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Dynamics for Spherical Spin Glasses: Disorder Dependent Initial Conditions

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Abstract
We derive the thermodynamic limit of the empirical correlation and response functions in the Langevin dynamics for spherical mixed p-spin disordered mean-field models, starting uniformly within one of the spherical bands on which the Gibbs measure concentrates at low temperature for the pure p-spin models and mixed perturbations of them. We further relate the large time asymptotics of the resulting coupled non-linear integro-differential equations, to the geometric structure of the Gibbs measures (at low temperature), and derive their FDT solution (at high temperature).

Keywords Interacting random processes · Disordered systems · Statistical mechanics · Langevin dynamics · Aging · Spin glass models

Mathematics Subject Classification Primary: 82C44 · Secondary: 82C31 · 60H10 · 60F15 · 60K35

1 Introduction

The thermodynamic limits of a wide class of Markovian dynamics with random interactions, exhibit complex long time behavior, which is of much interest in out of equilibrium statistical physics (c.f. the surveys [14,15,22] and the references therein). This work is about the ther-

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modynamic ($N \to \infty$), long time ($t \to \infty$), behavior of a certain class of systems composed of N Langevin particles $\mathbf{x}_t = (x_t^i)_{1 \le i \le N} \in \mathbb{R}^N$, interacting with each other through a random potential. More precisely, one considers a diffusion of the form

$$d\mathbf{x}_t = -f'(||\mathbf{x}_t||^2/N)\mathbf{x}_t dt - \beta \nabla H_{\mathbf{J}}(\mathbf{x}_t)dt + d\mathbf{B}_t, \tag{1.1}$$

where $\mathbf{B}_t$ is an N-dimensional Brownian motion, $||\mathbf{x}||$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^N$ and differentiable fast growing functions $f = f_L$ such that $e^{-f_L(r)}$ approximates as $L \to \infty$ the indicator on $r = 1$, effectively restricting $\mathbf{x}_t$ to the sphere $\mathbb{S}_N := \mathbb{S}^{N-1}(\sqrt{N})$ of radius $\sqrt{N}$. In particular, the spherical, mixed p-spin model (with $p \le m$), has a centered Gaussian potential $H_{\mathbf{J}} : \mathbb{R}^N \longrightarrow \mathbb{R}$ of non-negative definite covariance structure

$$\mathrm{Cov}\big(H_{\mathbf{J}}(\mathbf{x}), H_{\mathbf{J}}(\mathbf{y})\big) = N\nu\big(N^{-1}\langle \mathbf{x}, \mathbf{y}\rangle\big), \qquad \nu(r) := \sum_{p=2}^{m} b_p^2 r^p \tag{1.2}$$

(see Remark 1.8 on a possible extension to $m = \infty$). Hereafter we shall realize this potential as

$$H_{\mathbf{J}}(\mathbf{x}) = \sum_{p=2}^{m} b_p \sum_{1 \le i_1 \le \cdots \le i_p \le N} J_{i_1 \cdots i_p} x^{i_1} \cdots x^{i_p}, \quad b_m \ne 0 \tag{1.3}$$

for independent centered Gaussian coupling constants $\mathbf{J} = \{J_{i_1 \cdots i_p}\}$, such that

$$\mathrm{Var}(J_{i_1 \ldots i_p}) = N^{-p+1} \frac{p!}{\prod_k l_k!}, \tag{1.4}$$

where $(l_1, l_2, \ldots)$ are the multiplicities of the different elements of the set $\{i_1, \ldots, i_p\}$ (so having $i_1 \ne i_2 \cdots \ne i_p$ yields variance larger by a factor $p!$ from the variance in case $i_1 = i_2 = \cdots = i_p$).

Given a realization of the coupling constants, the dynamics of (1.1) is invariant (and moreover, reversible), for the (random) Gibbs measure $\mu_{2\beta,\mathbf{J}}^N$ on $\mathbb{R}^N$, where $\mu_{\beta,\mathbf{J}}^N$ has the density

$$\frac{d\mu_{\beta,\mathbf{J}}^N}{d\mathbf{x}} = Z_{\beta,\mathbf{J}}^{-1} e^{-\beta H_{\mathbf{J}}(\mathbf{x}) - N f(N^{-1}\|\mathbf{x}\|^2)} \tag{1.5}$$

(with respect to Lebesgue measure). The normalization factor $Z_{\beta,\mathbf{J}} = \int e^{-\beta H_{\mathbf{J}}(\mathbf{x}) - N f(N^{-1}\|\mathbf{x}\|^2)} d\mathbf{x}$ is finite if

$$\inf_{r \ge 0}\{f'(r) - A r^{2k-1}\} > -\infty \tag{1.6}$$

for some $A > 0$ and $k > m/4$. Similar random measures have been extensively studied in mathematics and physics over the last three decades (see e.g. [17,36], for the rigorous analysis of the asymptotic of $N^{-1} \log Z_{\beta,\mathbf{J}}$ for the hard spherical constraint of having $||\mathbf{x}||^2 = N$).

Large dimensional Langevin or Glauber dynamics often exhibit very different behavior at various time-scales (as functions of system size, c.f. [9] and references therein). Following the physics literature (see [15,20,22,23]), we study (1.1) for the potential $H_{\mathbf{J}}(\mathbf{x})$ of (1.3) at the shortest possible time-scale, where $N \to \infty$ first, holding $t \in [0, T]$. While it is too short to allow any escape from meta-stable states, considering the hard spherical constraint, Cugliandolo-Kurchan have nevertheless predicted a rich picture for the limiting dynamics when starting out of equilibrium, say at $\mathbf{x}_0$ distributed uniformly over $\mathbb{S}_N$. Such limiting dynamics involve the coupled integro-differential equations relating the non-random limits $C(s,t)$ and

$$\chi(s,t) = \int_0^t R(s,u)du\,, \tag{1.7}$$

of the *empirical covariance function*

$$C_N(s,t) = \frac{1}{N}\langle \mathbf{x}_s, \mathbf{x}_t\rangle = \frac{1}{N}\sum_{i=1}^N x_s^i x_t^i, \qquad s \ge t \tag{1.8}$$

and the *integrated response function*

$$\chi_N(s,t) = \frac{1}{N}\langle \mathbf{x}_s, \mathbf{B}_t\rangle = \frac{1}{N}\sum_{i=1}^N x_s^i B_t^i\,, \tag{1.9}$$

respectively. Specifically, it is predicted that for large β the asymptotic of $C(s,t)$ strongly depends on the way t and s tend to infinity, exhibiting *aging* behavior (where the older it gets, the longer the system takes to forget its current state, see e.g. [23,28]). A detailed analysis of such aging properties is given in [8] for the case of $m=2$ in (1.3) (noting that $\{J_{ij}\}$ form the GOE random matrix, whose semi-circle limiting spectral measure determines the asymptotic of $C(s,t)$). For $m>2$, assuming hereafter that f' is locally Lipschitz, satisfying (1.6) and such that for some $\kappa<\infty$,

$$\sup_{r\ge 0}|f'(r)|(1+r)^{-\kappa} < \infty\,, \tag{1.10}$$

we have from [10, proof of Proposition 2.1] that for each N, any finite disorder $\mathbf{J}$ and initial condition $\mathbf{x}_0$, there exists a unique strong solution in $\mathcal{C}(\mathbb{R}^+,\mathbb{R}^N)$ of (1.1) (for a.e. path $t\mapsto \mathbf{B}_t$). For such f the closed equations for C and R are rigorously derived in [10] when $\mathbf{x}_0$ *is independent of* $\mathbf{J}$ and satisfies the concentration of measure property of [10, Hypothesis 1.1], provided in addition $N\mapsto \mathbb{E}[C_N(0,0)^k]$ is uniformly bounded for each fixed $k<\infty$, the limit

$$\lim_{N\to\infty}\mathbb{E}C_N(0,0) = C(0,0)\,, \tag{1.11}$$

exists and $\mathbb{P}(|C_N(0,0)-C(0,0)|>x)$ decay exponentially fast in N. Building on it, [24, Proposition 1.1] proves that for integer $k>m/4$ and $\boldsymbol{\varphi}=1$, in the limit $L\to\infty$, the resulting equations of [10] for

$$f_L(r) := L(r-1)^2 + \frac{\boldsymbol{\varphi}}{4k}r^{2k}\,, \tag{1.12}$$

coincide for the pure m-spin case $\nu(r)=\frac{1}{8}r^m$ with the CKCHS-equations, derived independently by Cugliandolo-Kurchan [23] (who consider instead $C(2\cdot,2\cdot)$ and $R(2\cdot,2\cdot)$), and by Crisanti-Horner-Sommers [20].

The CKCHS-equations are for the Langevin dynamics of $\mathbf{x}_t$ on the sphere $\mathbb{S}_N$, reversible with respect to the pure spherical m-spin Gibbs measure $\widetilde{\mu}^N_{2\beta,\mathbf{J}}$ of density $\widetilde{Z}^{-1}_{2\beta,\mathbf{J}}e^{-2\beta H_{\mathbf{J}}(\mathbf{x})}$ with respect to the uniform measure on $\mathbb{S}_N$. According to the Thouless-Anderson-Palmer (TAP) approach [38], the local magnetizations of each pure state [31,37] approximately minimize the mean-field TAP free energy. For the pure spherical m-spin models [21,29] and β in the low temperature phase, the (stable) minimizers $\boldsymbol{\sigma}$ of the TAP free energy roughly have radius $\sqrt{N}q_\star$ with $q_\star^2 = q_{\rm EA}$ the Edwards-Anderson parameter, i.e. the right-most point in the support of the Parisi measure. As the TAP free energy only depends on $\|\boldsymbol{\sigma}\|$, such $\boldsymbol{\sigma}$ also approximately minimize the energy

$$H_{\mathbf{J}}(\boldsymbol{\sigma}) \approx \min_{\boldsymbol{\sigma}' \in q_\star \mathbb{S}_N} \{H_{\mathbf{J}}(\boldsymbol{\sigma}')\}. \tag{1.13}$$

More generally, it was recently rigorously proved [33] that for all spherical mixed p-spin models and β in the low temperature phase, for any $q_\star \in (0,1)$ such that $q_\star^2$ belongs to the support of the Parisi measure, $\boldsymbol{\sigma} \in q_\star \mathbb{S}_N$ satisfies (1.13) if and only if the probability under the Gibbs measure $\widetilde{\mu}^N_{\beta,\mathbf{J}}$ of sampling many (slowly diverging with N) i.i.d. points $\boldsymbol{\sigma}^i$ from the narrow band

$$\left\{\boldsymbol{\sigma}' \in \mathbb{S}_N : \frac{1}{N}\langle \boldsymbol{\sigma}' - \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle \approx 0\right\}$$

such that $\frac{1}{N}\langle \boldsymbol{\sigma}^i - \boldsymbol{\sigma}, \boldsymbol{\sigma}^j - \boldsymbol{\sigma}\rangle \approx 0$ for $i \neq j$ is not exponentially small. Moreover, any point $\boldsymbol{\sigma}$ in the ultrametric tree [30,32], and not only the barycenters of pure states, satisfies (1.13) with $q_\star = \|\boldsymbol{\sigma}\|/\sqrt{N}$. In fact, even for models with Ising spins [18,19], the above holds if one adds an appropriate deterministic correction depending on the empirical measure $N^{-1}\sum_{i\le N}\delta_{\sigma_i}$ to the Hamiltonian in both sides of (1.13).

For the pure m-spin models [35] and their 1-RSB mixed perturbations [13] with $\beta \gg 1$ an explicit pure states decomposition was proved by an investigation of the local structure around critical points. In particular, it was shown there that the Gibbs measure $\widetilde{\mu}^N_{\beta,\mathbf{J}}$ of the complement of the bands of small macroscopic width around all critical points with energy within small macroscopic distance from the minimal energy is exponentially small in N. Hence, in steady state the path $\mathbf{x}_t$ spends an exponentially small in N proportion of the time outside of those bands, hinting that they play the role of meta-stable states in the conjectured aging picture (see also [12,26] for spectral gap estimates and what they reveal about the Langevin dynamical phase transition parameter). If the initial distribution is independent of the disorder $\mathbf{J}$, one may expect an exponentially in N long time to reach bands around deep critical points and a plausible aging mechanism is having the path $\mathbf{x}_t$ decompose to time intervals spent in bands around deeper and deeper critical points, connected by excursions of much shorter length, having typically $\mathbf{x}_t$ within the deepest band it has yet reached by time $t \gg 1$. With initial distribution independent of the disorder $\mathbf{J}$, the CKCHS-equations discussed above concern (fixed) times not long enough (exponential in N) to be relevant to such meta-stability induced aging. However, to investigate the short-time dynamics as $\mathbf{x}_t$ enters meta-stable states (of different levels) it is natural to consider initial conditions that depend on $\mathbf{J}$. Specifically, having a random starting point at a fixed distance on the sphere from a critical point, which by itself is chosen randomly. Restricting to critical points at which $H_{\mathbf{J}}$ is near a fixed deep energy level $-E_\star$ allows us to probe the different 'layers' of wells in the landscape as we vary $E_\star$.

Provided that the number of such critical points is within a fixed factor off its mean (currently proved only for pure m-spin [34] and small mixed perturbation of them [13]), the Kac-Rice formula (see [1]), allows us to translate the study of dynamics under such disorder dependent random initial distribution to an investigation of dynamics driven by a modified, conditional Hamiltonian and deterministic initial distribution. To this end, our first result extends [10, Theorem 1.2] to the latter initial measures and conditional potentials.[1]

[1] The conditioning on (1.16) is interpreted in the usual way: the conditional law of $\mathbf{J}$ has density given, up to normalization, by the restriction of its original density to the appropriate affine subspace, and the conditional law of the independent $\mathbf{B}$ is identical to the unconditional one.

Specifically, fixing $q_\star > 0$ and $\boldsymbol{\sigma} \in q_\star \mathbb{S}_N$ (around which we center the law of $\mathbf{x}_0$), let

$$H_N(s) = -\frac{1}{N} H_{\mathbf{J}}(x_s), \qquad q_N^{\boldsymbol{\sigma}}(s) = \frac{1}{N}\langle \mathbf{x}_s, \boldsymbol{\sigma}\rangle = \frac{1}{N}\sum_{i=1}^N x_s^i \sigma^i. \tag{1.14}$$

For $|q| \le q_\star$ denote by $\mu_{\boldsymbol{\sigma}}^q$ the uniform measure on the sub-sphere

$$\mathbb{S}_{\boldsymbol{\sigma}}(q) := \left\{ \mathbf{x} \in \mathbb{S}_N : \quad \frac{1}{N}\langle \mathbf{x}, \boldsymbol{\sigma}\rangle = q \right\}, \tag{1.15}$$

with $\mathbb{P}_{\mathbf{J},\boldsymbol{\sigma}}^{N,q}$ denoting the joint law (on $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^{2N})$), of the Brownian motion $\mathbf{B}$ and the corresponding strong solution $\mathbf{x}$ of (1.1) for $\mathbf{x}_0$ of law $\mu_{\boldsymbol{\sigma}}^q$ and given $\mathbf{J}, \boldsymbol{\sigma}$ (see Proposition 3.8 for the existence of such a solution).

Theorem 1.1 *For $\boldsymbol{\sigma} \in q_\star \mathbb{S}_N$, $q_\star > 0$, consider $\mathbf{J}$ conditional upon the event*[2]

$$\mathsf{CP}(E_\star, G_\star, \boldsymbol{\sigma}) := \{H_{\mathbf{J}}(\boldsymbol{\sigma}) = -NE_\star,\ \nabla_{\rm sp} H_{\mathbf{J}}(\boldsymbol{\sigma}) = \mathbf{0},\ \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}) = -\|\boldsymbol{\sigma}\| G_\star\}, \tag{1.16}$$

where $\nabla_{\rm sp}$ and $\partial_\perp$ denote, respectively, the gradient WRT *the standard differential structure on $q_\star \mathbb{S}_N$, and the directional derivative normal to $q_\star \mathbb{S}_N$.*[3] *Setting $q_o \in [-q_\star, q_\star]$ let $\mathbf{x}_0$ be distributed according to $\mu_{\boldsymbol{\sigma}}^{q_o}$. Then, for fixed $T < \infty$, as $N \to \infty$ the random functions $(C_N, \chi_N, q_N^{\boldsymbol{\sigma}}, H_N)$ converge uniformly on $[0,T]^2$, almost surely and in L_p with respect to $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$, to non-random functions $C(s,t) = C(t,s)$, $\chi(s,t) = \int_0^t R(s,u)du$, $q(s)$ and $H(s)$, such that $q(0) = q_o$, $C(0,0) = 1$, $R(s,t) = 0$ for $t > s$, $R(s,s) \equiv 1$, and for $s > t$ the absolutely continuous functions C, R, $q(s)$, $H(s)$ and $K(s) = C(s,s)$ are the unique solution in the space of bounded, continuous functions, of the integro-differential equations*

$$\partial_s R(s,t) = -f'(K(s))R(s,t) + \beta^2 \int_t^s R(u,t)R(s,u)\nu''(C(s,u))du, \tag{1.17}$$

$$\begin{aligned}\partial_s C(s,t) &= -f'(K(s))C(s,t) \\ &\quad + \beta^2 \int_0^s R(s,u)\Big[\nu''(C(s,u))C(u,t) - \frac{q(t)\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\Big]du \\ &\quad + \beta^2 \int_0^t R(t,u)\Big[\nu'(C(s,u)) - \frac{\nu'(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\Big]du + \beta q(t)\mathsf{v}'_\star(q(s)),\end{aligned} \tag{1.18}$$

$$\begin{aligned}\partial_s q(s) &= -f'(K(s))q(s) \\ &\quad + \beta^2 \int_0^s R(s,u)\Big[q(u)\nu''(C(s,u)) \\ &\quad - \frac{q_\star^2 \nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\Big]du + \beta q_\star^2 \mathsf{v}'_\star(q(s)),\end{aligned} \tag{1.19}$$

$$\begin{aligned}\partial_s K(s) &= 1 - 2f'(K(s))K(s) \\ &\quad + 2\beta^2 \int_0^s R(s,u) \\ &\quad \Big[\psi(C(s,u)) - \frac{\psi(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\Big]du + 2\beta q(s)\mathsf{v}'_\star(q(s)),\end{aligned} \tag{1.20}$$

[2] In the pure case, i.e. having $\nu(r) = b_m^2 r^m$, one has that $\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}) = \frac{m}{\|\boldsymbol{\sigma}\|} H_{\mathbf{J}}(\boldsymbol{\sigma})$, hence necessarily $G_\star = mE_\star/q_\star^2$, whereas in the mixed case the vector $(E_\star, G_\star)$ can take any value.

[3] Alternatively $\nabla H_{\mathbf{J}}(\boldsymbol{\sigma}) = -G_\star \boldsymbol{\sigma}$.

$$H(s) = \widehat{H}(s) + \mathsf{v}_\star(q(s)),$$
$$\widehat{H}(s) = \beta \int_0^s R(s,u) \left[\nu'(C(s,u)) - \frac{\nu'(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\right] du\,, \tag{1.21}$$

where $\psi(r) := r\nu''(r) + \nu'(r)$ *and*

$$\mathsf{v}(r) := \sum_{p=2}^{m} b_p^2 \langle \mathsf{v}_p, (E,G)\rangle r^p\,, \qquad \mathsf{v}_p := \begin{bmatrix} q_\star^2 \nu(q_\star^2) & q_\star^2 \nu'(q_\star^2) \\ q_\star^2 \nu'(q_\star^2) & \psi(q_\star^2) \end{bmatrix}^{-1} \begin{bmatrix} q_\star^2 \\ p \end{bmatrix}, \tag{1.22}$$

using $\mathsf{v}_\star(\cdot)$ *to denote the case of* $(E,G) = (E_\star, G_\star)$.[4]

Remark 1.2 The conditional on $\mathsf{CP}(E, G, \mathbf{x}_\star)$ solution of (1.1) at $\beta > 0$, is unchanged by embedding β into the coefficients $\{b_p\}$ of (1.3) while taking $(E,G) \mapsto \beta(E,G)$ and setting $\beta = 1$ in the SDS. This modifies $\nu \mapsto \beta^2 \nu$, while $\mathsf{v} \mapsto \beta \mathsf{v}$, preserving the stated limiting dynamics of Theorem 1.1, apart from multiplying $H(s)$ (and its derivatives) by β. It thus suffices to establish Theorem 1.1 for $\beta = 1$.

Remark 1.3 From (1.2) we see that for any non-random orthogonal matrix $\mathbf{O}$, the covariance and hence the law of the Gaussian field $\mathbf{x} \mapsto (H_\mathbf{J}(\mathbf{O}^{-1}\mathbf{x}), \mathbf{O}\nabla H_\mathbf{J}(\mathbf{O}^{-1}\mathbf{x}))$ matches that of $\mathbf{x} \mapsto (H_\mathbf{J}(\mathbf{x}), \nabla H_\mathbf{J}(\mathbf{x}))$. When combined with $\boldsymbol{\sigma} \mapsto \mathbf{O}\boldsymbol{\sigma}$ the same applies for the law of this field conditional on $\mathsf{CP}(E_\star, G_\star, \boldsymbol{\sigma})$. By the rotational symmetry of the Brownian motion $t \mapsto \mathbf{B}_t$ and of the law $\mu_{\boldsymbol{\sigma}}^{q_o}$ of $\mathbf{x}_0$, the law of $\{\boldsymbol{\sigma}, \mathbf{x}_t, \mathbf{B}_t, t \in [0,T]\}$ in Theorem 1.1, matches that of $\{\mathbf{O}\boldsymbol{\sigma}, \mathbf{O}\mathbf{x}_t, \mathbf{O}\mathbf{B}_t, t \in [0,T]\}$. In particular, the joint law of $(C_N, \chi_N, q_N^{\sigma}, H_N)$ is invariant under the mapping $\boldsymbol{\sigma} \mapsto \mathbf{O}\boldsymbol{\sigma}$, and so it suffices to prove Theorem 1.1 only for $\boldsymbol{\sigma} = \mathbf{x}_\star = (\sqrt{N}q_\star, 0, \ldots, 0)$.

Remark 1.4 Conditional on $\mathsf{CP}(E, G, \boldsymbol{\sigma})$, an easy Gaussian computation (see (3.33) in case $\boldsymbol{\sigma} = \mathbf{x}_\star$), yields

$$H_\mathbf{J}(\mathbf{x}) = H_{\mathbf{J}_o}(\mathbf{x}) - N\mathsf{v}\big(N^{-1}\langle \mathbf{x}, \boldsymbol{\sigma}\rangle\big)\,, \tag{1.23}$$

for the centered Gaussian vector $\mathbf{J}_o$ the corresponds to conditioning by $\mathsf{CP}(0,0,\boldsymbol{\sigma})$. Thus, $(E_\star, G_\star)$ only affects (1.1) by adding a deterministic drift, which gives rise to the terms involving $\mathsf{v}_\star(\cdot)$, or $\mathsf{v}'_\star(\cdot)$, in (1.17)–(1.21). The law of $\mathbf{J}_o$ is, for $N \gg 1$, well approximated by the Gaussian law of $\mathbf{J}$ conditional only on $\nabla_{\rm sp} H_\mathbf{J}(\boldsymbol{\sigma}) = \mathbf{0}$. It is not hard to verify that the latter law has the covariance

$$N\nu(N^{-1}\langle \mathbf{x}, \mathbf{y}\rangle) - \big[\langle \mathbf{x}, \mathbf{y}\rangle - \|\boldsymbol{\sigma}\|^{-2}\langle \mathbf{x}, \boldsymbol{\sigma}\rangle\langle \mathbf{y}, \boldsymbol{\sigma}\rangle\big]\frac{\nu'(N^{-1}\langle \mathbf{x}, \boldsymbol{\sigma}\rangle)\nu'(N^{-1}\langle \mathbf{y}, \boldsymbol{\sigma}\rangle)}{\nu'(N^{-1}\langle \boldsymbol{\sigma}, \boldsymbol{\sigma}\rangle)} \tag{1.24}$$

(c.f. (3.34) for essentially such computation when $\boldsymbol{\sigma} = \mathbf{x}_\star$). This change from (1.2) to (1.24) is behind the modification WRT the CKCHS equations, in the square brackets within the integral terms of (1.18)–(1.21).

[4] It is easy to verify that in the mixed case the matrix in (1.22) is positive definite for any $q_\star > 0$, while in the pure case taking $G = mE/q_\star^2$ yields $b_m^2 \langle \mathsf{v}_m, (E,G)\rangle = q_\star^{-2m} E$.

For $I,\ I' \subset \mathbb{R}$, denote by

$$\mathscr{C}_{N,q}(I, I') = \Big\{\boldsymbol{\sigma} \in q\mathbb{S}_N \,:\, \nabla_{\rm sp} H_{\mathbf{J}}(\boldsymbol{\sigma}) = \mathbf{0},\ H_{\mathbf{J}}(\boldsymbol{\sigma}) \in -NI,\ \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}) \in -\sqrt{N} q I'\Big\} \tag{1.25}$$

the set of critical points of the Hamiltonian $H_{\mathbf{J}}(\boldsymbol{\sigma})$ on the sphere of radius $\sqrt{N}q$ with value in $-NI$ and with directional derivative normal to the sphere $\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma})$ in $-\sqrt{N}qI'$. Our next result relates the dynamics of the unconditional model with random initial measure centered at such a critical point with the limiting dynamics of Theorem 1.1. Specifically, denoting by $\|U_N\|_\infty$ the supremum of $|U_N(s,t)|$ over $s, t \in [0, T]$, we associate to $\boldsymbol{\sigma} \in q_\star \mathbb{S}_N$ around which we center a 'band', the (random) error

$$\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) := \|C_N - C\|_\infty \wedge 1 + \|\chi_N - \chi\|_\infty \wedge 1 + \|q_N^{\boldsymbol{\sigma}} - q\|_\infty \wedge 1 + \|H_N - H\|_\infty \wedge 1 \tag{1.26}$$

for the non-random functions (C, R, q, H) from Theorem 1.1, which depend only on $E_\star$, $G_\star$, $q_\star$, q_o and the model parameters $f(\cdot)$, β and $\nu(\cdot)$.

