct consequence of Lemma 5.1 (see also properties IK3, IK4 of φ).

LEMMA 5.2. *The function $U(q,u)$ has piecewise continuous bounded first order partial derivatives with respect to q and u. Also,*

$$U(q,u) = U(q + m\cdot\beta_0, u - m),\ \forall m \in \mathbb{Z}^3, \tag{5.4}$$

and, for some $c > 0$,

$$U(q,u) \equiv 0 \text{ if } |u\cdot\beta_0 + q| > c. \tag{5.5}$$

Note that the derivative in the inversion formula (2.16) is with respect to t. Using that $t = \epsilon\tilde{t}$ (cf. (4.24)) and taking into account (4.29), (5.3), in the formula below we will acquire the factor $(\epsilon/\nu)^2$. Thus, in terms of U, the expression for $f^{(1)}_{\chi\epsilon}$ becomes after changing variables $\alpha^\perp \to \tilde\alpha^\perp$ (this brings the factor ϵ), setting $\tilde t = 0$, and using (4.28), (4.29), (5.3),

(5.6)
$$\begin{aligned} f^{(1)}_{\chi^\epsilon}(x_\epsilon) &= \frac{1}{4\pi^2}\frac{f_0(2\pi\epsilon/\nu)}{\sqrt{\det N_0}}\frac{\epsilon}{(\epsilon/\nu)^2}\int_{|\tilde\alpha^\perp|<A} U\left(q(\tilde\alpha^\perp), u_0(\tilde\alpha^\perp)\right) d\tilde\alpha^\perp + O(\epsilon^{1/2}),\\ u_0(\tilde\alpha^\perp) &:= \left(\nu\left(h - \frac{M\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}\right), -\frac{|\Phi'_x|}{\epsilon^{1/2}}(\Phi''_{x^\perp y^\perp})^{-1}\tilde\alpha^\perp + A_1(\tilde\alpha^\perp,\tilde\alpha^\perp) + A_3 h\right). \end{aligned}$$

Two simplifications have been made in deriving (5.6). First, using that first and second order derivatives of ψ are bounded, it follows from (4.29) that

$$\frac{\partial^2}{\partial\tilde t^2}\psi\left(\cdot, u(\tilde\alpha^\perp,\tilde t)\right)\Big|_{\tilde t=0} = \nu^2\,\frac{\partial^2}{\partial\tau^2}\psi(\cdot, u(\tilde\alpha^\perp,0) + \tau(1, A_2/\nu))\Big|_{\tau=0} + O(\epsilon^{1/2}). \tag{5.7}$$

Second, since the derivatives of U are bounded, the integral with respect to $\tilde\alpha^\perp$ is over a bounded set, and the coefficient in front of the integral is bounded, using (5.7) and then replacing $u(\tilde\alpha^\perp, 0)$ with $u_0(\tilde\alpha^\perp)$ in the arguments of U leads to the term $O(\epsilon^{1/2})$ outside the integral.

Approximate the domain $|\tilde\alpha^\perp| < A$ by a union of nonoverlapping small squares of size δ. Let these squares be denoted B_k, $k = 1, 2, \dots, O(\delta^{-2})$. By (5.6),

$$\begin{aligned} u_0(\tilde\alpha^\perp) &= u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp) + O(\delta),\ \tilde\alpha^\perp \in B_k,\\ \Delta u_k(\tilde\alpha^\perp) &:= \left(0, -|\Phi'_x|(\Phi''_{x^\perp y^\perp})^{-1}(\tilde\alpha^\perp - \tilde\alpha^\perp_k)/\epsilon^{1/2}\right) \in \beta_0^\perp, \end{aligned} \tag{5.8}$$

where $\tilde\alpha^\perp_k$ is the center of B_k. By Lemma 5.2,

$$U\left(q(\tilde\alpha^\perp), u_0(\tilde\alpha^\perp)\right) = U\left(q(\tilde\alpha^\perp_k), u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\right) + O(\delta),\ \tilde\alpha^\perp \in B_k. \tag{5.9}$$

Therefore

(5.10)
$$\begin{aligned} &\int_{B_k} U\left(q(\tilde\alpha^\perp), u_0(\tilde\alpha^\perp)\right) d\tilde\alpha^\perp\\ &= \int_{B_k}\left[U\left(q(\tilde\alpha^\perp_k), u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\right) + O(\delta)\right] d\tilde\alpha^\perp\\ &= \int_{B_k}\left[U\left(q(\tilde\alpha^\perp_k) + \lfloor u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\rfloor\cdot\beta_0, \{u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\}\right) + O(\delta)\right] d\tilde\alpha^\perp\\ &= \int_{B_k}\Big[U\Big(q(\tilde\alpha^\perp_k) + u_0(\tilde\alpha^\perp_k)\cdot\beta_0 - \{u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\}\cdot\beta_0,\\ &\qquad\qquad \{u_0(\tilde\alpha^\perp_k) + \Delta u_k(\tilde\alpha^\perp)\}\Big) + O(\delta)\Big] d\tilde\alpha^\perp, \end{aligned}$$

where we have used that $\Delta u_k(\tilde\alpha^\perp)\cdot\beta_0 = 0$. In (5.10) and below the fractional part of a vector is computed componentwise: $\{u\} = (\{u_1\}, \{u_2\}, \{u_3\})$, where $\{u_i\} = u_i - \lfloor u_i\rfloor$ and $\lfloor u_i\rfloor$ is the largest integer not exceeding u_i.

Pick any $m \in \mathbb{Z}^3$, $m \neq 0$. Let $m^\perp$ be the projection of m onto the plane $\beta_0^\perp$. Condition LG2 in Definition 2.2 implies that $m^\perp \neq 0$. By the local Bolker condition DF4, $\Phi''_{x^\perp y^\perp}$ is nondegenerate, so the vector $(\Phi''_{x^\perp y^\perp})^{-T} m^\perp \in \mathbb{R}^2$ is not zero. Using

(5.8), a Weyl-type argument (cf. [15]) implies that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{B_k} U\left(q(\tilde\alpha^\perp),u_0(\tilde\alpha^\perp)\right)d\tilde\alpha^\perp\\ =\left(\int_{[0,1]^3} U\left(q(\tilde\alpha_k^\perp)+u_0(\tilde\alpha_k^\perp)\cdot\beta_0-\omega\cdot\beta_0,\omega\right)d\omega+O(\delta)\right)\mathrm{Vol}(B_k).\end{aligned}\tag{5.11}$$

Indeed, consider the function $U_1(q,\omega):=U(q-\omega\cdot\beta_0,\omega)$. Clearly, $U_1(q,\omega)$ is periodic: $U_1(q,\omega)=U_1(q,\omega+m)$, $m\in\mathbb{Z}^3$. Expand $U_1(q,\omega)$ in a Fourier series:

$$U_1(q,\omega)=\sum_{m\in\mathbb{Z}^3}A_m(q)\exp(2\pi i m\cdot\omega).\tag{5.12}$$

By the argument preceding (5.11),

$$\begin{aligned}m\cdot\left(0,-|\Phi_x'|(\Phi''_{x^\perp y^\perp})^{-1}(\tilde\alpha^\perp-\tilde\alpha_k^\perp)\right)=-|\Phi_x'|(\Phi''_{x^\perp y^\perp})^{-T}m^\perp\cdot(\tilde\alpha^\perp-\tilde\alpha_k^\perp),\\ (\Phi''_{x^\perp y^\perp})^{-T}m^\perp\neq 0,|m|>0.\end{aligned}\tag{5.13}$$

Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0}\left|\int_{B_k}\exp\left(2\pi i m\cdot\left(u_0(\tilde\alpha_k^\perp)+\Delta u_k(\tilde\alpha^\perp)\right)\right)d\alpha^\perp\right|\\ =\lim_{\epsilon\to 0}\left|\int_{B_k}\exp\left(-\frac{2\pi i|\Phi_x'|}{\epsilon}\left[(\Phi''_{x^\perp y^\perp})^{-T}m^\perp\right]\cdot\tilde\alpha^\perp\right)d\tilde\alpha^\perp\right|=0,\ |m|>0.\end{aligned}\tag{5.14}$$

By (5.14), only the term corresponding to $m=0$ does not vanish as $\epsilon\to 0$. The desired assertion follows from the standard approximation argument.

