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Fluctuations of Time Averages Around Closed Geodesics in Non-Positive Curvature

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Abstract: We consider the geodesic flow for a rank one non-positive curvature closed manifold. We prove an asymptotic version of the Central Limit Theorem for families of measures constructed from regular closed geodesics converging to the Bowen-Margulis-Knieper measure of maximal entropy. The technique expands on ideas of Denker, Senti and Zhang, who proved this type of asymptotic Lindeberg Central Limit Theorem on periodic orbits for expansive maps with the specification property. We extend these techniques from the uniform to the non-uniform setting, and from discrete-time to continuous-time. We consider Hölder observables subject only to the Lindeberg condition and a weak positive variance condition. If we assume a natural strengthened positive variance condition, the Lindeberg condition is always satisfied. Our results extend to dynamical arrays of Hölder observables, and to weighted periodic orbit measures which converge to a unique equilibrium state.

1. Introduction

A goal in the study of dynamical systems with some hyperbolicity is to exhibit the kind of stochastic behavior obeyed by sequences of i.i.d. random variables. In settings with non-uniform hyperbolicity, we may be able to demonstrate this kind of stochastic behavior *within* the system even in situations where it is intractable to demonstrate globally. Our paper follows this philosophy. We consider the geodesic flow for a rank one non-positive curvature closed manifold. We exhibit sequences of measures constructed from regular closed geodesics whose first order behavior is that of the measure of maximal entropy, and whose second order behavior obeys, in the limit, the Lindeberg Central Limit Theorem.

The Lindeberg condition is a classical criteria from Probability Theory, which often gives a necessary and sufficient criteria for the Central Limit Theorem (CLT) to hold

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for sequences of independent random variables which are not identically distributed. Roughly, the Lindeberg condition guarantees that the variance of a single random variable is negligible in comparison to the sum of all the variances. This idea was recently explored by Denker, Senti and Zhang [8] in the setting of maps with the specification property. They showed that a Lindeberg condition on the sequence of periodic orbit measures is equivalent to a Central Limit Theorem in the limit.

The analysis of this paper extends the ideas of Denker, Senti and Zhang to the geodesic flow on a rank one non-positive curvature closed manifold. This is one of the main classes of examples of non-uniformly hyperbolic flows. While the theory of equilibrium states in this setting has been extended recently by [3], the statistical properties of these measures remain largely wide open, even for the Knieper-Bowen-Margulis measure of maximal entropy μ_{KBM}. This contrasts with the well-understood case of geodesic flow on negative curvature manifolds, for which the CLT was established by Ratner [20]. In particular, the CLT for the MME and other equilibrium states remains out of reach of current methods in the non-positive curvature setting.

In this paper, we show that for a Hölder observable, the time averages for certain measures constructed from regular closed geodesics asymptotically obey the Central Limit Theorem. This enriches the picture for these time averages, whose first order behavior is convergence to the integral with respect to the measure of maximal entropy. This result applies under the Lindeberg condition and a weak positive variance condition on the sequence of periodic orbit measures. This result is stated formally as Theorem 4.1. We show that the Lindeberg condition is always satisfied under a natural strengthening of the positive variance condition. This is carried out in Sect. 5. We now build up some notation to state and motivate our results, and give an idea of the constructions involved.

Recall that for an invariant measure μ, and an observable f, the dynamical variance for the flow (g_t), when it exists, is defined by

$$\sigma^2_{\mathrm{Dyn}}(f,\mu) = \lim_{T\to\infty} \int \left(\frac{F(\cdot,T) - \int F(\cdot,T)d\mu}{\sqrt{T}} \right)^2 d\mu, \tag{1.1}$$

where $F(x,T) = \int_0^T f(g_s x)ds$. In our setting, for a fixed $\eta > 0$, we construct a sequence of discrete probability measures (m_l) on closed orbits in T^1M corresponding to uniformly η-regular closed geodesics (which are defined in Sect. 2.4). We consider the collection of η-regular closed geodesics which have least period in the interval $(T_l - \delta_l, T_l]$, where $T_l \to \infty$ and $\delta_l \to 0$, which we denote $\mathrm{Per}^\eta_R(T_l - \delta_l, T_l]$. We define m_l by choosing one point in T^1M tangent to each such geodesic (we denote this set of points by E_l), and distributing mass equally over these points. By analogy with (1.1), it is natural for us to define the (lower) dynamical variance for the sequence of measures (m_l) to be

$$\underline{\sigma}^2_{\mathrm{Dyn}}(f,(m_l)) = \liminf_{l\to\infty} \int \left(\frac{F(\cdot,T_l) - \int F(\cdot,T_l)dm_l}{\sqrt{T_l}} \right)^2 dm_l.$$

We choose two more sequences $k_l \to \infty$, $C_l \to \infty$, and define another sequence of measures (ν_l). Each ν_l is given by constructing points out of the product $E_l^{k_l}$ by using a certain specification property on the η-regular closed geodesics to find an orbit segment which loops C_l times round each of the closed geodesics indexed by an element of $E_l^{k_l}$. We write S_l for the total length of an orbit segment specified in this way (precisely, $S_l = k_l(C_l T_l + M)$, where M is the transition time in applying our specification property).

The measure ν_l is given by putting mass equally along the initial segment of length T_l of all the orbit segments defined this way.

If the variance quantity $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l))$ is positive, we can choose k_l and C_l so that the family of measures (ν_l) satisfies an asymptotic central limit theorem for the observable f. We can state a simple version of our main results as follows.

Theorem A. *For any $\eta > 0$ and sequences $\delta_l \to 0$, $T_l \to \infty$, we define a sequence of discrete probability measures $(m_l)_{l\in\mathbb{N}}$ by choosing a point tangent to each element of $\mathrm{Per}^{\eta}_{R}(T_l - \delta_l, T_l]$, and assigning each of these points equal mass. We assume that T_l is chosen to increase sufficiently fast, depending on η and δ_l, to allow for our construction of (ν_l) (see Hypothesis 3.1). Suppose $f \in C(T^1M)$ is Hölder continuous with*

$$\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) > 0. \tag{1.2}$$

Then there exists sequences $k_l \to \infty$, $C_l \to \infty$, so that the sequence of measures (ν_l) defined by the data $(\delta_l, T_l, k_l, C_l)_{l\in\mathbb{N}}$ (see Sect. 3 for details of the construction), which converges weak to the measure of maximal entropy μ_{KBM}, satisfies the following asymptotic central limit theorem. For all $a \in \mathbb{R}$,*

$$\lim_{l\to\infty} \nu_l \left(\left\{ v : \frac{F(v, S_l) - S_l \int f d\nu_l}{\sigma_{\nu_l}(F(\cdot, S_l))} \le a \right\} \right) = N(a), \tag{1.3}$$

where N is the cumulative distribution function of the normal distribution $\mathcal{N}(0, 1)$, and $\sigma^2_\mu(\phi)$ denotes the usual 'static' variance $\sigma^2_\mu(\phi) = \int \left(\phi - \int \phi d\mu\right)^2 d\mu$.

The sequences (k_l) and (C_l) are determined by $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l))$. Thus, given $\alpha > 0$, we can find a sequence of measures (ν_l), defined by the data $(\delta_l, T_l, k_l, C_l)_{l\in\mathbb{N}}$, so that any Hölder continuous observable f with $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) > \alpha$ satisfies (1.3). We comment on this positive variance condition. If the manifold has strictly negative curvature, (m_l) places mass on each closed periodic orbit whose length is in the interval $(T_l - \delta_l, T_l]$, and we expect that the variances for (m_l) converge to the variance of the MME, along the lines of the basic argument in [19, Theorem 1]. Thus, in negative curvature, we expect that $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) = \sigma^2_{\mathrm{Dyn}}(f, \mu_{\mathrm{KBM}})$. In negative curvature, the variance $\sigma^2_{\mathrm{Dyn}}(f, \mu_{\mathrm{KBM}})$ vanishes if and only if the observable is a coboundary [18]. It would be interesting to characterize the class of observables for which $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) = 0$ in the current context, although this will require some substantial new ideas and techniques. Although rigorous analysis of this question is beyond the scope of this paper, by analogy with the negative curvature case, our intuition is that the positive variance condition (1.2) should be the 'typical' case.

Our result extends to arrays of observable functions, and a large class of equilibrium states. Furthermore, the arguments of this paper will apply for other classes of systems with enough hyperbolicity to yield some non-uniform specification properties. We do not attempt to formalize an abstract general statement, but we hope that our proof makes clear what the roadmap should be in other related settings. We discuss these generalizations in Sect. 6.

The technique is an extension of Denker, Senti and Zhang (DSZ) [8]. The idea is to build ϵ-independent collections of regular closed periodic orbits whose growth rate is the topological entropy. Classical probability theory allows us to conclude that the Lindeberg CLT holds for certain uniform measures on parameter spaces associated to these collections. The analysis of the paper relies on using the specification property to propagate this result to measures with support in T^1M, modeled on closed geodesics. For the analysis to work, we must restrict to closed periodic orbits with some uniform regularity. For this, we use structure provided by the work of Burns, Climenhaga, Fisher, and the first named author [3]. To obtain the first order behavior of measures on these orbits, we need their growth rate to be comparable to the entropy, and that there is a unique measure of maximal entropy. The first point is provided by [3] and the second point was originally proved by Knieper [16].

While we are indebted to DSZ for the strategy and philosophy of this paper, our analysis requires several novelties. In DSZ, the focus is on discrete-time dynamical systems with uniform specification. They establish the Lindeberg CLT in their general setting, but do not explore how to verify the Lindeberg condition in examples. The novelty in the current work is that we deal with with non-uniformity and continuous-time, we apply it to geodesic flow in non-positive curvature, and we verify the Lindeberg condition from a natural positive variance condition. To achieve this, there are significant technical differences. A key difference is that our construction involves looping round closed geodesics multiple times. The reason that this is necessary is because in the flow case, it is necessary to construct the measures using segments of orbit rather than point masses. We lose independence between adjacent orbit segments due to the types of averages we are forced to consider. The looping construction is designed to compensate for this loss of independence, which is key to the whole approach. Looping brings new technical issues—notably, the small differences in periods of the closed geodesics in $\mathrm{Per}^{\eta}_{R}(T_l - \delta_l, T_l]$ add up. This is why we require $\delta_l \to 0$, and is one reason that the choice of constants in our construction is subtle. A by-product of our construction is that it easily generalizes to the case of equilibrium states, which was not clear in DSZ.

The paper is structured as follows. In Sect. 2, we recall relevant background information. In Sect. 3, we describe our construction of measures from closed geodesics. In Sect. 4, we state and prove our main results. In Sect. 5, we show how to check the Lindeberg condition under a suitable positive variance condition. In Sect. 6, we discuss various extensions of our main results.

2. Background

2.1. Preliminaries, entropy, and pressure. We consider a continuous flow (g_t) on a compact metric space (X, d). For $\epsilon > 0$ and $t > 0$, and $x \in X$, we define the dynamical (Bowen) ball to be

$$B_t(x, \epsilon) = \{y \in X : d(f_s x, f_s y) < \epsilon \text{ for all } 0 \leq s \leq t\}.$$

For a continuous function $f : X \to \mathbb{R}$, we write

$$F(x, t) = \int_0^t f(g_\tau x) d\tau.$$

We also write

$$F(x, [s, t]) = F(g_s x, t - s) = \int_s^t f(g_\tau x) d\tau$$

We use analogous notation when we use other lower case letters for an observable. Thus, for example, for an observable h, we write $H(x, t) = \int_0^t h(g_\tau x) d\tau$.

We consider collections of finite-length orbit segments $\mathcal{C} \subset X \times [0, \infty)$, where (x, t) is identified with the orbit segment $\{g_s x : s \in [0, t)\}$. For $t > 0$, we define $\mathcal{C}_t = \{x \in X, (x, t) \in \mathcal{C}\}$. We say $E \subset Z$ is (t, ϵ)-separated for Z if for all $x, y \in E$, $y \notin \overline{B_t(x, \epsilon)}$.

For $\mathcal{C} \subset X \times [0, \infty)$, the entropy $h(\mathcal{C}, \epsilon)$ at scale ϵ is defined as

$$h(\mathcal{C}, \epsilon) = \limsup_{t \to \infty} \frac{1}{t} \log \sup\{\#E : E \subset \mathcal{C}_t \text{ is } (t, \epsilon)\text{-separated}\},$$

and $h(\mathcal{C}) = \lim_{\epsilon \to 0} h(\mathcal{C}, \epsilon)$. For a set Z, we define $h(Z, \epsilon)$ as $h(\mathcal{C}_Z, \epsilon)$, where $\mathcal{C}_Z = \{(x, t) : x \in Z, t \in [0, \infty)\}$. In particular, $h(X, \epsilon)$ reduces to the standard definition of topological entropy, see [21]. The Variational Principle states that $h(X)$ is the supremum of the measure-theoretic entropies h_μ taken over flow-invariant probability measures. A measure achieving the supremum is called a measure of maximal entropy.