Theorem 1.5 *Let $E_\star$, $G_\star$, $T > 0$ and suppose $I_N = (a_N, b_N)$ and $I'_N = (a'_N, b'_N)$ with $a_N\,, b_N \to E_\star$ and $a'_N\,, b'_N \to G_\star > 2\sqrt{\nu''(q_\star^2)}$. Then, for any $\epsilon > 0$,*

$$\lim_{N\to\infty} \frac{1}{\mathbb{E}\#\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \mathbb{E}\Bigg\{ \sum_{\boldsymbol{\sigma}\in\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \mathbb{P}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}} \left\{\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) > \epsilon\right\} \Bigg\} = 0. \tag{1.27}$$

Further assuming that

$$\lim_{a\to 0^+} \liminf_{N\to\infty} \mathbb{P}\left\{\#\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right) > a\mathbb{E}\left\{\#\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)\right\}\right\} = 1, \tag{1.28}$$

we have that $\lim_{N\to\infty} \mathbb{P}\{\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right) \neq \emptyset\} = 1$, and, for any $\epsilon > 0$, conditionally on this event,

$$\frac{1}{\#\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \sum_{\boldsymbol{\sigma}\in\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \mathbb{P}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}} \left\{\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) > \epsilon\right\} \overset{N\to\infty}{\longrightarrow} 0\,, \quad \textit{in prob.} \tag{1.29}$$

The asymptotics of the expected number of critical points $\mathbb{E}\#\mathscr{C}_{N,q_\star}(I_N,\ I'_N)$ were computed for the pure m-spin models in [5] and for general mixed models in [4]. However, currently the concentration property of (1.28) is proved only for pure m-spin [34] with $G_\star > 2\sqrt{\nu''(q_\star^2)}$ (i.e. $E_\star > 2b_m q_\star^m \sqrt{1 - 1/m}$, see Footnote 2), or for mixed small perturbation of them [13] with large enough $E_\star, G_\star, q_\star$, and for I_N, I'_N of length asymptotically larger than $1/N$. In both cases, for large β the model is 1-RSB and the Gibbs measure concentrates on the set of spherical bands around the points in $\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)$, provided that $q_\star^2$ is set to be at the position of the non-zero atom of the Parisi measure, $-E_\star$ is set for the minimal normalized energy, and $G_\star$ chosen appropriately.

For arbitrary $\boldsymbol{\sigma} \in q_\star\mathbb{S}_N$, conditional on $\mathsf{CP}(E_\star, G_\star, \boldsymbol{\sigma})$ the eigenvalues of the spherical covariant Hessian of $H_{\mathbf{J}}$ at $\boldsymbol{\sigma}$ have the same distribution as those of a GOE matrix, scaled by $\sqrt{\nu''(q_\star^2)(N-1)/N}$ and shifted by $G_\star$. The value $2\sqrt{\nu''(q_\star^2)}$ is the threshold beyond which the Hessian is typically positive definite, i.e., $\boldsymbol{\sigma}$ is a local minimum. Consequently, as can be checked by an application of the Kac-Rice formula, if $G_\star > 2\sqrt{\nu''(q_\star^2)}$ then the ratio of the expected number of minima and the expected number of critical points of all indices in $\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)$ goes to 1. In the two situations mentioned above [13,34] where (1.28) holds, the latter also occurs with high probability and not just in expectation. On the other hand, if $G_\star < 2\sqrt{\nu''(q_\star^2)}$ then the expected number of minima in $\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)$ decays exponentially fast in N^2.

Considering Theorem 1.5 with $q_o = q_\star = 1$, corresponds to starting at a critical point $\mathbf{x}_0 = \boldsymbol{\sigma}$. This is related to some of the results of [11], where qualitative information about the limiting dynamics is gained from an approximate evolution for (only) the pair $(H_N(s), |\nabla_{\rm sp} H_{\mathbf{J}}(\mathbf{x}_s)|/\sqrt{N})$.

Extending [24, Proposition 1.1] to our context, we next establish the "hard spherical constraint" equations corresponding to the limit $L \to \infty$ and $f_L(\cdot)$ of (1.12).

Proposition 1.6 *For any $T < \infty$ the solutions $(R^{(L)}, C^{(L)}, q^{(L)}, H^{(L)})$ of (1.17)–(1.21) for potential $f_L(\cdot)$ as in (1.12) with positive $\boldsymbol{\varphi} = 1+2\beta q_o \mathsf{v}'_\star(q_o)$, converge as $L \to \infty$, uniformly in $[0,T]^2$, towards (R, C, q, H), for $H(\cdot)$ of (1.21). Further, $q(0) = q_o \in [-q_\star, q_\star]$, $R(t,t) = C(t,t) = 1$ for all $t \geq 0$, $R(s,t) = 0$ and $C(s,t) = C(t,s)$ when $s < t$, while (R, C, q) is for $T \geq s \geq t \geq 0$ the unique bounded solution of*

$$\partial_s R(s,t) = -\mu(s)R(s,t) + \beta^2 \int_t^s R(u,t)R(s,u)\nu''(C(s,u))du, \tag{1.30}$$

$$\begin{aligned}\partial_s C(s,t) &= -\mu(s)C(s,t) \\ &\quad + \beta^2 \int_0^s R(s,u)\Big[\nu''(C(s,u))C(u,t) - \frac{q(t)\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\Big]du \\ &\quad + \beta^2 \int_0^t R(t,u)\Big[\nu'(C(s,u)) - \frac{\nu'(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\Big]du + \beta q(t)\mathsf{v}'_\star(q(s))\,,\end{aligned} \tag{1.31}$$

$$\begin{aligned}\partial_s q(s) &= -\mu(s)q(s) \\ &\quad + \beta^2 \int_0^s R(s,u)\Big[q(u)\nu''(C(s,u)) \\ &\quad - \frac{q_\star^2\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\Big]du + \beta q_\star^2 \mathsf{v}'_\star(q(s))\,,\end{aligned} \tag{1.32}$$

$$\mu(s) = \frac{1}{2} + \beta^2 \int_0^s R(s,u)\Big[\psi(C(s,u)) - \frac{\psi(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\Big]du + \beta q(s)\mathsf{v}'_\star(q(s))\,. \tag{1.33}$$

In addition, $\bar{C}(s,t) := C(s,t) - q(s)q(t)/q_\star^2$ is a non-negative definite kernel, and

$$\Big|\int_{t_1}^{t_2} R(s,u)du\Big|^2 \leq t_2 - t_1\,, \qquad 0 \leq t_1 \leq t_2 \leq s < \infty\,. \tag{1.34}$$

Remark 1.7 Since $\mathsf{v}(0) = \mathsf{v}'(0) = 0$, taking $q_o = 0$ yields the solution $q(s) \equiv 0$ in both (1.19) and (1.32). The values of $(E_\star, G_\star, q_\star)$ are then irrelevant, and the system of equations (1.17)–(1.20), (1.30)–(1.33) reduces to the CKCHS-equations, as in [10, Theorem 1.2] and [24, Proposition 1.1], respectively. All terms involving $\mathsf{v}_\star(\cdot)$ disappear also when $E_\star = G_\star = 0$, but for $q_o \neq 0$ the equations (1.19) and (1.32) nevertheless yield non-zero solutions. Unlike the special case of [24, Proposition 1.1], here (R, C, q) may take negative values, but with $C(s,s) = 1$ and $\bar{C}(\cdot,\cdot)$ non-negative definite, necessarily $|q(\cdot)| \leq q_\star$ and $|C(\cdot,\cdot)| \leq 1$.

Remark 1.8 Any $\boldsymbol{\varphi} \in (0,\infty)$ in (1.12) result with equations (1.30)–(1.33) when $L \to \infty$, but since $\mu(0) = \boldsymbol{\varphi}/2$, taking $\boldsymbol{\varphi} = 1 + 2\beta q_o \mathsf{v}'_\star(q_o)$ (when it is positive), simplifies our derivation (otherwise, one merely has to use $\mu(0^+)$ when $s = 0$). The representation (1.3) with $m = \infty$ applies for any real-analytic $\nu(\cdot)$ such that $\nu(0) = \nu'(0) = 0$, $\nu^{(p)}(0) \geq 0$, $p \geq 2$, with a unique strong solution to (1.1) for locally Lipschitz $f'(r)$ growing fast enough as $r \to \infty$. While not pursued here, we expect Theorem 1.1 to hold for any such $f(\cdot)$ and upon considering $f_L(r) = L(r-1)^2 + f(r)$, to further arrive at the conclusions of Proposition 1.6.

Remark 1.9 Whenever $\nu(\cdot)$ is an even polynomial, so is $\mathsf{v}_\star(\cdot)$, resulting with (1.17)–(1.21) invariant under $(C, R, q, H) \mapsto (C, R, -q, H)$. The same applies to (1.30)–(1.33) and in such cases $q_o \mapsto -q_o$ yields the same solution apart from a global sign change in $q(s)$. Indeed, our realization is such that an even $\nu(\cdot)$ results with an even potential $H_{\mathbf{J}}(-\mathbf{x}) = H_{\mathbf{J}}(\mathbf{x})$ per given $\mathbf{J}$, hence also with $\mathsf{CP}(E, G, \boldsymbol{\sigma}) = \mathsf{CP}(E, G, -\boldsymbol{\sigma})$ and thereby a sign change $q_o \mapsto -q_o$ being equivalent to $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$.

In Sect. 2 we study the large time asymptotic of the solution (R, C, q, μ) of (1.30)–(1.33), establishing the FDT regime at high temperature (ie β small), and further analyzing the plausible FDT solutions for somewhat lower temperatures. While doing so, we observe a sharp distinction between the m-pure case and the mixed case, in terms of the emergence of aging. Such distinction was realized recently in [25], by a numerical solution of the CKCHS-equations for initial conditions from the Gibbs measure at different temperatures, suggesting, for example, more than one dynamical phase transition in the mixed case only. In Sect. 3 we prove Theorem 1.1 by adapting [10, Section 2] to our more challenging setting (where $\mathbf{x}_0$ is related to $\mathbf{J}$ via (1.15)–(1.16)). The key to our derivation are Propositions 3.5 and 3.6, whose proofs are deferred to Sects. 4.1 and 4.2 (adapting [10, Section 3] and [10, Section 4], respectively). From Proposition 3.5 one further has the limit dynamics (as $N \to \infty$), for other functions of interest (such as those given in (3.3)–(3.4)). Section 5 is devoted to proving our main result, Theorem 1.5, whereas Proposition 1.6 and Proposition 2.1 are established in Sects. 6 and 7, respectively, by adapting [24, Section 2] and [24, Section 4], to our more involved setting.

2 Large Time Asymptotic: The FDT Regime

At high enough temperature one has that $q(s) \to 0$ for $s \to \infty$. Our next proposition (which is comparable to [24, Theorem 1.3]), shows that the FDT regime of the solution of (1.30)–(1.33) then coincides with that of the CKCHS-equations.

Proposition 2.1 *For β small enough and $\alpha = 0$, the solution of (1.30)–(1.33) is such that $\underline{\lim}\{\mu(\tau)\} > 0$, $(R(t+\tau, t), \bar{C}(t+\tau, t), q(\tau)) \to (0, 0, \alpha\, q_\star)$ exponentially fast in $\tau \to \infty$, uniformly in t, and for any $\tau \geq 0$,*

$$\lim_{t\to\infty} (R(t+\tau, t), C(\tau + t, t), q(t)) = (R_{\mathrm{fdt}}(\tau), C_{\mathrm{fdt}}(\tau), \alpha\, q_\star) \,. \tag{2.1}$$

In such case, necessarily $R_{\mathrm{fdt}}(\tau) = -2C'_{\mathrm{fdt}}(\tau)$. Further, setting $\gamma = 1/2$ and

$$\phi(x) := \gamma + 2\beta^2 \nu'(x)\,, \tag{2.2}$$

we have that $\mu(t) \to \phi(1)$, and $C_{\mathrm{fdt}}(\cdot)$ is the unique $[0, 1]$-valued, continuously differentiable solution of

$$D'(s) = -\int_0^s \phi(D(v)) D'(s-v) dv - \frac{1}{2}\,, \qquad D(0) = 1\,. \tag{2.3}$$

More generally, if the solution (R, C, q) *of* (1.30)–(1.33) *is uniformly bounded, with* $\{R(t + \cdot, t), t \geq T_0\}$ *uniformly integrable (*WRT *Lebesgue measure),* $\underline{\lim}\{\mu(\tau)\} > 0$ *and* (2.1) *holds for some* $\alpha \in [-1, 1]$*, then necessarily* $\mu(t) \to \mu$ *such that* $(R_{\rm fdt}, C_{\rm fdt}, \mu)$ *satisfy [24, (4.15)–(4.17)], with*

$$\mu\,\alpha q_\star = \beta q_\star^2 \mathsf{v}_\star'(\alpha q_\star) - \beta^2 q_\star^2 \frac{\nu''(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}\kappa_2 + \beta^2 \alpha q_\star \kappa_1\,, \quad \kappa_1 := \int_0^\infty R(\theta)\nu''(C(\theta))d\theta\,, \tag{2.4}$$

$$\mathbf{I} = \beta\alpha q_\star \mathsf{v}_\star'(\alpha q_\star) - \beta^2 \frac{\psi(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}\kappa_2 + \beta^2\kappa_3\,, \qquad \kappa_2 := \int_0^\infty R(\theta)d\theta\,, \quad \kappa_3 := 0\,. \tag{2.5}$$

One such solution is $(-2D'(\cdot), D(\cdot), \phi(1))$ *for* (ϕ, D) *of* (2.2)–(2.3) *and* $D_\infty \in [0, 1)$*,* $\gamma \in \mathbb{R}$ *such that*

$$\mathbf{I} = \gamma - \frac{1}{2} + 2\beta^2 D_\infty \nu'(D_\infty)\,, \tag{2.6}$$

$$D_\infty = \sup\{x \in [0, 1] : (\gamma + 2\beta^2\nu'(x))(1 - x) \geq 1/2\}\,, \tag{2.7}$$

yielding in turn the values $\kappa_1 = 2(\nu'(1) - \nu'(D_\infty))$ *and* $\kappa_2 = 2(1 - D_\infty)$*.*

Remark 2.2 Our proof of (2.1) relies on $\Psi(\cdot)$ of (7.2)–(7.4) being a contraction on a suitable set $\mathcal{A}$ (and for uniqueness of $(R_{\rm fdt}, C_{\rm fdt})$ we require that the induced map $\Psi_{\rm fdt}(\cdot)$ be a contraction at the given α). In particular, a global contraction requires that $\alpha = 0$ be the unique solution of (2.4), which in turn depends not only on β and $q_\star$ but also on $(E_\star, G_\star)$. Nevertheless, at least when $b_2 = 0$ (so $\mathsf{v}_\star''(0) = 0$), we expect the FDT solution of Proposition 2.1 with $\alpha = 0$, $\gamma = 1/2$, to apply for all $\beta < \beta_c$ of [24, (1.23)], provided $q_o = q_o(\beta, E_\star, G_\star)$ is small enough.

Remark 2.3 For pure m-spins, [6] consider the diffusion (1.1) starting at $\mathbf{x}_0$ of law $\mu^N_{2\beta',\mathbf{J}}$ for various choices of $\beta' \in [0, \infty)$. Employing the mathematically non-rigorous replica method (in particular, its 1RSB picture for the Gibbs measure), they predict the resulting limit equations for (R, C) and their solution in the FDT regime. Building on it (and using again the replica method), [7] considers in this setting also the limit dynamics of the overlap $q(t)$.

Remark 2.4 The limit α of $q(t)/q_\star$ provides information on the state $\mathbf{x}_t$ in the limit $N \to \infty$, at $t \gg 1$ which does not scale with N. The case $\alpha = 0$ represents an escape from the energy well about the critical point $\boldsymbol{\sigma}$ to a point which is orthogonal to $\boldsymbol{\sigma}$. In contrast, $\alpha = 1$ implies convergence to the projection $q_\star^{-1}\boldsymbol{\sigma} \in \mathbb{S}_N$ of the critical point around which the state was initialized. Note also that for $\alpha = q_\star$ the eventual support $\mathbb{S}_{\boldsymbol{\sigma}}(q_\star^2)$ of the state, is precisely the sphere of co-dimension 1 and radius $\sqrt{N(1 - q_\star^2)}$, *centered at* the critical point $\boldsymbol{\sigma}$.

While Proposition 2.1 is limited to small β, we do expect (2.1) to hold at all β, albeit having $\alpha \neq 0$ for some $(E_\star, G_\star)$ and q_o close enough to $q_\star$, as soon as $\beta > \beta_+(G_\star)$, where as we detail in the sequel, β_+ is in general *lower than* β_c of [24, (1.23)]. To this end, we first briefly review the physics prediction for the (large time) asymptotic for the CKCHS-equations, namely when $q_o = 0$, or alternatively, when all terms involving $q(\cdot)$ are omitted from (1.30)–(1.33) (see Remark 1.7). Recall that for this limiting CKCHS dynamics, aging amounts to having a non-identically constant $C_{\rm aging}(\cdot)$ such that $C(\tau + t, t) \to C_{\rm aging}(0)$ as $t \to \infty$ followed by $\tau \to \infty$, whereas $C(s, \lambda s) \to C_{\rm aging}(\lambda)$ as $s \to \infty$. Now, in the absence of aging, such prediction is given by the FDT solution from Proposition 2.1, for $\alpha = 0$

and parameters which solve (2.5)–(2.7) assuming the limit D_∞ of $C_{\rm fdt}(\tau)$ as $\tau \to \infty$ is zero. As explained before, doing so amounts to setting $\mathbf{I} = 0$ and $\gamma = 1/2$, whereas (2.7) holds for such values iff $\beta < \beta_c$ of [24, (1.23)].

In contrast, when $\beta > \beta_c$ the limit D_∞ of $C_{\rm fdt}(\tau)$ must be strictly positive, which for $\alpha = 0$ indicates the onset of aging and in particular having $R_{\rm fdt}(\tau) \to 0$ at a sub-exponential rate. Such slow decay is expected in turn to require the additional relation

$$\gamma = 2\beta^2[\nu''(D_\infty)(1 - D_\infty) - \nu'(D_\infty)] \tag{2.8}$$

(see [24, (1.22)]), which together with (2.7) dictate the values of $\gamma > 1/2$ and of $D_\infty = D_\star(\beta) > 0$, with

$$D_\star(\beta) := \sup\{x \in [0,1] : 4\beta^2 g(x) \geq 1\}, \qquad \text{for} \qquad g(x) := \nu''(x)(1-x)^2 \tag{2.9}$$

(as in [24, (1.24)]). While (2.6) thereby determines $\mathbf{I}$, our expressions for κ_i in (2.5) (and in (2.4)), relied on the uniform in t, integrability of $\tau \mapsto R(t+\tau, t)$, which is no longer expected. To rectify this, at $\beta \geq \beta_c$ one adds to these formulas the contribution from the aging regime, namely having $\lambda = u/s$ bounded away from zero and one, to the integrals on the RHS of (1.31)–(1.33). As explained after [24, (1.24)], the physics ansatz of a single aging regime with $R_{\rm aging}(\lambda) = AC'_{\rm aging}(\lambda)$ starting at $C_{\rm aging}(1) = D_\infty$ and ending at $C_{\rm aging}(0) = \alpha^2$ (ie, having $\bar{C}_{\rm aging}(0) = 0$), implies the increase

$$\begin{aligned}
\kappa_1 &\leftarrow \kappa_1 + A(\nu'(D_\infty) - \nu'(\alpha^2))\,,\\
\kappa_2 &\leftarrow \kappa_2 + A(D_\infty - \alpha^2)\,,\\
\kappa_3 &\leftarrow \kappa_3 + A(D_\infty \nu'(D_\infty) - \alpha^2\nu'(\alpha^2))\,,
\end{aligned} \tag{2.10}$$

of the coefficients in the identity (2.5), which in turn determines the value of A. Finally, should the self-consistency requirement of $A > 0$ and $\bar{C}_{\rm aging}(0) = 0$ fail, one moves from the latter ansatz into the richer hierarchy of multiple aging regimes.

Recall Remark 2.4, that for $\alpha = 0$ and $\beta > \beta_c$ aging occurs for a state which is already orthogonal to the critical point $\boldsymbol{\sigma}$ around which we initialized the system, i.e. after the escape from the energy well about it. Here we consider another alternative, of having a still localized state, namely a solution with $\alpha \neq 0$ that in addition satisfies (2.4). Indeed, recall [24, Proposition 6.1] that the FDT regime of the CKCHS-equations must be given by (2.3) as soon as a key integral $I(t+\cdot, t)$ converges for $t \to \infty$ (uniformly on compacts), to some constant (which in terms of our notations, turns out to be $\widehat{\mathbf{I}} := \gamma - \frac{1}{2} - \mathbf{I} + \beta^2\kappa_3$). Assuming in addition that such convergence to constants $(\mathbf{I}_1^{(q)}, \mathbf{I}_2^{(q)})$ applies also for the integrals

$$I_1^{(q)}(s) := \int_0^s R(s,u)q(u)\nu''(C(s,u))du\,, \qquad I_2^{(q)}(s) := \int_0^s R(s,u)\nu'(q(u))du\,,$$

we have in (1.32), we can approximate the latter dynamics (at $s \gg 1$), by the much simpler ODE

$$\begin{aligned}
q'(s) &= -\mu(s)q(s) + \mathbf{Q}(q(s))\,, \quad \text{for} \quad \mu(s) = \mathbf{P}(q(s))\,,\\
\mathbf{Q}(x) &= \beta q_\star^2 \mathsf{v}_\star'(x) - \beta^2 q_\star^2 \frac{\nu''(x)}{\nu'(q_\star^2)}\mathbf{I}_2^{(q)} + \beta^2 \mathbf{I}_1^{(q)}\,,\\
\mathbf{P}(x) &= \beta x \mathsf{v}_\star'(x) - \beta^2 \frac{\psi(x)}{\nu'(q_\star^2)}\mathbf{I}_2^{(q)} + \rho + \widehat{\mathbf{I}}\,.
\end{aligned} \tag{2.11}$$

Such an ODE has no limit sets beyond its finitely many limit points, which are at the isolated solutions of

$$\mathbf{P}(x)x = \mathbf{Q}(x)\,, \qquad x \in [-q_\star, q_\star]\,. \tag{2.12}$$

Hence our earlier prediction that (2.1) remains valid at all β. Further, a convergence of $q(u)$ to some limit point $x = \alpha q_\star$ implies by self-consistency the values $\mathbf{I}_1^{(q)} = \alpha q_\star \kappa_1$ and $\mathbf{I}_2^{(q)} = \nu'(\alpha q_\star)\kappa_2$, which upon substitution in (2.11)–(2.12) yield the requirements (2.4)–(2.5) on α and $\mathbf{I}$.

The analysis of the FDT regime in the presence of aging starts precisely as for CKCHS-equations with $\beta > \beta_c$, $D_\infty = D_\star(\beta) > 0$ of (2.9) and the corresponding values of $(\gamma, \mathbf{I})$ (as determined by (2.6)–(2.8)). The only difference is that now we can try beyond the CKCHS-solution $\alpha = 0$ and $\mathbf{I} = \beta^2\kappa_3$, also any $A > 0$ and $\alpha^2 = C_{\text{aging}}(0) < D_\infty$ which satisfy (2.4)–(2.5) for κ_i of (2.10). Since $D_\star(\beta) \uparrow 1$, taking β large provides access to all solutions of (2.12) (but we do not expect a simple, explicit way to determine which interval of q_o values is attracted to each stable solution).

The most interesting case is that of a localized state with *no-aging at* $\alpha \neq 0$. Specifically, seeking $(R_{\text{fdt}}(\tau), C_{\text{fdt}}(\tau), \mu)$ as in Proposition 2.1 for $\gamma \neq 1/2$ such that $\bar{C}_{\text{fdt}}(\tau) \to 0$, i.e. with $D_\infty = \alpha^2$. Plugging such a solution in (2.4) gives

$$\gamma\alpha = \beta q_\star \mathsf{v}_\star'(\alpha q_\star) - 2\beta^2 \frac{q_\star \nu''(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}(1-\alpha^2) - 2\beta^2\alpha\nu'(\alpha^2)\,. \tag{2.13}$$

Similarly, plugging it in (2.5) and comparing with (2.6) results with

$$\gamma - \frac{1}{2} = \beta\alpha q_\star \mathsf{v}_\star'(\alpha q_\star) - 2\beta^2 \frac{\psi(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}(1-\alpha^2) - 2\beta^2\alpha^2\nu'(\alpha^2)\,. \tag{2.14}$$

Recall (2.7), that having $D_\infty = \alpha^2$ requires in addition to the preceding that

$$(\gamma + 2\beta^2\nu'(\alpha^2))(1-\alpha^2) - \frac{1}{2} = \frac{2\beta^2}{\nu'(q_\star^2)}[\nu'(\alpha^2)\nu'(q_\star^2) - \nu'(\alpha q_\star)^2](1-\alpha^2) = 0\,. \tag{2.15}$$

In the pure case the RHS of (2.15) always holds, while otherwise it holds only[5] for $\alpha = q_\star$. Proceeding first with the m-pure case, utilizing Footnotes 2 and 4, we get that both (2.13) and (2.14) hold for $\alpha \neq 0$ iff

$$4\beta^2 g(\alpha^2) = y^2 \quad \text{and} \quad G_\star = \sqrt{\nu''(q_\star^2)}(y + y^{-1})\,. \tag{2.16}$$

In view of (2.7), only the smaller positive root $y \in (0,1]$ for the RHS of (2.16) is relevant, with the condition $G_\star > 2\sqrt{\nu''(q_\star^2)}$ for existence of such $y \in (0,1)$ matching our assumption in Theorem 1.5 (alternatively, the latter inequality amounts to $\widehat{E}_\star > 2\sqrt{1-\frac{1}{m}}$ where $\widehat{E}_\star := E_\star/(b_m q_\star^m)$ denotes the given energy level, measured in standard deviations of $H_{\mathbf{J}}(\boldsymbol{\sigma})$). Moreover, the LHS of (2.16) can not hold for some $y < 1$, unless

$$\frac{1}{\beta} > 2\sqrt{\nu''(\alpha^2)}(1-\alpha^2)\,, \tag{2.17}$$

which is precisely the stability condition for TAP solutions on $\alpha\,\mathbb{S}_N$ (see [29, Eq. (25)]). Fixing $\widehat{E}_\star$ as above, namely $y \in (0,1)$ via the RHS of (2.16), here $g(\cdot)$ attains its maximum over

[5] Except for $\alpha = -q_\star$ equivalently holding whenever $\nu(\cdot)$ is an even polynomial

$[0, 1]$ at $\alpha_m^2 := 1 - \frac{2}{m}$, and by the same reasoning as for the CKCHS-equations, one should choose the larger solution α^2 in (2.16), namely take

$$D_\infty = D_\star(\beta/y) \qquad \text{provided} \qquad \beta > \beta_+ := y/(2\sqrt{g(\alpha_m^2)})\,, \tag{2.18}$$

where $\beta_+ < \beta_c$ of [24, (1.23)], for any $m \geq 2$ and all $\widehat{E}_\star$ as above.