By (4.23), (4.28), and (5.6),

$$q(\tilde\alpha_k^\perp)+u_0(\tilde\alpha_k^\perp)\cdot\beta_0=\nu h-\frac{\nu N_0^{-1}\tilde\alpha_k^\perp\cdot\tilde\alpha_k^\perp}{2}.\tag{5.15}$$

Add the integrals over all the squares B_k and use (5.6), (5.11), and (5.15):

$$\begin{aligned}&\lim_{\epsilon\to 0}f^{(1)}_{\chi^\epsilon}(x_\epsilon)\\ &=\kappa\sum_k\int_{B_k}U\left(q(\tilde\alpha^\perp),u_0(\tilde\alpha^\perp)\right)d\tilde\alpha^\perp+O(\delta)\\ &=\kappa\sum_k\left(\int_{[0,1]^3}U\left(\nu h-\frac{\nu N_0^{-1}\tilde\alpha_k^\perp\cdot\tilde\alpha_k^\perp}{2}-\omega\cdot\beta_0,\omega\right)d\omega+O(\delta)\right)\mathrm{Vol}(B_k)+O(\delta),\\ &\kappa:=\frac{\nu}{2\pi}\frac{f_0}{\sqrt{\det N_0}}.\end{aligned}\tag{5.16}$$

The term $O(\delta)$ outside the integral on the second line of (5.16) appears because of the error of approximating the domain of integration $|\tilde\alpha^\perp|<A$ in (5.6) by the union of B_k's. Since $\delta>0$ can be as small as we like, (5.16) implies

$$\lim_{\epsilon\to 0}f^{(1)}_{\chi^\epsilon}(x_\epsilon)=\kappa\int_{|\tilde\alpha^\perp|<A}\int_{[0,1]^3}U\left(\nu h-\frac{\nu N_0^{-1}\tilde\alpha^\perp\cdot\tilde\alpha^\perp}{2}-\omega\cdot\beta_0,\omega\right)d\omega\, d\tilde\alpha^\perp.\tag{5.17}$$

The second argument of U is bounded and N_0 is negative definite, so by (5.5) the integral with respect to $\tilde{\alpha}^{\perp}$ over the set $|\tilde{\alpha}^{\perp}| < A$, when $A > 0$ is large enough (but fixed), can be replaced by the integral over all $\mathbb{R}^2$. Changing variables and integrating in spherical coordinates gives

$$\begin{aligned}\lim_{\epsilon\to 0} f^{(1)}_{\chi^\epsilon}(x_\epsilon) &= \kappa \int_{\mathbb{R}^2}\int_{[0,1]^3} U\left(\nu h - \frac{\nu N_0^{-1}\tilde{\alpha}^{\perp}\cdot\tilde{\alpha}^{\perp}}{2} - \omega\cdot\beta_0, \omega\right) d\omega\, d\tilde{\alpha}^{\perp} \\ &= \kappa\frac{2\sqrt{\det N_0}}{\nu}\int_{\mathbb{R}^2}\int_{[0,1]^3} U\left(\nu h + v\cdot v - \omega\cdot\beta_0, \omega\right) d\omega dv \\ &= f_0 \int_0^{\infty}\int_{[0,1]^3} U\left(\nu h + \tau - \omega\cdot\beta_0, \omega\right) d\omega\, d\tau.\end{aligned} \tag{5.18}$$

The integral with respect to ω can be evaluated explicitly. Recall that in our coordinates, $\beta_0 = (1, 0, 0)$ (cf. (3.4)). From (4.26) and (5.3),

$$\begin{aligned}&-\int_{[0,1]^3} U(q - \omega\cdot\beta_0, \omega) d\omega \\ &= \int_{[0,1]^3} \frac{\partial^2}{\partial\tau^2}\psi(q - \omega\cdot\beta_0, \tau(1, A_2/\nu) + \omega)\bigg|_{\tau=0} d\omega \\ &= \frac{\partial^2}{\partial\tau^2}\int_{[0,1]^3}\sum_{j\in r+\mathbb{Z}^3}(q - \beta_0\cdot(\omega - j))_+\varphi(\tau(1, A_2/\nu) + \omega - j)d\omega\bigg|_{\tau=0} \\ &= \frac{\partial^2}{\partial\tau^2}\int_{\mathbb{R}^3}(q - \beta_0\cdot\omega)_+\varphi(\tau(1, A_2/\nu) + \omega)d\omega\bigg|_{\tau=0} \\ &= \frac{\partial^2}{\partial\tau^2}\int_{\mathbb{R}^3}(q - \beta_0\cdot\omega)_+\varphi(\tau(1, 0) + \omega)d\omega\bigg|_{\tau=0} = \hat{\varphi}(\beta_0, q).\end{aligned} \tag{5.19}$$

Substitute (5.19) into (5.18)

$$\lim_{\epsilon\to 0} f^{(1)}_{\chi^\epsilon}(x_\epsilon) = -f_0\int_0^{\infty}\hat{\varphi}(\beta_0, \nu h + \tau)d\tau = f_0\left(-\int_{\nu h}^{\infty}\hat{\varphi}(\beta_0, \tau)d\tau\right). \tag{5.20}$$

Since φ is normalized and compactly supported,

$$-\int_{\nu h}^{\infty}\hat{\varphi}(\beta_0, \tau)d\tau = \begin{cases}0, & h > c,\\ -1, & h < -c,\end{cases} \tag{5.21}$$

for some $c > 0$.

Using the definition (3.5) and some simple transformations, we can rewrite the integral in (5.20) in two different forms:

$$\int_{\nu h}^{\infty}\hat{\varphi}(\beta_0, \tau)d\tau = \int_{|\Phi_x'|h}^{\infty}\hat{\varphi}(\Phi_y', \tau)d\tau = \int_{\Phi_x'\tilde{x}+\Phi_y'\tilde{y}>0}\varphi(\tilde{y})d\tilde{y}. \tag{5.22}$$

6. Analysis of the term $f^{(2)}_{\chi^\epsilon}$.

LEMMA 6.1. *One can find $\omega > 0$, $\epsilon_0 > 0$ small enough and $A > 0$ large enough so that $g(y)$ is smooth in a neighborhood of all y such that $(y - Y(\alpha, 0; x_\epsilon))/\epsilon \in \operatorname{supp}(\varphi)$ for any $\alpha \in \Omega_2, \tilde{x} \in \tilde{U}$ (cf. (3.2)), and $0 < \epsilon < \epsilon_0$.*