2.2. Central limit Theorem. The Central Limit Theorem in dynamics describes the second order behavior of the sequence of ergodic sums/integrals. The classical CLT for a continuous flow equipped with an ergodic measure μ says that for a Hölder observable f with $\int f d\mu = 0$, the sequence $\frac{1}{\sqrt{t}} F(\cdot, t)$ converges in distribution to the normal distribution. This result was proved for hyperbolic flows by Ratner [20], and strengthened by Denker and Phillip in [7]. See also Parry and Pollicott [18].

The classical Central Limit Theorem can be obtained using a variety of techniques. We do not attempt to survey the literature here, but we recommend recent papers by [1,5,8,9,14,17] for an excellent paper trail. One might expect the classical CLT to hold in the setting of this paper, but none of these proof techniques are currently known to apply. We also mention an interesting recent related result—an asymptotic central limit theorem for lengths of closed geodesics in hyperbolic surfaces was recently proved by Gekhtman, Taylor and Tiozzo [12].

Our result is based on the Lindeberg CLT, which is one of the most famous generalizations of the classical CLT. We recall its statement in its original context of a sequence of independent random variables. First we define the Lindeberg function for a probability measure ν and an observable h, and a constant $c \geq 0$.

Definition 2.1. Let $Z(c) = Z(c, h, \nu) = \{x : |h - \int h d\nu| > c\}$. The *Lindeberg* function is

$$L_\nu(h, c) := \int (h - \int h d\nu)^2 \mathbb{1}_{Z(c)}(v) d\nu(v)$$

Recall that for a probability measure ν on a space Ω and a function $f : \Omega \to \mathbb{R}$, the variance $\sigma_\nu(f)$ is defined by

$$\sigma_\nu^2(f) = \int \left(f - \int f d\nu\right)^2 d\nu = \int f^2 d\nu - \left(\int f d\nu\right)^2. \tag{2.1}$$

Theorem 2.2. *(Lindeberg CLT for independent random variables) Let (Ω, ν) be a probability space and let $(X_i)_{i=1}^\infty$ be an independent sequence of random variables. Let σ_i be the variance of X_i, and let $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Suppose that for every $\epsilon > 0$*

$$\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{i=1}^n L_\nu(X_i, \epsilon s_n) = 0. \tag{2.2}$$

Then $\frac{1}{s_n}\sum_{i=1}^n (X_i - \int X_i d\mu)$ *converges in distribution to the standard normal distribution.*

The hypothesis (2.2) is called the Lindeberg condition, see e.g. [11]. We will formulate our results using a dynamical version of the Lindeberg condition on periodic orbits, following Denker, Senti and Zhang [8].

2.3. Geometry and dynamics of the geodesic flow. We recall the necessary background from [3] on geodesic flow for non-positive curvature manifolds. The arguments in this paper use the dynamical structure obtained there, rather than direct geometric arguments. We refer to [2,10] for general geometric background.

We consider a compact, connected, boundaryless smooth manifold M equipped with a smooth Riemannian metric g, with non-positive sectional curvatures at every point. For each v in the unit tangent bundle T^1M there is a unique constant speed geodesic denoted γ_v such that $\dot{\gamma}_v(0) = v$. The *geodesic flow* $(g_t)_{t\in\mathbb{R}}$ acts on T^1M by $g_t(v) = (\dot{\gamma}_v)(t)$. We equip T^1M with a metric d given by

$$d(v, w) = \max\{d_M(\gamma_v(t), \gamma_w(t)) \mid t \in [0, 1]\}, \tag{2.3}$$

where d_M is the Riemannian distance on M. The flow is entropy expansive, which implies that for sufficiently small ϵ, $h(X) = h(X, \epsilon)$. We call such a scale an *expansivity constant*. Any positive ϵ which is less than one third of the injectivity radius of M is an expansivity constant.

Given $v \in T^1M$, stable and unstable horospheres H^s_v and H^u_v can be defined locally using Jacobi fields or using a standard geometric construction in the universal cover. The horospheres are C^2 manifolds. The (strong) stable and unstable manifolds W^s_v, W^u_v are defined as normal vector fields to H^s_v, H^u_v, and we can define the stable and unstable subspaces $E^s_v, E^u_v \subset T_vT^1M$ to be the tangent spaces of W^s_v, W^u_v respectively. The weak stable manifold W^{cs}_v is defined in the obvious way so that its tangent space is $E^s_v \oplus E^0_v$, where E^0_v is the space given by the flow direction. The bundles E^s, E^u are invariant, and depend continuously on v, see [10,13].

We define the *singular set* Sing as the set of $v \in T^1M$ so that the geodesic determined by v has a parallel orthogonal Jacobi field, and Reg to be the complement of Sing. We say that M is rank one if Reg $\neq \emptyset$. The Jacobi field formalism is used extensively in [3], and we refer there for full definitons.

A key piece of geometric data which is at the heart of our analysis is a continuous function $\lambda\colon T^1M \to [0,\infty)$ defined in [3]. Roughly, $\lambda(v)$ is the smallest normal curvature at v (with sign chosen to be non-negative) of the stable and unstable horospheres centered at v. If $\lambda(v) > 0$, then $v \in$ Reg. We refer to [3] for the precise definition and more geometric context. Let $\mathrm{Reg}(\eta) = \{v : \lambda(v) \geq \eta\}$. If $v \in \mathrm{Reg}(\eta)$, then we have various uniform estimates at the point v, for example on how distance scales in the local stable and unstable manifolds. These are the properties exploited in this paper. We recall the precise statement obtained on local product structure.

Lemma 2.3 [3, Lemma 4.4]. *For every $\eta > 0$, there exist $\delta > 0$ and $\kappa \geq 1$ such that at every $v \in \mathrm{Reg}(\eta)$, the foliations W^u, W^{cs} have local product structure with constant κ in a δ-neighborhood of v. That is, for every $\epsilon \in (0, \delta]$ and all $w_1, w_2 \in B(v, \epsilon)$, the intersection $W^u_{\kappa\epsilon}(w_1) \cap W^{cs}_{\kappa\epsilon}(w_2)$ contains a single point, which we denote by $[w_1, w_2]$, and*

$$d^u(w_1, [w_1, w_2]) \leq \kappa d(w_1, w_2),$$

$$d^{cs}(w_2, [w_1, w_2]) \leq \kappa d(w_1, w_2).$$

Uniformity of the local product structure on $\mathrm{Reg}(\eta)$ is used to obtain the specification property for orbit segments starting and ending in $\mathrm{Reg}(\eta)$. Precisely, we define the collection of orbit segments

$$\mathcal{C}(\eta) := \{(v, t) : \lambda(v) \geq \eta,\ \lambda(g_t v) \geq \eta\}.$$

We have the following result.

Theorem 2.4 [3, Theorem 4.1]. *For each $\eta > 0$, the collection of orbit segments $\mathcal{C}(\eta)$ has the specification property. That is, given $\rho > 0$, there exists $\tau = \tau(\rho)$ such that for every $(x_1, t_1), \ldots, (x_N, t_N) \in \mathcal{C}$ and* every *collection of times $\tau_1, \ldots, \tau_{N-1}$ with $\tau_i \geq \tau$ for all i, there exists a point $y \in X$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^{j} t_i + \sum_{i=0}^{j-1} \tau_i$, we have*

$$f_{s_{j-1}+\tau_{j-1}}(y) \in B_{t_j}(x_j, \rho)$$

for every $j \in \{1, \ldots, N\}$.

We recall some other results that we will use from [3] and [6]. We often consider the following set of orbit segments

$$\mathcal{B}(\eta) := \{(v, t) : \frac{\int_0^t \lambda(g_u(v))du}{t} < \eta\}.$$

Note that λ vanishes on Sing, so any orbit segment in $\mathrm{Sing} \times [0, \infty)$ is a member of $\mathcal{B}(\eta)$. It was shown in [3, §5] that $\lim_{\eta \to 0} h(\mathcal{B}(\eta)) = h(\mathrm{Sing})$. For the class of geodesic flows under consideration, it is known that

$$h(\mathrm{Sing}) < h(T^1 M).$$

This is easy in the case that M is a surface, since $h(\mathrm{Sing}) = 0$. However, this entropy gap is a highly non-trivial fact in higher dimensions. It was first proved as a consequence of Knieper's work [16], and a direct proof is given in [3]. The geodesic flow has a unique measure of maximal entropy, known as the Knieper-Bowen-Margulis measure, which we denote by μ_{KBM}.

2.4. Counting closed regular geodesics. For a small $\delta > 0$, we define $\mathrm{Per}_R(T - \delta, T]$ to be the set of closed regular geodesics which have length in the interval $(T - \delta, T]$. For $\gamma \in \mathrm{Per}_R(T - \delta, T]$, we write $|\gamma|$ for its length, and v_γ for an element of $T^1 M$ chosen to be tangent to γ.

Recall from Proposition 6.4 in [3], for any $\delta > 0$, there exists $T_\delta > 0$ and

$$\beta = \beta(\delta) \approx e^{-hT_\delta} \tag{2.4}$$

such that for all $T > T_\delta$, we have

$$\frac{\beta}{T} e^{Th} \leq \#\mathrm{Per}_R(T - \delta, T] \leq \beta^{-1} e^{Th}. \tag{2.5}$$

We take $T_\delta \to \infty$ as $\delta \to 0$ (and the proofs of Proposition 4.5, Lemma 4.7 and Proposition 6.4 in [3] show that this is necessary). For $\eta > 0$, we define the *uniformly regular closed geodesics* as

$$\mathrm{Per}_R^\eta(T-\delta, T] := \{\gamma \in \mathrm{Per}_R(T-\delta, T] : \int_0^{|\gamma|} \lambda(g_s v_\gamma) ds \geq |\gamma|\eta\},$$

that is the collection of elements in $\mathrm{Per}_R(T-\delta, T]$ whose average of λ is at least η. Writing $h' := \frac{h(\mathrm{Sing})+h}{2}$, we fix $\eta > 0$ throughout the rest of the paper such that $h(\mathcal{B}(2\eta)) < h' < h$. We also choose ϵ so that 4ϵ is an expansivity constant. In particular, $h(T^1M, 4\epsilon) = h$. Notice that we can choose ϵ smaller if necessary.

Define $\delta' := \frac{\eta}{\lambda_{\max}}$ where $\lambda_{\max} := \max\{\lambda(v) : v \in T^1M\}$. We now argue that for δ sufficiently small, $\#\mathrm{Per}_R^\eta(T-\delta, T]$ is bounded uniformly from below.

Lemma 2.5 *For any $\delta < \delta'$, there exists $T_0 = T_0(\delta, \eta)$ such that for all $T > T_0$,*

$$\#\mathrm{Per}_R^\eta(T-\delta, T] \geq \frac{\beta}{2T} e^{Th}. \tag{2.6}$$

Proof Recall that we fix η such that $h(\mathcal{B}(2\eta)) < h'$. It follows that there exists $T_0' = T_0'(\eta) > 0$ so for all $T > T_0'$, there are maximal $(T, 4\epsilon)$-separated sets E_T for $\mathcal{B}(2\eta)$ so that $\#E_T < e^{Th'}$ and also so that $e^{Th'} < \frac{\beta}{2T}e^{Th}$. Given $\delta \in (0, \delta')$, define $T_0(\delta, \eta) := \max\{T_0'(\eta), T_\delta, 1\}$. With a fixed η, since $T_\delta \to \infty$ as $\delta \to 0$, we observe that $T_0(\delta, \eta) = T_\delta$ when δ is sufficiently small. We write $\mathrm{Per}_R^{<\eta}(T-\delta, T] := \mathrm{Per}_R(T-\delta, T] \setminus \mathrm{Per}_R^\eta(T-\delta, T]$. For $T > T_0$ and any $\gamma \in \mathrm{Per}_R^{<\eta}(T-\delta, T]$, we choose a vector $v_\gamma \in T^1M$ such that it is tangent to γ at some point. Due to the difference in the period of elements in $\mathrm{Per}_R(T-\delta, T]$, different choices of v_γ may lead to variations in the precise value of $\int_0^T \lambda(g_s v_\gamma) ds$. However, we have

$$\int_0^T \lambda(g_s v_\gamma) ds \leq |\gamma|\eta + \delta' \lambda_{\max} < T\eta + \eta < 2T\eta, \tag{2.7}$$

which shows that we always have $(v_\gamma, T) \in \mathcal{B}(2\eta)$. By the choice of 4ϵ and Sect. 6 in [16] we know elements in $\mathrm{Per}_R(T-\delta, T]$ are $(T, 4\epsilon)$-separated, which in turns shows that $\mathrm{Per}_R^{<\eta}(T-\delta, T] < e^{Th'} < \frac{\beta}{2T}e^{Th}$. As a consequence, we have

$$\begin{aligned}\#\mathrm{Per}_R^\eta(T-\delta, T] &= \#\mathrm{Per}_R^\eta(T-\delta, T] - \#\mathrm{Per}_R^{<\eta}(T-\delta, T] \\ &> \frac{\beta}{T}e^{Th} - \frac{\beta}{2T}e^{Th} = \frac{\beta}{2T}e^{Th}.\end{aligned}$$

□

From now on we always assume that δ and T satisfy the conditions in Lemma 2.5. By the definition of $\mathrm{Per}_R^\eta(T-\delta, T]$, if γ is an element in $\mathrm{Per}_R^\eta(T-\delta, T]$, there must be some $t \in [0, T)$ such that $v = \dot\gamma(t) \in \mathrm{Reg}(\eta)$. Since v is periodic, we know that $(v, |\gamma|) \in \mathcal{C}(\eta)$.