Turning to the mixed case, first note that $\mathsf{v}'_\star(q_\star^2) = G_\star$ (see (3.33) at $\mathbf{x}_t = \mathbf{x}_\star$). Upon plugging the generic solution $\alpha = q_\star$ of (2.15) into (2.13), it follows that no-aging with $\alpha \neq 0$ requires the RHS of (2.16) to hold for $y \leq 1$ and $q_\star^2 = D_\star(\beta/y)$ of (2.9). Taken together, we see that (2.16) must hold at $\alpha = q_\star$, yielding the relation

$$G_\star = G_\star(\alpha, \beta) := 2\beta\nu''(\alpha^2)(1-\alpha^2) + \frac{1}{2\beta(1-\alpha^2)}\,, \tag{2.19}$$

where the restriction to $y < 1$ amounts to the inequality (2.17).

It is easy to check that having such $(R_{\text{fdt}}(\tau), C_{\text{fdt}}(\tau), \mu)$ as in Proposition 2.1, except for possibly $\gamma \neq 1/2$, and with the no-aging condition $D_\infty = \alpha^2$ in place, implies the convergence of $H(s)$ of (1.21) as $s \to \infty$, to the limiting (macroscopic) energy

$$H(\infty) := \mathsf{v}_\star(\alpha q_\star) + 2\beta\theta(\alpha^2)\,, \qquad \text{where} \qquad \theta(q) := \nu(1) - \nu(q) - \nu'(q)(1-q) \tag{2.20}$$

(and to arrive at (2.20) we also use the RHS of (2.15)).

For $\boldsymbol{\sigma} \in \alpha\mathbb{S}_N$, similarly to the proof of Lemma 3.7, one can check that conditionally on $\mathsf{CP}(E_\star, G_\star, \boldsymbol{\sigma})$ the Gaussian field $H_{\mathbf{J}}(\mathbf{x})$ has expectation $-NE_\star$ and variance $N\theta(\alpha^2)$ at any $\mathbf{x}$ in the sub-sphere $\mathbb{S}_{\boldsymbol{\sigma}}(\alpha^2)$ of (1.15). Using this conditional field, one has the spherical model WRT the uniform measure $\mu_{\boldsymbol{\sigma}}^{\alpha^2}(\mathbf{x})$ on $\mathbb{S}_{\boldsymbol{\sigma}}(\alpha^2)$, its Gibbs measure $\mu_{\beta_0,\mathbf{J}}^{\boldsymbol{\sigma}}$ of density $(Z_{\beta_0,\mathbf{J}}^{\boldsymbol{\sigma}})^{-1}e^{-\beta_0 H_{\mathbf{J}}(\mathbf{x})}$ and the corresponding free energy $F_{\beta_0}(\boldsymbol{\sigma})$ to which $N^{-1}\log Z_{\beta_0,\mathbf{J}}^{\boldsymbol{\sigma}}$ converges. If for any β_0 near 2β this model is replica symmetric, then $F_{\beta_0}(\boldsymbol{\sigma}) = \beta_0 E_\star + \frac{\beta_0^2}{2}\theta(\alpha^2)$ and most of the mass of $\mu_{2\beta,\mathbf{J}}^{\boldsymbol{\sigma}}$ is indeed typically carried at the energy $E_\star + 2\beta\theta(\alpha^2)$. In the mixed case we know that $\alpha = q_\star$ hence the state $\mathbf{x}_t$ is supported for $t \gg 1$ on that same sub-sphere $\mathbb{S}_{\boldsymbol{\sigma}}(\alpha q_\star) = \mathbb{S}_{\boldsymbol{\sigma}}(\alpha^2)$ (see Remark 2.4). Further, in the m-pure case $\mathsf{CP}(E_\star, G_\star, \boldsymbol{\sigma}) = \mathsf{CP}(r^m E_\star, r^{m-2}G_\star, r\boldsymbol{\sigma})$ for any $r > 0$, with $r = \alpha/q_\star$ eliminating the effect of $q_\star$ and allowing us to take again WLOG $\|\boldsymbol{\sigma}\| = \alpha\sqrt{N} = q_\star\sqrt{N}$. Recall that $\mathsf{v}_\star(\alpha q_\star) = \mathsf{v}_\star(q_\star^2) = E_\star$ (see (3.33) at $\mathbf{x}_t = \mathbf{x}_\star$), so the energy $\alpha^m\widehat{E}_\star + 2\beta\theta(\alpha^2)$ carrying most of the mass of the spherical model $\mu_{2\beta,\mathbf{J}}^{\boldsymbol{\sigma}}$ is for such $\boldsymbol{\sigma}$ precisely the limit $H(\infty)$ of (2.20). Further, re-writing the conditional Gaussian field of $\mu_{2\beta,\mathbf{J}}^{\boldsymbol{\sigma}}$ as a polynomial in the re-centered coordinates $\mathbf{x} - \boldsymbol{\sigma}$ gives a new spherical mixed model, see [13, Lemma 7.1], whose 2-spin interaction part is in the replica symmetric regime precisely when (2.17) holds (c.f. [13, (7.6) and (8.8)]). Finally, in the m-pure case, the relation (2.19) determines from the energy $\widehat{E}_\star$ a limiting sub-sphere height α which is a local maximum of the free energy $F_{2\beta}(\boldsymbol{\sigma})$ plus the entropy $\frac{1}{2}\log(1-\alpha^2)$.

2.1 Limiting Dynamics for Spherical SK-Model

While of less interest from the physics point of view, for the spherical **SK**-model, namely $m = 2$, one can solve (1.30)–(1.33) and thereby confirm our predictions. Specifically, for $\nu(x) = \frac{x^2}{8}$ (hence $\psi(x) = 2\nu'(x) = \frac{x}{2}$, $\nu''(x) = \frac{1}{4}$, $\mathsf{v}'_\star(x) = \frac{G_\star}{q_\star^2}x$), starting at $R(s,s) = 1$, $\bar{C}(s,s) = 1 - q(s)^2/q_\star^2$ and $q(0) = q_o$ these equations are for $s > t$,

$$\partial_s R(s,t) = -\mu(s)R(s,t) + \frac{\beta^2}{4}\int_t^s R(s,u)R(u,t)du\,, \qquad q'(s) = -(\mu(s)-\beta G_\star)q(s)\,, \tag{2.21}$$

$$\partial_s \bar{C}(s,t) = -\mu(s)\bar{C}(s,t) + \frac{\beta^2}{4}\Big[\int_0^s R(s,u)\bar{C}(u,t)du + \int_0^t R(t,u)\bar{C}(u,s)du\Big],$$
$$\mu(s) = \frac{1}{2} + \frac{\beta^2}{2}\int_0^s R(s,u)\bar{C}(s,u)du + \beta G_\star \frac{q^2(s)}{q_\star^2}\,. \tag{2.22}$$

Further, in this case we get from (1.21) and (2.22) that

$$H(s) = \frac{1}{2\beta}\Big[\frac{\beta^2}{2}\int_0^s R(s,u)\bar{C}(s,u)du + \beta G_\star \frac{q^2(s)}{q_\star^2}\Big] = \frac{\mu(s)}{2\beta} - \frac{1}{4\beta}\,. \tag{2.23}$$

Setting $\Lambda(s) := q_\star e^{\int_0^s (\mu(u)-\beta G_\star)du}$ the solution of (2.21) must be

$$q(s) = \frac{q_\star q_o}{\Lambda(s)}\,, \qquad R(s,t) = \frac{\Lambda(t)}{\Lambda(s)}\mathcal{L}_{G_\star}(s-t)\,,$$

where $\mathcal{L}_G(\theta) = e^{-\beta G\theta}\mathcal{L}(\theta)$ for $\mathcal{L}(\theta) := \frac{2}{\pi}\int_{-1}^1 e^{\beta\theta x}\sqrt{1-x^2}\,dx$ (see [24, (4.9)]). Substituting this in (2.22), the symmetric $\mathsf{M}(s,t) := \bar{C}(s,t)\Lambda(s)\Lambda(t)$, is the positive, unique solution of

$$\begin{aligned}\partial_s M(s,t) &= -\beta G_\star M(s,t)\\ &\quad + \frac{\beta^2}{4}\Big[\int_0^s \mathcal{L}_{G_\star}(s-u)M(u,t)du\\ &\quad + \int_0^t \mathcal{L}_{G_\star}(t-u)M(u,s)du\Big], \quad \forall s > t\,,\\ M'(t) &= q_o^2 + (1-2\beta G_\star)M(t) + \beta^2\int_0^t \mathcal{L}_{G_\star}(t-u)M(t,u)du\,, \quad M(t,t) = M(t)\,,\end{aligned} \tag{2.24}$$

starting at $M(0) = q_\star^2 - q_o^2$, and with $\Lambda(t) = \sqrt{q_o^2 + M(t)}$. By the super-position principle for this linear system

$$M(s,t) = (q_\star^2 - q_o^2)e^{-\beta G_\star(s+t)}\,M_{\rm ck}(s,t) + q_o^2\,M_{G_\star}(s,t)\,, \tag{2.25}$$

where $M_{\rm ck}$ denotes the CKCHS-type solution of (2.24) with $q_o = G_\star = 0$, starting at $M_{\rm ck}(0) = 1$, while $M_{G_\star}$ is the solution of (2.24) for $q_o^2 = 1$ and $M_{G_\star}(0) = 0$. The spherical **SK**-model is somewhat degenerate, as in view of (2.25), having $q(t) \to \alpha \neq 0$, or equivalently a finite limit for $M(t)$ as $t \to \infty$, does not depend on the value of $0 < |q_o| < q_\star$ and when such non-zero limit exists, the same invariance to q_o applies to the issue of no-aging (i.e. having $M(t+\tau,t) \to 0$ as $t \to \infty$ followed by $\tau \to \infty$). The analog of $M_{\rm ck}$ for (1.17)–(1.20) at $q(\cdot) \equiv 0$ and linear $f'(x) = cx$, is studied in [8, Section 3]. A similar but finer analysis shows that $M_{\rm ck}(s,t)$ grows as $s,t \to \infty$, up to some polynomial pre-factors, at the exponential rate $\mu_\star(s+t)$, where $\mu_\star = \beta$ for $\beta > 1$ and otherwise $\mu_\star = (1+\beta^2)/2$. Focusing on the case of a stable energy well around the critical point $\boldsymbol{\sigma}$, namely $G_\star > 1$ as in Theorem 1.5, we have that $\beta G_\star > \mu_\star$ iff $\beta > y$, with $y \in (0,1)$ as in the RHS of (2.16). We thus have the dichotomy predicted earlier, that $q_o = 0$ requires $\alpha = 0$, with the onset of aging at β_c determined by the asymptotic of $M_{\rm ck}(s,t)/\sqrt{M_{\rm ck}(s)M_{\rm ck}(t)}$, whereas for any $q_o \neq 0$, $G_\star > 1$ and $\beta > y$ we have a localized state, with $\alpha^{-2} - 1$ given by the finite limit of $M_{G_\star}(t)$, and $C_{\rm fdt}(\tau)$ being the limit as $t \to \infty$ of $(1 + M_{G_\star}(t+\tau, t))/(1 + M_{G_\star}(t))$. We get these limits by replacing $M_{G_\star}(s,t)$ with the stationary solution $M^{(\rm st)}_{G_\star}(s,t)$ of (2.24) when all the integrals start at $-\infty$ (instead of at zero). By translation invariance, $\mathsf{M}^{(\rm st)}_{G_\star}(s,t)$ must be of the form $\Gamma(s-t)$ for symmetric $\Gamma(\cdot)$ such that

$$\begin{aligned}\Gamma'(\tau) &= -\beta G_\star \Gamma(\tau) + \frac{\beta^2}{4}\Big[\int_0^\infty \mathcal{L}_{G_\star}(u)\Gamma(u+\tau)du + \int_0^\infty \mathcal{L}_{G_\star}(u)\Gamma(u-\tau)du\Big], \\ 0 &= 1 + (1-2\beta G_\star)\Gamma(0) + \beta^2 \int_0^\infty \mathcal{L}_{G_\star}(u)\Gamma(u)du\,. \end{aligned} \tag{2.26}$$

Next, recall that $y \in (0,1)$ on the RHS of (2.16) satisfies

$$1 - 2G_\star y + y^2 = 0\,, \qquad \text{that is} \qquad y = G_\star - \sqrt{G_\star^2 - 1} \tag{2.27}$$

and hence (see [8, Page 16]), also

$$y = \frac{1}{2\pi}\int_{-2}^{2} \frac{\sqrt{4-x^2}}{(2G_\star) - x}\,dx = \frac{\beta}{2}\int_0^\infty \mathcal{L}_{G_\star}(\theta)d\theta\,. \tag{2.28}$$

Further, utilizing (2.27), (2.28), with $\mathcal{L}_{G_\star}(0) = 1$ and having

$$\mathcal{L}'_G(\tau) = -\beta G \mathcal{L}_G(\tau) + \frac{\beta^2}{4}\int_0^\infty \mathcal{L}_G(u)\mathcal{L}_G(\tau - u)du \tag{2.29}$$

(compare with the LHS of (2.21)), one can verify that

$$\Gamma(\tau) = \frac{1}{c}\int_\tau^\infty \mathcal{L}_{G_\star}(u)du\,, \qquad c := 2 - \int_0^\infty \mathcal{L}_{G_\star}(u)du = 2\Big(1 - \frac{y}{\beta}\Big),$$

satisfies (2.26). Consequently, in this case

$$\alpha^{-2} - 1 = \Gamma(0) = \frac{2}{c} - 1\,, \qquad \text{that is} \qquad \alpha^2 = \frac{c}{2} = 1 - \frac{y}{\beta} \tag{2.30}$$

in agreement with our prediction on the LHS of (2.16), whereas

$$C_{\rm fdt}(\tau) = \frac{1+\Gamma(\tau)}{1+\Gamma(0)} = 1 - \frac{1}{2}\int_0^\tau \mathcal{L}_{G_\star}(u)du\,, \tag{2.31}$$

is precisely $D(\tau)$ of (2.3) for $\phi(x) = \beta G_\star + \frac{\beta^2}{2}(x-1)$, and converges to $D_\infty = \alpha^2$ (i.e. with no-aging). In addition, having here $\mu(s) \to G_\star\beta$ we get from (2.23) that $H(s) \to H(\infty) = \frac{G_\star}{2} - \frac{1}{4\beta}$ (matching the expression $H(\infty) = \alpha^2\frac{G_\star}{2} + \frac{\beta}{4}(1-\alpha^2)^2$ of (2.20)).

3 Proof of Theorem 1.1 at $\boldsymbol{\sigma} = \mathbf{x}_\star$, $\beta = 1$

In view of Remarks 1.2–1.3, WLOG we fix throughout this section $\beta = 1$ and $\boldsymbol{\sigma} = \mathbf{x}_\star = (\sqrt{N}q_\star, 0, \ldots, 0)$. Fixing also T and letting $d(N,m)$ be the length of the coupling vector $\mathbf{J}$, following [10] we equip the product space $\mathcal{E}_N = \mathbb{R}^N \times \mathbb{R}^{d(N,m)} \times \mathbb{C}([0,T], \mathbb{R}^N)$ with the norm

$$\|(\mathbf{x}_0, \mathbf{J}, \mathbf{B})\|^2 = \sum_{i=1}^N (x_0^i)^2 + \sum_{p=2}^m \sum_{1\le i_1 \le \ldots \le i_p \le N} (N^{\frac{p-1}{2}} J_{i_1\cdots i_p})^2 + \sup_{0\le t\le T}\sum_{i=1}^N (B_t^i)^2 \tag{3.1}$$

and denote by $\widetilde{\mathbb{P}} = \mu_{\mathbf{x}_\star}^{q_o} \otimes \gamma_N^{(E,G,q_\star)} \otimes P_N$ the product probability measure of $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ on $\mathcal{E}_N$, where $\mathbf{x}_0$ follows the law $\mu_{\mathbf{x}_\star}^{q_o}$ (defined above (1.15)), $\gamma_N^{(E,G,q_\star)}$ denotes the (Gaussian) distribution of $\mathbf{J}$ conditional upon $\mathsf{CP}(E, G, \mathbf{x}_\star)$[6] and P_N stands for the distribution of N-dimensional Brownian motion. Next, for $C_N(s,t)$ of (1.8) and $q_N(s) = q_N^{\mathbf{x}_\star}(s)$ of (1.14), we let

$$\bar{C}_N(s,t) := C_N(s,t) - q_\star^{-2} q_N(s) q_N(t) = \frac{1}{N}\sum_{i=2}^N x_s^i x_t^i\,, \qquad q_N(s) = \frac{q_\star x_s^1}{\sqrt{N}}\,. \quad (3.2)$$

Setting $G^i(\mathbf{x}) := -\partial_{x^i} H_{\mathbf{J}}(\mathbf{x})$, the derivation of Theorem 1.1 builds on the proof of [10, Thm. 1.2], which utilizes beyond C_N and χ_N of (1.8)–(1.9), two auxiliary functions A_N and F_N (see [10, (1.15)]). Having here a distinguished first coordinate, those four functions of [10] are replaced by $\mathcal{U}_N^\dagger := \{C_N, \chi_N, \bar{C}_N, \bar{\chi}_N, \bar{A}_N, \bar{F}_N\}$, for $\bar{C}_N$ of (3.2) and

$$\bar{\chi}_N(s,t) := \frac{1}{N}\sum_{i=2}^N x_s^i B_t^i\,, \quad \bar{A}_N(s,t) := \frac{1}{N}\sum_{i=2}^N G^i(\mathbf{x}_s) x_t^i\,, \quad \bar{F}_N(s,t) := \frac{1}{N}\sum_{i=2}^N G^i(\mathbf{x}_s) B_t^i\,. \quad (3.3)$$

Beyond $\mathcal{U}_N^\dagger$, our derivation clearly has to also involve q_N of (3.2), the pre-limit of $\widehat{H}$ from (1.21), and the (centered) contribution of the first coordinate to A_N, given respectively by

$$\widehat{H}_N(s) := -\frac{1}{N}[H_{\mathbf{J}}(\mathbf{x}_s) - \bar{H}(\mathbf{x}_s)]\,, \qquad V_N(s) := \frac{q_\star}{\sqrt{N}}(G^1(\mathbf{x}_s) - \bar{G}^1(\mathbf{x}_s))\,, \quad (3.4)$$

where $\bar{G}(\mathbf{x}) := -\nabla \bar{H}(\mathbf{x})$ and $\bar{H}(\mathbf{x}) := \mathbb{E}[H_{\mathbf{J}}(\mathbf{x}) \,|\, \mathsf{CP}(E, G, \mathbf{x}_\star)]$. Analogously to D_N and E_N [10, (1.16)], it is convenient to define in addition to V_N, $\bar{A}_N$ and $\bar{F}_N$, also their contribution to the incremental changes in q_N, $\bar{C}_N$ and $\bar{\chi}_N$, which for $K_N(t) := C_N(t,t)$ are given respectively by

$$Q_N(s) := -f'(K_N(s)) q_N(s) + q_\star^2 \mathsf{v}'(q_N(s)) + V_N(s)\,, \quad (3.5)$$

$$\bar{D}_N(s,t) := -f'(K_N(t))\bar{C}_N(t,s) + \bar{A}_N(t,s),\ \bar{E}_N(s,t) := -f'(K_N(s))\bar{\chi}_N(s,t) + \bar{F}_N(s,t). \quad (3.6)$$

We shall establish limit equations for $\mathcal{U}_N = \mathcal{U}_N^\dagger \cup \{q_N, \widehat{H}_N, V_N, Q_N, \bar{D}_N, \bar{E}_N, \Upsilon_N, \Phi_N, \Phi_N^1, \Psi_N, \Psi_N^1\}$, where

$$\Upsilon_N(s,u) := \nu(C_N(s,u)) - \bar{C}_N(s,u)\frac{\nu'(q_N(s))\nu'(q_N(u))}{\nu'(q_\star^2)}\,,$$

$$\Phi_N(s,u) := \nu'(C_N(s,u)) - \frac{\nu'(q_N(u))\nu'(q_N(s))}{\nu'(q_\star^2)}\,, \quad (3.7)$$

$$\Phi_N^1(s,u) := q_N(u)\nu'(C_N(s,u)) - \bar{C}_N(s,u)\frac{q_\star^2\nu'(q_N(u))\nu''(q_N(s))}{\nu'(q_\star^2)}\,,$$

$$\Psi_N(s,u) := \nu''(C_N(s,u))\big(\bar{D}_N(s,u) + \frac{q_N(s)}{q_\star^2}Q_N(u)\big) - \frac{\nu'(q_N(s))\nu''(q_N(u))}{\nu'(q_\star^2)}Q_N(u)\,,$$

$$\Psi_N^1(s,u) := \bar{D}_N(s,u)\big[\nu''(C_N(s,u))q_N(u) - \frac{q_\star^2\nu'(q_N(u))\nu''(q_N(s))}{\nu'(q_\star^2)}\big] \quad (3.8)$$
$$+ Q_N(u)[\nu'(C_N(s,u)) + \frac{q_N(s)q_N(u)}{q_\star^2}\nu''(C_N(s,u)) - q_\star^2\bar{C}_N(s,u)\frac{\nu''(q_N(s))\nu''(q_N(u))}{\nu'(q_\star^2)}]\,.$$

6 Which in the pure case is restricted to $G = mE/q_\star^2$; see Footnote 2.

The functions Υ_N, Φ_N, Φ^1_N, Ψ_N and Ψ^1_N, which arise out of conditional covariances (see (3.34), (4.20) and (4.26)), are used in approximating certain conditional expectations of $\widehat{H}_N$, V_N and $\bar{A}_N$.

For convenience we refer hereafter to all elements of $\mathcal{U}_N$ as functions on $[0, T]^2$, with the obvious modification in force for q_N, $\widehat{H}_N$, V_N and Q_N. Adopting this convention, our proof of Theorem 1.1 relies on pre-compactness and self-averaging of functions from $\mathcal{U}_N$. Specifically, in Sect. 3.1 we establish the following analog of [10, Prop. 2.3 and 2.4].

Proposition 3.1 *For any $U_N \in \mathcal{U}_N$, fixed finite T and k,*

$$\sup_{|E|,|G|\le\alpha}\ \sup_N \widetilde{\mathbb{E}}\Big[\sup_{s,t\le T} |U_N(s,t)|^k\Big] < \infty\,, \tag{3.9}$$

with the sequence of continuous functions $U_N(s,t)$ being pre-compact almost surely and in expectation, WRT *the uniform topology on* $[0,T]^2$*. Moreover, for any $U_N \in \mathcal{U}_N$, $T < \infty$ and $\rho > 0$,*

$$\sum_N \sup_{|E|,|G|\le\alpha} \widetilde{\mathbb{P}}[\sup_{s,t\le T} |U_N(s,t) - \widetilde{\mathbb{E}}\,U_N(s,t))| \ge \rho] < \infty \tag{3.10}$$

and hence by (3.9)*, also*

$$\lim_{N\to\infty}\ \sup_{|E|,|G|\le\alpha}\ \sup_{s,t\le T} \widetilde{\mathbb{E}}\Big[|U_N(s,t) - \widetilde{\mathbb{E}}\,U_N(s,t)|^2\Big] = 0\,. \tag{3.11}$$

In view of (3.9) and (3.11) we thus deduce the following, exactly as in [10, proof of Corollary 2.8].