Proof. Fix some $c > 0$ sufficiently large. Using that $N(x_0)$ is negative definite, (4.20) implies that we can find $A > 0$ large enough and $\omega > 0$ small enough so that $P(Y(\alpha, 0; x_\epsilon)) > c\epsilon$ for all $\alpha \in \Omega_2$ and all $\tilde{x} \in \tilde{U}$ provided that ϵ is small enough. Since φ is compactly supported and $P(y)$ is smooth, $P(y) > 0$ for all y such that $(y - Y(\alpha, 0; x_\epsilon))/\epsilon \in \text{supp}(\varphi)$ (this is where we use that $c > 0$ is sufficiently large). Therefore, g is a smooth function in a neighborhood of all such y because in this case we can drop the subscript "+" from $P_+(y)$ in (3.18). ◻

Lemma 6.1 implies that the inversion formula (2.16) and its discrete analogue do not see singularities in the data when $\alpha \in \Omega_2$ provided that ω, A are selected as in the proof of Lemma 6.1. Since $|\Omega_1| = O(\epsilon)$, the limit $\lim_{\epsilon\to 0^+} f^{(2)}_{\chi\epsilon}(x_\epsilon)$ exists, is independent of $\tilde{x} \in \tilde{\mathcal{U}}$, and

$$f^{(2)}_{\chi\epsilon}(x_\epsilon) - f^{(2)}_{\chi\epsilon}(x_0) = O(\epsilon). \tag{6.1}$$

Using again that $N(x_0)$ is negative definite, there exist sufficiently small neighborhoods $\mathcal{U}_1$ of x_0 and $\mathcal{V}_1$ of y_0 such that $\mathcal{S}_y \cap \mathcal{U}_1$ is on the *exterior* side of $\mathcal{S}$ whenever $\mathcal{S}_y$ is tangent to $\mathcal{S}$ and $y \in \mathcal{V}_1$. This implies that if $x \in \mathcal{U}_1$ and x is on the *interior* side of $\mathcal{S}$, then there is no $y \in \mathcal{V}_1$ such that $\mathcal{S}_y$ contains x and is tangent to $\mathcal{S}$. In turn, this implies that the data $g(y)$, $y = Y(\alpha, t = 0; x) \in \mathcal{V}_1$, which is used to compute $f_\chi(x)$ is also smooth. Hence, assumptions IK1–IK3 imply that the L^∞ error of computing the second derivative $\partial^2/\partial t^2$ in (2.16) from the interpolated data is of order $O(\epsilon)$. The integral in the discrete data inversion formula can be viewed as a Riemann sum with step-size $O(\epsilon)$ for the integral in the continuous data inversion formula. The integrand in the latter is smooth when x is on the interior side of $\mathcal{S}$, so we have using again that $|\Omega_1| = O(\epsilon)$

$$f^{(2)}_{\chi\epsilon}(x_0 + \epsilon h\alpha_0) - f_\chi(x_0 + \epsilon h\alpha_0) = O(\epsilon) \tag{6.2}$$

for any $h > 0$. Combining (6.1) and (6.2) gives

$$f^{(2)}_{\chi\epsilon}(x_\epsilon) = f_\chi(x_{0^+}) + O(\epsilon). \tag{6.3}$$

This concludes the proof of Theorem 3.2.

7. Contribution of remote singularities.

Suppose $x_0 \in \mathcal{S}_{y_0}$, $\mathcal{S}_{y_0}$ is tangent to $\mathcal{S}$ at some $z_0 \in \mathcal{S}_{y_0}$, $z_0 \neq x_0$, $\mathcal{S}$ is smooth at z_0, and $N(z_0)$ is either positive definite or negative definite. Set $\alpha_0 = \Phi'_x(z_0, y_0)/|\Phi'_x(z_0, y_0)|$, so that $y_0 = Y(\alpha_0, 0; x_0)$. As before, $\mathcal{V}_1$ is a small neighborhood of y_0, and $\text{supp}(\chi) \subset \mathcal{V}_1$.

As follows from assertion 1 of Lemma 3.4 (with x_0 replaced by z_0 as the point of tangency), $\mathcal{T}_\mathcal{S}$ is a smooth submanifold of $\mathcal{V}_1$ through y_0, and the vector $\Phi'_y(z_0, y_0)$ is normal to $\mathcal{T}_\mathcal{S}$ at y_0. The local equation of $\mathcal{T}_\mathcal{S}$ is $P(y) = 0$, where the function P is the same as in Lemma 3.4. By assertion 2 of the lemma, $P'_y(y_0) \neq 0$.

Another locally smooth submanifold through y_0 is $\mathcal{T}_{x_0}$, and $\Phi'_y(x_0, y_0)$ is normal to it at y_0. By the assumption DF3 of no conjugate points, $\Phi'_y(x_0, y_0)$ and $\Phi'_y(z_0, y_0)$ are not parallel, so the intersection of the two submanifolds is a smooth curve Γ_{x_0} through y_0. This is the same curve Γ_{x_0}, which was introduced in (2.13). From this argument it is easy to see that Γ_x depends smoothly on x near $x = x_0$.

THEOREM 7.1. *Pick a globally generic pair* $(x_0, y_0) \in \mathcal{C}$ *such that* $x_0 \notin \mathcal{S}$ *and* $\mathcal{S}_{y_0}$ *is tangent to* $\mathcal{S}$ *at* $z_0 \in \mathcal{S}$. *Suppose* $\text{supp}(f)$ *is contained in a sufficiently small neighborhood of* z_0. *One has*

$$\lim_{\epsilon\to 0} f_{\chi\epsilon}(x_\epsilon) = f_\chi(x_0). \tag{7.1}$$

Proof. Suppose first that the reconstruction point is x_0. Since the reconstruction point is fixed, the dependence of various quantities on x_0 is omitted from notation in most places when there is no risk of confusion. Then

$$\tag{7.2} \begin{aligned} f_{\chi\epsilon}(x_0) &= \int_{S^2_+} B(\alpha) \sum_j g(\epsilon j) \left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon}\right)\Bigg|_{t=0} d\alpha, \\ B(\alpha) &:= -\frac{\chi(Y(\alpha,0))}{4\pi^2}\frac{1}{b(x_0,Y(\alpha,0))}. \end{aligned}$$

By Lemma 3.5,

$$\tag{7.3} g(y) := P_+(y)G(y),\ G(y) := G(Z(y),P(y)).$$

Since φ is compactly supported, we can expand the factor $P(y)$ in (7.3) in the Taylor series centered at $Y(\alpha,0)$. Let $L(y)$ be its linear term:

$$\tag{7.4} L(y) := P(Y(\alpha,0)) + P'_y(Y(\alpha,0))\cdot(y-Y(\alpha,0)).$$

We begin by looking at the expression, which is obtained by ignoring the second and higher order terms in the expansion of P:

$$\tag{7.5} J^{(1)}_\epsilon := \int_{S^2_+} B(\alpha)\sum_j G(\epsilon j)L_+(\epsilon j)\left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon}\right)\Bigg|_{t=0} d\alpha.$$

Clearly,

$$\tag{7.6} \begin{aligned} &\left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon}\right)\Bigg|_{t=0} d\alpha \\ &\quad = \frac{1}{\epsilon^2}\nabla^2_{u(\alpha)}\varphi\left(\frac{Y(\alpha,0)}{\epsilon}-j\right) + \frac{1}{\epsilon}\varphi'_y\left(\frac{Y(\alpha,0)}{\epsilon}-j\right)\cdot Y''_{tt}(\alpha,0), \\ &u(\alpha) := Y'_t(\alpha,0),\ \nabla^2_u\varphi(y) = \left(\frac{\partial}{\partial t}\right)^2 \varphi(y+tu)\Bigg|_{t=0}. \end{aligned}$$