For each $\gamma \in \mathrm{Per}_R^\eta(T-\delta, T]$, we choose $v = v(\gamma)$ such that $v \in \gamma \cap \mathrm{Reg}(\eta)$. We let

$$E_\delta(T) = \{v(\gamma) : \gamma \in \mathrm{Per}_R^\eta(T-\delta, T]\},$$

recalling that $E_\delta(T)$ is a $(T, 4\epsilon)$-separated set. From the definition of $E_\delta(T)$ and (2.6), we know that

$$\#E_\delta(T) = \#\mathrm{Per}_R^\eta(T-\delta, T] \geq \frac{\beta}{2T} e^{Th}. \tag{2.8}$$

We will often work with the collection of orbit segments

$$\{(v, T) : v \in E_\delta(T)\}.$$

Here we use the same T across all $v \in E_\delta(T)$ so we can compare lengths uniformly—note that T differs from the least period of $\gamma(v)$ by at most δ.

2.5. Growth of variations on $\mathcal{C}(\eta)$. For a collection of orbit segments $\mathcal{C}$, any $\delta > 0$, $T > 0$ and $h \in C(T^1M)$ we define

$$\omega(h, T, \delta, \mathcal{C}) := \sup_{(u,T)\in\mathcal{C}, v\in B_T(u,\delta)} |H(u, T) - H(v, T)|.$$

The following analogy of Lemma 5.6 in [22] holds for ω, and is a crucial estimate in the construction given in Sect. 3.

Lemma 2.6 . *Let $\mathcal{C}(\eta)$ be the collection previously defined. Then for sufficiently small δ_0, for any $h \in C(T^1M)$, we have*

$$\lim_{T\to\infty} \frac{\omega(h, T, \delta_0, \mathcal{C}(3\eta/4))}{T} = 0. \tag{2.9}$$

Proof This proof is parallel to the one of Lemma 5.6 in [22]. Choose the same η as before and δ_0 such that

(1) $\mathrm{Reg}(\frac{3\eta}{4})$ has local product structure at scale $4\delta_0'$, with coefficient $\kappa = \kappa(\frac{3\eta}{4}, 4\delta_0') > 1$. Take $\delta_0 = \delta_0'/\kappa$.
(2) for any $u, v \in T^1M$ such that $d(u, v) < \kappa\delta_0$, we have $|\lambda(u) - \lambda(v)| < \frac{\eta}{4}$. In particular, we have $B(\mathrm{Reg}(\frac{3\eta}{4}), \kappa\delta_0) \subset \mathrm{Reg}(\frac{\eta}{2})$ and $B(\mathrm{Reg}(\frac{\eta}{2}), \kappa\delta_0) \subset \mathrm{Reg}(\frac{\eta}{4})$.

Consider $(u, T) \in \mathcal{C}(3\eta/4)$ and $v \in B_T(u, \delta_0)$. By the local product structure, there is a vector $u_0 \in T^1M$ such that $u_0 \in W^s_{\kappa\delta_0}(u) \cap W^{cu}_{\kappa\delta_0}(v)$. Then there exists $s \in (-\kappa\delta_0, \kappa\delta_0)$ such that $g_s(u_0) \in W^u_{\kappa\delta_0}(v)$. Observe that $d(g_Tu, g_Tv) < \delta_0$, $d(g_Tu, g_Tu_0) < \kappa\delta_0$, $d(g_Tu_0, g_{T+s}u_0) < \kappa\delta_0$. Combining the above three inequalities together we have $d(g_Tv, g_{T+s}u_0) < 3\kappa\delta_0$. As $g_{T+s}u_0 \in W^u(g_Tv)$, we know $d^u(g_{T+s}u_0, g_Tv) < 3\kappa^2\delta_0$, therefore $d^{cu}(g_Tu_0, g_Tv) < 4\kappa^2\delta_0 = 4\kappa\delta_0'$. In other words, $g_Tu_0 \in W^s_{4\kappa\delta_0'}(g_Tu) \cap W^{cu}_{4\kappa\delta_0'}(g_T(v))$. As $d(g_Tu, g_Tv) < \delta_0$ and $g_Tu \in \mathrm{Reg}(\frac{3\eta}{4})$, by the local product structure we know $g_Tu_0 \in W^s_{\kappa\delta_0}(g_Tu) \cap W^{cu}_{\kappa\delta_0}(g_Tv)$. In particular, $g_{T+s}u_0 \in W^u_{\kappa\delta_0}(g_Tv)$. We conclude that $g_{t+s}u_0 \in W^u_{\kappa\delta_0}(g_tv)$ and $g_tu_0 \in W^s_{\kappa\delta_0}(g_tu) \cap W^{cu}_{\kappa\delta_0}(g_tv)$ for all $t \in [0, T]$. Now, for any fixed $h \in C(T^1M)$, we can bound the variation of h over (u, T) and (v, T) by variations along the stable, central and unstable directions. To be more precise, we have $|H(u, T) - H(v, T)| \leq |H(u, T) - H(u_0, T)| + |H(u_0, T) - H(g_su_0, T)| + |H(g_su_0, T) - H(v, T)|$. From the definition of $\mathcal{C}(3\eta/4)$ and property (2) of δ_0, we know $\lambda(u), \lambda(g_Tu) > \frac{3\eta}{4}$ and

$\lambda(v), \lambda(g_T v) > \frac{\eta}{2}$, so $(v, T) \in \mathcal{C}(\frac{\eta}{2})$. Therefore, to prove (2.1), it suffices to prove the following

$$\lim_{T\to\infty} \frac{\omega_s(h, T; \kappa\delta_0, 3\eta/4)}{T} = 0 \tag{2.10}$$

and

$$\lim_{T\to\infty} \frac{\omega_u(h, T; \kappa\delta_0, \eta/2)}{T} = 0, \tag{2.11}$$

where

$$\omega_s(h, T; \kappa\delta_0, 3\eta/4) := \sup_{g_T(u)\in \mathrm{Reg}(3\eta/4),\, v\in W^s_{\kappa\delta_0}(u)} |H(u, T) - H(v, T)|$$

and

$$\omega_u(h, T; \kappa\delta_0, \eta/2) := \sup_{u\in \mathrm{Reg}(\eta/2),\, v\in g_{-T}(W^u_{\kappa\delta_0}(g_T(u)))} |(H(u, T) - H(v, T)|.$$

Let us prove (2.10). Consider $u' \in T^1M$ such that $g_T u' \in \mathrm{Reg}(3\eta/4)$ and $v' \in W^s_{\kappa\delta_0}(u')$. For any $\hat{\epsilon} > 0$, by uniform continuity of h on T^1M we know there exists $\hat{\delta} > 0$ such that if $v_1, v_2 \in T^1M$, $d(v_1, v_2) < \hat{\delta}$, then $|h(v_1) - h(v_2)| < \hat{\epsilon}$. Meanwhile, property (2) of δ_0 shows that any vector $\hat{v}$ lying on the local stable arc connecting u' and v' satisfies $\lambda(g_T\hat{v}) > \frac{\eta}{2}$. Following the proof of Lemma 3.10 in [3], for any $0 \le t_1 \le t_2 \le T$ we have

$$d^s(g_{t_1}u', g_{t_1}v') \ge e^{\eta(t_2-t_1)/2} d^s(g_{t_2}u', g_{t_2}v'). \tag{2.12}$$

Since $d^s(u', v') < \kappa\delta_0$, by (2.12) we have $d^s(g_t u', g_t v') < \kappa\delta_0 e^{-\eta t/2}$. By writing $(2\log(\frac{\kappa\delta_0}{\hat{\delta}}))/\eta$ as $\hat{T}$ and assuming that $T > \hat{T}$ (which is possible since the choice on $\hat{T}$ does not depend on T and T approaches ∞), it is easy to see that $d(g_t u', g_t v') \le d^s(g_t u', g_t v') < \hat{\delta}$ for $t \in [\hat{T}, T]$. Therefore, we have $|H(u', T) - H(v', T)| \le |H(u', \hat{T}) - H(v', \hat{T})| + |H(g_{\hat{T}}u', T - \hat{T}) - H(g_{\hat{T}}v', T - \hat{T})| \le 2\hat{T}||h|| + (T - \hat{T})\hat{\epsilon}$, and this holds for all such u', v' and $T > \hat{T}$, which shows that $\lim_{T\to\infty} \frac{\omega_s(h,T;\kappa\delta_0,3\eta/4)}{T} \le \hat{\epsilon}$. By making $\hat{\epsilon}$ arbitrarily small, (2.10) is proved. (2.11) is proved similarly by replacing $3\eta/4, \eta/2$ with $\eta/2, \eta/4$. □

The small δ_0 in the above lemma can be chosen so that $\epsilon < \delta_0 < \delta'$, and we will do so in §3. Recall ϵ is a choice of scale so that 4ϵ is an expansivity constant, and can be chosen arbitrarily small, so we can ensure that it is chosen smaller than δ_0.

3. Construction of Measures

We now construct sequences of measures that converge to μ_{KBM}, and are reference measures for our CLT. Recall T_0 and δ_0 are chosen as in Lemmas 2.5 and 2.6. We start by constructing a sequence of 4-tuples $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$ as follows.

Hypothesis 3.1 *We choose sequences* $T_l \in (0, \infty)$, $k_l \in \mathbb{N}$, $\delta_l \in (0, \delta_0)$, *and* $C_l \in \mathbb{N}$ *which satisfy the following relationships:*

(1) For all $l \in \mathbb{N}$, $T_l > \max\{T_0(\delta_l, \eta), 1\}$
(2) $T_l \uparrow \infty$, $\frac{T_l}{T_0(\delta_l,\eta)} \uparrow \infty$ *and* $k_l \uparrow \infty$
(3) $k_l\delta_l^2 \downarrow 0$ *and* $\frac{\sqrt{k_l}T_l}{C_l} \downarrow 0$.

Sequences which satisfy these conditions can be easily found by first choosing δ_l, then T_l, then k_l and C_l. For each l, let

$$E_l := E_{\delta_l}(T_l)$$

be a $(T_l, 4\epsilon)$-separated set chosen by following the procedure described in Sect. 2.4. Each $x \in E_l$ corresponds to a regular closed geodesic $\gamma(x)$ with least period in the interval $(T_l - \delta_l, T_l]$. We write $t = t(x)$ for the period of $\gamma(x)$, and we recall that by construction $(x, t) \in \mathcal{C}(\eta)$.

For each $l \in \mathbb{N}$, we consider $E_l^{k_l}$, which is the Cartesian product of E_l of order k_l. By the specification property on $\mathcal{C}(\eta)$ at scale ϵ, we define a sequence of maps $\{\pi_l\}_{l\in\mathbb{N}}$: $E_l^{k_l} \to T^1M$ as follows. The map π_l sends $\underline{x} = (x_1, x_2, \dots, x_{k_l})$ to $\pi_l(\underline{x})$ by finding a point which tracks the periodic orbit defined by x_1 for C_l times, and then tracks the periodic orbit defined by x_2 for C_l times, etc. The transition times (which depend on the choice of $\underline{x}$) are chosen so that times line up correctly at the start of each prescribed periodic orbit, independent of the choice of $\underline{x}$.

More precisely, let $\underline{x} = (x_1, x_2, \dots, x_{k_l}) \in E_l^{k_l}$. Since each (x_i, t_i) with $x_i \in E_l$ is a member of $\mathcal{C}(\eta)$, each such orbit segment has the specification property at scale ϵ. We use this property to construct a point $z = \pi_l(\underline{x})$ such that

(1) $d_{C_l t_1}(z, x_1) < \epsilon$,
(2) $d_{C_l t_2}(g_{C_l T_l + M} z, x_2) < \epsilon$,
(3) $d_{C_l t_3}(g_{2(C_l T_l + M)} z, x_3) < \epsilon$,

and continue this way so that

$$d_{C_l t_i}(g_{(i-1)(C_l T_l + M)} z, x_i) < \epsilon$$

for all $1 \le i \le k_l$. In the above, $M = M(\eta, \epsilon)$ is the transition time in specification for $\mathcal{C}(\eta)$. We note that the transition time between looping around one periodic orbit to the next is bounded by M from below and $C_l T_l - C_l(T_l - \delta_l) + M = C_l\delta_l + M$ from above. We define

$$P_l = \pi_l(E_l^{k_l}).$$

Since E_l is $(T_l, 4\epsilon)$-separated and we are applying specification at scale ϵ, for $\underline{x} = (x_1, x_2, \dots, x_{k_l})$ and $\underline{y} = (y_1, y_2, \dots, y_{k_l}) \in (E_l)^{k_l}$, we have

(1) If $x_1 = y_1$, then $d_{C_l(T_l - \delta_l)}(\pi_l(\underline{x}), \pi_l(\underline{y})) < 2\epsilon$,
(2) If $x_1 \neq y_1$, then $d_{T_l}(\pi_l(\underline{x}), \pi_l(\underline{y})) > 2\epsilon$,

and similarly for each $i \in \{2, \dots, k_l\}$. In particular, $\#P_l = \#E_l^{k_l}$ and the set P_l is $(k_l C_l T_l + (k_l - 1)M, 2\epsilon)$-separated.