Corollary 3.2 *Suppose $\Psi : \mathbb{R}^\ell \to \mathbb{R}$ is locally Lipschitz with $|\Psi(z)| \le M'\|z\|_k^k$ for some $M', \ell, k < \infty$, and $\mathbf{Z}_N \in \mathbb{R}^\ell$ is a random vector, where for $j = 1, \ldots, \ell$, the j-th coordinate of $\mathbf{Z}_N$ is of the form $U_N(s_j, t_j)$, for some $U_N \in \mathcal{U}_N$ and some $(s_j, t_j) \in [0, T]^2$. Then,*

$$\lim_{N\to\infty}\ \sup_{|E|,|G|\le\alpha}\ \sup_{s_j,t_j} |\widetilde{\mathbb{E}}\Psi(\mathbf{Z}_N) - \Psi(\widetilde{\mathbb{E}}\mathbf{Z}_N)| = 0\,. \tag{3.12}$$

As explained in Remark 1.4, the expectation $\widetilde{\mathbb{E}}$ amounts to taking $\mathbf{J} = \mathbf{J}_o$ of the Gaussian law $\gamma_N^{(0,0,q_\star)}$, while adding to (1.1) the drift corresponding to (1.23), provided that we add back to $(G^1, H_{\mathbf{J}})$ the relevant constant shift $(\bar{G}^1, \bar{H})$. For $\beta = 1$, $\boldsymbol{\sigma} = \mathbf{x}_\star$, this provides an alternative representation via the diffusion

$$x^i_s = x^i_0 + B^i_s - \int_0^s f'(K_N(u))x^i_u du + \int_0^s G^i(\mathbf{x}_u)du + \mathbf{1}_{\{i=1\}}\sqrt{N}q_\star \int_0^s \nu'(q_N(u))du\,, \tag{3.13}$$

starting at $\mathbf{x}_0$ of law $\mu^{q_o}_{\mathbf{x}_\star}$ independently of $\mathbf{B}$ and $\mathbf{J}$, while in studying $\mathcal{U}_N$ we re-adjust to have $(\bar{G}^1, \bar{H}) \equiv (0, 0)$ in (3.4). Adopting hereafter the latter setting, it is more convenient to consider the solution of (3.13) under the joint law $\mathbb{P}_\star$ of $\mathbf{x}_0$, $\mathbf{B}$ and the disorder $\mathbf{J}$ conditional only upon $\mathsf{CP}_\star := \big\{\forall i \ge 2 : \partial_{x^i} H_{\mathbf{J}}(\mathbf{x}_\star) = 0\big\}$ (whose covariance is given by (1.24) at $\boldsymbol{\sigma} = \mathbf{x}_\star$). Indeed, our next proposition, whose proof is deferred to Sect. 3.2, relates $\widetilde{\mathbb{P}}$ to $\mathbb{P}_\star$ and further extends the conclusions of Proposition 3.1 to $\mathbb{P}_\star$.

Proposition 3.3 *Proposition 3.1 applies for $\mathbb{P}_\star$ instead of $\widetilde{\mathbb{P}}$. Further, for Ψ and $\mathbf{Z}_N$ of Corollary 3.2,*

$$\lim_{N\to\infty}\ \sup_{|E|,|G|\le\alpha}\ \sup_{s_j,t_j\le T} |\widetilde{\mathbb{E}}\Psi(\mathbf{Z}_N) - \mathbb{E}_\star\Psi(\mathbf{Z}_N)| = 0\,. \tag{3.14}$$

Setting hereafter for the filtration $\mathcal{F}_u = \sigma(\mathbf{x}_v : v \in [0, u])$, $U_N \in \mathcal{U}_N$ *and* $\tau \in [0, T]$,

$$U_N(s, t|\tau) := \mathbb{E}_\star[U_N(s, t)|\mathcal{F}_\tau], \tag{3.15}$$

Corollary 3.2 *applies for* $\mathbb{E}_\star$, *with coordinates of* $\mathbf{Z}_N$ *taken from* $\mathcal{U}_N^\star := \mathcal{U}_N \cup \{U_N(\cdot|\tau), U_N \in \mathcal{U}_N, \tau \in [0, T]\}$.

Our next result, whose proof is deferred to Sect. 3.3, shows that the limiting dynamics of (1.17)–(1.21) admits at most one solution.

Proposition 3.4 *Let* $T < \infty$ *and* $\Delta_T = \{s, t \in (\mathbb{R}^+)^2 : 0 \le t \le s \le T\}$. *There exists at most one solution* $(R, C, q, K, H) \in \mathcal{C}_b^1(\Delta_T)^2 \times \mathcal{C}_b^1([0, T])^3$ *to* (1.17)–(1.21) *at* $\beta = 1$ *with* $C(s, t) = C(t, s)$ *and boundary conditions*

$$R(s, s) \equiv 1 \qquad \forall s \ge 0 \tag{3.16}$$

$$C(s, s) = K(s) \qquad \forall s \ge 0 \tag{3.17}$$

$$K(0) = 1, \qquad q(0) = q_o \quad \text{known.} \tag{3.18}$$

Our next proposition, whose proof is deferred to Sect. 4.1, plays here the role of [10, Prop. 1.3].

Proposition 3.5 *Let* $U_N^a := \mathbb{E}_\star U_N$. *Fixing* $T < \infty$, *any limit point of the sequence* $\mathcal{U}_N^a := \{U_N^a, U_N \in \mathcal{U}_N\}$ *with respect to uniform convergence on* $[0, T]^2$, *satisfies the integral equations in* $\mathcal{C}_b([0, T]^2)$,

$$C(s, t) = \bar{C}(s, t) + \frac{q(s)q(t)}{q_\star^2}, \chi(s, t) = \bar{\chi}(s, t), \tag{3.19}$$

$$Q(s) = -f'(K(s))q(s) + q_\star^2 \nu'(q(s)) + V(s), q(s) = q(0) + \int_0^s Q(u)du, \tag{3.20}$$

$$\bar{D}(s, t) = -f'(K(t))\bar{C}(t, s) + \bar{A}(t, s), \bar{E}(s, t) = -f'(K(s))\bar{\chi}(s, t) + \bar{F}(s, t), \tag{3.21}$$

$$\Upsilon(s, t) = \nu(C(s, t)) - \bar{C}(s, t)\frac{\nu'(q(s))\nu'(q(t))}{\nu'(q_\star^2)}, \tag{3.22}$$

$$\Phi(s, t) = \nu'(C(s, t)) - \frac{\nu'(q(s))\nu'(q(t))}{\nu'(q_\star^2)} \tag{3.23}$$

$$\Phi^1(s, u) = q(u)\nu'(C(s, u)) - \bar{C}(s, u)\frac{q_\star^2\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}, \tag{3.24}$$

$$\Psi(s, u) = \nu''(C(s, u))\big(\bar{D}(s, u) + \frac{q(s)}{q_\star^2}Q(u)\big) - \frac{\nu'(q(s))\nu''(q(u))}{\nu'(q_\star^2)}Q(u), \tag{3.25}$$

$$\begin{aligned}\Psi^1(s, u) = \bar{D}(s, u)&\left[\nu''(C(s, u))q(u) - \frac{q_\star^2\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\right] \\ &+ Q(u)\left[\nu'(C(s, u)) + \frac{q(s)q(u)}{q_\star^2}\nu''(C(s, u)) - \bar{C}(s, u)\frac{q_\star^2\nu''(q(s))\nu''(q(u))}{\nu'(q_\star^2)}\right],\end{aligned} \tag{3.26}$$

$$\bar{C}(s, t) = \bar{C}(s, 0) + \bar{\chi}(s, t) + \int_0^t \bar{D}(s, u)du, \bar{\chi}(s, t) = s \wedge t + \int_0^s \bar{E}(u, t)du, \tag{3.27}$$

$$V(s) = \Phi^1(s, s) - \Phi^1(s, 0) - \int_0^s \Psi^1(s, u)du, \tag{3.28}$$

$$\bar{A}(t, s) = \bar{C}(s, \tau)\Phi(t, \tau) - \bar{C}(s, 0)\Phi(t, 0)$$

$$-\int_0^{\tau}\Big\{\bar{D}(s,u)\Phi(t,u)+\bar{C}(s,u)\Psi(t,u)\Big\}du\,, \tag{3.29}$$

$$\bar{F}(s,t)=\ \bar{\chi}(s,t)\Phi(s,s)-\int_0^{t\wedge s}\Phi(s,u)du$$
$$-\int_0^{s}\bar{E}(u,t)\Phi(s,u)du-\int_0^{s}\bar{\chi}(u,t)\Psi(s,u)du\,, \tag{3.30}$$

$$\widehat{H}(s)=\ \Upsilon(s,s)-\Upsilon(s,0)-\int_0^{s}\Big\{\bar{D}(s,u)\Phi(u,s)+\frac{Q(u)}{q_\star^2}\Phi^1(u,s)\Big\}du\,, \tag{3.31}$$

where $\tau = t\vee s$, *subject to the symmetry* $C(s,t) = C(t,s)$ *and boundary conditions* $q(0) = q_o$, $K(0) = 1$, $K(s) = C(s,s)$, $\bar{E}(s,0) = 0$ *for all* s, *and* $\bar{E}(s,t) = \bar{E}(s,s)$ *for all* $t \geq s$.

Our final ingredient for Theorem 1.1 is the following link between (3.19)–(3.31) and (1.17)–(1.21), whose proof we defer to Sect. 4.2.

Proposition 3.6 *Fixing* $T < \infty$, *if* $(C,\chi,q,\widehat{H}) \in \mathcal{C}_b([0,T]^2;\mathbb{R}^4)$ *satisfies* (3.19)–(3.31), *with* $\mathsf{v}_\star(\cdot)$ *instead of* $\mathsf{v}(\cdot)$, *subject to the symmetry and boundary conditions of Proposition 3.5, then* $\chi(s,t) = \int_0^t R(s,u)du$ *where* $R(s,t) = 0$ *for* $t > s$, $R(s,s) = 1$ *and on* Δ_T *the bounded and absolutely continuous functions* $(C,R,q,\widehat{H})$ *satisfy the integro-differential equations* (1.17)–(1.21) *(at* $\beta = 1$*).*

Proof of Theorem 1.1 Setting WLOG $\beta = 1$ and $\boldsymbol{\sigma} = \mathbf{x}_\star$, recall from Proposition 3.3 that all conclusions of Proposition 3.1 apply for $\mathbb{P}_\star$. In particular, we thus have pre-compactness of $(U_N^a, U_N \in \mathcal{U}_N) : [0,T]^2 \to \mathbb{R}^{17}$ in the topology of uniform convergence on $[0,T]^2$, implying the existence of limit points of this sequence as $N \to \infty$. By Proposition 3.5 any such limit point must be a solution of the integral equations (3.19)–(3.31) with the stated symmetry and boundary conditions. Further, by Proposition 3.6, for $(E,G) = (E_\star, G_\star)$ any such solution results with $(C,R,q,\widehat{H})$ that satisfy the integro-differential equations (1.17)–(1.21) (at $\beta = 1$). In view of Proposition 3.4 the latter system admits at most one solution per given boundary conditions. Hence, we conclude that the sequence $(\chi_N^a, C_N^a, q_N^a, \widehat{H}_N^a)$ converges as $N \to \infty$, uniformly in $[0,T]^2$ to the unique solution determined by (1.17)–(1.21) subject to the appropriate boundary conditions. Thanks to Proposition 3.3, the same applies to $\widetilde{\mathbb{E}}[(\chi_N, C_N, q_N, \widehat{H}_N)]$. Further, by (3.10) of Proposition 3.1, almost surely $|(\chi_N, C_N, q_N, \widehat{H}_N) - \widetilde{\mathbb{E}}(\chi_N, C_N, q_N, \widehat{H}_N)| \to \mathbf{0}$ as $N \to \infty$, uniformly on $[0,T]^2$. In addition, $H_N(s) = \widehat{H}_N(s) + \mathsf{v}_\star(q_N(s))$ (see (1.23) and the LHS of (1.14), (3.4)). Thus, the function (χ, C, q, H) determined from (1.17)–(1.21) is also the unique almost sure uniform (in s,t) limit of (χ_N, C_N, q_N, H_N), as stated in Theorem 1.1. The L_p convergence follows by the uniform moments bounds of Proposition 3.1, thereby completing the proof of the theorem. □

3.1 Proof of Proposition 3.1

We start by computing the covariances conditional on the event $\mathsf{CP}_\star$, which replace here the unconditional covariances of [10, Lemma 3.2].

Lemma 3.7 *For* v_p, $p \geq 2$, *of* (1.22) *one has the following conditional expectations*

$$\mathbb{E}[J^{(p)}_{1\dots1}|CP(E,G,\mathbf{x}_\star)] = -b_p N^{1-\frac{p}{2}} q_\star^p\,\langle \mathsf{v}_p, (E,G)\rangle\,. \tag{3.32}$$

Letting $\mathbb{E}_{\mathbf{J}}$ *denote the expectation with respect to the Gaussian law* $\mathbb{P}_{\mathbf{J}}$ *of the disorder* $\mathbf{J}$*, it follows that for* $\mathsf{v}(\cdot)$ *of* (1.22)*, any* $\mathbf{x} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)$ *which is independent of* $\mathbf{J}$ *and all* $s, t \in [0, T]$*,* $i, j \in \{1, \dots, N\}$*,*

$$\begin{aligned} \bar{H}(\mathbf{x}_t) &:= \mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_t) \mid CP(E, G, \mathbf{x}_\star)\} = -N\mathsf{v}(q_N(t)), \\ \bar{G}^i(\mathbf{x}_t) &:= \mathbb{E}_{\mathbf{J}}\left\{G^i(\mathbf{x}_t) \mid CP(E, G, \mathbf{x}_\star)\right\} = \mathbf{1}_{\{i=1\}}\sqrt{N}q_\star \mathsf{v}'(q_N(t)). \end{aligned} \tag{3.33}$$

Further, for $CP_\star = \left\{\forall i \geq 2 : \partial_{x^i} H_{\mathbf{J}}(\mathbf{x}_\star) = 0\right\}$*, we have that* $\mathbb{E}_{\mathbf{J}}\left\{G^i(\mathbf{x}_t) \mid CP_\star\right\} = 0$ *for any* (t, i)*, while*

$$\begin{aligned} k^{ij}_{ts}(\mathbf{x}) &:= \mathbb{E}_{\mathbf{J}}\left\{G^i(\mathbf{x}_t)G^j(\mathbf{x}_s) \mid CP_\star\right\} = \partial_{x^i_t}\partial_{x^j_s}\{\widetilde{k}(\mathbf{x}_t, \mathbf{x}_s)\}, \\ \widetilde{k}(\mathbf{x}_s, \mathbf{x}_t) &:= \mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_s)H_{\mathbf{J}}(\mathbf{x}_t) \mid CP_\star\} = N\Upsilon_N(s, t), \end{aligned} \tag{3.34}$$

for $\Upsilon_N(s, t)$ *of* (3.7).

Proof Fix two points $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbb{R}^N$. Recall that [1, Eq. (5.5.4)]

$$\begin{aligned} \mathbb{E}\left\{\partial_{\bar{x}^i} H^N_{\mathbf{J}}(\bar{\mathbf{x}})H^N_{\mathbf{J}}(\bar{\mathbf{y}})\right\} &= \partial_{\bar{x}^i}\mathrm{Cov}\left(H^N_{\mathbf{J}}(\bar{\mathbf{x}})H^N_{\mathbf{J}}(\bar{\mathbf{y}})\right) \\ &= \bar{y}^i\nu'(N^{-1}\langle\bar{\mathbf{x}}, \bar{\mathbf{y}}\rangle), \\ \mathbb{E}\left\{\partial_{\bar{x}^i} H^N_{\mathbf{J}}(\bar{\mathbf{x}})\partial_{\bar{y}^j} H^N_{\mathbf{J}}(\bar{\mathbf{y}})\right\} &= \partial_{\bar{x}^i}\partial_{\bar{y}^j}\mathrm{Cov}\left(H^N_{\mathbf{J}}(\bar{\mathbf{x}})H^N_{\mathbf{J}}(\bar{\mathbf{y}})\right) \\ &= \frac{\bar{x}^j\bar{y}^i}{N}\nu''(N^{-1}\langle\bar{\mathbf{x}}, \bar{\mathbf{y}}\rangle) + \mathbf{1}_{\{i=j\}}\nu'(N^{-1}\langle\bar{\mathbf{x}}, \bar{\mathbf{y}}\rangle). \end{aligned} \tag{3.35}$$

In particular, $\mathbf{w} = (q_\star H_{\mathbf{J}}(\mathbf{x}_\star), \sqrt{N}\partial_{x^1} H_{\mathbf{J}}(\mathbf{x}_\star))$ and $\mathbf{z} = \sqrt{N}(\partial_{x^i} H_{\mathbf{J}}(\mathbf{x}_\star))_{i>1}$ are independent. Therefore, from the well-known formula for conditional Gaussian distributions [1, pages 10-11],

$$\bar{H}(\mathbf{x}_t) = \left\langle \mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_t)\mathbf{w}\}(\mathbb{E}_{\mathbf{J}}\{\mathbf{w}^T\mathbf{w}\})^{-1}, (-Nq_\star E, -Nq_\star G)\right\rangle$$

which by substitution yields the top line of (3.33). Recall that $\bar{G} = -\nabla_{\mathbf{x}}\bar{H}$ to complete the derivation of (3.33). The formula (3.32) for the conditional expectations of $J^{(p)}_{1\cdots 1}$ is similarly verified from

$$\begin{aligned} \mathbb{E}\left\{J^{(p)}_{1\cdots 1}H^N_{\mathbf{J}}(\mathbf{x}_\star)\right\} &= b_p q^p_\star N^{\frac{p}{2}}\mathbb{E}\left\{(J^{(p)}_{1\cdots 1})^2\right\} = b_p q^p_\star N^{1-\frac{p}{2}} \\ \mathbb{E}\left\{J^{(p)}_{1\cdots 1}(\partial_{x^1} H^N_{\mathbf{J}}(\mathbf{x}_\star))\right\} &= b_p p q^{p-1}_\star N^{\frac{p-1}{2}}\mathbb{E}\left\{(J^{(p)}_{1\cdots 1})^2\right\} = b_p p q^{p-1}_\star N^{\frac{1-p}{2}}. \end{aligned}$$

Next, recall that any centered Gaussian field, conditional on a linear map being zero, remains centered. In particular, $\mathbb{E}_{\mathbf{J}}\left\{G^i(\mathbf{x}_t) \mid \mathrm{CP}_\star\right\} = 0$ for any choice of $\mathbf{x}_\star$ and (t, i). Further, with $z_k = \sqrt{N}\partial_{x^k} H_{\mathbf{J}}(\mathbf{x}_\star)$ independent for different k, the formula for the conditional covariance of $H_{\mathbf{J}}(\cdot)$, simplifies to

$$\widetilde{k}(\mathbf{x}_t, \mathbf{x}_s) = \mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_t)H_{\mathbf{J}}(\mathbf{x}_s)\} - \sum_{k=2}^N \mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_t)z_k\}\left\{\mathbb{E}_{\mathbf{J}}z^2_k\right\}^{-1}\mathbb{E}_{\mathbf{J}}\{H_{\mathbf{J}}(\mathbf{x}_s)z_k\},$$

from which (3.34) follows by substitution (and comparison with the definition of Υ_N in (3.7)). □

Preparing to adapt [10, Section 2], recall $K_N(t) = C_N(t,t)$ and set hereafter $B_N(t) := \frac{1}{N}\sum_{i=1}^N |B_t^i|^2$ and $G_N(t) := \frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_t)|^2$. Using throughout the corresponding sup-norms $\|K_N\|_\infty := \sup\{K_N(t) : 0 \le t \le T\}$, $\|B_N\|_\infty$ and $\|G_N\|_\infty$ as well as the N-dependent disorder-norms

$$\|\mathbf{J}\|_\infty^N := \max_{1\le p\le m} \sup_{\|\mathbf{u}^i\|\le 1, 1\le i\le p} \Big|\sqrt{N}^{-1} \sum_{1\le i_k\le N, 1\le k\le p} N^{\frac{p-1}{2}} J_{i_1\cdots i_p} u_{i_1}^1 \cdots u_{i_p}^p\Big| \tag{3.36}$$

of [10, (2.1)], we first mimic [10, Proposition 2.1] about the strong solution $\mathbf{x}_t$ of (1.1).

Proposition 3.8 *Assume that f' is locally Lipschitz, satisfying (1.6). Then, for any $N \in \mathbb{N}$, $\mathbf{x}_0$, $\mathbf{J}$ there exists a unique strong solution of (1.1) for a.e. Brownian path $\mathbf{B}$. Denoting by $\mathbb{P}^N_{\mathbf{J},\mathbf{x}_0}$ the (unique) law of $\{\mathbf{B}_t, \mathbf{x}_t\}$ as $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^{2N})$-valued variable, we have that for some c, κ finite, all N, $z > 0$, $\mathbf{J}$ and $\mathbf{x}_0$,*

$$\mathbb{P}^N_{\mathbf{J},\mathbf{x}_0}\Big(\sup_{t\in\mathbb{R}^+} K_N(t) \ge K_N(0) + \kappa(1 + \|\mathbf{J}\|_\infty^N)^c + z\Big) \le e^{-zN}. \tag{3.37}$$

In particular, for some $D_o(k,M)$ finite, any k, M and all N,

$$\sup_{\{\mathbf{J},\mathbf{x}_0: K_N(0)+\|\mathbf{J}\|_\infty^N \le M\}} \Big\{\mathbb{E}^N_{\mathbf{J},\mathbf{x}_0}\big[\sup_{t\in\mathbb{R}^+} K_N(t)^k\big]\Big\} \le D_o(k,M). \tag{3.38}$$

Further, for any finite positive $q_\star$, k and α

$$\sup_{|E|,|G|\le\alpha} \sup_N \widetilde{\mathbb{E}}\big[\,(||\mathbf{J}||_\infty^N)^k\,\big] < \infty \tag{3.39}$$

and there exist finite $\widetilde{\kappa} \ge 1$ such that for any $t \ge 0$,

$$\sup_{|E|,|G|\le\alpha} \sup_N \widetilde{\mathbb{P}}\big[\|\mathbf{J}\|_\infty^N > \widetilde{\kappa} + t\big] \le \widetilde{\kappa} e^{-Nt^2/\widetilde{\kappa}}. \tag{3.40}$$

Consequently, for any $|q_o| \le q_\star$ positive, finite k and α,

$$\sup_{|E|,|G|\le\alpha} \sup_N \widetilde{\mathbb{E}}\big[\sup_{t\in\mathbb{R}^+} K_N(t)^k\big] < \infty \tag{3.41}$$

and for any finite L there exist $z = z(L)$ finite such that

$$\sup_{|E|,|G|\le\alpha} \sup_N \widetilde{\mathbb{P}}\big[\sup_{t\in\mathbb{R}^+} K_N(t) \ge z\,\big] \le 2\widetilde{\kappa} e^{-LN}. \tag{3.42}$$

Proof From [10, Proposition 2.1] we have the existence of a unique strong solution as well as the bound (3.37) (while stated in [10] for a.e. $\mathbf{J}$, $\mathbf{x}_0$, examining their proof we see that it holds for all $\mathbf{J}$ and $\mathbf{x}_0$). Clearly, (3.38) and (3.39) are immediate consequences of (3.37) and (3.40), respectively. Further, taking $\mathbf{x}_0 \in \mathbb{S}_N$ amounts to $K_N(0) = 1$, yielding (3.41) and (3.42) upon combining (3.37) with (3.39) and (3.40), respectively. Turning to the only remaining task, of proving (3.40), recall [10, (B.7)] that for some $\widetilde{\kappa}$ and all $t \ge 0$,

$$\sup_N \mathbb{P}\big[\|\mathbf{J}\|_\infty^N > \widetilde{\kappa} + t\big] \le \widetilde{\kappa} e^{-Nt^2/\widetilde{\kappa}}. \tag{3.43}$$

Since $\|\mathbf{J}\|_\infty^N$ is a symmetric, convex function of $\mathbf{J}$, by Anderson's inequality [2, Corollary 3], the bound (3.43) holds when $\mathbf{J}$ is replaced by the centered Gaussian vector $\mathbf{J}_o$ having the law

$\gamma_N^{(0,0,q_\star)}$. Further, conditionally on $\mathsf{CP}(E, G, \mathbf{x}_\star)$, we have that $\mathbf{J} = \mathbf{J}_o + \bar{\mathbf{J}}_{E,G}$ for the non-random vector $\bar{\mathbf{J}}_{E,G} := \mathbb{E}[\mathbf{J} \mid \mathsf{CP}(E, G, \mathbf{x}_\star)]$. The only non-zero entries of $\bar{\mathbf{J}}_{E,G}$ correspond to $\{J^{(p)}_{1\dots1}\}$ and are given by (3.32). Consequently,

$$\|\bar{\mathbf{J}}_{E,G}\|_\infty^N = \max_{2\le p\le m} \{|b_p q_\star^p \langle \mathsf{v}_p, (E, G)\rangle|\}, \tag{3.44}$$

is bounded, uniformly over $|E|, |G| \le \alpha$ by some $\widehat{\kappa}(\alpha, q_\star)$ finite. In conjunction with the triangle inequality for $\|\cdot\|_\infty^N$, this yields (3.40) (upon adding $\widehat{\kappa}$ to $\widetilde{\kappa}$). □

The same reasoning as in proving [10, Proposition 2.3], but with (3.39)–(3.42) of Proposition 3.8 replacing [10, Eqn. (2.12), (B.7), (2.13), (2.3)], respectively, yields for $U_N \in \mathcal{U}_N^\dagger$ both (3.9) and the stated pre-compactness. Along the way we also find that for some $M = M(L, T, \alpha) < \infty$ the subsets

$$\mathcal{L}_{N,M} := \{(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) \in \mathcal{E}_N \ : \ \|\mathbf{J}\|_\infty^N + \|B_N\|_\infty + \|K_N\|_\infty + \|G_N\|_\infty \le M\} \tag{3.45}$$

of $\mathcal{E}_N$ are such that for any finite L, T, α and all N,

$$\sup_{|E|,|G|\le\alpha} \widetilde{\mathbb{P}}(\mathcal{L}_{N,M}^c) \le M e^{-LN}. \tag{3.46}$$

Next, similarly to [10, (2.10)],

$$\frac{1}{\sqrt{N}} |H_{\mathbf{J}}(\mathbf{x}) - H_{\mathbf{J}}(\widetilde{\mathbf{x}})| \le c\, \|\mathbf{J}\|_\infty^N \big(1 + (N^{-1}\|\mathbf{x}\|^2)^r\big)\big(1 + (N^{-1}\|\widetilde{\mathbf{x}}\|^2)^r\big)\|\mathbf{x} - \widetilde{\mathbf{x}}\|, \tag{3.47}$$

for $r = (m-1)/2$, $c = m\sqrt{\nu'(1)}$ and any $\mathbf{x}, \widetilde{\mathbf{x}} \in \mathbb{R}^N$. In particular,

$$|H_N(t) - H_N(t')| \le c\, \|\mathbf{J}\|_\infty^N (1 + K_N(t)^r)(1 + K_N(t')^r) \frac{\|\mathbf{x}_t - \mathbf{x}_{t'}\|}{\sqrt{N}}. \tag{3.48}$$

The uniform moment bound (3.9) then extends to all $\mathcal{U}_N$ since $q_N^2(s) \le q_\star^2 C_N(s,s)$ and $V_N^2(s) \le q_\star^2 G_N(s) + q_\star^4(\mathsf{v}'(q_N(s))^2$, with the locally Lipschitz $f'(\cdot)$, $\nu''(\cdot)$ and $\mathsf{v}'(\cdot)$ having at most a polynomial growth. In addition, from [10, (2.18)] adapted to our setting of $\widetilde{\mathbb{P}}$, we have for any $\epsilon > 0$, some $L'(\delta, \epsilon) \to \infty$ as $\delta \to 0$, and all N,

$$\sup_{|E|,|G|\le\alpha} \widetilde{\mathbb{P}}\Big[\sup_{|t-t'|<\delta} \{|q_N(t) - q_N(t')|\} > q_\star\sqrt{\epsilon}\,\Big] \le e^{-L'(\delta,\epsilon)N}$$

$$\sup_{|E|,|G|\le\alpha} \sup_N \widetilde{\mathbb{E}}\Big[\sup_{|t-t'|<\delta} |q_N(t) - q_N(t')|^4\Big] \le L'(\delta,\epsilon)^{-1}.$$

The same holds also for $\widehat{H}_N(\cdot)$ (see (3.48)), and for $V_N(\cdot)$ (c.f. [10, display preceding (2.18)]). Such bounds yield the equi-continuity of $q_N(\cdot)$, $V_N(\cdot)$ and $\widehat{H}_N(\cdot)$ (a.s. and in expectation), from which we deduce the pre-compactness, first of $q_N, V_N, \widehat{H}_N$, then of $Q_N, \bar{D}_N, \bar{E}_N$ and finally of $\Upsilon_N, \Phi_N, \Phi_N^1, \Psi_N, \Psi_N^1$ (by the uniform moments control (3.9) and the Arzela-Ascoli theorem). In particular, this way we have further established that for some $\widetilde{L}(\delta, \epsilon) \to \infty$ as $\delta \to 0$, any $\epsilon > 0$ and $U_N \in \mathcal{U}_N$

$$\begin{aligned} &\sup_{|E|,|G|\le\alpha} \widetilde{\mathbb{P}}\left(\sup_{|s-s'|+|t-t'|<\delta} |U_N(s,t) - U_N(s',t')| > \epsilon\right) \le e^{-\widetilde{L}(\delta,\epsilon)N}, \\ &\sup_{|E|,|G|\le\alpha} \sup_N \sup_{|s-s'|+|t-t'|<\delta} |\widetilde{\mathbb{E}}\, U_N(s,t) - \widetilde{\mathbb{E}}\, U_N(s',t')| \le \widetilde{L}(\delta,\epsilon)^{-1}. \end{aligned} \tag{3.49}$$

Turning to the self-averaging property (3.10), similarly to [10, Proposition 2.4] our proof relies on the following pointwise Lipschitz estimate on $\mathcal{L}_{N,M}$ of (3.45).