Consider the most singular part of $J^{(1)}_\epsilon$, which is obtained by using the first term on the right in (7.6) and replacing $G(\epsilon j)$ with $G(Y(\alpha,0))$:

$$\tag{7.7} \begin{aligned} J^{(1a)}_\epsilon &:= \int_{S^2_+} B_1(\alpha)\sum_j L_+(\epsilon j)\frac{1}{\epsilon^2}\nabla^2_{u(\alpha)}\varphi\left(\frac{Y(\alpha,0)}{\epsilon}-j\right)d\alpha, \\ B_1(\alpha) &:= B(\alpha)G(Y(\alpha,0)). \end{aligned}$$

In view of (7.4) and (7.7), similarly to (4.26) and (5.3), introduce the function

$$\tag{7.8} \psi(q,v;\alpha) := \sum_j \left(e(\alpha)\cdot(j-v)+q\right)_+ \nabla^2_{u(\alpha)}\varphi\left(v-j\right),\ e(\alpha) := P'_y(Y(\alpha,0)).$$

Clearly,

1. ψ is compactly supported in q (by (2.14)),
2. ψ has bounded first order partial derivatives, and
3. $\psi(q,v;\alpha) = \psi(q,v-m;\alpha)$ for any $m\in\mathbb{Z}^3$.

Using (7.8) in (7.7) yields

$$J_\epsilon^{(1a)} = \frac{1}{\epsilon}\int_{S_+^2} B_1(\alpha)\psi\left(\frac{P(Y(\alpha,0))}{\epsilon}, \frac{Y(\alpha,0)}{\epsilon};\alpha\right) d\alpha. \tag{7.9}$$

Introduce local coordinates $s=(s_1,s_2)$ on S_+^2 so that $s_1 \equiv P(Y(\alpha,0))$ in a neighborhood of α_0. As is shown at the beginning of this section, Γ is the transverse intersection of the submanifolds $\mathcal{T}_{x_0}$ and $\mathcal{T}_{\mathcal{S}}$. By the first equation in (4.2), $\alpha \to Y(\alpha,0) \in \mathcal{T}_{x_0}$ is a regular parametrization near α_0. In (4.2), $y_0 = Y(\alpha_0, 0; x_0)$, and $Y^\perp$ is determined by the projection onto the plane $(\Phi_y'(x_0,y_0))^\perp$ (as opposed to $(\Phi_y'(z_0,y_0))^\perp$). Hence $P_y'(y_0)\neq 0$ (cf. (3.12)) implies $\partial P(Y(\alpha,0))/\partial\alpha^\perp \neq 0$ near α_0. Therefore the preimage of $\Gamma\cap\mathcal{V}_1$, given by $\{\alpha\in S_+^2 : P(Y(\alpha,0))=0,\, Y(\alpha,0)\in\mathcal{V}_1\}$, is also a smooth curve, and local coordinates (s_1,s_2) with the required property do exist. Then

(7.10)

$$\begin{aligned} J_\epsilon^{(1a)} &= \frac{1}{\epsilon}\int_{\mathbb{R}^2} B_1(\alpha(s))\psi\left(\frac{P(Y(\alpha(s),0))}{\epsilon}, \frac{Y(\alpha(s),0)}{\epsilon};\alpha(s)\right)\left|\frac{\partial\alpha}{\partial s}\right| ds + O(\epsilon)\\ &= \int_{\mathbb{R}^2} B_2(s_2)\psi\biggl(\tilde{s}_1, \frac{Y(\alpha(0,s_2),0)}{\epsilon} + \frac{\partial Y(\alpha(s_1,s_2),0)}{\partial s_1}\bigg|_{s_1=0}\tilde{s}_1;\alpha(0,s_2)\biggr) d\tilde{s}_1 ds_2\\ &\qquad + O(\epsilon),\\ B_2(s_2) &:= B_1(\alpha(0,s_2))\left|\frac{\partial\alpha(s_1=0,s_2)}{\partial s}\right|,\ \tilde{s}_1 = s_1/\epsilon. \end{aligned}$$

In the first line, the integral is over the bounded set $\{s\in\mathbb{R}^2 : Y(\alpha(s),0)\in\operatorname{supp}(\chi)\}$. In the second line, the integral can be confined to a bounded set $\{(\tilde{s}_1,s_2)\in\mathbb{R}^2 : |\tilde{s}_1| < \tilde{A}, Y(\alpha(0,s_2))\in\operatorname{supp}(\chi)\}$ for some $\tilde{A}>0$ large enough.

LEMMA 7.2. *Let D be a rectangle $D := [a_1,b_1]\times[a_2,b_2]$. Consider a function $\psi\in C(D\times\mathbb{R}^3)$. Suppose ψ is periodic: $\psi(s,y)=\psi(s,y+m)$ for any $m\in\mathbb{Z}^3$ and $(s,y)\in D\times\mathbb{R}^3$. Let $Y:[a_2,b_2]\to\mathbb{R}^3$ be a C^1 function with the following properties. For any $m\in\mathbb{Z}^3$, $|m|>0$,*

1. *the set $\{s_2\in[a_2,b_2] : |m\cdot Y'(s_2)|\le\delta\}$ is contained in a finite number of intervals for all $\delta>0$ sufficiently small (this number may depend on m and δ), and*
2. *the sum of the lengths of these intervals goes to zero as $\delta\to 0$.*

Then one has

$$\lim_{\epsilon\to 0^+}\int_D \psi\left(s,\frac{Y(s_2)}{\epsilon}\right) ds = \int_D\int_{[0,1]^3}\psi(s,y)dyds. \tag{7.11}$$

Proof. Pick any $\delta_1>0$. Let $\Psi_m(s)$ denote the coefficients of the Fourier expansion of $\psi(s,y)$ with respect to y. We can find $M>0$ large enough and a partition of D into sufficiently small rectangles such that

$$\sup_{s\in D, y\in\mathbb{R}^3}\left|\psi(s,y) - \sum_{|m|\le M}\tilde{\Psi}_m(s)\exp(2\pi i m\cdot y)\right| \le \delta_1. \tag{7.12}$$

Here $\tilde{\Psi}_m$ is an approximation of Ψ_m, which is constant on each rectangle of the partition. Thus, the lemma will be proven if we show that

$$\int_a^b \exp\left(2\pi i m\cdot Y(s_2)/\epsilon\right) ds_2 \to 0,\ \epsilon\to 0,\ \text{for any } [a,b]\subset[a_2,b_2],\ m\in\mathbb{Z}^3, |m|>0. \tag{7.13}$$

Using assumptions 1 and 2 of the lemma, partition $[a,b]$ into a finite collection of nonoverlapping intervals so that (i) their union is as close to $[a,b]$ as we like, and (ii) in each of these intervals $m\cdot Y'(s_2)$ is bounded away from zero. The result now follows immediately. ∎