We define a measure m_l by uniformly distributing mass over E_l, i.e. we let

$$m_l = \frac{1}{\#E_l} \sum_{v \in E_l} \delta_v.$$

Now define μ_l to be the self-product measure of m_l on $E_l^{k_l}$, equivalently the uniform measure on $E_l^{k_l}$, which is written as

$$\mu_l := \frac{1}{\#E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \delta_{\underline{x}}.$$

We write $L(v, t)$ for the natural measure along the orbit segment (v, t), in the sense that for any continuous function ϕ, $\int \phi \, dL(v, t) = \int_0^t \phi(g_s v) ds$. For each l, define a sequence of probability measures ν_l on $T^1 M$ by

$$\nu_l = \frac{1}{\#P_l} \sum_{y \in P_l} \frac{1}{T_l} L(y, T_l) = \frac{1}{\#E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} L(\pi_l(\underline{x}), T_l).$$

Note that although ν_l puts mass along only the orbit segment of length T_l, we will be interested in evaluating potentials of the form $F(\cdot, [s, t])$ with respect to ν_l with $s < t$ taking carefully chosen values in the interval $[0, k_l C_l T_l + (k_l - 1)M]$. Integrals of these functions thus incorporate information along the whole prescribed length of the orbit segment. We often state our results with time running up to $k_l(C_l T_l + M)$, since this is a slightly simpler expression and the extra run of time M makes no difference. This is the time S_l denoted in Theorem A.

Lemma 3.2 . *Given $(T_l, k_l, \delta_l, C_l)_{l \in \mathbb{N}}$ satisfying Hypothesis 3.1 the corresponding sequence of measures ν_l converges to the measure of maximal entropy μ_{KBM}.*

Proof It is convenient to define another sequence of probability measures on $T^1 M$, $\{\mu_l^*\}_{l \in \mathbb{N}}$ by $\mu_l^* = \frac{1}{\#E_l} \sum_{v \in E_l} \frac{1}{T_l} L(v, T_l)$. We show that μ_l^* converges to μ_{KBM} when $l \to \infty$. It is not hard to observe that any weak* limit measure of μ_l^* is g_t-invariant for all t since $\delta_l \downarrow 0$ and $T_l \uparrow \infty$. Recall that T_{δ_l} is the time defined at (2.4). We know E_l is $(T_l, 2\epsilon)$-separated, $h(2\epsilon) = h$ and

$$\liminf_{l \to \infty} \frac{1}{T_l} \log \#E_l \geq \lim_{l \to \infty} \frac{1}{T_l} \log(\frac{\beta(\delta_l)}{2T_l} e^{T_l h}) \geq \lim_{l \to \infty} \frac{1}{T_l} \log(\frac{e^{-T_{\delta_l} h}}{2CT_l} e^{T_l h}).$$

Observe from the proof of Lemma 2.5 that $T_0(\delta_l, \eta) = T_{\delta_l}$ for l sufficiently large. Thus by Hypothesis 3.1, for any small $\epsilon' > 0$, $T_{\delta_l}/T_l < \epsilon'$ for large enough l, so $e^{-T_{\delta_l} h} > e^{-\epsilon' T_l h}$. Thus

$$\lim_{l \to \infty} \frac{1}{T_l} \log(\frac{e^{-T_{\delta_l} h}}{2CT_l} e^{T_l h}) \geq \lim_{l \to \infty} \frac{1}{T_l} \log(\frac{e^{(1-\epsilon')T_l h}}{2CT_l}) = (1 - \epsilon')h.$$

From the choice of ϵ', it follows that $\lim_{l \to \infty} \frac{1}{T_l} \log \#E_l = h$. The proof of the second half of variational principle in [21] will imply that μ_l^* converges to μ_{KBM} in the weak*-topology. Therefore, to prove the statement in the lemma, it suffices to show that for any $f \in C(T^1 M)$, we have $\lim_{t \to \infty} \int f d\mu_l^* = \lim_{t \to \infty} \int f d\nu_l$.

Notice that for a fixed $x_1 \in E_l$ and any $(x_2, \ldots, x_{k_l}) \in E_l^{k_l - 1}$, we have

$$|F(\pi_l(x_1, \ldots, x_{k_l}), T_l) - F(x_1, T_l)| \leq \omega(f, T_l, \epsilon, \mathcal{C}(3\eta/4)). \tag{3.1}$$

Here the scale $3\eta/4$ is by property (2) in the choice of δ_0.

Averaging over all $(x_2, \ldots, x_{k_l})$, we have

$$\begin{aligned}&\left|\frac{1}{(\#E_l)^{k_l-1}}\sum_{x_2,\ldots,x_{k_l}}(F(\pi_l(x_1,x_2,\ldots,x_{k_l}),T_l)-F(x_1,T_l))\right|\\&\leq\frac{1}{(\#E_l)^{k_l-1}}\sum_{x_2,\ldots,x_{k_l}}\left|F(\pi_l(x_1,x_2,\ldots,x_{k_l}),T_l)-F(x_1,T_l)\right|\\&\leq\frac{\omega(f,T_l,\epsilon,\mathcal{C}(3\eta/4))}{T_l}\end{aligned} \tag{3.2}$$

where the second inequality follows from (3.1). On the other hand, it is not hard to show that

$$\lim_{l\to\infty}\left|\int f d\mu_l^* - \int f d\nu_l\right| = \lim_{l\to\infty}\left|\frac{1}{\#E_l}\sum_{x_1\in E_l}V_l(x_1)\right| \tag{3.3}$$

where the variation term $V_l(x_1)$ is defined as

$$V_l(x_1) := \frac{1}{(\#E_l)^{k_l-1}}\sum_{x_2,\ldots,x_{k_l}}(F(\pi_l(x_1,x_2,\ldots,x_{k_l}),T_l)-F(x_1,T_l)).$$

By (3.2), for each $x_1 \in E_l$ we have $|V_l(x_1)| \leq \frac{\omega(f,T_l,\epsilon,\mathcal{C}(3\eta/4))}{T_l}$. By plugging this into (3.3), observing $\epsilon < \delta_0$ by property (3) of δ_0 and applying Lemma 2.6, we get

$$\lim_{l\to\infty}\left|\int f d\mu_l^* - \int f d\nu_l\right| \leq \lim_{l\to\infty}\frac{\omega(f,T_l,\epsilon,\mathcal{C}(3\eta/4))}{T_l} = 0,$$

which concludes the proof of the lemma. □

3.1. Variance. Given a function $f \in C(T^1M)$, we consider $F(\cdot, T_l) : E_l \to \mathbb{R}$ defined in the obvious way, i.e. $F(v, T_l) = \int_0^{T_l} f(g_t v)dt$ for $v \in E_l$, and we consider the variances

$$\sigma_l^2 := \sigma_{m_l}^2(F(\cdot,T_l)) = \frac{1}{\#E_l}\sum_{x\in E_l}\left(F(x,T_l)-\frac{1}{\#E_l}\sum_{x\in E_l}F(x,T_l)\right)^2.$$

Terms of the form $k_l\sigma_l^2$ appear in our version of the Lindeberg condition. We let

$$Q_l := \left\lfloor\frac{(T_l-\delta_l)C_l}{T_l}\right\rfloor - 1.$$

The interpretation of this constant is that it is chosen so that if we spend Q_lT_l time looping around one of the closed geodesics then we have definitely not exceeded C_l times the actual length of the geodesic, which is the time at which we move on to approximating the next closed geodesic. For fixed $l \geq 2$ and each $p \in [1, k_l]$, we let

$t_p = (p-1)(C_l T_l + M)$. For $q \in [0, Q_l - 1]$, we define a family of function $F^l_{p,q}$ by averaging f over the time interval $[t_p + qT_l, t_p + (q+1)T_l]$, that is

$$F^l_{p,q}(v) := F(g_{t_p} v, [qT_l, (q+1)T_l]) = F(g_{t_p+qT_l} v, T_l) = \int_{t_p+qT_l}^{t_p+(q+1)T_l} f(g_t v)dt.$$

We also consider this function summed over the range of q. Note that

$$\sum_q F^l_{p,q}(v) = \sum_{q=0}^{Q_l-1} \int_{t_p+qT_l}^{t_p+(q+1)T_l} f(g_t v)dt = \int_{t_p}^{t_p+Q_l T_l} f(g_t v)dt.$$

Thus, we define F^l_p by averaging f over the time interval $[t_p, t_p + Q_l T_l]$, that is

$$F^l_p := F(g_{t_p} v, Q_l T_l)$$

We define

$$s_l^2 = \sum_p \sigma^2_{\nu_l}(F^l_p) = \sum_p \sigma^2_{\nu_l}(\sum_q F^l_{p,q}),$$

where $1 \le p \le k_l$, $0 \le q \le Q_l - 1$. This quantity is the relevant variance quantity for the measures ν_l, recording the sum of variances of each prescribed closed geodesic for Q_l times. To emphasize what goes into this variance quantity, we observe that it can be easily computed that

$$\sigma^2_{\nu_l}(F^l_p) = \frac{1}{\#P_l} \sum_{y\in P_l} \frac{1}{T_l} \int_0^{T_l} \left(\int_{t_p}^{t_p+Q_l T_l} f(g_{s+t} y)ds \right)^2 dt$$
$$- \left(\frac{1}{\#P_l} \sum_{y\in P_l} \frac{1}{T_l} \int_0^{T_l} \int_{t_p}^{t_p+Q_l T_l} f(g_{s+t} y)dsdt \right)^2,$$

and summing the above expression over p from 1 to k_l gives s_l^2.

3.2. Basic estimates. We have the following comparison between averages along the total number of loops around a fixed x_i, and the corresponding orbit segment along $\pi((x_1, \ldots, x_{k_l}))$.

Lemma 3.3 . *For a Hölder continuous potential function f, any $\underline{x} \in E_l^{k_l}$, any $t \in [0, T_l]$ and $p \in \{1, \ldots, k_l\}$, there exists $K = K(f)$ such that*

$$|F^l_p(g_t(\pi_l(\underline{x}))) - Q_l F(x_p, T_l)| \le 2KT_l + (2\kappa\epsilon + 2\delta_l Q_l)\|f\|.$$

Proof Let $\underline{x} = (x_1, \ldots, x_{k_l}) \in E_l^{k_l}$, and let $z = \pi_l(\underline{x})$, and we fix $1 \le p \le k_l$. By construction, we have

$$d_{C_l t(x_p)}(g_{(p-1)(C_l T_l+M)} z, x_p) < \epsilon.$$

Recall that for each $x \in E_l$, we have $(x, t(x)) \in \mathcal{C}(\eta)$, where $t(x) \in [T_l - \delta_l, T_l]$ is the least period of the periodic orbit defined by x. In particular, since each lap round such a periodic orbit will carry a definite amount of hyperbolicity, the distance between the orbit of $(g_{(p-1)(C_l T_l+M)}z, C_l T_l)$ and $(x_p, C_l T_l)$ is much smaller than ϵ when l is large. This is the key idea in getting the desired estimate.

More precisely, for each $l \geq 2$, $1 \leq p \leq k_l$ and $\underline{x} \in E_l^{k_l}$, following the proof of Lemma 2.6 we know there is some $u_p = u_p(\underline{x})$ such that $u_p \in T^1M$ and

$$g_s u_p \in W^s_{\kappa\epsilon}(g_{s+t_p} z \cap W^{cu}_{\kappa\epsilon}(g_s x_p)$$

for all $s \in [0, C_l t_{p+1}]$. In particular, it holds for all $s \in [0, (Q_l+1)T_l]$ by the definition of Q_l. There is $s_p = s_p(\underline{x})$ such that $s_p \in [-\kappa\epsilon, \kappa\epsilon]$ and $g_{s_p+s}u_p \in W^u_{\kappa\epsilon}(g_s x_p)$. Fix such l, p and for any $t \in [0, T_l]$ we want to control

$$|F_p^l(g_t z) - Q_l F(x_p, T_l)|,$$

which is bounded above by the sum of the following four terms

(1) $|F_p^l(g_t z) - F(g_t u_p, Q_l T_l)|$
(2) $|F(g_t u_p, Q_l T_l) - F(g_{t+s_p} u_p, Q_l T_l)|$
(3) $|F(g_{t+s_p} u_p, Q_l T_l)) - F(g_t x_p, Q_l T_l)|$
(4) $|F(g_t x_p, Q_l T_l) - Q_l F(x_p, T_l)|$.