Lemma 3.9 *Let $\mathbf{x}, \widetilde{\mathbf{x}}$ be the two strong solutions of (1.1) constructed from $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ and $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})$, respectively. If $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ and $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})$ are both in $\mathcal{L}_{N,M}$, then we have the Lipschitz estimate for each $U_N \in \mathcal{U}_N$,*

$$\sup_{s,t \le T} |U_N(s,t) - \widetilde{U}_N(s,t)| \le \frac{D(M,T)}{\sqrt{N}} \|(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})\|\,, \tag{3.50}$$

where the constant $D(M,T)$ depends only on M and T and not on N.

Further, for $e_N(s) := N^{-1}\|\mathbf{x}_s - \widetilde{\mathbf{x}}_s\|_2^2$ any N and T, if $\widetilde{\mathbf{B}} = \mathbf{B}$ and $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}) \to (\mathbf{x}_0, \mathbf{J})$, then

$$\mathbb{E}\big[\, 1 \wedge \|e_N\|_\infty \,|\, \widetilde{\mathbf{J}}, \mathbf{J}, \widetilde{\mathbf{x}}_0, \mathbf{x}_0 \big] \to 0\,. \tag{3.51}$$

Proof For $U_N \in \mathcal{U}_N^\dagger$ the bound (3.50) is precisely the statement of [10, Lemma 2.7], while for $U_N = q_N$ it follows upon taking the square-root of the bound

$$\|e_N\|_\infty \le \frac{D_1(M,T)}{N} \|(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})\|^2 \tag{3.52}$$

from [10, Lemma 2.6]. Further, while proving [10, Lemma 2.7] it is shown that on $\mathcal{L}_{N,M}$

$$\|G(\mathbf{x}_s) - \widetilde{G}(\widetilde{\mathbf{x}}_s)\|_2 \le D_2(M,T)\|(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})\|$$

(where $\widetilde{G}(\cdot) := -\nabla H_{\widetilde{\mathbf{J}}}(\cdot)$, see [10, Page 636]). Utilizing (3.47) instead of [10, (2.10)] yields the same bound for $\frac{1}{\sqrt{N}}|H_{\mathbf{J}}(\mathbf{x}_s) - H_{\widetilde{\mathbf{J}}}(\widetilde{\mathbf{x}}_s)|$. Recall (3.45) that $\|q_N\|_\infty \le q_\star \|K_N\|_\infty^{1/2} \le q_\star\sqrt{M}$ on $\mathcal{L}_{N,M}$, which thus in view of (3.33) for the locally Lipschitz $\mathsf{v}'(\cdot)$, thus results with (3.50) holding for $U_N = V_N$ and $U_N = \widehat{H}_N$. Similarly, having $f'(\cdot)$, $\nu''(\cdot)$ locally Lipschitz and $\|K_N\|_\infty \le M$ on $\mathcal{L}_{N,M}$, extends the validity of (3.50) first to $U_N \in \{Q_N, \bar{D}_N, \bar{E}_N\}$, then also to $U_N \in \{\Upsilon_N, \Phi_N, \Phi_N^1, \Psi_N, \Psi_N^1\}$.

In case $\widetilde{\mathbf{B}} = \mathbf{B}$ we see from [10, Proof of Lemma 2.6] that (3.52) holds when $\|\mathbf{J}\|_\infty^N + \|K_N\|_\infty + \|\widetilde{K}_N\|_\infty \le M$. With $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}) \to (\mathbf{x}_0, \mathbf{J})$, the RHS of (3.52) decays to zero and $\widetilde{K}_N(0) + \|\widetilde{\mathbf{J}}\|_\infty^N$ is uniformly bounded. Such uniform boundedness implies in view of (3.38) that as $M \to \infty$,

$$\mathbb{P}(\|\mathbf{J}\|_\infty^N + \|K_N\|_\infty + \|\widetilde{K}_N\|_\infty > M \,|\, \widetilde{\mathbf{J}}, \mathbf{J}, \widetilde{\mathbf{x}}_0, \mathbf{x}_0) \to 0\,,$$

uniformly in $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}})$, from which we deduce by bounded convergence that (3.51) holds. □

We next verify that $\widetilde{\mathbb{P}}$ satisfies the Lipschitz concentration of measure, as in [10, Hypothesis 1.1], uniformly over $|E|, |G| \le \alpha$.

Proposition 3.10 *For some $C > 0$, any $(E, G, q_\star)$, function $V : \mathcal{E}_N \mapsto \mathbb{R}$ of Lipschitz constant K and all $\rho > 0$,*

$$\widetilde{\mathbb{P}}\left\{|V - \widetilde{\mathbb{E}}V| \ge \rho\right\} \le C^{-1}\exp\left(-C\rho^2/K^2\right)\,. \tag{3.53}$$

Proof Assume first that $\widetilde{\mathbb{E}}V = 0$. Recall that $\widetilde{\mathbb{P}} = \mu_{\mathbf{x}_\star}^{q_o} \otimes \gamma_N^{(E,G,q_\star)} \otimes P_N$. Denoting a generic point in $\mathcal{E}_N$ by $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$, let $\mathbb{E}_{\mathbf{x}_0}$ denote the expectation WRT $\mu_{\mathbf{x}_\star}^{q_o}$ and the variable $\mathbf{x}_0$ only, and for fixed $\mathbf{x}_0$, let $\widetilde{\mathbb{P}}_{\mathbf{J},\mathbf{B}} = \gamma_N^{(E,G,q_\star)} \otimes P_N$. By conditioning on $\mathbf{x}_0$,

$$\widetilde{\mathbb{P}}(V > \rho) \le \mathbb{E}_{\mathbf{x}_0}\widetilde{\mathbb{P}}_{\mathbf{J},\mathbf{B}}(V - \widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}}V > \rho/2) + \mathbb{P}_{\mathbf{x}_0}(\widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}}V > \rho/2). \tag{3.54}$$

For any fixed $\mathbf{x}_0$, $(\mathbf{J}, \mathbf{B}) \mapsto V(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ has Lipschitz constant K WRT the norm

$$\|(\mathbf{J}, \mathbf{B})\|^2 = \sum_{p=2}^{m} \sum_{1 \le i_1 \le \ldots \le i_p \le N} (N^{\frac{p-1}{2}} J_{i_1 \cdots i_p})^2 + \sup_{0 \le t \le T} \sum_{i=1}^{N} (B_t^i)^2.$$

Next, set $\mathbb{P}_{\mathbf{J},\mathbf{B}} := \gamma_N \otimes P_N$ for the unconditional Gaussian law γ_N of $\mathbf{J}$, and $W(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) := (\mathbf{x}_0, \bar{W}(\mathbf{J}), \mathbf{B})$, for the orthogonal projection $\bar{W}$ to the affine subspace of $\mathbb{R}^{d(N,m)}$ defined by $\mathsf{CP}(E, G, \mathbf{x}_\star)$. The composition $V \circ W$ necessarily has at most the Lipschitz constant K. Hence, for some $C > 0$, any N, $V(\cdot)$, $\rho > 0$ and all $\mathbf{x}_0$, by the concentration of measure of the Gaussian measure (see, e.g. [3]),

$$\widetilde{\mathbb{P}}_{\mathbf{J},\mathbf{B}}(V - \widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}} V > \rho/2) = \mathbb{P}_{\mathbf{J},\mathbf{B}}(V \circ W - \mathbb{E}_{\mathbf{J},\mathbf{B}} V \circ W > \rho/2) \le C^{-1} \exp(-C\rho^2/K^2)\,.$$

Further, by Jensen's inequality, $\mathbf{x}_0 \mapsto \widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}} V$ has Lipschitz constant K WRT the Euclidean norm on $\mathbb{R}^N$. Moreover, $\mathbb{E}_{\mathbf{x}_0} \widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}} V = \widetilde{\mathbb{E}} V = 0$, so by the concentration of measure of the uniform measure on the sphere [16, Theorem 1.7.9], for some $C > 0$ and any N, $V(\cdot)$, $\rho > 0$,

$$\mathbb{P}_{\mathbf{x}_0}(\widetilde{\mathbb{E}}_{\mathbf{J},\mathbf{B}} V > \rho/2) < C^{-1} \exp(-C\rho^2/K^2)\,.$$

Combining the above we deduce from (3.54) that for some $C > 0$ any K-Lipschitz V and $\rho > 0$,

$$\widetilde{\mathbb{P}}(V > \rho) \le C^{-1} \exp(-C\rho^2/K^2)\,.$$

Considering this bound for $\pm(V - \widetilde{\mathbb{E}} V)$ yields (3.53). □

Equipped with Lemma 3.9 and Proposition 3.10 we establish (3.10) via the same reasoning as in [10, proof of Proposition 2.4]. Specifically, fixing $(s, t) \in [0, T]^2$, we use [10, Lemma 2.5] to extend (thanks to (3.46)), the tail control of Proposition 3.10 to $V = U_N(s, t)$ for U_N satisfying only (3.9) and (3.50). With constants C, K, $M(L)$, $D = D(M(L), T)$ in [10, (2.21)] which are independent of s, t, ρ, N (and uniform over $|E|, |G| \le \alpha$), we get by the union bound that (3.10) holds whenever the supremum is restricted to s, t in some (arbitrary) finite subset $\mathcal{A}$ of $[0, T]^2$. The preceding quantitative equi-continuity control of (3.49), further allow for strengthening to the full summability result (3.10) by considering a finite δ-net $\mathcal{A}$ of $[0, T]^2$ (say with $\delta > 0$ small, so $\widetilde{L}(2\delta, \rho/3) > 3/\rho$).

3.2 Proof of Proposition 3.3

Under both $\widetilde{\mathbb{P}}$ and $\mathbb{P}_\star$ the vector $\mathbf{J}$ has the Gaussian law $\mathbb{P}_{\mathbf{J}}$ of independent coordinates, conditioned on $\mathsf{CP}_\star$. Indeed, the only difference between $\widetilde{\mathbb{P}}$ and $\mathbb{P}_\star$ is that $\widetilde{\mathbb{P}}$ imposes on $\mathbf{J}$ an *additional* conditioning via $\mathsf{CP}_1 := \{H_{\mathbf{J}}(\mathbf{x}_\star) = \partial_{x^1} H_{\mathbf{J}}(\mathbf{x}_\star) = 0\}$. Having a conditional law for $\mathbf{J}$ enters twice throughout the whole derivation of Proposition 3.1 (via Propositions 3.8 and 3.10): first in upgrading (3.43) from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$ via Andreson's inequality, then in proving Proposition 3.10 by representing the conditional disorder as $\bar{W}(\mathbf{J})$ (for some orthogonal projection $\bar{W}$). Both arguments are applicable also for $\mathbb{P}_\star$ (namely, without conditioning on CP_1), hence so are all the conclusions of Proposition 3.1 (and of Proposition 3.8).

Turning to (3.14), we set $\widetilde{J}_p := N^{\frac{p-1}{2}} J^{(p)}_{\{1\cdots 1\}}$, noting that $\mathsf{CP}_\star$ is independent of the standard Gaussian vector $\underline{\widetilde{J}} := (\widetilde{J}_p, 2 \le p \le m)$, whereas

$$\mathsf{CP}_1 = \left\{ \underline{\widetilde{J}} : \sum_{p=2}^{m} b_p \widetilde{J}_p q_\star^p = \sum_{p=2}^{m} b_p p \widetilde{J}_p q_\star^{p-1} = 0 \right\}. \tag{3.55}$$

Denoting by $\widetilde{W}$ the orthogonal projection sending $\underline{\widetilde{J}}$ to the linear subspace determined by (3.55), leaving the remainder of $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ unchanged, we thus have that $\widetilde{\mathbb{E}} V = \mathbb{E}_\star V \circ \widetilde{W}$ for any $V : \mathcal{E}_N \mapsto \mathbb{R}$. Further, with

$$\|\widetilde{W}(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\mathbf{x}_0, \mathbf{J}, \mathbf{B})\| \le \|\underline{\widetilde{J}}\| \le \frac{\sqrt{m}}{\sqrt{N}} \|\mathbf{J}\|_\infty^N ,$$

we deduce from (3.50) that when $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ and $\widetilde{W}(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ are both in $\mathcal{L}_{N,M}$

$$\sup_{E,G} \sup_{s_j,t_j \le T} \|\mathbf{Z}_N \circ \widetilde{W} - \mathbf{Z}_N\|_2 \le \frac{D'}{N} , \tag{3.56}$$

where $D' := \sqrt{\ell m} D(M, T) M$. With $|\Psi(z)| \le M' \|z\|_k^k$ and c_r denoting the finite Lipschitz constant of $\Psi(\cdot)$ (with respect to $\| \cdot \|_2$), on the compact set $\Gamma_r := \{z : \|z\|_k \le r\}$, we thus have that for any $E, G, M, r < \infty$ and $s_j, t_j \le T$,

$$\begin{aligned} |\widetilde{\mathbb{E}}\Psi(\mathbf{Z}_N) - \mathbb{E}_\star \Psi(\mathbf{Z}_N)| &\le \mathbb{E}_\star |\Psi(\mathbf{Z}_N \circ \widetilde{W}) - \Psi(\mathbf{Z}_N)| \\ &\le M' \widetilde{\mathbb{E}}[\|\mathbf{Z}_N\|_k^k (\mathbf{1}_{\mathcal{L}_{N,M}^c} + \mathbf{1}_{\|\mathbf{Z}_N\|_k > r})] + M' \mathbb{E}_\star[\|\mathbf{Z}_N\|_k^k (\mathbf{1}_{\mathcal{L}_{N,M}^c} + \mathbf{1}_{\|\mathbf{Z}_N\|_k > r})] + c_r \frac{D'}{N} , \end{aligned}$$

The last term on the RHS vanishes when $N \to \infty$. Recall (3.9), that both $\widetilde{\mathbb{E}}\|\mathbf{Z}_N\|_k^{2k}$ and $\mathbb{E}_\star \|\mathbf{Z}_N\|_k^{2k}$ are bounded, uniformly over $|E|, |G| \le \alpha$ and $s_j, t_j \le T$. Thus, by Cauchy-Schwartz, considering (3.46) for $\widetilde{\mathbb{P}}$ and $\mathbb{P}_\star$, the contribution to the RHS from the pair of terms with $\mathcal{L}_{N,M}^c$ also vanishes as $N \to \infty$. Now, to arrive at (3.14), simply combine (3.9) with Markov's inequality, to deduce that $\widetilde{\mathbb{P}}(\|\mathbf{Z}_N\|_k > r) + \mathbb{P}_\star(\|\mathbf{Z}_N\|_k > r) \to 0$ as $r \to \infty$, uniformly in $N, |E|, |G| \le \alpha$ and $s_j, t_j \le T$. Finally, combining (3.12) and (3.14) we deduce that

$$\lim_{N \to \infty} \sup_{|E|,|G| \le \alpha} \sup_{s_j,t_j} |\mathbb{E}_\star \Psi(\mathbf{Z}_N) - \Psi(\mathbb{E}_\star \mathbf{Z}_N)| = 0 \tag{3.57}$$

whenever the coordinates of $\mathbf{Z}_N$ are from $\mathcal{U}_N$. Clearly, $\mathbb{E}_\star |U_N(\cdot|\tau) - \mathbb{E}_\star U_N| \le \mathbb{E}_\star |U_N - \mathbb{E}_\star U_N|$ and $\mathbb{E}_\star U_N(\cdot|\tau) = \mathbb{E}_\star U_N$ for any $U_N \in \mathcal{U}_N$, $\tau \in [0, T]$, thereby extending the validity of (3.57) to coordinates of $\mathbf{Z}_N$ from $\mathcal{U}_N^\star$.

3.3 Proof of Proposition 3.4

Fixing $T < \infty$ note that $H(\cdot)$ does not affect (R, C, q, K). With $H(\cdot)$ uniquely determined by (R, C, q) via (1.21), it suffices to prove the uniqueness of the solution (R, C, q, K) of the reduced system (S):=(1.17,1.18,1.19,1.20). To this end, fixing two solutions (R, C, q, K), $(\widetilde{R}, \widetilde{C}, \widetilde{q}, \widetilde{K})$, of (S) at $\beta = 1$ of the same boundary condition (BC):=(3.16,3.17,3.18), let

$$\underline{\Delta U} := (\Delta R, \Delta C, \Delta q, \Delta K) = |(R, C, q, K) - (\widetilde{R}, \widetilde{C}, \widetilde{q}, \widetilde{K})| .$$

From (BC) we have that $\Delta C(s, s) = \Delta K(s)$ and $\Delta R(s, s) \equiv 0$, $\Delta K(0) = \Delta q(0) = 0$. Denoting all constants by M (which may depend on T and the uniform bound on both solutions), even though they may change from line to line, we arrive at $\underline{\Delta U} \equiv 0$ by adapting the Gronwall's type argument leading to [10, Proposition 4.2]. To this end, (1.17) yields, exactly as in [10, (4.9)] that for all $(s, t) \in \Delta_T$,

$$\Delta R(s, t) \le M \int_t^s \Delta K(u) du + M \int_{t \le t_2 \le t_1 \le s} \Delta C(t_1, t_2) dt_1 dt_2 := I_2(s, t) + I_8(s, t) . \tag{3.58}$$

Next, integrating (1.19) yields that

$$q(t)=q(0)-\int_0^t f'(K(u))q(u)du+\int_0^t q_\star^2 \mathsf{v}_\star'(q(u))du$$
$$+\int_0^t du\int_0^u R(u,v)\Big[q(v)\nu''(C(u,v))-\frac{q_\star^2\nu'(q(v))\nu''(q(u))}{\nu'(q_\star^2)}\Big]dv\,.$$

The same identity holds for $(\widetilde{R},\widetilde{C},\widetilde{q},\widetilde{K})$. With $f'(\cdot)$, $\mathsf{v}_\star'(\cdot)$ locally Lipschitz, considering the difference between that identity for our two uniformly bounded on Δ_T solutions of (S), yields that

$$\Delta q(t)\le M\Big[\int_0^t \Delta q(u)du+\int_0^t du\int_0^u \Delta R(u,v)dv$$
$$+\int_0^t du\int_0^u \Delta C(u,v)dv+\int_0^t \Delta K(u)du\Big].$$

By Gronwall's lemma, upon suitably increasing the value of M we can eliminate the first term on the RHS, whereas by (3.58) the second term on the RHS is controlled by the remaining two terms. Hence,

$$\Delta q(t)\le I_2(t,0)+I_8(t,0)\,,\qquad \forall t\in[0,T]\,. \tag{3.59}$$

Likewise, integrating (1.18) yields that each solution of (S) satisfies for $s\ge t$,

$$C(s,t)=K(t)-\int_t^s f'(K(u))C(u,t)du+\int_t^s du\int_0^t dv\nu'(C(u,v))R(t,v)$$
$$+\int_t^s du\int_0^t dvR(u,v)\nu''(C(u,v))C(t,v)$$
$$+\int_t^s du\int_t^u dvR(u,v)\nu''(C(u,v))C(v,t)$$
$$-q(t)\int_t^s du\frac{q_\star^2\nu''(q(u))}{\nu'(q_\star^2)}\int_0^t dvR(u,v)\nu'(q(v))$$
$$-\int_t^s du\frac{\nu'(q(u))}{\nu'(q_\star^2)}\int_0^t dv\nu'(q(v))R(t,v)+q(t)\int_t^s \mathsf{v}_\star'(q(u))du\,. \tag{3.60}$$

By (3.59), the terms on the RHS which involve $q(\cdot)$, contribute to $\Delta C(s,t)$ at most

$$M\Big[\Delta q(t)+\int_t^s \Delta q(u)du+\int_t^s du\int_0^t \Delta q(v)dv$$
$$+\int_t^s du\int_0^t dv\Delta R(u,v)+\int_t^s du\int_0^t dv\Delta R(t,v)\Big]$$
$$\le I_2(s,0)+I_8(s,0)+I_7(s,t)+I_6(s,t)$$

(see (3.61) for I_6 and I_7). Utilizing [10, (4.10)] to bound the effect on $\Delta C(s,t)$ from the rest of (3.60), yields

$$\Delta C(s,t)\le M\Big[\Delta K(t)+\int_0^s \Delta K(u)du+\int_t^s \Delta C(u,t)du+\int_t^s du\int_0^t dv\Delta C(u,v)$$
$$+\int_t^s du\int_0^t dv\Delta C(t,v)+\int_t^s du\int_0^t dv\Delta R(t,v)+\int_t^s du\int_0^t dv\Delta R(u,v)$$

$$+\int_0^s du \int_0^u dv \Delta C(u,v) + \int_t^s du \int_t^u dv \Delta C(v,t) + \int_t^s du \int_t^u dv \Delta R(u,v)\Big]$$
$$:= I_1(s,t) + I_2(s,0) + I_3(s,t) + \cdots + I_7(s,t) + I_8(s,0) + I_9(s,t) + I_{10}(s,t)\,. \tag{3.61}$$

Similarly, by (1.20) we have for each solution of (S) and any $t \in [0,T]$,

$$\begin{aligned} K(t) - K(0) - t = {} & -2\int_0^t f'(K(u))K(u)du + 2\int_0^t du \int_0^u dv \psi(C(u,v))R(u,v) \\ & - \frac{2}{\nu'(q_\star^2)} \int_0^t du\, \psi(q(u)) \int_0^u dv\, \nu'(q(v))R(u,v) \\ & + 2\int_0^t q(u)\mathsf{v}_\star'(q(u))du\,. \end{aligned} \tag{3.62}$$

Clearly, the terms involving $q(\cdot)$ on the RHS contribute to $\Delta K(t)$ at most $M\int_0^t \Delta q(u)du + I_{10}(t,0)$. Further, with $\Delta K(0) = 0$, utilizing (3.59) and bounding the effect of the rest of (3.62) as in [10, (4.11)], yields here

$$\Delta K(t) \le I_2(t,0) + I_8(t,0) + I_{10}(t,0)\,. \tag{3.63}$$

We follow the derivation of [10, (4.13)], by first plugging (3.58) into (3.63) to eliminate $I_{10}(t,0)$, then by Gronwall's lemma eliminating $I_2(t,0)$. Setting $\mathsf{D}(s) := \int_0^s \Delta C(s,v)dv$, we thereby get, as in [10, (4.13)], that

$$\Delta K(t) \le I_8(t,0) = M\int_0^t \mathsf{D}(u)du\,. \tag{3.64}$$

Plugging (3.64) into (3.58) and (3.59), yields in turn that

$$\Delta R(s,t) \le M\int_0^s \mathsf{D}(u)du\,, \qquad \Delta q(s) \le M\int_0^s \mathsf{D}(u)du\,, \qquad \forall (s,t) \in \Delta_T\,. \tag{3.65}$$

With (3.61) differing from [10, (4.10)] only in having $I_2(s,0) + I_8(s,0)$ instead of $I_2(s,t) + I_8(s,t)$, upon integrating both sides of (3.61) with respect to $t \in [0,s]$, we deduce from (3.64) to (3.65), exactly as in [10, Page 652], that

$$\mathsf{D}(s) \le M\int_0^s \mathsf{D}(u)du\,, \qquad \forall s \in [0,T]\,.$$

Recall that $s \mapsto \mathsf{D}(s)$ is non-negative and non-decreasing. Hence, by yet another Gronwall argument we conclude that $\mathsf{D} \equiv 0$. In particular, $\Delta C(s,t) = 0$ for almost every $(s,t) \in \Delta_T$, while from (3.64)–(3.65)

$$\Delta K \equiv 0, \qquad \Delta R \equiv 0, \qquad \Delta q \equiv 0, \qquad \text{on} \quad \Delta_T\,.$$

Going back to (3.61), this suffices for its RHS to be zero at any $t \le s \le T$, thereby having $\Delta C \equiv 0$ on Δ_T.