Clearly, condition GG2 in Definition 2.1 is independent of the choice of the vector field $\dot{\Gamma}$ as long as it does not vanish at any point of Γ. In the s-coordinates, $Y(\alpha(0,s_2),0)$ is a regular parametrization of $\Gamma\cap\mathcal{V}_1$ because $\det(\partial Y^\perp/\partial\alpha^\perp)_{\alpha=\alpha_0}\neq 0$ and $\det(\partial\alpha^\perp/\partial s)\neq 0$ (see also the argument following (7.9)). The latter determinant is computed at s such that $\alpha(s)=\alpha_0$. Therefore $\partial Y/\partial s_2$ never vanishes on $\Gamma\cap\mathcal{V}_1$. Condition GG2 implies that $Y(\alpha(0,s_2))$ satisfies conditions 1 and 2 in Lemma 7.2. Set

$$\psi_1(s,y) := B_2(s_2)\psi\left(\tilde{s}_1, y+\frac{\partial Y(\alpha(s_1,s_2),0)}{\partial s_1}\bigg|_{s_1=0}\tilde{s}_1;\alpha(0,s_2)\right),\ s=(\tilde{s}_1,s_2). \tag{7.14}$$

Using the properties 1–3 of ψ, we see that Lemma 7.2 applies to ψ_1. Also, ψ_1 is compactly supported. Compact support along $\tilde{s}_1$ is due to the property 1 of ψ, and along s_2 is due to the cut-off χ. Substituting ψ_1 into (7.10), using Lemma 7.2, and then expressing ψ_1 in terms of ψ yields

$$\lim_{\epsilon\to 0} J_\epsilon^{(1a)} = \int_{\mathbb{R}^2} B_2(s_2)\left[\int_{[0,1]^3}\psi\left(\tilde{s}_1, v;\alpha(0,s_2)\right)dv\right]d\tilde{s}_1 ds_2. \tag{7.15}$$

By (7.8), similarly to (5.19),

$$\begin{aligned}&\int_{[0,1]^3}\sum_j \left(e\cdot(j-v)+q\right)_+ \nabla_u^2\varphi\left(v-j\right)dv\\ &\quad=\int_{\mathbb{R}^3}\left(-e\cdot v+q\right)_+\nabla_u^2\varphi(v)dv=(e\cdot u)^2\hat{\varphi}(e,q).\end{aligned} \tag{7.16}$$

As $|e|$ not necessarily equals one, (7.16) assumes the extended definition of the CRT; cf. (3.5). By (7.6) and (7.8),

$$e(\alpha)\cdot u(\alpha) = \partial P(Y(\alpha,t))/\partial t|_{t=0} =: P_t'(\alpha),\ \alpha=\alpha(0,s_2). \tag{7.17}$$

With φ normalized, using (7.16) with $q=\tilde{s}_1$ and (7.17) in (7.15) gives

$$\begin{aligned}\lim_{\epsilon\to 0} J_\epsilon^{(1a)} &= \int_{\mathbb{R}} B_2(s_2)(P_t'(\alpha(0,s_2)))^2\int\hat{\varphi}(e(\alpha(0,s_2)),\tilde{s}_1)d\tilde{s}_1 ds_2\\ &= \int_{\mathbb{R}} B_2(s_2)(P_t'(\alpha(0,s_2)))^2 ds_2.\end{aligned} \tag{7.18}$$

Consequently, from (7.10) we get

$$
\begin{aligned}
\lim_{\epsilon\to 0} J_\epsilon^{(1a)} &= \int_{\mathbb{R}} B_1(\alpha(0,s_2)) \left|\frac{\partial\alpha(0,s_2)}{\partial s}\right| (P_t'(\alpha(0,s_2)))^2 ds_2 \\
&= \int_{\mathbb{R}^2} B_1(\alpha(s))(P_t'(\alpha(s)))^2 \delta(P(Y(\alpha(s),0))) \left|\frac{\partial\alpha(s)}{\partial s}\right| ds \\
&= \int_{S_+^2} B_1(\alpha) \left(\frac{\partial P(Y(\alpha,t))}{\partial t}\bigg|_{t=0} \right)^2 \delta(P(Y(\alpha,0)))d\alpha.
\end{aligned}
\tag{7.19}
$$

In the second line we used that $s=(s_1,s_2)$ and $s_1 \equiv P(Y(\alpha(s),0))$.

Next, consider the second part of $J_\epsilon^{(1)}$, which is obtained by using the second term on the right in (7.6) and replacing $G(\epsilon j)$ with $G(Y(\alpha,0))$:

$$
J_\epsilon^{(1b)} := \int_{S_+^2} B_1(\alpha) \sum_j L_+(\epsilon j) \frac{1}{\epsilon} \varphi_y' \left(\frac{Y(\alpha,0)}{\epsilon} - j \right) \cdot Y_{tt}''(\alpha,0) d\alpha. \tag{7.20}
$$

The function $L_+(y)$ has bounded first derivatives, hence the limit of $J_\epsilon^{(1b)}$ can be easily found:

(7.21)
$$
\begin{aligned}
\lim_{\epsilon\to 0} J_\epsilon^{(1b)} &= \lim_{\epsilon\to 0} \int_{S_+^2} B_1(\alpha) \sum_j L_+(\epsilon j) \frac{1}{\epsilon} \varphi_y' \left(\frac{Y(\alpha,0)-\epsilon j}{\epsilon} \right) \cdot Y_{tt}''(\alpha,0) d\alpha \\
&= \lim_{\epsilon\to 0} \int_{S_+^2} B_1(\alpha)\, \partial_y \sum_j L_+(\epsilon j) \varphi \left(\frac{Y(\alpha,0)+y-\epsilon j}{\epsilon} \right) \bigg|_{y=0} \cdot Y_{tt}''(\alpha,0) d\alpha \\
&= \int_{S_+^2} B_1(\alpha)\, \partial_y L_+(Y(\alpha,0)+y)|_{y=0} \cdot Y_{tt}''(\alpha,0) d\alpha \\
&= \int_{S_+^2} B_1(\alpha) P_y'(Y(\alpha,0)) \cdot Y_{tt}''(\alpha,0) \theta(P(Y(\alpha,0))) d\alpha.
\end{aligned}
$$

Recall that $P>0$ on the interior side of $\mathcal{S}$.

The final piece of $J_\epsilon^{(1)}$ is

$$
\begin{aligned}
J_\epsilon^{(1c)} := \int_{S_+^2} & B(\alpha) \sum_j (G(\epsilon j) - G(Y(\alpha,0))) L_+(\epsilon j) \\
& \times \left(\frac{\partial}{\partial t}\right)^2 \varphi \left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon} \right) \bigg|_{t=0} d\alpha.
\end{aligned}
\tag{7.22}
$$

In an $O(\epsilon)$ neighborhood of Γ, we have $|L(y)| = O(\epsilon)$. For any fixed $y = Y(\alpha,0) \in \mathcal{V}_1$, $P(y)>0$, we compute by dropping the subscript "+" from L:

$$
\begin{aligned}
& \sum_j (G(\epsilon j) - G(Y(\alpha,0))) L(\epsilon j) \left(\frac{\partial}{\partial t}\right)^2 \varphi \left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon} \right) \bigg|_{t=0} \\
&= \left(\frac{\partial}{\partial t}\right)^2 [(G(Y(\alpha,t)) - G(Y(\alpha,0)))L(Y(\alpha,t))] \bigg|_{t=0} + O(\epsilon).
\end{aligned}
\tag{7.23}
$$

Substitution into (7.22) gives

$$\lim_{\epsilon\to 0} J_\epsilon^{(1c)} = \int_{S_+^2} B(\alpha)\left(\frac{\partial}{\partial t}\right)^2 [(G(Y(\alpha,t)) - G(Y(\alpha,0)))L(Y(\alpha,t))]\bigg|_{t=0} \theta(P(Y(\alpha,0)))d\alpha. \tag{7.24}$$

We can apply the limit as $\epsilon \to 0$ inside the integral in (7.22) to obtain (7.24) because the integrand is uniformly bounded. This follows because the function $(G(y) - G(Y(\alpha,0)))L(y)$ is smooth away from an $O(\epsilon)$ neighborhood of Γ, and the integrand is $O(1)$ within that neighborhood.