Let us analyze these four terms. We begin with the first term. Suppose f satisfies $|f(x) - f(y)| \leq L_0 d(x, y)^\alpha$. We know for any $u, v \in T^1M$, $|\lambda(u) - \lambda(v)| < \frac{\eta}{4}$ whenever $d(u, v) < \delta_0$. Therefore, for each $0 \leq q \leq Q_l - 1$, by Lemma 3.10 in [3] we have

$$\begin{aligned}
|F^l_{p,q}(g_t z) - F(g_t u_p, [qT_l, (q+1)T_l])| &\leq \int_{qT_l}^{(q+1)T_l} |f(g_{t+s+t_p} z) - f(g_{t+s} u_p)| ds \\
&\leq L_0 \int_{qT_l}^{(q+1)T_l} d(g_{t+s+t_p} z, g_{t+s} u_p)^\alpha ds \\
&\leq L_0 T_l d(g_{qT_l+t+t_p} z, g_{qT_l+t} u_p)^\alpha \\
&\leq L_0 T_l \kappa\epsilon e^{-\frac{qT_l\eta\alpha}{2}}.
\end{aligned}$$

We obtain

$$|\sum_{q=0}^{Q_l-1} F^l_{p,q}(g_t z - F(g_t u_p, Q_l T_l)| \leq \sum_{q=0}^{Q_l-1} L_0 T_l \kappa\epsilon e^{-\frac{qT_l\eta\alpha}{2}} \leq \frac{L_0 T_l \kappa\epsilon}{1 - e^{-\frac{T_l\eta\alpha}{2}}} \leq K T_l,$$

where $K := \frac{L_0\kappa\epsilon}{1-e^{-\frac{\eta\alpha}{2}}}$ is a constant (Recall that $T_l > 1$ for all l). This gives an upper bound on the first term.

The above argument can be repeated along the unstable direction to control the third term. We get

$$|F(g_{t+s_p} u_p, [qT_l, (q+1)T_l])) - F(g_t x_p, [qT_l, (q+1)T_l])| \leq L_0 T_l \kappa\epsilon e^{-\frac{(Q_l-1-q)T_l\eta\alpha}{2}},$$

and thus

$$\begin{aligned}
|F(g_{t+s_p} u_p, Q_l T_l)) - F(g_t x_p, Q_l T_l)| &\leq \sum_{q=0}^{Q_l-1} L_0 T_l \kappa\epsilon e^{-\frac{(Q_l-1-q)T_l\eta\alpha}{2}} \\
&\leq \frac{L_0 T_l \kappa\epsilon}{1 - e^{-\frac{T_l\eta\alpha}{2}}} \leq K T_l
\end{aligned}$$

To estimate the second term, we observe that

$$|F(g_t u_p, Q_l T_l) - F(g_{t+s_p} u_p, Q_l T_l)| \leq 2||f||s_p \leq 2\kappa\epsilon||f||.$$

To estimate the fourth term, we observe that $|F(g_t x_p, Q_l T_l) - Q_l F(x_p, T_l)|$ is bounded above by

$$\sum_{q=0}^{Q_l-1} \left| \int_{qT_l}^{(q+1)T_l} f(g_{t+s} x_p) ds - \int_0^{T_l} f(g_t x_p) dt \right| \leq 2\delta_l Q_l ||f||.$$

where the last inequality follows because x_p has period $t(x_p) \in [T_l - \delta_l, T_l]$. By summing the estimates on these four terms, the lemma is proved. □

We also have the following basic comparison between integrals using ν_l and m_l.

Lemma 3.4 . *We have*

$$|\int F_p^l d\nu_l - Q_l \int F(\cdot, T_l) dm_l| \leq 2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l \|f\|. \tag{3.4}$$

Proof The expression $|\int F_p^l d\nu_l - Q_l \int F(\cdot, T_l) dm_l|$ can be rewritten as

$$\frac{1}{(\#E_l)^{k_l}} \left| \sum_{\underline{x} \in E_l^{k_l}} \left(\frac{1}{T_l} \int_0^{T_l} F_p^l(g_s(\pi_l(\underline{x})) ds - Q_l F(x_p, T_l) \right) \right|.$$

The results follows using Lemma 3.3. □

4. Main Theorem

Recall $L_\nu(h, c)$ is the Lindeberg function from Definition 2.1.

Theorem 4.1 *Let $(\nu_l)_{l \in \mathbb{N}}$ be a sequence of measures as constructed in the previous section. Suppose $f \in C(T^1 M)$ is Hölder continuous with*

$$\liminf_{l \to \infty} \sigma_l^2 > 0. \tag{4.1}$$

Then the Lindeberg-type condition

$$\lim_{l \to \infty} \frac{\sum_{1 \leq p \leq k_l,} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} = 0 \tag{4.2}$$

for any $\gamma > 0$, implies that for all $a \in \mathbb{R}$,

$$\lim_{l \to \infty} \nu_l \left(\left\{ v : \frac{F(v, k_l(C_l T_l + M)) - \int F(\cdot, k_l(C_l T_l + M)) d\nu_l}{s_l} \leq a \right\} \right) = N(a), \tag{4.3}$$

where N is the cumulative distribution function of the normal distribution $\mathcal{N}(0, 1)$. Conversely, under the hypothesis (4.1), (4.3) implies (4.2).

We also show in Lemma 4.6 that the conclusion (4.3) is equivalent to

$$\lim_{l\to\infty} \nu_l \left(\left\{v : \frac{F(v, k_l(C_l T_l + M)) - k_l(C_l T_l + M) \int f d\nu_l}{\sigma_{\nu_l}(F(\cdot, k_l(C_l T_l + M)))} \le a\right\}\right) = N(a), \quad (4.4)$$

which has the advantage of being in the most elementary possible terms.

The proof of Theorem 4.1 is based on comparing μ_l and ν_l. The measures μ_l are product measures supported on $E_l^{k_l}$ (more precisely, the measures μ_l are products of the measures m_l on E_l) and the version of the results we want can be obtained there by considering these objects as sequences of independent random variables and appealing to classical probability theory. Our main theorem is proved by showing that the relevant quantities for ν_l are comparable to corresponding quantities for μ_l. The starting point for our main theorem is thus the following theorem on the sequence of measures (μ_l). We give the Lindeberg condition in terms of the uniform measure m_l on E_l since this is the most elementary object under consideration.

Theorem 4.2 . *The condition*

$$\lim_{l\to\infty} \frac{L_{m_l}(F(\cdot, T_l), \gamma\sqrt{k_l}\sigma_l)}{\sigma_l^2} = 0. \quad (4.5)$$

holds for any $\gamma > 0$ *if any only if*

$$\lim_{l\to\infty} \mu_l \left(\left\{(v_1, \ldots, v_{k_l}) : \frac{\sum_{i=1}^{k_l} F(v_i, T_l) - k_l \int F(\cdot, T_l) dm_l}{\sqrt{k_l \sigma_l^2}} \le a\right\}\right) = N(a). \quad (4.6)$$

This result can be obtained formally using an analogous statement of Denker-Senti-Zhang for dynamical arrays. It can be obtained quite easily from the classical Lindeberg CLT [11]. We give a short formal proof based on verifying Denker-Senti-Zhang's hypotheses [8, Proposition 3.3].

Proof We follow the terminology of [8, Proposition 3.3]. We consider the product of k_l copies of a finite set. We consider a function $G_{l,i} : E_l^{k_l} \to \mathbb{R}$ which depends on the i^{th} component. That is,

$$G_{l,i}(x_1, \ldots, x_{k_l}) = G_{l,i}(x_i).$$

Let $\hat{s}_l^2 = \sum_{i=1}^{k_l} \sigma_{\mu_l}^2(G_{l,i})$. As stated in [8, Proposition 3.3], it follows from Lindeberg's CLT for independent random variables that the Lindeberg condition holds

$$\lim_{l\to\infty} \frac{\sum_{i=1}^{k_l} L_{\mu_l}(G_{l,i}, \gamma\hat{s}_l)}{\hat{s}_l^2} = 0 \quad \text{for any } \gamma > 0 \quad (4.7)$$

if and only if $(G_{l,i})$ is asymptotically negligible

$$\lim_{l\to\infty} \max_{1\le i\le k_l} \frac{\sigma_{\mu_l}^2(G_{l,i})}{\hat{s}_l^2} = 0 \quad (4.8)$$

and for all $a \in \mathbb{R}$,

$$\lim_{l\to\infty} \mu_l \left(\left\{ (x_1, \dots, x_{k_l}) : \frac{\sum_{i=1}^{k_l} G_{l,i} - \int (\sum_{i=1}^{k_l} G_{l,i}) d\mu_l}{\hat{s}_l} \le a \right\} \right) = N(a). \quad (4.9)$$

For our statement, we set $G_{l,i}(x_1, \dots, x_{k_l}) = F(x_i, T_l)$ for all $1 \le i \le k_l$. We observe that for each i, we have

$$\sigma^2_{\mu_l}(\underline{x} \to F(x_i, T_l)) = \sigma^2_{m_l}(F(\cdot, T_l)).$$

This is because the first expression is

$$\frac{1}{\# E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \left(F(x_i, T_l) - \frac{1}{\# E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} F(x_i, T_l) \right)^2,$$

so using that $\# E_l^{k_l} = (\# E_l)(\# E_l^{k_l - 1})$, and that

$$\sum_{\underline{x} \in E_l^{k_l}} F(x_i, T_l) = \# E_l^{k_l - 1} \sum_{x_i \in E_l} F(x_i, T_l),$$

the result follows. Thus, $\hat{s}_l^2 = k_l \sigma_l^2$. The expression $\hat{s}_l^{-2} \sigma^2_{\mu_l}(G_{l,i})$ in (4.8) reduces to $\sigma_l^2 / k_l \sigma_l^2$. Thus asymptotic negligibility is trivially satisfied. The condition (4.7) clearly simplifies to (4.5). We obtain the desired statement, noting in (4.9) that $\int (\sum_{i=1}^{k_l} G_{l,i}) d\mu_l = \sum_{i=1}^{k_l} \int \left(\underline{x} \to F(x_i, T_l) \right) d\mu_l = k_l \int F(\cdot, T_l) dm_l$. □

The normalization quantities in (4.6) are stated in terms of m_l to keep them in the most elementary terms. We note that these quantities can be thought of as quantities depending on the product measure μ_l (rather than m_l) since the proof above makes clear that

$$\int \sum_{i=1}^{k_l} F(v_i, T_l) d\mu_l = k_l \int F(\cdot, T_l) dm_l,$$

$$\sum_{i=1}^{k_l} \sigma^2_{\mu_l}(\underline{x} \to F(x_i, T_l)) = k_l \sigma_l^2.$$

We now prove a key lemma we will need in order to use Theorem 4.2 to describe behavior of the sequence (ν_l). For $x \in E_l$, define

$$D_{m_l}(x) := F(x, T_l) - \int F(\cdot, T_l) dm_l.$$

For $1 \le p \le k_l$, $t \in [0, T_l]$ and $\underline{x} \in E_l^{k_l}$, define

$$D_{\nu_l}(\underline{x}, t; p) := F_p^l(g_t(\pi_l(\underline{x}))) - \int F_p^l d\nu_l,$$

and define

$$\Delta_p^l(\underline{x}, t) := D_{\nu_l}(\underline{x}, t; p) - Q_l D_{m_l}(x_p).$$

These quantities satisfy the following properties.

Lemma 4.3 . *For all $t \in [0, T_l]$ and $\underline{x} \in E_l^{k_l}$, we have*

$$|\Delta_p^l(\underline{x}, t)| \leq 2(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l \|f\|), \tag{4.10}$$

$$D_{\nu_l}(\underline{x}, t; p)^2 = \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)] + Q_l^2 D_{m_l}(x_p)^2. \tag{4.11}$$

Proof The first property follows directly from Lemmas 3.3 and 3.4. The second property holds because

$$\begin{aligned} D_{\nu_l}(\underline{x}, t; p)^2 &= D_{\nu_l}(\underline{x}, t; p)^2 - Q_l^2 D_{m_l}(x_p)^2 + Q_l^2 D_{m_l}(x_p)^2 \\ &= \Delta_p^l(\underline{x}, t)[D_{\nu_l}(\underline{x}, t; p) + Q_l D_{m_l}(x_p)] + Q_l^2 D_{m_l}(x_p)^2 \\ &= \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)] + Q_l^2 D_{m_l}(x_p)^2. \end{aligned}$$

□

Lemma 4.4 $\lim_{l\to\infty} \frac{\sigma_{\nu_l}^2(F_p^l)}{Q_l^2\sigma_l^2} = 1$ *uniformly in* $1 \leq p \leq k_l$.

Proof Observe that we can write

$$\begin{aligned} Q_l^2\sigma_l^2 &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \left(Q_l F(x_p, T_l) - Q_l \int F(\cdot, T_l) dm_l\right)^2 \\ &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} (Q_l D_{m_l}(x_p))^2 = \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} (Q_l D_{m_l}(x_p))^2 dt. \end{aligned}$$

We thus observe using (4.10) and (4.11) that

$$\begin{aligned} \sigma_{\nu_l}^2(F_p^l) - Q_l^2\sigma_l^2 &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} (D_{\nu_l}(\underline{x}, t; p))^2 dt - Q_l^2\sigma_l^2 \\ &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} ((D_{\nu_l}(\underline{x}, t; p))^2 - Q_l^2 D_{m_l}(x_p)^2) dt \\ &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} \Delta_p(\underline{x}, t)(\Delta_p(\underline{x}, t) + 2Q_l D_{m_l}(x_p)) dt \\ &= \int \left(\frac{1}{T_l}\int_0^{T_l} \Delta_p^l(\underline{x}, t)^2 dt\right) d\mu_l + \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} 2Q_l \Delta_p^l(\underline{x}, t) D_{m_l}(x_p) dt \\ &\leq \int \left(\frac{1}{T_l}\int_0^{T_l} \Delta_p^l(\underline{x}, t)^2 dt\right) d\mu_l + 2Q_l \sup_{\underline{x}, t}\{|\Delta_p^l(\underline{x}, t)|\} \int D_{m_l} dm_l \\ &\leq \int \left(\frac{1}{T_l}\int_0^{T_l} (\Delta_p^l(\underline{x}, t))^2 dt\right) d\mu_l + 2Q_l\sigma_l \sup_{\underline{x}, t}\{|\Delta_p^l(\underline{x}, t)|\} \\ &\leq 4(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l\|f\|)^2 + 4Q_l\sigma_l(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l\|f\|). \end{aligned}$$

By Hypothesis 3.1 on $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$ and our hypothesis that $\liminf_{l\to\infty} \sigma_l > 0$, we conclude that

$$\lim_{l\to\infty} \frac{\sigma_{\nu_l}^2(F_p^l) - Q_l^2\sigma_l^2}{Q_l^2\sigma_l^2} = 0.$$

Notice that the above upper bound on $\sigma^2_{\nu_l}(F^l_p) - Q^2_l\sigma^2_l$ is independent of p. As a result, the convergence is uniform in p and this ends the proof of Lemma 4.4. □

We obtain the following lemma as an immediate corollary.