4 Proof of Propositions 3.5 and 3.6

4.1 Proof of Proposition 3.5

Consider the limit $N \to \infty$ of the $\mathbb{P}_\star$-expectation of the identities

$$C_N(s,t) = \bar{C}_N(s,t) + \frac{q_N(s)q_N(t)}{q_\star^2}, \qquad \chi_N(s,t) = \bar{\chi}_N(s,t) + q_N(s)\frac{B_t^1}{q_\star\sqrt{N}}.$$

From (3.57) we see that any limit point $(C, \chi, q, \bar{C}, \bar{\chi})$ must satisfy (3.19) (with $\chi = \bar{\chi}$ as both $\mathbb{E}_\star[q_N^2(s)]$ and $\mathbb{E}[|B_t^1|^2]$ are bounded uniformly in N and on $[0, T]$). The $\mathbb{P}_\star$-expectation of (3.13) at $i = 1$, amounts in view of (3.5), to $q_N^a(s) = q_N^a(0) + \int_0^s Q_N^a(u)du$, from which, by utilizing again (3.57) as $N \to \infty$, we deduce the validity of the RHS of (3.20). By the same reasoning, each limit point of the $\mathbb{P}_\star$-expectation of (3.5)–(3.8) must satisfy (3.20)–(3.26), respectively. Observing that $\bar{\chi}_N^a(0,t) = 0$, and having as in [10, Eqn. (3.2)–(3.3)],

$$\bar{C}_N(s,t) = \bar{C}_N(s,0) + \bar{\chi}_N(s,t) + \int_0^t \bar{D}_N(u,s)du\,,$$

$$\bar{\chi}_N(s,t) = \bar{\chi}_N(0,t) + \frac{1}{N}\sum_{i=2}^N B_s^i B_t^i + \int_0^s \bar{E}_N(u,t)du \tag{4.1}$$

(recall the definition (3.6) of $\bar{D}_N$ and $\bar{E}_N$), we likewise deduce that (3.27) holds. Recall that by the $\mathbb{P}_\star$-independence of the standard Brownian increments

$$U_N^a(s,0) = 0, \qquad U_N^a(s,t) = U_N^a(s, t\wedge s), \qquad U_N \in \{\bar{F}_N, \bar{\chi}_N, E_N\} \tag{4.2}$$

(c.f. [10, Page 638]), hence our stated boundary conditions on the limit point. The key to the proof is Proposition 4.1, which approximates $(V_N^a, \bar{A}_N^a, \bar{F}_N^a, \widehat{H}_N^a)$ for $N \to \infty$, by a combination of functions from $\mathcal{U}_N^a$ (where expressions involving ν, ν' and ν'' emerge via the covariance kernels of Lemma 3.7). Indeed, with Proposition 4.1 replacing [10, Prop. 3.1], we get (3.28)–(3.31) (and thereby establish Proposition 3.5), by following the derivation of [10, Prop. 1.3], while utilizing (3.57) and the pre-compactness results of Proposition 3.1 (for $\mathbb{P}_\star$), instead of [10, Cor. 2.8] and [10, Prop. 2.3], respectively.

Proposition 4.1 *Set $a_N \simeq b_N$ when $|a_N(\cdot) - b_N(\cdot)| \to 0$ as $N \to \infty$, uniformly on $[0, T]^2$. Then, for $\tau = t \vee s$,*

$$V_N^a(s) \simeq \Phi_N^{1,a}(s,s) - \Phi_N^{1,a}(s,0) - \int_0^s \Psi_N^{1,a}(s,u)du\,, \tag{4.3}$$

$$\bar{A}_N^a(t,s) \simeq \bar{C}_N^a(s,\tau)\Phi_N^a(t,\tau) - \bar{C}_N^a(s,0)\Phi_N^a(t,0) - \int_0^\tau \Big\{\bar{D}_N^a(s,u)\Phi_N^a(t,u) + \bar{C}_N^a(s,u)\Psi_N^a(t,u)\Big\}du\,, \tag{4.4}$$

$$\bar{F}_N^a(s,t) \simeq \bar{\chi}_N^a(s,t)\Phi_N^a(s,s) - \int_0^{t\wedge s} \Phi_N^a(s,v)dv - \int_0^s \Big\{\Phi_N^a(s,u)\bar{E}_N^a(u,t) + \bar{\chi}_N^a(u,t)\Psi_N^a(s,u)\Big\}du\,, \tag{4.5}$$

$$\widehat{H}_N^a(s) \simeq \Upsilon_N^a(s,s) - \Upsilon_N^a(s,0) - \int_0^s \Big\{\bar{D}_N^a(s,u)\Phi_N^a(u,s) + \frac{Q_N^a(u)}{q_\star^2}\Phi_N^{1,a}(u,s)\Big\}du\,. \tag{4.6}$$

Towards proving Proposition 4.1 we fix a continuous path $\mathbf{x}$ satisfying (3.13). Then, for any operator k_t of kernel $k^{ij}_{uv}(\mathbf{x})$ on $L_2(\{1,\cdots N\}\times[0,t])$ and $f\in L_2(\{1,\cdots N\}\times[0,t])$, let

$$[k_t f]^i_u := \sum_{j=1}^{N}\int_0^t k^{ij}_{uv} f^j_v\, dv\,, \qquad (i,u)\in\{1,\cdots,N'\}\times[0,t]\,, \tag{4.7}$$

which is clearly in $L_2(\{1,\cdots N'\}\times[0,t])$. Assuming that each $k^{ij}_{uv}(\mathbf{x})$ is the finite sum of terms such as $x^{i_1}_u\cdots x^{i_a}_u x^{j_1}_v\cdots x^{j_b}_v$ (for some non-random a,b and $i_1,\ldots,i_a,j_1,\ldots,j_b$), we further extend (4.7) to stochastic integrals of the form

$$[k_t\circ dZ]^i_u = \sum_{j=1}^{N}\int_0^t k^{ij}_{uv}\, dZ^j_v\,, \tag{4.8}$$

where Z_v is a continuous $\mathcal{F}_v$-semi-martingale (composed for each j, of a squared-integrable continuous martingale and a continuous, adapted, squared-integrable finite variation part). Adopting the conventions of [10, Page 640] for interpreting $\int_0^t k^{ij}_{uv} dZ^j_v$ in terms of Itô integrals, note that $[k_t\circ dZ]^i_u\in L_2(\{1,\cdots,N'\}\times[0,t])$ (recall (3.41) that $\mathbf{x}_t$ has uniformly over time, bounded moments of all orders under $\mathbb{P}_\star$, hence so does the kernel $k^{ij}_{ts}(\mathbf{x})$), with the following extension of [10, Lemma 3.3].

Lemma 4.2 *Fixing $\tau\in\mathbb{R}_+$ there exist a version of $V^i_{s;\tau} := \mathbb{E}_\star[G^i(\mathbf{x}_s)|\mathcal{F}_\tau]$ and $Z^i_{s;\tau} := \mathbb{E}_\star[B^i_s|\mathcal{F}_\tau]$ with*

$$Z^i_{s;\tau} = x^i_s - x^i_0 - \int_0^s Q^i_{u;\tau}\, du\,, \qquad Q^i_{s;\tau} := V^i_{s;\tau} - f'(K_N(s))x^i_s + \mathbf{1}_{\{i=1\}}\sqrt{N}q_\star \mathsf{V}'(q_N(s))\,, \tag{4.9}$$

such that $s\mapsto Z^i_{s;\tau}$ are continuous semi-martingales with respect to the filtration $(\mathcal{F}_s, s\le\tau)$, composed of squared-integrable continuous martingales and finite variation parts. If $\{S^i(\mathbf{x}),\ i\le N'\}$ are linear forms in $\mathbf{J}$ with covariance kernels

$$k^{ij}_{st}(\mathbf{x}) := \mathbb{E}_{\mathbf{J}}\left\{S^i(\mathbf{x}_s)G^j(\mathbf{x}_t)\,|\,CP_\star\right\},\quad \widetilde{k}^{il}_{st}(\mathbf{x}) := \mathbb{E}_{\mathbf{J}}\left\{S^i(\mathbf{x}_s)S^l(\mathbf{x}_t)\,|\,CP_\star\right\},$$
$$1\le i,l\le N',\ 1\le j\le N\,, \tag{4.10}$$

consisting of polynomials in $\mathbf{x}$, then

$$Y^i_{s;\tau} := \mathbb{E}_\star[S^i(\mathbf{x}_s)|\mathcal{F}_\tau] = [k_\tau\circ dZ]^i_s = [k_\tau\circ dx]^i_s - [k_\tau Q]^i_s\,,$$
$$\forall (i,s)\in\{1,\cdots,N'\}\times[0,\tau]\,. \tag{4.11}$$

Further, there exist then a version of

$$\widetilde{\Gamma}^{il}_{st;\tau} := \mathbb{E}_\star\Big[(S^i(\mathbf{x}_s)-Y^i_{s;\tau})(S^l(\mathbf{x}_t)-Y^l_{t;\tau})|\mathcal{F}_\tau\Big],\qquad i,l\in\{1,\cdots,N'\},$$
$$\Gamma^{jl}_{st;\tau} := \mathbb{E}_\star\Big[(G^j(\mathbf{x}_s)-V^j_{s;\tau})(S^l(\mathbf{x}_t)-Y^l_{t;\tau})|\mathcal{F}_\tau\Big],\qquad s,t\in[0,\tau],\ j\in\{1,\cdots,N\}, \tag{4.12}$$

such that

$$\widetilde{\Gamma}^{il}_{st;\tau} = \widetilde{k}^{il}_{st} - \sum_{j=1}^{N}\int_0^\tau k^{ij}_{su}\Gamma^{jl}_{ut;\tau}\, du\,,\qquad \forall s,t\in[0,\tau],\quad i,l\in\{1,\cdots,N'\}. \tag{4.13}$$

Proof The right equality in (4.11) follows from the relation (4.9) between $\mathbb{E}_\star[B_s^i|\mathcal{F}_\tau]$ and $\mathbb{E}_\star[G^i(\mathbf{x}_u)|\mathcal{F}_\tau]$, which in turn is a consequence of having in (3.13),

$$U_s^i := x_s^i - x_0^i + \int_0^s f'(K_N(u))x_u^i du - \mathbf{1}_{\{i=1\}}\sqrt{N}q_\star \int_0^s \mathsf{v}'(q_N(u))du$$
$$= \int_0^s G^i(\mathbf{x}_u)du + B_s^i. \tag{4.14}$$

The latter relation implies the stated continuity and integrability properties of the semimartingales U_s^i and $Z_{s;\tau}^i = U_s^i - \int_0^s V_{u;\tau}^i du$. By Girsanov formula (see [10, Eqn. (3.16)]), the restriction to $\mathcal{F}_\tau$ satisfies

$$\mathbb{P}_{\mathbf{J},\mathbf{x}_0}^N|_{\mathcal{F}_\tau} = \Lambda_\tau^N \,\mathbb{P}_{\mathbf{0},\mathbf{x}_0}^N|_{\mathcal{F}_\tau},$$
$$\Lambda_\tau^N := \exp\Big\{\sum_{i=1}^N \int_0^\tau G^i(\mathbf{x}_s)dU_s^i - \frac{1}{2}\sum_{i=1}^N \int_0^\tau (G^i(\mathbf{x}_s))^2 ds\Big\}, \tag{4.15}$$

with U_s^i a standard Brownian motion under $\mathbb{P}_{\mathbf{0},\mathbf{x}_0}^N$. Setting $\mathbb{P}_{\mathbf{J}}^\star$ for the law of $\mathbf{J}$ conditional on $\mathsf{CP}_\star$, we thus have (as in the proof of [10, Lemma 3.3]), that

$$Y_{s;\tau}^i = \frac{\mathbb{E}_{\mathbf{J}}^\star[S^i(\mathbf{x}_s)\Lambda_\tau^N]}{\mathbb{E}_{\mathbf{J}}^\star[\Lambda_\tau^N]},$$
$$\widetilde{\Gamma}_{st;\tau}^{il} = \frac{\mathbb{E}_{\mathbf{J}}^\star\Big[(S^i(\mathbf{x}_s) - Y_{s;\tau}^i)(S^l(\mathbf{x}_t) - Y_{t;\tau}^l)\Lambda_\tau^N\Big]}{\mathbb{E}_{\mathbf{J}}^\star[\Lambda_\tau^N]}. \tag{4.16}$$

The centered Gaussian law $\mathbb{P}_{\mathbf{J}}^\star$ is not a product measure, but the arguments used in proving [10, Proposition C.1] still apply. Specifically, here $G^j(\mathbf{x}_s) = \sum_\alpha J_\alpha^o L_s^j(\alpha)$ and $S^i(\mathbf{x}_t) = \sum_\alpha J_\alpha^o M_t^i(\alpha)$ for some independent centered Gaussian $\{J_\alpha^o\}$ of positive variances υ_α, with $k_{su}^{ij} = \sum_\alpha M_s^i(\alpha)\upsilon_\alpha L_u^j(\alpha)$. Our Radon-Nikodym derivative Λ_τ^N is given in terms of $\mathbf{R} = \{R_{\alpha\gamma}\}$ of [10, (C.4)] and $\mathbf{J}^o := \{J_\alpha^o\}$, by the display following [10, (C.4)]. Under such a change of measure the Gaussian law of $\mathbf{J}^o$ has the covariance matrix $(\mathbf{D}^{-1} + \mathbf{R})^{-1}$ for $\mathbf{D}$ =diag(υ_α) and the mean vector $\mathbf{q} = (\mathbf{D}^{-1} + \mathbf{R})^{-1}\boldsymbol{\mu}$ of [10, (C.5)]. From the LHS of (4.16) we have that $Y_{s;\tau}^i = \sum_\alpha M_s^i(\alpha)q_\alpha$ and $V_{u;\tau}^j = \sum_\alpha L_u^j(\alpha)q_\alpha$. Further, by definition $[k_\tau \circ dU]_s^i = \sum_\alpha M_s^i(\alpha)\upsilon_\alpha\mu_\alpha$ and $[k_\tau V]_s^i = \sum_{\alpha,\gamma} M_s^i(\alpha)\upsilon_\alpha R_{\alpha\gamma}q_\gamma$ (thanks to [10, (C.4)]), with the identity $Y_{s;\tau}^i = [k_\tau \circ Z]_s^i$ of (4.11) thus a direct consequence of [10, (C.5)]. Next, note that $\widetilde{k}_{st}^{il} = \sum_\alpha M_s^i(\alpha)\upsilon_\alpha M_t^l(\alpha)$, whereas from the RHS of (4.16) we have that

$$\Gamma_{ut;\tau}^{jl} = \sum_{\alpha,\gamma} L_u^j(\alpha)[(\mathbf{D}^{-1} + \mathbf{R})^{-1}]_{\alpha\gamma} M_t^l(\gamma),$$
$$\widetilde{\Gamma}_{st;\tau}^{il} = \sum_{\alpha,\gamma} M_s^i(\alpha)[(\mathbf{D}^{-1} + \mathbf{R})^{-1}]_{\alpha\gamma} M_t^l(\gamma).$$

By [10, (C.4)] we thus get (4.13) out of $(\mathbf{D}^{-1} + \mathbf{R})^{-1} = \mathbf{D} - \mathbf{DR}(\mathbf{D}^{-1} + \mathbf{R})^{-1}$ (as in the proof of [10, (C.3)]). □

Proof of Proposition 4.1 In view of (3.3)–(3.6) and (3.15), one has as in [10, Pg. 642], for any $\tau \in [t \vee s, T]$,

$$\bar{A}_N(t,s|\tau) = \frac{1}{N}\sum_{i=2}^N V_{t;\tau}^i x_s^i, \quad \bar{D}_N(s,t|\tau) = \frac{1}{N}\sum_{i=2}^N Q_{t;\tau}^i x_s^i,$$

$$V_N(s|\tau) = \frac{q_\star}{\sqrt{N}} V^1_{s;\tau}\,, \quad Q_N(s|\tau) = \frac{q_\star}{\sqrt{N}} Q^1_{s;\tau}\,. \tag{4.17}$$

Recall Itô's formula for $u \mapsto \widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)$,

$$\sum_{j=1}^N \int_0^\tau \partial_{x_u^j} \{\widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)\} dx_u^j = \widetilde{k}(\mathbf{x}_s, \mathbf{x}_\tau) - \widetilde{k}(\mathbf{x}_s, \mathbf{x}_0) - \frac{1}{2}\int_0^\tau \{\Delta_{\mathbf{x}_u} \widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)\} du. \tag{4.18}$$

Thus, for the operator k_τ corresponding to $S^i \equiv G^i$ in Lemma 4.2, we get from the first identity of (3.34) that

$$[k_\tau \circ dx]^i_s = \sum_{j=1}^N \int_0^\tau \partial_{x_s^i} \partial_{x_u^j} \{\widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)\} dx_u^j = \varphi^i_N(s,\tau) - \varphi^i_N(s,0) - \int_0^\tau \delta^i_N(s,u) du\,, \tag{4.19}$$

for any $(i, s) \in [0, \tau] \times \{1, \ldots, N\}$, where

$$\varphi^i_N(s,u) := \partial_{x_s^i} \widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)\,, \qquad \delta^i_N(s,u) := \frac{1}{2} \partial_{x_s^i} \{\Delta_{\mathbf{x}_u} \widetilde{k}(\mathbf{x}_s, \mathbf{x}_u)\}\,.$$

By the second identity of (3.34) we arrive at

$$\varphi^i_N(s,u) = \partial_{x_s^i} \widetilde{k}(\mathbf{x}_s, \mathbf{x}_u) = x_u^i \mathbf{1}_{\{i \neq 1\}} \Phi_N(s,u) + \frac{\sqrt{N}}{q_\star} \mathbf{1}_{\{i=1\}} \Phi^1_N(s,u)\,, \tag{4.20}$$

in terms of $\Phi_N(\cdot,\cdot)$ and $\Phi^1_N(\cdot,\cdot)$ of (3.7). Consequently,

$$\begin{aligned} k^{ij}_{su} =& \partial_{x_u^j} \{\varphi^i_N(s,u)\} = \frac{x_s^j x_u^i}{N} \nu''(C_N(s,u)) + \mathbf{1}_{\{i=j\neq 1\}} \Phi_N(s,u) \\ &+ \mathbf{1}_{\{i=j=1\}} \Big[\nu'(C_N(s,u)) - \bar{C}_N(s,u) \frac{q_\star^2 \nu''(q_N(s))\nu''(q_N(u))}{\nu'(q_\star^2)} \Big] \\ &- \mathbf{1}_{\{i=1\}} \mathbf{1}_{\{j\neq 1\}} \frac{q_\star x_s^j}{\sqrt{N}} \frac{\nu''(q_N(s))\nu'(q_N(u))}{\nu'(q_\star^2)} - \mathbf{1}_{\{j=1\}} \mathbf{1}_{\{i\neq 1\}} \frac{q_\star x_u^i}{\sqrt{N}} \frac{\nu''(q_N(u))\nu'(q_N(s))}{\nu'(q_\star^2)}\,. \end{aligned} \tag{4.21}$$

Similarly,

$$\Delta_{\mathbf{x}_u}\widetilde{k}(\mathbf{x}_s,\mathbf{x}_u) = K_N(s)\nu''(C_N(s,u)) - \frac{q_\star^2\nu'''(q_N(u))}{\nu'(q_\star^2)}\nu'(q_N(s))\bar{C}_N(s,u)\,, \tag{4.22}$$

resulting after some algebra with

$$\begin{aligned}\delta_N^i(s,u) &= \frac{x_s^i}{N}\nu''(C_N(s,u)) + \frac{x_u^i}{2N}K_N(s)\nu'''(C_N(s,u)) \\ &\quad - \frac{q_\star^2\nu'''(q_N(u))}{2\nu'(q_\star^2)}\Big[\mathbf{1}_{\{i\neq 1\}}\frac{x_u^i\nu'(q_N(s))}{N} + \mathbf{1}_{\{i=1\}}\frac{q_\star\nu''(q_N(s))}{\sqrt{N}}\bar{C}_N(s,u)\Big]\,. \end{aligned}\tag{4.23}$$

Next, with $\varphi_N^j(u,s) = \partial_{x_u^j}\{\widetilde{k}(\mathbf{x}_s,\mathbf{x}_u)\}$ it follows from (3.34) and (4.7), that

$$[k_\tau Q]_s^i = \int_0^\tau \psi_N^i(s,u|\tau)du\,, \qquad \psi_N^i(s,u|\tau) := \partial_{x_s^i}\Big[\sum_{j=1}^N \varphi_N^j(u,s)Q_{u;\tau}^j\Big]. \tag{4.24}$$

Combining (4.17) and (4.20), we have

$$\sum_{j=1}^N \varphi_N^j(u,s)Q_{u;\tau}^j = N\bar{D}_N(s,u|\tau)\Phi_N(u,s) + \frac{N}{q_\star^2}Q_N(u|\tau)\Phi_N^1(u,s)\,, \tag{4.25}$$

which in view of (3.7), (3.8) and the symmetry of $\Phi_N(\cdot,\cdot)$ yields that

$$\psi_N^i(s,u|\tau) = \begin{cases} \frac{\sqrt{N}}{q_\star}\Psi_N^1(s,u|\tau)\,, & i=1\,, \\ Q_{u;\tau}^i\Phi_N(s,u) + x_u^i\Psi_N(s,u|\tau)\,, & i\neq 1\,. \end{cases} \tag{4.26}$$

In this case $Y_{s;\tau}^i = V_{s;\tau}^i$, so by (4.11), (4.19) and (4.24) we get

$$\begin{aligned} V_{s;\tau}^i &= [k_\tau \circ dx]_s^i - [k_\tau Q]_s^i \\ &= \varphi_N^i(s,\tau) - \varphi_N^i(s,0) - \int_0^\tau [\psi_N^i(s,u|\tau) + \delta_N^i(s,u)]du \qquad \forall s\in[0,\tau]\,. \end{aligned}\tag{4.27}$$

In particular, for $\epsilon_N(s) := \frac{q_\star}{\sqrt{N}}\int_0^s \delta_N^1(s,u)du$, by (4.17), (4.20) and (4.26),

$$V_N(s|s) + \epsilon_N(s) = \Phi_N^1(s,s) - \Phi_N^1(s,0) - \int_0^s \Psi_N^1(s,u|s)du\,.$$

We now consider the $\mathbb{E}_\star$-expected value of the preceding identity. From (4.23) we have that $\epsilon_N^a \simeq 0$, so with $U_N^a(s,t) = \mathbb{E}_\star U_N(s,t|\tau)$ we arrive at (4.3). Turning to the derivation of (4.4), for $\tau = t\vee s$ and

$$\widetilde{\epsilon}_N(t,s) := \int_0^\tau \Big\{\frac{1}{N}\sum_{i=2}^N x_s^i\delta_N^i(t,u)\Big\}du\,,$$

we have in view of (4.17), (4.20), (4.26) and (4.27), that

$$\bar{A}_N(t,s|\tau) + \widetilde{\epsilon}_N(t,s) = \frac{1}{N}\sum_{i=2}^N \varphi_N^i(t,\tau)x_s^i - \frac{1}{N}\sum_{i=2}^N \varphi_N^i(t,0)x_s^i$$

$$- \int_0^\tau \Big\{ \frac{1}{N} \sum_{i=2}^N \psi_N^i(t, u|\tau) x_s^i \Big\} du$$
$$= \bar{C}_N(s,\tau)\Phi_N(t,\tau) - \bar{C}_N(s,0)\Phi_N(t,0)$$
$$- \int_0^\tau \Big\{ \bar{D}_N(s,u|\tau)\Phi_N(t,u) + \bar{C}_N(s,u)\Psi_N(t,u|\tau) \Big\} du\,.$$

Since $\widetilde{\epsilon}_N^a \simeq 0$, we get (4.4) from the preceding identity (upon applying (3.57) for the function $z_1 z_2$).