The final term to be considered arises because of the difference between $P(y)$ (cf. (7.3)) and its linear approximation $L(y)$ (cf. (7.4)):

$$J_\epsilon^{(2)} := \int_{S_+^2} B(\alpha)\sum_j G(\epsilon j)\,[P_+(\epsilon j) - L_+(\epsilon j)]\left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon}\right)\bigg|_{t=0} d\alpha. \tag{7.25}$$

In the domain where $P(\epsilon j)$ and $L(\epsilon j)$ are both positive, we have

$$P_+(\epsilon j) - L_+(\epsilon j) = \frac{1}{2}P''_{yy}(Y(\alpha,0))(\epsilon j - Y(\alpha,0))\cdot(\epsilon j - Y(\alpha,0)) + O(\epsilon^3). \tag{7.26}$$

This difference is zero if $P(\epsilon j)$ and $L(\epsilon j)$ are both negative. Thus,

$$\begin{aligned}&\lim_{\epsilon\to 0}\sum_j G(\epsilon j)\,[P(\epsilon j) - L(\epsilon j)]\left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha,t)-\epsilon j}{\epsilon}\right)\bigg|_{t=0}\\ &\qquad = G(Y(\alpha,0))P''_{yy}(Y(\alpha,0))Y'_t(\alpha,0)\cdot Y'_t(\alpha,0),\ P(Y(\alpha,0))>0.\end{aligned} \tag{7.27}$$

In the region where $P(Y(\alpha,0)) < 0$, the limit is obviously zero. Hence

$$\lim_{\epsilon\to 0} J_\epsilon^{(2)} = \int_{S_+^2} B_1(\alpha)P''_{yy}(Y(\alpha,0))Y'_t(\alpha,0)\cdot Y'_t(\alpha,0)\theta(P(Y(\alpha,0)))d\alpha. \tag{7.28}$$

As before, we can apply the limit as $\epsilon \to 0$ inside the integral in (7.25) to obtain (7.28) because the integrand is uniformly bounded. Indeed,

$$P_+(y) - L_+(y) = (L(y) + O(\epsilon^2))_+ - L_+(y) = O(\epsilon^2), \tag{7.29}$$

so the integrand in (7.25) remains bounded as $\epsilon \to 0$. The domain where $P(y)$ and $L(y)$ are of different signs is a shrinking $O(\epsilon)$ neighborhood of Γ, and the desired result follows.

Combining (7.19), (7.21), (7.24), and (7.28) gives the result, which, in compact form, can be written as follows:

$$\begin{aligned}&\lim_{\epsilon\to 0} f_{\chi^\epsilon}(x_0)\\ &= \int_{S_+^2} B(\alpha)[G_0(\partial_t P)^2\delta(P_0) + (G_0P'_yY''_{tt} + \partial_t^2((G-G_0)L) + G_0P''_{yy}Y'_t\cdot Y'_t)\theta(P_0)]d\alpha\\ &= \int_{S_+^2} B(\alpha)[G_0(\partial_t P)^2\delta(P_0) + (G_0\partial_t^2 P + \partial_t^2((G-G_0)L))\theta(P_0)]d\alpha.\end{aligned} \tag{7.30}$$

Here $G_0 := G(Y(\alpha,0))$, $G := G(Y(\alpha,t))$, $P := P(Y(\alpha,t))$, $P_0 := P(Y(\alpha,0))$, and the derivatives with respect to t are evaluated at $t=0$. This coincides with what we get by substituting $g = P_+G$ into the continuous inversion formula (2.16). Indeed, representing $P_+G = (GP)\theta(P)$, we have

$$\begin{aligned}\partial_t^2((GP)\theta(P)) &= \partial_t[\partial_t(GP)\theta(P) + (GP)\delta(P)\partial_t P] = \partial_t[\partial_t(GP)\theta(P)]\\ &= \partial_t(GP)\partial_t P\delta(P_0) + \partial_t^2(GP)\theta(P_0)\\ &= G_0(\partial_t P)^2\delta(P_0) + \partial_t^2(GP)\theta(P_0).\end{aligned} \tag{7.31}$$

The coefficients in front of the delta-function in (7.30) and (7.31) match. Subtracting the coefficients in front of the Heaviside function gives

$$[G_0\partial_t^2 P + \partial_t^2((G-G_0)L)] - \partial_t^2(GP) = -\partial_t^2[(G-G_0)(P-L)] = 0. \tag{7.32}$$

Here we have used that G_0 is independent of t, and the expression under the derivative has a zero of third order at $t=0$. Thus the theorem is proven in the case $x = x_0$.

Next, consider the case of a general $x_\epsilon := x_0+\epsilon\tilde{x}$ (cf. (7.1)). We begin by repeating the steps (7.3)–(7.9), where all the auxiliary functions, such as Y, are computed using x_ϵ instead of x_0. It is clear that in any place where an auxiliary function is not divided by ϵ, e.g., $B(\alpha)$ in (7.5) and $e(\alpha)$, $u(\alpha)$ in (7.8), replacing x_ϵ with x_0 introduces an error of magnitude $O(\epsilon)$. Here we also used the property 2 of ψ. Consequently, the analogue of (7.9) for x_ϵ becomes

$$J_\epsilon^{(1a)}(x_\epsilon) = \frac{1}{\epsilon}\int_{S_+^2} B_1(\alpha)\psi\left(\frac{P(Y(\alpha,0;x_\epsilon))}{\epsilon}, \frac{Y(\alpha,0;x_\epsilon)}{\epsilon};\alpha\right)d\alpha + O(\epsilon), \tag{7.33}$$

where only Y is different from the analogous function in (7.9). Note that $P(y)$ depends only on the shape of $\mathcal{S}$ in a neighborhood of z_0 and, therefore, is independent of x_ϵ. We have

$$\begin{aligned}Y(\alpha,0;x_\epsilon) &= Y(\alpha,0) + \epsilon W(\alpha,\tilde{x}) + O(\epsilon^2),\\ P(Y(\alpha,0;x_\epsilon)) &= P(Y(\alpha,0)) + \epsilon P_y'(Y(\alpha,0))W(\alpha,\tilde{x}) + O(\epsilon^2),\end{aligned} \tag{7.34}$$

for some smooth and bounded W. Here $Y(\alpha,0)$ is the same as in (7.9). Substituting into (7.33) gives

$$\begin{aligned}J_\epsilon^{(1a)}(x_\epsilon) = \frac{1}{\epsilon}\int_{S_+^2} B_1(\alpha)\psi\Bigg(&\frac{P(Y(\alpha,0))}{\epsilon} + P_y'(Y(\alpha,0))W(\alpha,\tilde{x}),\\ &\frac{Y(\alpha,0)}{\epsilon} + W(\alpha,\tilde{x});\alpha\Bigg)d\alpha + O(\epsilon).\end{aligned} \tag{7.35}$$