Lemma 4.5 . *The sequence* $s^2_l = \sum_p \sigma^2_{\nu_l}(F^l_p)$, *satisfies*

$$\lim_{l\to\infty} \frac{s_l}{Q_l\sigma_l\sqrt{k_l}} = 1. \tag{4.12}$$

We might also consider $s'^2_l := \sigma^2_{\nu_l}(\sum_p F^l_p)$ or $s''^2_l := \sigma^2_{\nu_l}(F(\cdot, k_l(C_lT_l + M)))$ as natural substitutes for s^2_l. We have the following result

Lemma 4.6 . $\lim_{l\to\infty} \frac{s'^2_l}{s^2_l} = \lim_{l\to\infty} \frac{s''^2_l}{s^2_l} = 1.$

Proof We begin by verifying $\lim_{l\to\infty} \frac{s'^2_l}{s^2_l} = 1$. For any $l > 1$ and $1 \le p_1 < p_2 \le k_l$, we have

$$\begin{aligned}&\int D_{\nu_l}(\underline{x}, t; p_1)D_{\nu_l}(\underline{x}, t; p_2)d\nu_l\\ &= \int ((Q_lD_{m_l}(x_{p_1}) + \Delta^l_{p_1}(\underline{x}, t))(Q_lD_{m_l}(x_{p_2}) + \Delta^l_{p_2}(\underline{x}, t)))d\nu_l.\end{aligned}$$

The right hand side is the sum of four terms, among which $\int Q^2_lD_{m_l}(x_{p_1})D_{m_l}(x_{p_2})d\nu_l = 0$, and $\int (Q_lD_{m_l}(x_{p_1})|\Delta^l_{p_2}(\underline{x}, t)|d\nu_l \le 2Q_l\sigma_l(2KT_l+2\kappa\epsilon\|f\|+2\delta_lQ_l\|f\|)$, which also holds true when p_1 and p_2 are switched, and $\int |\Delta^l_{p_1}(\underline{x}, t)\Delta^l_{p_2}(\underline{x}, t)|d\nu_l \le 4(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_lQ_l\|f\|)^2$. As a result, we have

$$\begin{aligned}&\lim_{l\to\infty}\left|\frac{s'^2_l}{s^2_l} - 1\right| = \lim_{l\to\infty}\left|\frac{2\sum_{1\le p_1<p_2\le k_l}\int (D_{\nu_l}(\underline{x}, t; p_1)D_{\nu_l}(\underline{x}, t; p_2))}{s^2_l}d\nu_l\right|\\ &\le \lim_{l\to\infty} \frac{8Q_l\sigma_l(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_lQ_l\|f\|) + 4(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_lQ_l\|f\|)^2}{s^2_l}\\ &= \lim_{l\to\infty} \frac{8Q_l\sigma_l(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_lQ_l\|f\|) + 4(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_lQ_l\|f\|)^2}{Q^2_l\sigma^2_lk_l}\\ &= 0,\end{aligned}$$

where in the second last equality we use Lemma 4.5. The limit being 0 follows from Hypothesis 3.1 on $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$ and $\liminf_{l\to\infty}\sigma_l > 0$.

Now to show $\lim_{l\to\infty} \frac{s''^2_l}{s^2_l} = 1$, it suffices to show that $\lim_{l\to\infty} \frac{s''^2_l}{s'^2_l} = 1$. Write $\Delta'_l(\underline{x}, t) := \sum_{p=1}^{k_l} \int_{(p-1)(C_lT_l+M)+Q_lT_l}^{p(C_lT_l+M)} f(g_{s+t}\pi_l(\underline{x}))ds$. Notice that by the definition of Q_l, for any $l \ge 1$ and $\underline{x} \in E^{k_l}_l$ we have

$$|\Delta'_l(\underline{x}, t)| \le k_l((C_l - Q_l)T_l + M)||f|| \le k_l(C_l\delta_l + 2T_l + M)||f||. \tag{4.13}$$

We write $D_{\nu_l}(\underline{x}, t) := \sum_{p=1}^{k_l}(F_p^l(g_t\pi_l(\underline{x})) - \int F_p^l(g_t\pi_l(\underline{x}))d\nu_l) = \sum_{p=1}^{k_l} D_{\nu_l}(\underline{x}, t; p)$. As in the proof of Lemma 4.4, we have

$$\begin{aligned}
&|s_l''^2 - s_l'^2| \\
&= \left| \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} \Delta_l'(\underline{x}, t)(\Delta_l'(\underline{x}, t) + 2D_{\nu_l}(\underline{x}, t))dt \right| \\
&= \left| \int \left(\frac{1}{T_l} \int_0^{T_l} (\Delta_l'(\underline{x}, t))^2 dt \right) d\mu_l + \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} 2\Delta_l'(\underline{x}, t) D_{\nu_l}(\underline{x}, t)dt \right| \\
&\le (k_l(C_l\delta_l + 2T_l + M)||f||)^2 + 2(k_l(C_l\delta_l + 2T_l + M)||f||)s_l',
\end{aligned}$$

which in turns shows that

$$\begin{aligned}
&\lim_{l\to\infty} \left| \frac{s_l''^2}{s_l'^2} - 1 \right| \\
&\le \lim_{l\to\infty} \frac{(k_l(C_l\delta_l + 2T_l + M)||f||)^2 + 2(k_l(C_l\delta_l + 2T_l + M)||f||)s_l'}{s_l'^2} \\
&= \lim_{l\to\infty} \frac{(k_l(C_l\delta_l + 2T_l + M)||f||)^2 + 2(k_l(C_l\delta_l + 2T_l + M)||f||)Q_l\sigma_l\sqrt{k_l}}{Q_l^2\sigma_l^2 k_l} \\
&= \lim_{l\to\infty} \frac{k_l^2 C_l^2 \delta_l^2 + 2k_l^{3/2} C_l Q_l \sigma_l \delta_l}{Q_l^2 \sigma_l^2 k_l} = 0,
\end{aligned}$$

and therefore concludes the proof of the lemma. □

Applying Lemma 4.6, we can freely replace s_l in (4.2) and (4.3) by s_l' or s_l''. Although this is not used in proving Theorem 4.1, it allows us to reinterpret the conclusion. We also observe that, in the conclusion, one can easily see that terms of the form $\int F(\cdot, k_l(C_lT_l + M))d\nu_l$ can be replaced with $k_l(C_lT_l + M)\int f d\nu_l$.

We now prove the following statement where we compare the average of F along the orbit segment of v over the time interval $[0, k_l(C_lT_l + M)]$ to its average over the sum of the time intervals $[t_p, t_p + Q_lT_l]$.

Lemma 4.7 . *For each $l \ge 2$, define the functions*

$$A_l(v) := \frac{F(v, k_l(C_lT_l + M)) - \int F(\cdot, k_l(C_lT_l + M))d\nu_l}{s_l},$$

$$B_l(v) := \frac{\sum_p F_p^l(v) - \int \sum_p F_p^l d\nu_l}{s_l},$$

where the sum is over $1 \le p \le k_l$. For any $a > 0$, we have

$$\lim_{l\to\infty} \nu_l(v : |A_l - B_l| > a) = 0.$$

Proof For any constant $a > 0$, we have

$$\begin{aligned}
\lim_{l\to\infty} \nu_l(v : |A_l(v) - B_l(v)| > a) &\leq \lim_{l\to\infty} \frac{\int |A_l - B_l| d\nu_l}{a} \\
&\leq \lim_{l\to\infty} \frac{2k_l(C_l\delta_l + M + 2T_l)\|f\|}{as_l} \\
&= \lim_{l\to\infty} \frac{2k_l(C_l\delta_l + M + 2T_l)\|f\|}{a\sqrt{\sum_p \sigma_{\nu_l}^2(F_p^l)}} \\
&= \lim_{l\to\infty} \frac{2k_l(C_l\delta_l + M + 2T_l)\|f\|}{a\sqrt{k_l Q_l^2 \sigma_l^2}} \\
&= \lim_{l\to\infty} \frac{2k_l\delta_l\|f\|}{a\sqrt{k_l}\sigma_l} + \frac{(2M + 4T_l)\sqrt{k_l}\|f\|}{aQ_l\sigma_l} = 0.
\end{aligned}$$

In the above calculation, the second line follows from

$$\begin{aligned}
\left| F(v, k_l(C_lT_l + M)) - \sum_p F_p^l \right| &\leq k_l(C_lT_l + M - Q_lT_l)\|f\| \\
&\leq k_l(C_lT_l + M - ((T_l - \delta_l)C_lT_l^{-1} - 2)T_l)\|f\| \\
&= k_l(C_l\delta_l + M + 2T_l)\|f\|,
\end{aligned}$$

the fourth line follows from Lemma 4.4, and the fifth line converges to 0 by Hypothesis 3.1 and $\liminf \sigma_l^2 > 0$. □

Lemma 4.7 is the reason we consider sums of the form $\sum_{p=1}^{k_l} F_p^l$. We now show that the CLT conclusions for μ_l and ν_l are equivalent. For the following proofs, we define a function $Y_p : g_{[0,T_l]}\pi_l(E_l^{k_l}) \to \mathbb{R}$ by

$$Y_p(g_s\pi_l(\underline{x})) = F(x_p, T_l) - \int (\underline{x} \to F(x_i, T_l))d\mu_l = D_{m_l}(x_p),$$

and we note that $\sum_{p=1}^{k_l} Y_p(g_s\pi_l(\underline{x})) = \sum_{p=1}^{k_l} F(x_p, T_l) - k_l \int F(\cdot, T_l)dm_l$.

Lemma 4.8 . *The sequence* (ν_l) *satisfies the CLT* (4.3)

$$\lim_{l\to\infty} \nu_l(\{x : \frac{F(x, k_l(C_lT_l + M)) - \int F(\cdot, k_l(C_lT_l + M))d\nu_l}{s_l} \leq a\}) = N(a).$$

if and only if the sequence (μ_l) *satisfies the CLT* (4.6)

$$\lim_{l\to\infty} \mu_l\left(\left\{(x_1, \ldots, x_{k_l}) : \frac{\sum_{p=1}^{k_l} F(x_p, T_l) - k_l \int F(\cdot, T_l)dm_l}{\sqrt{k_l\sigma_l^2}} \leq a\right\}\right) = N(a).$$

Proof First we observe that by Lemma 4.5 and Lemma 4.7, and the fact that ν_l only gives mass to points in $g_{[0,T_l]}\pi_l(E_l^{k_l})$, that the CLT (4.3) holds if and only if

$$\lim_{l\to\infty}\nu_l\left(\left\{g_s(\pi_l(\underline{x})):\underline{x}\in E_l^{k_l},\, s\in[0,T_l],\ \frac{\sum_{p=1}^{k_l}(F_p^l-\int F_p^l d\nu_l)}{\sqrt{k_l}Q_l\sigma_l}\le a\right\}\right)=N(a).$$

Observe that by (4.10) we have

$$\left|\left(\sum_{p=1}^{k_l}(F_p^l-\int F_p^l d\nu_l)-Q_l\sum_{p=1}^{k_l}Y_p\right)(g_s(\pi_l(\underline{x})))\right|=\left|\sum_{p=1}^{k_l}\Delta_p(\underline{x},t)\right|$$
$$\le 2k_l(2KT_l+2\kappa\epsilon\|f\|+2\delta_l Q_l\|f\|).$$

Fix $b>0$. By Hypothesis 3.1 and (4.1), for sufficiently large l,

$$\frac{2k_l(2KT_l+2\kappa\epsilon\|f\|+2\delta_l Q_l\|f\|)}{\sqrt{k_l}Q_l\sigma_l}<b, \tag{4.14}$$

and it thus follows that for sufficiently large l,

$$\left\{g_s\pi_l(x):\frac{|\sum_{p=1}^{k_l}(F_p^l-\int F_p^l d\nu_l)-Q_l\sum_{p=1}^{k_l}Y_p|}{\sqrt{k_l}Q_l\sigma_l}>b\right\}=\emptyset.$$