Moving to (4.5), by (4.2) it suffices to consider hereafter $t \in [0, s]$. Further, $B_t^i = U_t^i - \int_0^t G^i(\mathbf{x}_v) dv$ with U_t^i measurable on $\mathcal{F}_\tau$ (c.f. (4.14)). Hence, in view of (4.12),

$$\mathbb{E}_\star[(V_{s;\tau}^i - G^i(\mathbf{x}_s))B_t^i|\mathcal{F}_\tau] = \mathbb{E}_\star[(V_{s;\tau}^i - G^i(\mathbf{x}_s)) \int_0^t (V_{v;\tau}^i - G^i(\mathbf{x}_v))dv|\mathcal{F}_\tau]$$
$$= \int_0^t \Gamma_{sv;\tau}^{ii} dv\,. \tag{4.28}$$

In particular, setting

$$\Gamma_N(s,t|\tau) := \frac{1}{N}\sum_{i=2}^N \int_0^t \Gamma_{sv;\tau}^{ii} dv$$

we deduce that

$$\Gamma_N(s,t|\tau) = \frac{1}{N}\sum_{i=2}^N V_{s;\tau}^i Z_{t;\tau}^i - \bar{F}_N(s,t|\tau) = \frac{1}{N}\sum_{i=2}^N Q_{s;\tau}^i Z_{t;\tau}^i - E_N(s,t|\tau),$$
$$\bar{\chi}_N(s,t|\tau) = \frac{1}{N}\sum_{i=2}^N x_s^i Z_{t;\tau}^i\,. \tag{4.29}$$

From (4.20), (4.26) and (4.29) (at $\tau = s$), we also have that

$$\Theta_N(s,u;t) := \frac{1}{N}\sum_{i=2}^N \varphi_N^i(s,u) Z_{t;s}^i = \bar{\chi}_N(u,t|s)\Phi_N(s,u),$$
$$\Pi_N(s,u;t) := \frac{1}{N}\sum_{i=2}^N \psi_N^i(s,u|s) Z_{t;s}^i = \big[\Gamma_N(u,t|s)$$
$$+ \bar{E}_N(u,t|s)\big]\Phi_N(s,u) + \bar{\chi}_N(u,t|s)\Psi_N(s,u|s). \tag{4.30}$$

Further, from (4.27) we get

$$\Gamma_N(s,t|s) + \bar{F}_N(s,t|s) = \frac{1}{N}\sum_{i=2}^N V_{s;s}^i Z_{t;s}^i$$
$$= \Theta_N(s,s;t) - \Theta_N(s,0;t) - \int_0^s \Pi_N(s,u;t)du - \widehat{\epsilon}_N(s,t)\,, \tag{4.31}$$

where $\widehat{\epsilon}_N(s,t) := \frac{1}{N}\sum_{i=2}^N \int_0^s \delta_N^i(s,u) Z_{t;s}^i du$ is such that $\widehat{\epsilon}_N^a \simeq 0$ (see (4.23)). Next, setting

$$\phi_N(s,v) := \Phi_N(s,v) - \frac{1}{N}\sum_{i=2}^{N}\Gamma^{ii}_{sv;s} - \frac{1}{N}\sum_{i=2}^{N}\int_0^s \Phi_N(s,u)\Gamma^{ii}_{uv;s}du\,, \quad v\in[0,s], \tag{4.32}$$

we see that

$$\int_0^t \Phi_N(s,v)dv = \Gamma_N(s,t|s) + \int_0^s \Phi_N(s,u)\Gamma_N(u,t|s)du + \int_0^t \phi_N(s,v)dv\,,$$

so combining (4.30) and (4.31) results with

$$\int_0^t \Phi_N(s,v)dv + \bar{F}_N(s,t|s) = \bar{\chi}_N(s,t|s)\Phi_N(s,s) - \bar{\chi}_N(0,t|s)\Phi_N(s,0) + \int_0^t \phi_N(s,v)dv$$
$$- \widehat{\epsilon}_N(s,t) - \int_0^s \Big\{\bar{E}_N(u,t|s)\Phi_N(s,u) + \bar{\chi}_N(u,t|s)\Psi_N(s,u|s)\Big\}du\,. \tag{4.33}$$

Recalling that $\bar{\chi}^a_N(0,t)=0$, we thus get (4.5) by employing (3.57) on the $\mathbb{E}_\star$-expectation of the RHS of (4.33) and relying on the following analog of [10, Lemma 3.4].

Lemma 4.3 *For $\phi_N(s,v)$ of (4.32),*

$$\lim_{N\to\infty}\sup_{(s,v)\in\Delta_T}|\phi^a_N(s,v)| = 0.$$

Proof of Lemma 4.3 Recall that $\widetilde{\Gamma}=\Gamma$ and $\widetilde{k}_s = k_s$ in our special case of Lemma 4.2. Thus, setting

$$\bar{\gamma}_N(u,v|s) := \frac{1}{N^2}\sum_{i,j=2}^{N} x^j_s x^i_u \Gamma^{ji}_{uv;s}\,, \qquad \gamma^1_N(u,v|s) := \frac{q_\star}{\sqrt{N}}\frac{1}{N}\sum_{i=2}^{N} x^i_u\Gamma^{1i}_{uv;s},$$

we deduce from (4.13), (4.21) and (4.32) that for any $v,u\in[0,s]$,

$$\begin{aligned}
0 &= \frac{1}{N}\sum_{i=2}^{N}\Big[k^{ii}_{sv} - \Gamma^{ii}_{sv;s} - \sum_{j=1}^{N}\int_0^s k^{ij}_{su}\Gamma^{ji}_{uv;s}du\Big]\\
&= \phi_N(s,v) + \frac{1}{N}\Big[\nu''(C_N(s,v))\bar{C}_N(s,v) - \Phi_N(s,v)\Big]\\
&\quad - \int_0^s \nu''(C_N(s,u))\bar{\gamma}_N(u,v|s)du\\
&\quad - \int_0^s \Big[\frac{q_N(s)}{q_\star^2}\nu''(C_N(s,u)) - \frac{\nu''(q_N(u))\nu'(q_N(s))}{\nu'(q_\star^2)}\Big]\gamma^1_N(u,v|s)du\,.
\end{aligned}$$

Recalling Proposition 3.3 that the uniform moment bounds (3.9) apply for $\mathbb{P}_\star$ and any $U_N\in\mathcal{U}_N$, it thus suffices to show that $\mathbb{E}_\star[(\bar{\gamma}_N)^2]\simeq 0$ and $\mathbb{E}_\star[(\gamma^1_N)^2]\simeq 0$. To this end, from the definitions of $\bar{A}_N$, $\Gamma^{ji}_{uv;s}$ (see (3.3), (4.12)), and the LHS of (4.17), we find that

$$\begin{aligned}
\bar{\gamma}_N(u,v|s) &= \mathbb{E}_\star\Big[\frac{1}{N^2}\sum_{i,j=2}^{N} x^j_s x^i_u (G^j(\mathbf{x}_u) - V^j_{u;s})(G^i(\mathbf{x}_v) - V^i_{v;s})|\mathcal{F}_s\Big]\\
&= \mathbb{E}_\star\Big[(\bar{A}_N(u,s) - \bar{A}_N(u,s|s))(\bar{A}_N(v,u) - \bar{A}_N(v,u|s))|\mathcal{F}_s\Big].
\end{aligned}$$

In particular, by Cauchy-Schwarz

$$\sup_{u,v\in[0,s]}\Big\{\mathbb{E}_\star\big[\bar{\gamma}_N(u,v|s)^2\big]\Big\} \le \sup_{u,v\in[0,s]}\Big\{\mathbb{E}_\star\Big[(\bar{A}_N(v,u)-\bar{A}_N(v,u|s))^2\Big]\Big\}$$

which goes to zero as $N\to\infty$ (apply Corollary 3.2 for $\Psi(z)=(z_1-z_2)^2$ and $\mathbf{Z}_N=(\bar{A}_N(\cdot),\bar{A}_N(\cdot|s))$ with $\Psi(\mathbb{E}_\star\mathbf{Z}_N)=0$). Similarly, we get from (3.3), (4.12) and the right-most identity in (4.17) that

$$\begin{aligned}\gamma_N^1(u,v|s) &= \mathbb{E}_\star\Big[\frac{q_\star}{\sqrt{N}}(G^1(\mathbf{x}_u)-V^1_{u;s})\frac{1}{N}\sum_{i=2}^N x_u^i(G^i(\mathbf{x}_v)-V^i_{v;s})|\mathcal{F}_s\Big]\\ &= \mathbb{E}_\star\Big[(V_N(u)-V_N(u|s))(\bar{A}_N(v,u)-\bar{A}_N(v,u|s))|\mathcal{F}_s\Big].\end{aligned}$$

Thus, as before, the uniform convergence to zero of $\mathbb{E}_\star[(\gamma_N^1(u,v|s))^2]$ follows by combining Cauchy-Schwarz and Corollary 3.2 for $\mathbb{P}_\star$ (taking here $\mathbf{Z}_N=(V_N(u),V_N(u|s))$). □

Proceeding to establish (4.6), we compute $\widehat{H}_N(s|s)$ by employing Lemma 4.2 for $H=\widehat{H}_N$ (with $N'=1$). This corresponds to having covariance kernel $\hat{k}^{1j}_{su}=\frac{1}{N}\partial_{x_u^j}\widetilde{k}(\mathbf{x}_s,\mathbf{x}_u)$. In view of our definition of $\varphi_N^j(u,s)$, we then get from the RHS of (4.11) at $\tau=s$, upon utilizing (4.18) and (4.25), that

$$\begin{aligned}\widehat{H}_N(s|s) &= \frac{1}{N}\sum_{j=1}^N\int_0^s\partial_{x_u^j}\{\widetilde{k}(\mathbf{x}_s,\mathbf{x}_u)\}dx_u^j-\int_0^s\frac{1}{N}\Big[\sum_{j=1}^N\varphi_N^j(u,s)Q^j_{u:s}\Big]du\\ &= \frac{1}{N}\widetilde{k}(\mathbf{x}_s,\mathbf{x}_s)-\frac{1}{N}\widetilde{k}(\mathbf{x}_s,\mathbf{x}_0)\\ &\quad-\int_0^s\Big[\bar{D}_N(s,u|s)\Phi_N(u,s)+q_\star^{-2}Q_N(u|s)\Phi_N^1(u,s)\Big]du-\epsilon_N^\dagger(s)\,,\end{aligned}$$

for $\epsilon_N^\dagger(s):=\frac{1}{2N}\int_0^s\{\Delta_{\mathbf{x}_u}\widetilde{k}(\mathbf{x}_s,\mathbf{x}_u)\}du$ such that $(\epsilon_N^\dagger)^a\simeq 0$ (see (4.22)). In view of the second identity of (3.34), considering $\mathbb{E}_\star\widehat{H}_N(s|s)$ yields (4.6) (upon applying (3.57) for the function z_1z_2), thereby completing the proof of Proposition 4.1. □

4.2 Proof of Proposition 3.6

We first show that $t\mapsto\chi(s,t)=\bar{\chi}(s,t)$ is continuously differentiable on $s\ge t$. Indeed, per fixed t we have from (3.30) and the RHS of (3.21) that $\bar{E}(s,t)=[k_C\bar{E}(\cdot,t)](s)+h(s,t)$, with

$$h(s,t):=[\Phi(s,s)-f'(K(s))]\chi(s,t)-\int_0^s\chi(u,t)\Psi(s,u)du-\int_0^{t\wedge s}\Phi(s,u)du$$

in $\mathcal{C}_b([0,T]^2)$, and integral operator k_C on $\mathcal{C}([0,T])$ of uniformly bounded kernel $\Phi(s,u)$ on $[0,T]^2$. As in the proof of [10, Lemma 4.1], Picard iterations yield that

$$\bar{E}(s,t)=\sum_{n\ge0}[k_C^n h(\cdot,t)](s)=h(s,t)+\int_0^s\kappa_C(s,v)h(v,t)dv\,, \tag{4.34}$$

with a uniformly bounded kernel κ_C. Plugging (4.34) into the RHS of (3.27), we find by Fubini's theorem that

$$\chi(s,t) = s\wedge t + \int_0^s [\int_0^{t\wedge v} \Phi(v,u)du]\kappa_1(s,v)dv + \int_0^s \chi(v,t)\kappa_2(s,v)dv\,,$$

for some uniformly bounded κ_1 and κ_2 (which depend only on Φ, Ψ and $f'(K(\cdot))$). Applying Picard's iterations now with respect to the integral operator $[\kappa_2 g](s) = \int_0^s \kappa_2(s,v)g(v)dv$, we deduce that

$$\chi(s,t) = s\wedge t + \int_0^s \Big[(u\wedge t)\kappa_3(s,u) + [\int_0^{t\wedge u} \Phi(u,v)dv]\kappa_4(s,u)\Big]du\,,$$

for some uniformly bounded κ_3 and κ_4. With $s\wedge t = t$ continuously differentiable on $s \ge t$, we conclude by Fubini's theorem that $\chi(s,t) = \int_0^t R(s,u)du$, for the bounded continuous

$$R(s,t) = 1 + \int_t^s [\kappa_3(s,u) + \Phi(u,t)\kappa_4(s,u)]du\,.$$

In particular, $R(s,s) = 1$ for all s. Next, having that $\bar{E}(s,0) = 0$ for all s and $\bar{E}(s,t) = \bar{E}(s,s)$ for all $t \ge s$, imply the same for $\chi(s,t)$ (see the RHS of (3.27)), and in particular $R(s,t) = (\partial_2\chi)(s,t) = 0$ when $t > s$. From the LHS of (3.27) we see that $\partial_2\bar{C}(s,t) = R(s,t) + \bar{D}(s,t)$, hence also $\partial_1\bar{C}(s,t) = \partial_2\bar{C}(t,s) = \bar{D}(t,s) + R(t,s)$ (by the symmetry of $\bar{C}$). From the RHS of (3.20) we have $Q(t) = \partial q(t)$, so by the LHS of (3.19)

$$\partial_2 C(s,t) = \bar{D}(s,t) + R(s,t) + \frac{q(s)Q(t)}{q_\star^2}\,. \tag{4.35}$$

These imply in turn that the symmetric $\Upsilon(\cdot,\cdot)$ of (3.22) is differentiable and by (3.23), (3.24),

$$\begin{aligned}\partial_2\Upsilon(s,u) &= \bar{D}(s,u)\Phi(u,s) + \frac{Q(u)}{q_\star^2}\Phi^1(u,s)\\ &\quad + R(s,u)\left[\nu'(C(s,u)) - \frac{\nu'(q(s))\nu'(q(u))}{\nu'(q_\star^2)}\right],\end{aligned}$$

with (1.21) a consequence of (3.31). Similarly, the symmetric $\Phi(\cdot,\cdot)$ of (3.23) is differentiable and by (3.25),

$$\begin{aligned}\partial_2\Phi(s,t) &= \Psi(s,t) + \nu''(C(s,t))R(s,t)\,, \qquad (4.36)\\ \partial_2[\bar{C}(s,u)\Phi(t,u)] &= \bar{D}(s,u)\Phi(t,u) + \bar{C}(s,u)\Psi(t,u)\\ &\quad + \bar{C}(s,u)\nu''(C(t,u))R(t,u) + R(s,u)\Phi(t,u).\end{aligned}$$

Combining the latter with (3.29), then substituting into the LHS of (3.21) we get that for all $t, s \in [0,T]^2$,

$$\begin{aligned}\bar{D}(s,t) &= -f'(K(t))\bar{C}(t,s) + \int_0^{t\vee s} \Phi(t,u)R(s,u)du\\ &\quad + \int_0^{t\vee s} R(t,u)\nu''(C(t,u))\bar{C}(s,u)du\,.\end{aligned} \tag{4.37}$$

Similarly, comparing (3.24) and (3.26) it is easy to check that

$$\partial_2\Phi^1(s,u) - \Psi^1(s,u) = R(s,u)\Big[q(u)\nu''(C(s,u)) - \frac{q_\star^2\nu'(q(u))\nu''(q(s))}{\nu'(q_\star^2)}\Big],$$

which together with (3.28) and (3.20) (with $\mathsf{v}(\cdot) = \mathsf{v}_\star(\cdot)$), results with (1.19) (at $\beta = 1$). Further, combining (1.19) at $\beta = 1$, (4.35) and (4.37) at $t > s$ leads to

$$\partial_2 C(s,t) = R(s,t) - f'(K(t))C(t,s) + \int_0^t \Phi(t,u)R(s,u)du \\ + \int_0^t R(t,u)\Big[\nu''(C(t,u))C(s,u) - \frac{q(s)\nu'(q(u))\nu''(q(t))}{\nu'(q_\star^2)}\Big]du + \beta q(s)\mathsf{v}_\star'(q(t)) . \tag{4.38}$$

Noting that $R(s,u) = 0$ when $u > s$, whereas $\partial_1 C(s,t) = \partial_2 C(t,s)$, interchanging t and s in (4.38) results for $s > t$ with (1.18) at $\beta = 1$.

Since $K(s) = C(s,s)$, with $C(s,t) = C(t,s)$ and $\partial_2 C = D + R$ for $D := \bar{D} + q(s)Q(t)/q_\star^2$ (see (4.35)), it follows that for all $h > 0$,

$$K(s) - K(s-h) = \int_{s-h}^s (D(s,u) + R(s,u))du + \int_{s-h}^s (D(s-h,u) + R(s-h,u))du .$$

Recall that $R(s,u) = 0$ for $u > s$, hence, dividing by h and taking $h \downarrow 0$, we thus get by the continuity of D and that of R for $s \geq t$ that $K(\cdot)$ is differentiable, with

$$\partial_s K(s) = 2D(s,s) + R(s,s) = 2D(s,s) + 1 , \tag{4.39}$$

resulting by (4.37) with (1.20) for $\beta = 1$.

From the RHS of (3.27) we know that $(\partial_1 \bar{\chi})(u,t) = \bar{E}(u,t) + \mathbf{1}_{\{u<t\}}$, which together with (4.35) results for $s \geq t$, with

$$\bar{\chi}(u,t)\Phi(s,u)\mid_0^s = \int_0^s \Big[(\partial_1\chi)(u,t)\Phi(s,u) + \bar{\chi}(u,t)(\partial_2\Phi)(s,u)\Big]du \\ = \int_0^t \Phi(s,u)du + \int_0^s \bar{E}(u,t)\Phi(s,u)du \\ + \int_0^s \bar{\chi}(u,t)\big[\Psi(s,u) + \nu''(C(s,u))R(s,u)\big]du .$$

It thus follows from (3.30) and the LHS of (3.21) that for any $s \in [t,T]$,

$$\bar{E}(s,t) = -f'(K(s))\bar{\chi}(s,t) + \int_0^s \bar{\chi}(u,t)\nu''(C(s,u))R(s,u)du \tag{4.40}$$

(recall that $\bar{\chi}(0,t) = 0$). Thus, setting as in [10, (4.4)],

$$g(s,t) := -f'(K(s))R(s,t) + \int_0^s R(u,t)\nu''(C(s,u))R(s,u)du \tag{4.41}$$

for $s,t \in [0,T]^2$, we get (1.17) (at $\beta = 1$), by following [10, Page 31] (now with (4.40) and the RHS of (3.27) instead of [10, (4.3)] and [10, (1.18)], respectively).

5 Critical Points and the Conditional Model

In this section, using the Kac-Rice formula, we relate the dynamics of Theorem 1.5 corresponding to initial conditions distributed according to $\mu_{\boldsymbol{\sigma}}^{q_o}$ around a uniformly chosen critical point $\boldsymbol{\sigma}$ from $\mathscr{C}_{N,q_\star}(I_N, I'_N)$ to those of Theorem 1.1 that correspond to initial conditions distributed according to $\mu_{\mathbf{x}_\star}^{q_o}$ and the conditional disorder given $\mathsf{CP}(E, G, \mathbf{x}_\star)$.

Setting

$$\omega_N := \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

for the surface area of the $(N-1)$-dimensional unit sphere, we start with the following consequence of the Kac-Rice formula (of [1, Theorem 12.1.1]).

Proposition 5.1 *Let $(\boldsymbol{\sigma}, \mathbf{J}) \mapsto g_{\mathbf{J}}(\boldsymbol{\sigma})$ be a continuous mapping in $\mathbf{J}$ such that $\mathbb{E}[g_{\mathbf{J}}(\boldsymbol{\sigma})^2] < \infty$ and the field*

$$q_\star \mathbb{S}_N \ni \boldsymbol{\sigma} \mapsto (H_{\mathbf{J}}(\boldsymbol{\sigma}),\ \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}),\ g_{\mathbf{J}}(\boldsymbol{\sigma}))$$

has a.s. continuous sample functions and a law invariant to rotations. We then have for $\mathbf{x}_\star = (\sqrt{N}q_\star, 0, \ldots, 0)$, $CP(E, G, \mathbf{x}_\star)$ of (1.16), $\mathscr{C}_{N,q_\star}(I, I')$ of (1.25) and open intervals $I,\ I',\ I_0 \subset \mathbb{R}$, that

$$\begin{aligned} \mathbb{E}\#\left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I,\ I'\right) :\ g_{\mathbf{J}}(\boldsymbol{\sigma}) \in I_0\right\} &\le (\sqrt{N}q_\star)^{N-1}\omega_N\, \varphi_{\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\mathbf{x}_\star)}(0) \\ &\quad \times \int_{I\times I'} d\eta(E, G)\mathbb{E}\left\{\left|\det\left(\nabla^2_{\mathrm{sp}} H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right| \mathbf{1}\left\{g_{\mathbf{J}}(\mathbf{x}_\star) \in \bar{I}_0\right\} \,\middle|\, CP(E, G, \mathbf{x}_\star)\right\}, \end{aligned} \tag{5.1}$$

where $\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}) = \{F_i H_{\mathbf{J}}(\boldsymbol{\sigma})\}_{i=1}^{N-1}$ and $\nabla^2_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}) = \{F_i F_j H_{\mathbf{J}}(\boldsymbol{\sigma})\}_{i,j=1}^{N-1}$ for an arbitrary piecewise smooth orthonormal frame field $\{F_i\}$ on the sphere, with $\varphi_{\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\mathbf{x}_\star)}(0)$ denoting the Gaussian density of $\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\mathbf{x}_\star)$ at 0, while η denotes the joint law of $(-H_{\mathbf{J}}(\mathbf{x}_\star)/N, -\partial_\perp H_{\mathbf{J}}(\mathbf{x}_\star)/\|\mathbf{x}_\star\|)$ and $\bar{I}_0$ is the closure of I_0.

Remark 5.2 Under additional regularity conditions about $g_{\mathbf{J}}(\boldsymbol{\sigma})$, the variant of the Kac-Rice formula in [1, Theorem 12.1.1] would have implied that (5.1) holds with equality and with I_0 instead of $\bar{I}_0$ on the RHS.

Proof Recall that in the pure case of $\nu(r) = b_m r^m$ the value of $\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma})$ is determined by $H_{\mathbf{J}}(\boldsymbol{\sigma})$, whereas in the mixed case (i.e. any other $\nu(\cdot)$), the joint law of $(H_{\mathbf{J}}(\boldsymbol{\sigma}), \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}))$ is non-degenerate (c.f. the statement of Theorem 1.1). We assume hereafter that $\nu(\cdot)$ corresponds to a mixed case, leaving to the reader the modifications required for handling such degeneracy in the pure case.

Specifically, fixing $\epsilon, \delta > 0$ define $I_\delta = \{x + y : x \in I_0, |y| < \delta\}$ and $g^\epsilon_{\mathbf{J}}(\boldsymbol{\sigma}) = g_{\mathbf{J}}(\boldsymbol{\sigma}) + \epsilon Z$, where $Z \sim N(0, 1)$ is independent of $\boldsymbol{\sigma}$ and all other random variables. Note that $(\mathbf{J}, g^\epsilon_{\mathbf{J}}(\boldsymbol{\sigma}))$ has a continuous, strictly positive density $(J, x) \mapsto p_{\mathbf{J}}(J) p_Z(\frac{1}{\epsilon}(x - g_{\mathbf{J}}(\boldsymbol{\sigma})))$, where $p_{\mathbf{J}}$ and p_Z are the densities of $\mathbf{J}$ and Z. By [13, Section 4.1] the vector $(H_{\mathbf{J}}(\boldsymbol{\sigma}), \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}), \nabla_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}), \nabla^2_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}))$, which is measurable w.r.t $\mathbf{J}$, has a non-degenerate[7] Gaussian joint density. Therefore, the vector

$$(H_{\mathbf{J}}(\boldsymbol{\sigma}), \partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}), \nabla_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}), \nabla^2_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}), g^\epsilon_{\mathbf{J}}(\boldsymbol{\sigma}))$$

has a non-degenerate, strictly positive, continuous density.