Similarly to (7.14), introduce

$$\begin{aligned}&\psi_2(s,y) := B_1(\alpha)\psi(\tilde{s}_1 + P_y'(Y(\alpha,0))W(\alpha,\tilde{x}), y + W(\alpha,\tilde{x});\alpha),\\ &\qquad \alpha = \alpha(0,s_2), s = (\tilde{s}_1, s_2).\end{aligned} \tag{7.36}$$

The point $\tilde{x}$ is fixed, so we do not need to list it in the arguments of ψ_2. Clearly, ψ_2 satisfies the same properties 1–3 as ψ. Hence Lemma 7.2 applies to ψ_2 as well, and we get similarly to (7.10), (7.14), and (7.15)

$$\begin{aligned}\lim_{\epsilon\to 0} J_\epsilon^{(1a)}(x_\epsilon) &= \int_{\mathbb{R}^2}\left[\int_{[0,1]^3}\psi_2(\tilde{s}_1, v; \alpha(0,s_2))\,dv\right]d\tilde{s}_1 ds_2\\ &= \int_{\mathbb{R}^2} B_2(s_2)\left[\int_{[0,1]^3}\psi(\tilde{s}_1, v;\alpha(0,s_2))\,dv\right]d\tilde{s}_1 ds_2.\end{aligned} \tag{7.37}$$

Here we have used that the integrals with respect to $\tilde{s}_1$ and v are unaffected by the constant (with respect to $\tilde{s}_1$ and v) shifts in (7.36). Therefore, (7.19) holds with $J_\epsilon^{(1a)}(x_\epsilon)$ on the left.

To find the limit of $J_\epsilon^{(1b)}(x_\epsilon)$, consider the key step in (7.21),

$$\sum_j L_+(\epsilon j; x_\epsilon)\varphi\left(\frac{Y(\alpha,0;x_\epsilon)+y-\epsilon j}{\epsilon}\right) = L_+(Y(\alpha,0;x_\epsilon)+y;x_\epsilon), \tag{7.38}$$

which is rewritten with x_0 replaced by x_ϵ. This equality holds everywhere except in an $O(\epsilon)$ neighborhood of Γ $(=\Gamma_{x_0})$. Here we use that the curve Γ_{x_ϵ}, which is obtained by solving $P(Y(\alpha,0;x_\epsilon))=0$, depends smoothly on x_ϵ, and $\text{dist}(\Gamma_{x_\epsilon},\Gamma)=O(\epsilon)$ (see the argument preceding the statement of Theorem 7.1). Similarly to (7.21), the integrand is uniformly bounded, and we get

$$\lim_{\epsilon\to 0} J_\epsilon^{(1b)}(x_\epsilon) = \int_{S_+^2} B_1(\alpha)P_y'(Y(\alpha,0))\cdot Y_{tt}''(\alpha,0)\theta(P(Y(\alpha,0)))d\alpha. \tag{7.39}$$

The fact that the limits of $J_\epsilon^{(1c)}(x_\epsilon)$ and $J_\epsilon^{(2)}(x_\epsilon)$ as $\epsilon\to 0$ are independent of $\tilde{x}\in\tilde{\mathcal{U}}$ can be established in a similar way, and the theorem is proven. ∎

8. Illustrative example. As an example, consider the GRT that integrates a function supported in the half-space $x_3>0$ over spheres that are tangent to the plane $x_3=0$. For simplicity, we take $b(x,y)\equiv 1$ in (2.16). The family of such spheres is three-dimensional. We parametrize the spheres (and, consequently, the GRT) by the coordinates of their center y. Thus, $\mathcal{U}:=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3>0\}$ and $\mathcal{V}:=\{y=(y_1,y_2,y_3)\in\mathbb{R}^3 : y_3>0\}$. The surfaces $\mathcal{S}_y$ are spheres, and the defining function Φ in (2.9) becomes

$$\Phi(x,y) := y_3^2-(x_1-y_1)^2-(x_2-y_2)^2-(x_3-y_3)^2=0. \tag{8.1}$$

Clearly,

$$\Phi_x'(x,y)=2(y-x),\ \Phi_y'(x,y)=2(x_1-y_1,x_2-y_2,x_3). \tag{8.2}$$

Let us check the conditions DF1–DF4. From (8.2), $\Phi_{xy}''(x,y)=2I$, where I is the 3×3 identity matrix. In this case, by applying elementary row operations we conclude that the matrix in (2.5) is nonsingular if $\Phi_x'\cdot\Phi_y'\neq 0$. Using (8.1) and (8.2) we find that $\Phi_x'\cdot\Phi_y'=-4x_3y_3<0$. This verifies DF1 and DF4. Condition DF2 holds, because for any $\alpha\in S^2$ we can find $y\in\mathcal{V}$ such that $x\in\mathcal{S}_y$ and α is perpendicular to the sphere $\mathcal{S}_y$ at x. Denote $\hat{y}:=(y_1,y_2,0)$. The point $\hat{y}$ can be viewed as the south pole of the sphere $\mathcal{S}_y$, and $\Phi_y'(x,y)=2(x-\hat{y})$. Condition DF3 is violated if there are two distinct points on the sphere $x,z\in\mathcal{S}_y$ such that $x-\hat{y}$ and $z-\hat{y}$ are parallel. Since this is not possible, DF3 holds as well.

Details of applying the inversion formula (2.16) to discrete data, including computation of the function $Y(\alpha,t;x)$, are described in the next section.

9. Numerical experiment. We start by constructing an interpolation kernel with the required properties. To obtain φ, we first obtain an interpolation kernel $\varphi_{1\text{D}}$ that has properties IK1–IK5 in $\mathbb{R}$, and then extend it to $\mathbb{R}^3$ in a separable fashion. To obtain $\varphi_{1\text{D}}$ we use the result of [12], where such a kernel is obtained following the method in [3]:

$$\varphi_{1\text{D}}(t)=0.5(B_3(t)+B_3(t-2))+4B_3(t-1)-2(B_4(t)+B_4(t-1)). \tag{9.1}$$

Here B_n is the cardinal B-spline of degree n supported on $[0, n+1]$. Then the kernel φ becomes

$$\varphi(y) = \prod_{k=1}^{3} \varphi_{1\mathrm{D}}\left(\frac{y_k}{\Delta_k} + 3\right),\ y = (y_1, y_2, y_3), \tag{9.2}$$

where Δ_k is the data step-size along the kth axis. For simplicity, in this paper all the Δ_k are equal, i.e., $\Delta_k = \epsilon$, $k = 1, 2, 3$.

The GRT integrates over spheres, as described in section 8. The test object is the ball with center $x_c = (0, 0, 11)$, radius $R = 5$, and uniform density 1. The point on the boundary x_0, in a neighborhood of which we compute resolution, is given by

$$x_0 = x_c - R\alpha_0,\ \alpha_0 = (\sin(0.2\pi)\cos(0.7\pi), \sin(0.2\pi)\sin(0.7\pi), \cos(0.2\pi)). \tag{9.3}$$

In agreement with our convention, α_0 points into the interior of the ball.