In particular,

$$\lim_{l\to\infty}\nu_l\left(\left\{g_s\pi_l(x):\frac{|\sum_{p=1}^{k_l}(F_p^l-\int F_p^l d\nu_l)-Q_l\sum_{p=1}^{k_l}Y_p|}{\sqrt{k_l}Q_l\sigma_l}>b\right\}\right)=0.$$

Therefore (4.3) holds if and only if

$$\lim_{l\to\infty}\nu_l\left(\left\{g_s\pi_l(\underline{x}):\underline{x}\in E_l^{k_l},\, s\in[0,T_l],\ \frac{Q_l\sum_{p=1}^{k_l}Y_p}{\sqrt{k_l}Q_l\sigma_l}\le a\right\}\right)=N(a).$$

We are now in a position to reformulate in terms of μ_l. Since Y_p does not depend on the variable s, then either $g_s\pi_l(\underline{x})$ belongs to the above set for all $s\in[0,T_l]$ or for no $s\in[0,T_l]$. It thus follows from the definition of ν_l that

$$\nu_l\left(\left\{g_s\pi_l(\underline{x}):\frac{Q_l\sum_{p=1}^{k_l}Y_p}{\sqrt{k_l}Q_l\sigma_l}\le a\right\}\right)=\frac{1}{\#E_l^{k_l}}\#\left\{\pi_l(\underline{x}):\frac{Q_l\sum_{p=1}^{k_l}Y_p}{\sqrt{k_l}Q_l\sigma_l}\le a\right\}.$$

Furthermore, by the definition of Y_p, we see that

$$\left\{\pi_l(\underline{x}):\frac{Q_l\sum_{p=1}^{k_l}Y_p}{\sqrt{k_l}Q_l\sigma_l}\le a\right\}=\left\{\underline{x}\in E_l^{k_l}:\frac{\sum_{p=1}^{k_l}F(x_p,T_l)-k_l\int F(\cdot,T_l)dm_l}{\sqrt{k_l}\sigma_l}\le a\right\}.$$

We can thus conclude that

$$\lim_{l\to\infty}\mu_l\left(\left\{\underline{x}:\frac{\sum_{p=1}^{k_l}F(x_p,T_l)-k_l\int F(\cdot,T_l)dm_l}{\sqrt{k_l}\sigma_l}\le a\right\}\right)=N(a).$$

Thus, we conclude that (4.3) holds if and only if (4.6) holds. □

All that remains to show equivalence of the Lindeberg conditions in Theorem 4.1 on (ν_l) and in Theorem 4.2 on (μ_l).

Lemma 4.9 . *If* $\liminf_{l\to\infty} \sigma_l > 0$*, then the Lindeberg condition* (4.2)

$$\lim_{l\to\infty} \frac{\sum_{1\le p\le k_l,} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} = 0$$

holds for all $\gamma > 0$ *if and only if the Lindeberg condition* (4.5)

$$\lim_{l\to\infty} \frac{L_{m_l}(F(\cdot, T_l), \gamma\sqrt{k_l}\sigma_l)}{\sigma_l^2} = 0$$

holds for all $\gamma > 0$.

Proof Let $Z_l(c) = Z(c, F_p^l, \nu_l) = \{x : |F_p^l - \int F_p^l d\nu_l| > c\}$ be the set from the Lindeberg condition. Observe that

$$\begin{aligned} L_{\nu_l}(F_p^l, \gamma s_l) &= \int (F_p^l - \int F_p^l d\nu_l)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l \\ &= \frac{1}{E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} D_{\nu_l}(\underline{x}, t; p)^2 \mathbb{1}_{Z_l(\gamma s_l)}(g_t \pi_l(\underline{x})) dt. \end{aligned}$$

Using (4.11), we see that $L_{\nu_l}(F_p^l, \gamma s_l)$ is bounded above by the sum of the terms

$$\frac{1}{E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)] dt,$$

and

$$\frac{1}{E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} Q_l^2 D_{m_l}(x_p)^2 \mathbb{1}_{Z(\gamma s_l)}(g_t \pi_l(\underline{x})) dt.$$

The first of these terms is equal to $\sigma_{\nu_l}^2(F_p^l) - Q_l^2 \sigma_l$ as observed in the proof of Lemma 4.4. The second term can be written as

$$\int (Q_l Y_p)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l.$$

Since $s_l^{-2} \Sigma_p(\sigma_{\nu_l}^2(F_p^l) - Q_l^2 \sigma_l^2) \to 0$ by the proof of Lemma 4.4, it follows that

$$\lim_{l\to\infty} \frac{\sum_p L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} \le \lim_{l\to\infty} \frac{\sum_p \int (Q_l Y_p)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l}{s_l^2}.$$

We now work on the set $Z_l(\gamma s_l)$. Since $\nu_l(\{g_t(\pi_l(\underline{x})) : \underline{x} \in E_l^{k_l}, t \in [0, T_l]\}) = 1$, it suffices for our argument to consider the set

$$Z_l'(\gamma s_l) := \{g_t(\pi_l(\underline{x})) : \underline{x} \in E_l^{k_l}, t \in [0, T_l], |F_p^l - \int F_p^l d\nu_l| > \gamma s_l\}.$$

Note that $|F_p^l g_t(\pi_l(\underline{x})) - \int F_p^l d\nu_l| = |D_{\nu_l}(\underline{x}, t)| \le |\Delta_p^l(\underline{x}, t)| + Q_l |Y_p(g_t(\pi_l(\underline{x}))|$. Thus

$$Z_l'(\gamma s_l) \subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \ge Q_l^{-1}(\gamma s_l - |\Delta_p^l(\underline{x}, t)|)\}.$$

Recall that $\sup_{\underline{x},t}\{|\Delta_p^l(\underline{x}, t)|\} \le 2(2KT_l + \kappa\epsilon\|f\| + 2\delta_l Q_l \|f\|)$ and $\lim_{l\to\infty} \frac{s_l}{\sqrt{k_l} Q_l \sigma_l} = 1$. Therefore, by Hypothesis 3.1 and (4.1), for sufficiently large l, we have $|\Delta_p^l(\underline{x}, t)| \le \frac{\gamma s_l}{2}$ for all $t \in [0, T_l]$ and $\underline{x} \in E_l^{k_l}$. It follows that for sufficiently large l,

$$\begin{aligned} Z_l'(\gamma s_l) &\subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \ge \gamma s_l (2Q_l)^{-1}\} \\ &\subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \ge (\gamma \sigma_l \sqrt{k_l})/4\}. \end{aligned} \tag{4.15}$$

Thus for all large l,

$$\begin{aligned} \int (Q_l Y_p)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l &= \int (Q_l Y_p)^2 \mathbb{1}_{Z_l'(\gamma s_l)} d\nu_l \\ &\le \int (Q_l Y_p)^2 \mathbb{1}_{\{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \ge (\gamma\sigma_l\sqrt{k_l})/4\}} d\nu_l \\ &= Q_l^2 \int D_{m_l}((\underline{x} \to x_p))^2 \mathbb{1}_{\{\underline{x} : |D_{m_l}(x_p)| \ge (\gamma\sigma_l\sqrt{k_l})/4\}} d\mu_l, \\ &= Q_l^2 \int D_{m_l}(x)^2 \mathbb{1}_{\{x : |D_{m_l}(x)| \ge (\gamma\sigma_l\sqrt{k_l})/4\}} dm_l \\ &= Q_l^2 L_{m_l}(F(\cdot, T_l), \gamma\sigma_l\sqrt{k_l}/4). \end{aligned} \tag{4.16}$$

Combining the above calculations, and using (4.12), it follows that if we assume (4.5), then

$$\begin{aligned} \lim_{l\to\infty} \frac{\sum_p L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} &\le \lim_{l\to\infty} \frac{\sum_p \int (Q_l Y_p)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l}{s_l^2} \\ &\le \lim_{l\to\infty} \frac{k_l Q_l^2 L_{m_l}(F(\cdot, T_l), \gamma\sigma_l\sqrt{k_l}/4)}{s_l^2} \\ &= \lim_{l\to\infty} \frac{L_{m_l}(F(\cdot, T_l), \gamma\sigma_l\sqrt{k_l}/4)}{\sigma_l^2} = 0, \end{aligned}$$

and thus (4.2) is true.

To check (4.2) $\implies$ (4.5), note that $L_{\nu_l}(F_p^l, \gamma s_l)$ is bounded below by the sum of

$$-\frac{1}{E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)] dt,$$

and

$$\frac{1}{E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} Q_l^2 D_{m_l}(x_p)^2 \mathbb{1}_{Z(\gamma s_l)}(g_t \pi_l(\underline{x})) dt.$$

As in the discussion above, we have

$$\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}L_{\nu_l}(F_p^l,\gamma s_l)}{s_l^2}\geq\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}\int(Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l}{s_l^2}.$$

We also have $|F_p^l g_t(\pi_l(\underline{x}))-\int F_p^l d\nu_l|\geq -|\Delta_p^l(\underline{x},t)|+Q_l|Y_p(g_t(\pi_l(\underline{x}))|$, which implies that $\{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))|\geq Q_l^{-1}(\gamma s_l+|\Delta_p^l(\underline{x},t)|)\}\subset Z_l'(\gamma s_l)$.

Since $|\Delta_p^l(\underline{x},t)|\leq\gamma s_l$ for all $\underline{x}\in E_l^{k_l}$ and $t\in[0,T_l]$ when l is sufficiently large, we have $\{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))|\geq 2Q_l^{-1}\gamma s_l\}\subset Z_l'(\gamma s_l)$. By (4.12), we have

$$\{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))|\geq 4\gamma\sqrt{k_l}\sigma_l\}\subset Z_l'(\gamma s_l).$$

Then following the same argument as in (4.16), we have

$$\int(Q_lY_p)^2\mathbb{1}_{Z_l'(\gamma s_l)}d\nu_l\geq Q_l^2L_{m_l}(F(\cdot,T_l),4\gamma\sigma_l\sqrt{k_l}),$$

which shows that

$$\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}L_{\nu_l}(F_p^l,\gamma s_l)}{s_l^2}\geq\lim_{l\to\infty}\frac{L_{m_l}(F(\cdot,T_l),4\gamma\sigma_l\sqrt{k_l})}{\sigma_l^2}.$$

This shows that (4.2) implies (4.5). □

5. Verifying the Lindeberg Condition

Historically, the Lindeberg CLT is used in the case where an underlying probabilistic mixing structure is available (see condition (I) and (II) in [15] for definitions of mixing and K-property in probability). In those situations, given any L^1 random variable f, to evaluate the distribution of a sum S_nf, one observes its partial sums $(S_{a_i}^{b_i}f)_{i\in\mathbb{N}}$, where $0=a_0<b_0<a_1<\cdots$. Due to the mixing assumptions on the system, one can expect $S_{a_i}^{b_i}f$ to behave 'independently' for different $i\in\mathbb{N}$, if $a_{i+1}-b_i$, which is the gap between i-th and $i+1$-th segment, increases to ∞ uniformly for all $i\in\mathbb{N}$. To make S_nf well-approximated by the sum over $S_{a_i}^{b_i}f$, it is natural to consider $b_i-a_i\gg a_{i+1}-b_i$ for all $i\in\mathbb{N}$ so that the effect from the gap is negligible. See Theorem 1.3 in [15]. In particular, for f with finite $2+\delta$ moments and $\sigma^2(S_nf)$ tending to infinity, the Lindeberg condition is satisfied. The mixing structure of the system allows one to argue that the Lindeberg variance distributed by each segment individually is sub-linear compared to the total variance, while mixing also implies the growth of total variance is (almost) linear. Therefore, the overall Lindeberg variance is negligible.

In our situation, we do not have any strong mixing properties available for the measures (ν_l). However, each ν_l is weighted over concatenations of k_l segments of (repeated) independent closed geodesics with (approximately) T_l length, so one can study the global Lindeberg condition (4.2) via the local condition (4.5). Intuitively, if we can make k_l increase at an appropriate rate compared to T_l, eventually the Lindeberg variance contributed by individual terms becomes negligible, and thus the local condition (4.5) is satisfied.

From now on, we strengthen condition (4.1) to the following

$$\lim_{l\to\infty} \sigma_l^2 = \infty. \tag{5.1}$$

With this assumption, we can weaken the condition $k_l\delta_l^2 \downarrow 0$ in Hypothesis 3.1 to

$$\frac{k_l\delta_l^2}{\sigma_l^2} \downarrow 0, \tag{5.2}$$

and still obtain Theorem 4.1. This is because wherever the old condition $k_l\delta_l^2 \downarrow 0$ is applied, we are actually dealing with the limit of $k_l\delta_l^2/\sigma_l^2$ (see the last line in the proof of Lemma 4.6, the fifth line of Lemma 4.7, (4.14) in Lemma 4.8 and (4.15) in Lemma 4.9). With the new assumption (5.2), we can allow k_l to grow faster than before. If we can find k_l which satisfies (5.2) while simultaneously satisfying the hypothesis of the following lemma, we are done.