There can be two spheres that are tangent to the ball at x_0. As an example, we consider the sphere whose center y_0 satisfies $(x_0 - y_0)\cdot\alpha_0 > 0$. Thus, for reconstruction near x_0 we use the data in a neighborhood of y_0. With this choice of y_0, the condition $\Phi_x'(x_0, y_0)/|\Phi_x'(x_0, y_0)| = -\alpha_0$ (see the text following (3.1)) is satisfied with Φ given by (8.1). For the selected x_0, α_0, and y_0, we compute using (8.2)

$$\nu = |\Phi_x'|/|\Phi_y'| = 0.526. \tag{9.4}$$

To compute the GRT, we use the formula for the area of a spherical cap:

$$A = 2\pi R h, \tag{9.5}$$

where R is the radius of the sphere, and h is the height of the cap. The values of R and h can be computed once the center of the sphere $\mathcal{S}_y$ is chosen (e.g., $R = y_3$). To simulate discrete data, the GRT is computed at the points $y = r + \epsilon j$. The interpolated data g_ϵ is computed using (2.15), where the kernel is given by (9.2) with $\Delta_k = \epsilon$, $k = 1, 2, 3$.

To apply the inversion formula (2.16), we numerically integrate g_ϵ over a neighborhood of α_0 on the unit sphere. To compute $(\partial/\partial t)^2 g_\epsilon(Y(\alpha, t; x))$ at $t = 0$ we use (2.15) and the chain rule as in (7.6). Given x, α, and t, the center of the sphere containing the point $x + t\alpha$ and normal to α at that point (cf. the paragraph following (2.15)) is easily found to be

$$Y(\alpha, t; x) = (x + t\alpha) - \frac{x_3 + t\alpha_3}{1 + \alpha_3}\alpha. \tag{9.6}$$

Consequently,

$$\frac{\partial}{\partial t} Y(\alpha, t; x)\bigg|_{t=0} = \frac{1}{1 + \alpha_3}\alpha,\ \ \frac{\partial^2}{\partial t^2} Y(\alpha, t; x)\bigg|_{t=0} = 0, \tag{9.7}$$

and

(9.8)
$$\left(\frac{\partial}{\partial t}\right)^2 \varphi\left(\frac{Y(\alpha, t; x) - \epsilon j}{\epsilon}\right)\bigg|_{t=0} d\alpha = \frac{1}{\epsilon^2}\frac{1}{(1 + \alpha_3)^2}\sum_{i,k=1}^{3} \varphi_{ik}''\left(\frac{Y(\alpha, 0; x)}{\epsilon} - j\right)\alpha_i\alpha_k.$$

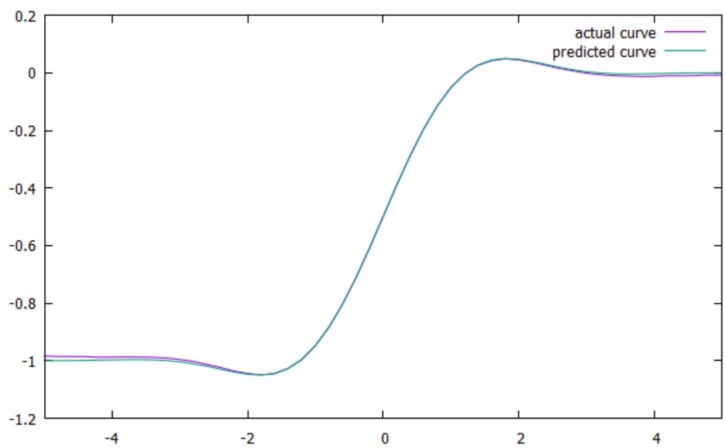

FIG. 4. *Comparison of the predicted and actual edge responses for* $\epsilon = 0.01$.

The cut-off function χ in (2.16) is constructed as follows. Let $\alpha^\perp$ run through the unit sphere in the plane $\alpha_0^\perp$. Then any $\alpha \in S_+^2$ (α_0 is the north pole of S_+^2) can be represented in the form $\alpha = (\cos\omega)\alpha_0 + (\sin\omega)\alpha^\perp$, $0 \le \omega \le \pi/2$. In the code we use

$$\chi(\alpha) = \begin{cases} 1, & 0 \le \omega < 0.8\omega_{\rm mx}, \\ \frac{1+\cos((\omega-0.8\omega_{\rm mx})/(0.2\omega_{\rm mx}))}{2}, & 0.8\omega_{\rm mx} \le \omega < \omega_{\rm mx}, \\ 0, & \omega \ge \omega_{\rm mx}. \end{cases} \tag{9.9}$$

Finally, the predicted response is computed using (3.6). The results corresponding to $\epsilon = 0.01$ are shown in Figure 4. We see a good match between the predicted and actual edge responses.

Appendix A. The Morse lemma and its application. For convenience of the reader we first present here the Morse lemma (see [9, p. 502]).

LEMMA A.1. *Let* $f(v,y)$ $(v \in \mathbb{R}^n, y \in \mathbb{R}^N)$ *be a real valued* C^∞ *function in a neighborhood of* $(0,0)$. *Assume that* $f'_v(0,0) = 0$ *and that* $A = f''_{vv}(0,0)$ *is nonsingular. Then the equation* $f'_v(v,y) = 0$ *determines in a neighborhood of* 0 *a* C^∞ *function* $v(y)$ *with* $v(0) = 0$, *and we have in a neighborhood of* $(0,0)$

$$f(v,y) = f(v(y),y) + Aw\cdot w/2, \tag{A.1}$$

where $w = v - v(y) + O(|v - v(y)|(|v| + |y|))$ *is a* C^∞ *function of* (v,y) *at* $(0,0)$.

To apply this lemma, we solve the equation $\Phi(x,y) = 0$ for x_1 in terms of $x^\perp$ and y in a neighborhood of (x_0, y_0). Because of the choice of coordinates (cf. (3.1)), this is possible. Thus, $x_1 = X_1(x^\perp, y)$ is a parametrization of $\mathcal{S}_y$ (see also (3.23)). Then we apply the Morse lemma with

$$v = x^\perp \in \mathbb{R}^2,\ y \in \mathbb{R}^3,\ f(v,y) = \Psi((X_1(x^\perp,y),x^\perp),y). \tag{A.2}$$

Since $\Psi'_{x_1}(x_0,y_0) = 0$, we have

$$f''_{v_j v_k}(0,0) = \Psi''_{x_1x_1}(x_0)\frac{\partial X_1}{\partial x_j^\perp}\frac{\partial X_1}{\partial x_k^\perp} + \Psi'_{x_1}(x_0)\frac{\partial^2 X_1}{\partial x_j^\perp \partial x_k^\perp} + \Psi''_{x_j^\perp x_k^\perp}(x_0),\ j,k = 1,2, \tag{A.3}$$

where the derivatives of X_1 are computed at $(x_0^\perp, y_0)$. By construction, $\partial X_1/\partial x_j^\perp = 0$, $j = 1, 2$, and $\Psi'_{x_1}(x_0) = |\Psi'(x_0)| > 0$. Hence

$$f''(0,0) = |\Psi'(x_0)|N(x_0), \tag{A.4}$$

and $f''(0,0)$ is negative definite. By the Morse lemma, there exists a smooth function $w(x^\perp, y)$ such that (A.1) holds. Diagonalizing A, and then rotating and rescaling w, we get another smooth function $\omega(x^\perp, y)$ such that $Aw \cdot w = -|\omega|^2$. Additionally, $\partial w(0,0)/\partial v = I$, where I is the 3×3 identity matrix. This follows easily from the proof of the Morse lemma (see [9, p. 503]). Hence, we can solve $w = w(x^\perp, y)$ for $x^\perp$: $x^\perp = X^\perp(w, y)$, which leads to the desired diffeomorphism $\omega \to x = x(\omega; y) \in \mathcal{S}_y$ for all y close to y_0.

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