Lemma 5.1 . *Suppose that we have chosen δ_l and T_l, and our observable f, and that $\sigma_l \to \infty$. Suppose we can find $k_l \to \infty$ so that $\frac{\sqrt{k_l}\sigma_l}{T_l} \to \infty$. Then the Lindeberg condition (4.5) is satisfied.*

Proof We consider the Lindeberg condition (4.5). For any fixed $\gamma > 0$ and $v \in T^1M$, the indicator function in the integral satisfies

$$\mathbb{1}_{|F(\cdot,T_l)-\int F(\cdot,T_l)dm_l|\geq\gamma\sqrt{k_l}\sigma_l}(v) \leq \mathbb{1}_{2T_l||f||\geq\gamma\sqrt{k_l}\sigma_l}(v) = \mathbb{1}_{K_{\gamma,f}\geq T_l^{-1}\sqrt{k_l}\sigma_l}(v) \tag{5.3}$$

where $K_{\gamma,f} := 2||f||\gamma^{-1}$ is a constant. Thus,

$$\begin{aligned}
&\lim_{l\to\infty} \frac{L_{m_l}(F(\cdot,T_l),\gamma\sqrt{k_l}\sigma_l)}{\sigma_l^2}\\
&= \lim_{l\to\infty} \frac{\int(F(\cdot,T_l)-\int F(\cdot,T_l)dm_l)^2\mathbb{1}_{|F(\cdot,T_l)-\int F(\cdot,T_l)dm_l|\geq\gamma\sqrt{k_l}\sigma_l}dm_l}{\sigma_l^2}\\
&\leq \lim_{l\to\infty} \frac{\int(F(\cdot,T_l)-\int F(\cdot,T_l)dm_l)^2\mathbb{1}_{K_{\gamma,f}\geq T_l^{-1}\sqrt{k_l}\sigma_l}dm_l}{\sigma_l^2}\\
&= 0
\end{aligned} \tag{5.4}$$

which verifies Lindeberg condition (4.5) □

Recall that we defined the (lower) dynamical variance for the sequence of measures (m_l) to be

$$\underline{\sigma}^2_{\mathrm{Dyn}}(f,(m_l)) = \liminf_{l\to\infty} \int \left(\frac{F(\cdot,T_l)-\int F(\cdot,T_l)dm_l}{\sqrt{T_l}}\right)^2 dm_l = \liminf_{l\to\infty} \frac{\sigma_l^2}{T_l} \tag{5.5}$$

See the introduction for a discussion of this quantity.

Theorem 5.2 . *Suppose that we have chosen δ_l and T_l, and our observable f. Suppose that $\underline{\sigma}^2_{Dyn}(f,(m_l)) > 0$. Then there exists sequences $k_l \to \infty$ and $C_l \to \infty$ so that the measures (ν_l) constructed from the data $(\delta_l, T_l, k_l, C_l)_{l\in\mathbb{N}}$ are valid for Theorem 4.1 to hold, and so that the Lindeberg condition (4.2) holds.*

Proof We let $k_l := \sigma_l^2/\delta_l$, which clearly tends to ∞. Observe that $\frac{k_l \delta_l^2}{\sigma_l^2} = \delta_l \downarrow 0$, and thus (5.2) is satisfied. Making any suitable choice of C_l, it follows from the discussion above that Theorem 4.1 is valid for the measures (ν_l) constructed from the data $(\delta_l, T_l, k_l, C_l)_{l\in\mathbb{N}}$.

Observe that from the hypothesis that $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) > 0$, the sequence $\frac{\sigma_l}{T_l}$ is eventually greater than some constant $\alpha > 0$, and thus we have

$$\frac{\sqrt{k_l}\sigma_l}{T_l} \to \infty.$$

Thus the hypothesis of Lemma 5.1 is satisfied, and we can conclude that the Lindeberg condition (4.5) on (m_l) holds. Thus, by Lemma 4.9, the Lindeberg condition (4.2) holds on (ν_l). $\square$

Combining Theorem 4.1 and Theorem 5.2 gives us Theorem A as an immediate consequence.

Remark 5.3 . One can investigate when the Lindeberg condition holds under the weaker condition that $\lim_{l\to\infty} \sigma_l^2 = \infty$ without assuming that $\underline{\sigma}^2_{\mathrm{Dyn}}(f, (m_l)) > 0$. It can be verified that a suitable sequence (k_l) satisfying Lemma 5.1 can be found if $\sigma_l^2/\delta_l T_l \to \infty$. To verify this condition, first recall from Hypothesis 3.1 that the choice on T_l is only determined by δ_l. Thus, we need information on how T_{δ_l} is related to δ_l as $\delta_l \to 0$. This information can be extracted in the uniform case using symbolic dynamics, and the issue does not appear at all in discrete-time analogues of this result. While it may be possible to use this criterion to slightly sharpen our results in some concrete examples where the relationship between δ_l and T_{δ_l} is tractable, we do not pursue this at present.

6. Extensions of Main Result

In this section, we extend our main result to dynamical arrays of observables. We also discuss how our techniques extend to equilibrium states and how they apply to other classes of dynamical system beyond geodesic flow.

6.1. Dynamical arrays. A benefit of the Lindeberg approach is that we can consider dynamical arrays in the CLT instead of a single function. In this section, our setup is as follows. We let $(f_l)_{l\in\mathbb{N}}$ be a sequence of Hölder continuous observables. We allow for different Hölder constants and exponents, not necessarily bounded away from ∞ and 0 respectively. We let L_l and α_l be the Hölder constant and exponent respectively for f_l, so that $|f_l(x) - f_l(y)| \leq L_l d(x, y)^{\alpha_l}$ for all $l \in \mathbb{N}$.

Given a sequence of 4-tuples $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$ to be chosen precisely later, and the sequence of observables (f_l), we write $F_l(v, T_l) := \int_0^{T_l} f_l(g_t(v))dt$, and $F^l_{p,q}(v) := \int_{t_p+qT_l}^{t_p+(q+1)T_l} f_l(g_t(v))dt$. Using these modified definitions, new definitions for σ_l^2, F_p^l and s_l^2 follow as in §3.1. We have the following analogy to the statement of Lemma 3.3, with only minor modifications to the proof.

Lemma 6.1 *For $(f_l)_{l\in\mathbb{N}}$ given as above and $\underline{x} \in E_l^{k_l}$, $1 \leq p \leq k_l$, we have*

$$|F_p^l(g_t(\pi_l(\underline{x}))) - Q_l F_l(x_p, T_l)| \leq 2K_l T_l + (\kappa\epsilon + 2\delta_l Q_l)\|f_l\|,$$

where $K_l := L_l\kappa\epsilon(1 - e^{-\frac{\eta\alpha_l}{2}})^{-1}$.

We need to modify our assumptions on the sequence of 4-tuples $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$.

Hypothesis 6.2 *We choose sequences* $T_l \in (0, \infty)$, $k_l \in \mathbb{N}$, $\delta_l \in (0, \delta_0)$, *and* $C_l \in \mathbb{N}$ *which satisfy the following relationships:*

1) For all $l \in \mathbb{N}$, $T_l > \max\{T_0(\delta_l, \eta), 1\}$,
2) $T_l \uparrow \infty$, $\frac{T_l}{T_0(\delta_l,\eta)} \uparrow \infty$ *and* $k_l \uparrow \infty$,
3) $k_l\delta_l^2 \max\{\|f_l\|, 1\} \downarrow 0$,
4) $\frac{\sqrt{k_l}T_l \max\{|K_l|,1\}}{Q_l} \downarrow 0$ *and* $\frac{\sqrt{k_l}T_l \max\{\|f_l\|,1\}}{Q_l} \downarrow 0$.

It is always possible to have such sequence of 4-tuples as we can first choose k_l, then δ_l and T_l, finally Q_l. We will demonstrate why we choose $(T_l, k_l, \delta_l, C_l)$ this way below. We have the following analogy to Theorem 4.1:

Theorem 6.3 . *Fix* $(f_l)_{l\in\mathbb{N}}$ *as above. Let* $(T_l, k_l, \delta_l, C_l)_{l\in\mathbb{N}}$ *be a sequence satisfying Hypothesis 6.2 and* $(\nu_l)_{l\in\mathbb{N}}$ *be the sequence of measures constructed as in §3. Suppose* $(f_l)_{l\in\mathbb{N}}$ *satisfies*

$$\liminf_{l\to\infty} \sigma_l^2 > 0. \tag{6.1}$$

Then the Lindeberg-type condition

$$\lim_{l\to\infty} \frac{\sum_{1\le p\le k_l,} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} = 0 \tag{6.2}$$

for any $\gamma > 0$, *implies that for all* $a \in \mathbb{R}$,

$$\lim_{l\to\infty} \nu_l(\{v : \frac{F_l(v, k_l(C_lT_l + M)) - \int F_l(\cdot, k_l(C_lT_l + M))d\nu_l}{s_l} \le a\}) = N(a), \tag{6.3}$$

where N *is the cumulative distribution function of the normal distribution* $\mathcal{N}(0, 1)$. *Conversely, under the hypotheses* (6.1)*,* (6.3) *implies* (6.2)*.*

The proof follows the arguments of §4, with F replaced by F_l and other notations referring to the array version of the definitions. We point out where the differences appear in the proofs between Theorem 6.3 and Theorem 4.1.

We inherit the definitions of $D_{m_l}(x)$, $D_{\nu_l}(\underline{x}, t; p)$ and $\Delta_p^l(\underline{x}, t)$ from §4, which all adapt to the dynamical array setting. Observe that as a direct consequence of Lemma 6.1, (4.10) in Lemma 4.3 now becomes

$$|\Delta_p^l(\underline{x}, t)| \le 2(2K_lT_l + 2\kappa\epsilon\|f_l\| + 2\delta_l Q_l\|f_l\|). \tag{6.4}$$

Therefore, to conclude the main lemma, which says that $\lim_{l\to\infty} \frac{\sigma_{\nu_l}^2(F_p^l)}{Q_l^2\sigma_l^2} = 1$ uniformly in $1 \le p \le k_l$, it suffices to show $\lim_{l\to\infty} \frac{2(2K_lT_l+\kappa\epsilon\|f_l\|+2\delta_l Q_l\|f_l\|)}{Q_l\sigma_l} = 0$. This can be observed from the proof of Lemma 4.4, using Hypothesis 6.2 and (6.1). As a simple follow-up we have

$$\lim_{l\to\infty} \frac{s_l^2}{Q_l^2\sigma_l^2 k_l} = 1. \tag{6.5}$$

To retrieve the content of Lemma 4.7, it suffices to show the last step of its proof holds true in the array case, which is that

$$\lim_{l\to\infty}\left(\frac{2k_l\delta_l\|f_l\|}{\sqrt{k_l}\sigma_l}+\frac{(2M+4T_l)\sqrt{k_l}\|f_l\|}{Q_l\sigma_l}\right)=0.$$

This is obtained by applying condition 3) in Hypothesis 6.2 to the first half, condition 4) to the second and applying (6.1).

To verify the equivalence between the CLT for (ν_l) and (μ_l), which is Lemma 4.8, it suffices to replace (4.14) by showing $\frac{2k_l(2K_lT_l+\kappa\epsilon\|f_l\|+2\delta_lQ_l\|f_l\|)}{\sqrt{k_l}Q_l\sigma_l} < b$ for any $b > 0$ when l is sufficiently large. Finally, to verify the equivalence of the Lindeberg conditions, analogous to Lemma 4.9, we invoke (6.4) and (6.5) along with Hypothesis 6.2 and (6.1). As a result, we are able to conclude that Theorem 6.3 holds.

6.2. Equilibrium states. We refer the reader to [3] for definitions and notations. We consider a potential function φ that is either Hölder continuous or $q\varphi^u$ with $q < 1$, where φ^u is the geometric potential. We assume that the pressure gap condition $P(\text{Sing}, \varphi) < P(\varphi)$ holds. Theorem A in [3] shows that the geodesic flow has a unique equilibrium state μ_φ. Our main result, Theorem 4.1, extends to equilibrium states of this type. The generalization is a natural one. In place of the measures (m_l), we use weighted measures

$$\hat{m}_l := \frac{1}{\sum_{v\in E_l} e^{\Phi(v,T_l)}}\sum_{v\in E_l} e^{\Phi(v,T_l)}\delta_v,$$

and we define a weighted sequence of measures $(\hat{\nu}_l)$ analogously to our definition of (ν_l). We can show that $(\hat{\nu}_l)$ converges to μ_φ, and that we have the analogue of Theorem 4.1: if the variance of an observable f with respect to the sequence $(\hat{m}_l)$ is positive, we can ensure that the sequence $(\hat{\nu}_l)$ satisfies (4.3). The details of the statement and proof can be found in the PhD thesis of T. Wang [23].

6.3. Systems with non-uniform specification. The reader will have observed that our arguments used dynamical structure proved in [3] rather than direct geometric arguments, and thus it is clear that the arguments of this paper will apply to a variety of systems other than the geodesic flow on non-positive curvature manifolds. We do not attempt to make an general statement abstracting the properties of the geodesic flow used in our analysis—a main point of course is the non-uniform specification structure obtained in [3]. The interested reader can infer from Sects. 2–4 exactly what properties are needed to obtain this Lindeberg-type CLT on periodic orbits for other systems. In [4], we defined λ-decompositions as an abstraction of the non-uniform structure enjoyed by rank one geodesic flows. Systems admitting this kind of structure are prime candidates for this kind of analysis. We note that our arguments are all given for flows, but could also be given in the simpler discrete-time case. In discrete-time, one advantage of our construction is that it extends easily from the MME case to equilibrium states.

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