Combining this with the assumptions made on $g_{\mathbf{J}}(\boldsymbol{\sigma})$, the formula (1.3) for the Hamiltonian and its rotation-invariant law, we conclude that with $f(\boldsymbol{\sigma}) = \nabla_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma})$, $\nabla f(\boldsymbol{\sigma}) = \nabla^2_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma})$,

$$h(\boldsymbol{\sigma}) = (-H_{\mathbf{J}}(\boldsymbol{\sigma})/N, -\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma})/(\sqrt{N}q_\star),\ g^\epsilon_{\mathbf{J}}(\boldsymbol{\sigma}))$$

[7] In the sense that the law of this array, when interpreting $\nabla^2_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma})$ as the corresponding upper triangular matrix, is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^{N(N-1)/2}$.

and $B = I \times I' \times I_\delta$ all the conditions of [1, Theorem 12.1.1] hold, except maybe the bound in condition (g) on the modulus of continuity of $g_{\mathbf{J}}^{\epsilon}(\boldsymbol{\sigma})$. However, in the current setting the latter condition is not necessary in order to conclude only the upper bound of [1, Eq. (12.1.4)], i.e., an inequality in the direction $\leq$, instead of an equality. Indeed, going through the proof of the upper bound of [1, Theorem 12.1.1] — which is based on the Euclidean version [1, Theorem 11.2.1] — one sees that the bound on the modulus of continuity of $h(\boldsymbol{\sigma})$ is only used when invoking [1, Lemma 11.2.12] to conclude that a.s. there is no point $\boldsymbol{\sigma}$ such that both $f(\boldsymbol{\sigma}) = 0$ and $h(\boldsymbol{\sigma}) \in \partial B$. However, the latter fact follows here directly from the definition of $g_{\mathbf{J}}^{\epsilon}(\boldsymbol{\sigma})$ and the fact the number of points such that $\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\boldsymbol{\sigma}) = 0$ is a.s. finite. Thanks to the assumed rotation-invariance, the upper bound of [1, Eq. (12.1.4)] that we have just stated simplifies to

$$\begin{aligned} \mathbb{E}\# \left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I,\, I'\right) :\, g_{\mathbf{J}}^{\epsilon}(\boldsymbol{\sigma}) \in I_\delta\right\} &\leq (\sqrt{N} q_\star)^{N-1} \omega_N\, \varphi_{\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\mathbf{x}_\star)}(0) \\ &\quad \times \mathbb{E}\left\{\left|\det\left(\nabla_{\mathrm{sp}}^2 H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right| \mathbf{1}\left\{h(\mathbf{x}_\star) \in B\right\} \,\middle|\, \nabla_{\mathrm{sp}} H_{\mathbf{J}}\left(\mathbf{x}_\star\right) = 0\right\}. \end{aligned} \tag{5.2}$$

Recalling [13, Section 4.1] that $(-H_{\mathbf{J}}(\mathbf{x}_\star)/N, -\partial_\perp H_{\mathbf{J}}\left(\mathbf{x}_\star\right)/(\sqrt{N} q_\star))$ and $\nabla_{\mathrm{sp}} H_{\mathbf{J}}\left(\mathbf{x}_\star\right)$ are independent, by further conditioning on the former we obtain from (5.2) that

$$\begin{aligned} \mathbb{E}\# \left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I,\, I'\right) :\, g_{\mathbf{J}}^{\epsilon}(\boldsymbol{\sigma}) \in I_\delta\right\} &\leq (\sqrt{N} q_\star)^{N-1} \omega_N\, \varphi_{\nabla_{\mathrm{sp}} H_{\mathbf{J}}(\mathbf{x}_\star)}(0) \\ &\quad \times \int_{I \times I'} d\eta(E, G) \mathbb{E}\left\{\left|\det\left(\nabla_{\mathrm{sp}}^2 H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right| \mathbf{1}\left\{g_{\mathbf{J}}^{\epsilon}(\mathbf{x}_\star) \in I_\delta\right\} \,\middle|\, \mathsf{CP}(E, G, \mathbf{x}_\star)\right\}. \end{aligned} \tag{5.3}$$

Let $\Xi_L(\epsilon, A)$ and $\Xi_R(\epsilon, A)$, respectively, denote the left- and right-hand side of (5.3), with general $A \subset \mathbb{R}$ instead of I_δ. Note that $\lim_{\epsilon \to 0^+} \mathbb{P}\{\epsilon Z < \delta\} = 1$ and

$$\mathbb{E}\# \left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I, I'\right) :\, g_{\mathbf{J}}(\boldsymbol{\sigma}) \in I_0\right\} \leq \frac{1}{\mathbb{P}\{\epsilon Z < \delta\}} \Xi_L\left(\epsilon, I_\delta\right).$$

Consequently, denoting by $\bar{I}_\delta$ the closure of I_δ, it follows from (5.3) that

$$\begin{aligned} \mathbb{E}\# \left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,1}\left(I, I'\right) :\, g_{\mathbf{J}}(\boldsymbol{\sigma}) \in I_0\right\} &\leq \lim_{\delta \to 0^+} \varlimsup_{\epsilon \to 0^+} \Xi_L\left(\epsilon, I_\delta\right) \leq \lim_{\delta \to 0^+} \varlimsup_{\epsilon \to 0^+} \Xi_R\left(\epsilon, \bar{I}_\delta\right) \\ &\leq \lim_{\delta \to 0^+} \Xi_R\left(0, \bar{I}_\delta\right) = \Xi_R\left(0, \bar{I}_0\right) \end{aligned}$$

where the last inequality holds since $g_{\mathbf{J}}^{\epsilon}(\boldsymbol{\sigma}) \overset{a.s.}{\to} g_{\mathbf{J}}\left(\boldsymbol{\sigma}\right)$, as $\epsilon \to 0$ and the indicator function of $\bar{I}_\delta$ is upper semi-continuous, while the equality holds due to monotone convergence. This completes the proof. □

For G large enough, the determinant on the RHS of (5.1) is uniformly integrable in N and the expectation of the determinant and the indicator can be separated, yielding the following lemma.

Lemma 5.3 *Assume that $g_{\mathbf{J}}(\boldsymbol{\sigma})$ satisfies (5.1). Let $\mathbf{x}_\star = (\sqrt{N} q_\star, 0, \ldots, 0)$, $I_N, I'_N \in \mathbb{R}$ be a pair of open intervals as in Theorem 1.5 and $I_0 \subset \mathbb{R}$ a fixed open interval. If it holds that*

$$\lim_{N \to \infty} \sup_{E \in I_N} \sup_{G \in I'_N} \mathbb{P}\left\{g_{\mathbf{J}}(\mathbf{x}_\star) \in \bar{I}_0 \,\middle|\, CP(E, G, \mathbf{x}_\star)\right\} = 0, \tag{5.4}$$

then in addition

$$\lim_{N \to \infty} \frac{\mathbb{E}\# \left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I_N, I'_N\right) :\, g_{\mathbf{J}}(\boldsymbol{\sigma}) \in I_0\right\}}{\mathbb{E}\# \mathscr{C}_{N,q_\star}\left(I_N, I'_N\right)} = 0. \tag{5.5}$$

Proof From (5.1) we have an upper bound for the numerator of (5.5). By an application of the Kac-Rice formula [1, Theorem 12.1.1], the denominator of (5.5) is equal to the RHS of (5.1) with the indicator omitted. Thus, to complete the proof it suffices to show that

$$\lim_{N\to\infty}\sup_{E\in I_N}\sup_{G\in I'_N}\frac{\mathbb{E}\left\{\left|\det\left(\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right|\mathbf{1}\left\{g_{\mathbf{J}}(\mathbf{x}_\star)\in\bar{I}_0\right\}\,\middle|\,\mathsf{CP}(E,G,\mathbf{x}_\star)\right\}}{\mathbb{E}\left\{\left|\det\left(\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right|\,\middle|\,\mathsf{CP}(E,G,\mathbf{x}_\star)\right\}}=0.$$

By (5.4) and the Cauchy-Schwarz inequality, it is therefore enough to show that

$$\limsup_{N\to\infty}\sup_{E\in I_N}\sup_{G\in I'_N}\frac{\mathbb{E}\left\{\left|\det\left(\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right|^2\,\middle|\,\mathsf{CP}(E,G,\mathbf{x}_\star)\right\}}{\left(\mathbb{E}\left\{\left|\det\left(\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\right)\right|\,\middle|\,\mathsf{CP}(E,G,\mathbf{x}_\star)\right\}\right)^2}<\infty. \tag{5.6}$$

To this end, recall [13, Section 4.1], that conditional on $\mathsf{CP}(E,G,\mathbf{x}_\star)$,

$$\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)\overset{d}{=}\sqrt{\frac{N-1}{N}\nu''(q_\star^2)}\,\mathbf{M}+G\,\mathbf{I},$$

where $\mathbf{M}$ is a normalized $(N-1)$-dimensional GOE matrix, i.e., a real symmetric matrix with independent centered Gaussian entries (up to symmetry), such that

$$\mathbb{E}\mathbf{M}_{ij}^2=\begin{cases}2/(N-1), & i=j\\ 1/(N-1), & i\neq j.\end{cases}$$

We have assumed that $\inf I'_N\to G_\star>2\sqrt{\nu''(q_\star^2)}$. Thus, the conditional distribution of $\nabla^2_{\mathrm{sp}}H_{\mathbf{J}}\left(\mathbf{x}_\star\right)$ is identical to that of a shifted (scaled) GOE matrix whose eigenvalues are bounded away from 0, uniformly in $G\in I'_N$ (and $E\in I_N$). Considering [34, Corollary 23] (at $k=2$), this yields (5.6), thereby completing the proof. □

Recall the joint law $\mathbb{P}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}}$ on $\mathcal{C}(\mathbb{R}^+,\mathbb{R}^{2N})$, of $\mathbf{B}_t$ and the corresponding strong solution $\mathbf{x}_t$ of (1.1) for initial conditions $\mathbf{x}_0$ distributed per $\mu^{q_o}_{\boldsymbol{\sigma}}$ (see Proposition 3.8), denoting by $\mathbb{E}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}}$ the corresponding expectation.

Lemma 5.4 *For* $\mathsf{Err}_{N,T}(\boldsymbol{\sigma})$ *of* (1.26), *the function*

$$(\boldsymbol{\sigma},\mathbf{J})\mapsto\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}):=\mathbb{E}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}}\left[\mathsf{Err}_{N,T}(\boldsymbol{\sigma})\right] \tag{5.7}$$

satisfies the conditions of Proposition 5.1. *Further,* (5.4) *then holds for any open intervals* I_N, I'_N *as in Theorem 1.5, and any fixed open interval* I_0 *such that* $0\notin\bar{I}_0$.

Proof Clearly $\bar{g}_{\mathbf{J}}\in[0,4]$, is uniformly bounded. The continuity of $\boldsymbol{\sigma}\mapsto(H_{\mathbf{J}}(\boldsymbol{\sigma}),\,\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}))$ follows for example from the representation (1.3). The invariance of the law of $(H_{\mathbf{J}}(\boldsymbol{\sigma}),\,\partial_\perp H_{\mathbf{J}}(\boldsymbol{\sigma}),\,\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}))$ under rotations follows by the argument detailed in Remark 1.3. Turning to show that $(\boldsymbol{\sigma},\mathbf{J})\mapsto\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma})$ is a.s. continuous, upon fixing N and the driving Brownian motion $\mathbf{B}$ we have by the triangle inequality and Cauchy-Schwarz, that

$$|\,\mathsf{Err}_{N,T}(\boldsymbol{\sigma}\,|\,\mathbf{x}_0,\mathbf{J})-\,\mathsf{Err}_{N,T}(\widetilde{\boldsymbol{\sigma}}\,|\,\widetilde{\mathbf{x}}_0,\widetilde{\mathbf{J}})|\leq L_1\|\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}\|_2+L_2\|\mathbf{J}-\widetilde{\mathbf{J}}\|+L_3\sqrt{1\wedge\|e_N\|_\infty}$$

where $e_N(s):=N^{-1}\|\mathbf{x}_s-\widetilde{\mathbf{x}}_s\|_2^2$, $L_1:=N^{-1/2}\|K_N\|_\infty^{1/2}$, $L_2:=\widetilde{c}\beta N^{-1/2}(1+\|K_N\|_\infty^{m/2})$ and

$$L_3:=4+\|K_N\|_\infty^{1/2}+\|\widetilde{K}_N\|_\infty^{1/2}+\|B_N\|_\infty^{1/2}+c\beta\|\widetilde{\mathbf{J}}\|_\infty^N(1+\|K_N\|_\infty^r)(1+\|\widetilde{K}_N\|_\infty^r)\,,$$

for $\widetilde{c} = \sqrt{\nu(1)}$, the finite constants c, r from (3.47) and with the L_2-norm $\|\mathbf{J}\|$ which is normalized as in (3.1). Next, fixing $\boldsymbol{\sigma} \in q_\star \mathbb{S}_N$, to jointly produce $\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma})$ and $\bar{g}_{\mathbf{J}}(\widetilde{\boldsymbol{\sigma}})$ for arbitrary $\widetilde{\boldsymbol{\sigma}} \in q_\star \mathbb{S}_N$, let $\widetilde{\mathbf{O}}$ be an orthogonal matrix which only rotates the space spanned by $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$ (i.e., $\widetilde{\mathbf{O}}\mathbf{x} = \mathbf{x}$ if $\langle \mathbf{x}, \boldsymbol{\sigma} \rangle = \langle \mathbf{x}, \widetilde{\boldsymbol{\sigma}} \rangle = 0$), such that $\widetilde{\mathbf{O}}\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}}$. Then,

$$\sup_{\mathbf{x} \in \mathbb{S}_N} \|\widetilde{\mathbf{O}}\mathbf{x} - \mathbf{x}\|_2 = \sup_{\mathbf{x} \in \mathbb{S}_N \cap \mathrm{sp}\{\boldsymbol{\sigma}, \widetilde{\boldsymbol{\sigma}}\}} \|\widetilde{\mathbf{O}}\mathbf{x} - \mathbf{x}\|_2 = \frac{1}{q_\star} \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_2 .$$

Drawing $\mathbf{x}_0$ from law $\mu_{\boldsymbol{\sigma}}^{q_o}$, we set $\widetilde{\mathbf{x}}_0 := \widetilde{\mathbf{O}}\mathbf{x}_0$ as the initial condition of laws $\mu_{\widetilde{\boldsymbol{\sigma}}}^{q_o}$, noting that by design $\|\mathbf{x}_0 - \widetilde{\mathbf{x}}_0\|_2 \le \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_2 / q_\star$. Utilizing this coupling and Cauchy-Schwarz, yields that

$$|\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}) - \bar{g}_{\widetilde{\mathbf{J}}}(\widetilde{\boldsymbol{\sigma}})| \le \int \Big\{ \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_2 \, \mathbb{E}[L_1 | \mathbf{J}, \mathbf{x}_0] + \|\mathbf{J} - \widetilde{\mathbf{J}}\| \, \mathbb{E}[L_2 | \mathbf{J}, \mathbf{x}_0] + \big\{ \mathbb{E}[1 \wedge \|e_N\|_\infty | \mathbf{J}, \widetilde{\mathbf{J}}, \mathbf{x}_0] \, \mathbb{E}[L_3^2 | \mathbf{J}, \widetilde{\mathbf{J}}, \mathbf{x}_0] \big\}^{1/2} \Big\} d\mu_{\boldsymbol{\sigma}}^{q_o}(\mathbf{x}_0) .$$

From (3.41) we deduce that $\int \mathbb{E}[L_i | \mathbf{J}, \mathbf{x}_0] d\mu_{\boldsymbol{\sigma}}^{q_o}(\mathbf{x}_0)$, $i = 1, 2$, are a.s. finite. Further, fixing a sequence $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\mathbf{J}}) \to (\boldsymbol{\sigma}, \mathbf{J})$, necessarily also $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}) \to (\mathbf{x}_0, \mathbf{J})$. In view of (3.38), this implies a uniform, over $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\mathbf{J}})$, bound on $\mathbb{E}[\|\widetilde{K}_N\|_\infty^k | \widetilde{\mathbf{J}}, \mathbf{x}_0]$. Thereby, such uniform bound applies also for $\int \mathbb{E}[L_3^2 | \mathbf{J}, \widetilde{\mathbf{J}}, \mathbf{x}_0] d\mu_{\boldsymbol{\sigma}}^{q_o}(\mathbf{x}_0)$, with (3.51) yielding the a.s. continuity of $\bar{g}_{\mathbf{J}}(\boldsymbol{\sigma})$.

Next, setting $\widetilde{g}_{\mathbf{J}}(\boldsymbol{\sigma}) := \mathbb{E}_{\mathbf{J},\boldsymbol{\sigma}}^{N,q_o}[\,\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) \mathbf{1}_{\mathcal{L}_{N,M}}]$, we have in view of (3.46) and (5.7), that

$$\lim_{N\to\infty} \sup_{E \in I_N} \sup_{G \in I'_N} \mathbb{E}\Big[|\bar{g}_{\mathbf{J}}(\mathbf{x}_\star) - \widetilde{g}_{\mathbf{J}}(\mathbf{x}_\star)| \,\Big|\, \mathsf{CP}(E, G, \mathbf{x}_\star) \Big] = 0 .$$

We thus establish (5.4) whenever $0 \notin \bar{I}_0$, once we show that in such a case

$$\lim_{N\to\infty} \sup_{E \in I_N} \sup_{G \in I'_N} \mathbb{P}\Big\{ \widetilde{g}_{\mathbf{J}}(\mathbf{x}_\star) \in \bar{I}_0 \,\Big|\, \mathsf{CP}(E, G, \mathbf{x}_\star) \Big\} = 0. \tag{5.8}$$

To this end, recall from our proof of Proposition 3.8, that given $\mathsf{CP}(E, G, \mathbf{x}_\star)$ one has $\mathbf{J} = \mathbf{J}_o + \bar{\mathbf{J}}_{E,G}$ where the law of $\mathbf{J}_o$ is independent of (E, G) and the only non-zero entries of $\bar{\mathbf{J}}_{E,G} = \mathbb{E}[\mathbf{J} \,|\, \mathsf{CP}(E, G, \mathbf{x}_\star)]$ are given by (3.32). Hence,

$$\lim_{N\to\infty} \sup_{E \in I_N} \sup_{G \in I'_N} \{ N^{-1} \|(\mathbf{x}_0, \mathbf{J}_o + \bar{\mathbf{J}}_{E,G}, \mathbf{B}) - (\mathbf{x}_0, \mathbf{J}_o + \bar{\mathbf{J}}_{E_\star,G_\star}, \mathbf{B})\|^2 \}$$
$$= \lim_{N\to\infty} \sup_{E \in I_N} \sup_{G \in I'_N} \sum_{p=2}^{m} (b_p q_\star^p \, \langle \mathsf{v}_p, (E - E_\star, G - G_\star) \rangle)^2 = 0.$$

The Lipschitz property (3.50) then implies that

$$\lim_{N\to\infty} \sup_{E \in I_N} \sup_{G \in I'_N} |\widetilde{g}_{\mathbf{J}_o + \bar{\mathbf{J}}_{E,G}}(\mathbf{x}_\star) - \widetilde{g}_{\mathbf{J}_o + \bar{\mathbf{J}}_{E_\star,G_\star}}(\mathbf{x}_\star)| = 0 ,$$

whereas from the L_1-convergence in Theorem 1.1 we deduce that

$$\lim_{N\to\infty} \mathbb{P}\Big\{ \widetilde{g}_{\mathbf{J}}(\mathbf{x}_\star) \in \bar{I}_0 \,\Big|\, \mathsf{CP}(E_\star, G_\star, \mathbf{x}_\star) \Big\} = 0.$$

Finally, note that combining the preceding two displays results with (5.8). □

Proof of Theorem 1.5 With $\bar{g}_{\mathbf{J}} \in [0, 4]$, by Markov's inequality, for any $\delta, \epsilon > 0$,

$$\mathbb{E}\Big\{ \sum_{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}(I_N, I'_N)} \mathbb{P}_{\mathbf{J},\boldsymbol{\sigma}}^{N,q_o}(\,\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) > \epsilon) \Big\} \le \frac{1}{\epsilon} \mathbb{E}\Big\{ \sum_{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}(I_N, I'_N)} \bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}) \Big\}$$

$$\le \frac{\delta}{\epsilon}\mathbb{E}\#\left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I_N, I'_N\right)\right\} + \frac{4}{\epsilon}\mathbb{E}\#\left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I_N, I'_N\right) : \bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}) > \delta\right\}.$$

In addition, for any $\delta > 0$ it follows from Lemmas 5.3 and 5.4, that

$$\lim_{N\to\infty} \frac{\mathbb{E}\#\left\{\boldsymbol{\sigma} \in \mathscr{C}_{N,q_\star}\left(I_N, I'_N\right) : \bar{g}_{\mathbf{J}}(\boldsymbol{\sigma}) > \delta\right\}}{\mathbb{E}\#\mathscr{C}_{N,q_\star}\left(I_N, I'_N\right)} = 0.$$

Combining the above and taking $N \to \infty$ followed by $\delta \to 0$ results with (1.27).

Next, denoting by Y_a the indicator of the event that

$$\#\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right) > a\mathbb{E}\left\{\#\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)\right\},$$

we have by Markov's inequality, that for any $\delta > 0$,

$$\mathbb{P}\Big\{\frac{Y_a}{\#\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \sum_{\boldsymbol{\sigma}\in\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \mathbb{P}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}}\left(\,\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) > \epsilon\right) > \delta\Big\}$$
$$\le \frac{1}{a\delta\mathbb{E}\left\{\#\mathscr{C}_{N,q_\star}\left(I_N,\ I'_N\right)\right\}}\mathbb{E}\Big\{\sum_{\boldsymbol{\sigma}\in\mathscr{C}_{N,q_\star}(I_N,\ I'_N)} \mathbb{P}^{N,q_o}_{\mathbf{J},\boldsymbol{\sigma}}\left(\,\mathsf{Err}_{N,T}(\boldsymbol{\sigma}) > \epsilon\right)\Big\} \overset{N\to\infty}{\longrightarrow} 0,$$

from which (1.29) follows. □

6 Proof of Proposition 1.6

As $\chi(s,t) = \int_0^t R(s,u)du$ is the limit of $\chi_N(s,t)$, it follows from the definition (1.9) of χ_N that

$$|\int_{t_1}^{t_2} R(s,u)du|^2 \le K(s)(t_2 - t_1)\,, \qquad 0 \le t_1 \le t_2 \le s < \infty\,. \tag{6.1}$$

Likewise, the limit $\bar{C}(s,t) = C(s,t) - q(s)q(t)/q_\star^2$ of the empirical correlation functions $\bar{C}_N(s,t)$ must be a non-negative definite kernel on $\mathbb{R}_+ \times \mathbb{R}_+$. In particular, $C(s,t)^2 \le K(s)K(t)$, whereas by (3.41) we have that $\sup_{t\ge 0} K(t) < \infty$. Unlike the special case considered in [24, Proposition 1.1], here the functions (C, R) may take negative values. Nevertheless, we next show that if $(R^{(L)}, C^{(L)}, q^{(L)}, K^{(L)})$ are solutions of the system (1.17)–(1.20) with $K^{(L)}(0) = 1$ and potential $f_L(\cdot)$ as in (1.12) with $\boldsymbol{\varphi} = 1 + 2\beta q_o \mathsf{v}'_\star(q_o) > 0$, then $K^{(L)}(s) \to 1$ as $L \to \infty$, uniformly over $s \ge 0$.

Lemma 6.1 *Assuming $K^{(L)}(0) = 1$, there exist $B < \infty$, such that for all $L \ge B$,*

$$\sup_{s\ge 0} |K^{(L)}(s) - 1| \le \frac{B}{2L}\,. \tag{6.2}$$

Proof First note that for some $B_0 = B_0(\boldsymbol{\varphi}, k)$ finite and any $B \in [B_0, L]$,

$$g_L(r) := 1 - 2f'_L(r)r = 1 + 4Lr(1-r) - \boldsymbol{\varphi}r^{2k} \tag{6.3}$$

satisfies $g_L(1 - B/(2L)) \ge B/2$ and $g_L(1 + B/(2L)) \le -B/2$. Further, from (4.39) and the LHS of (3.19)–(3.21) we see that

$$\partial_s K^{(L)}(s) = 1 + 2D^{(L)}(s,s) = g_L(K^{(L)}(s)) + 2\beta \mathsf{A}^{(L)}(s,s)\,, \tag{6.4}$$

where it is easy to verify that (in terms of $V(\cdot)$ and $\bar{A}(\cdot,\cdot)$ of (3.28) and (3.29)),

$$\mathsf{A}(s,t) := q(t)\mathsf{v}'_\star(q(s)) + \beta\bar{A}(s,t) + \beta q(t)V(s)/q_\star^2 = \lim_{N\to\infty} \widetilde{\mathbb{E}}\Big[\frac{1}{N}\sum_{i=1}^N G^i(\mathbf{x}_s)x_t^i\Big].$$

Recall [10, (2.15)], that for some universal constant $c < \infty$ any s, $\mathbf{J}$ and N,

$$G_N(s) := \frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_s)|^2 \le c(\|\mathbf{J}\|_\infty^N)^2[1 + K_N(s)^{m-1}].$$

Hence, by Cauchy-Schwarz inequality and (3.39) (at $k = 4$), it follows that for some other universal constant $\kappa < \infty$ (which is independent of L),

$$\begin{aligned}|\mathsf{A}(s,s)|^2 &\le \lim_{N\to\infty} \widetilde{\mathbb{E}}[G_N(s)K_N(s)]\\ &\le c \lim_{N\to\infty} \widetilde{\mathbb{E}}\big[(\|\mathbf{J}\|_\infty^N)^2(K_N(s) + K_N(s)^m)\big] \le \kappa(K(s) + K(s)^m)\end{aligned}$$

(in the last step we relied also on Corollary 3.2). We thus have, similarly to [24, (2.3)], that for all s and L,

$$|\partial_s K^{(L)}(s) - g_L(K^{(L)}(s))|^2 \le (2\beta)^2\kappa[K^{(L)}(s) + K^{(L)}(s)^m].$$

Our claim (6.2) then follows as in [24, proof of Lemma 2.2] (employing the argument used there for $K^{(L)} \ge 1$, to handle now also the case $K^{(L)} \le 1$). □

Adapting the proof of [24, Lemma 2.3], we next establish the equi-continuity and uniform boundedness of $(R^{(L)}, C^{(L)}, K^{(L)}, q^{(L)})$, which thereby admit limit points (R, C, K, q).

Lemma 6.2 *Set $\mu^{(L)}(s) := f'_L(K^{(L)}(s))$ and $\hat{h}^{(L)}(s) := \partial_s K^{(L)}(s)$. Then $(R^{(L)}, C^{(L)}, q^{(L)}, K^{(L)}, \mu^{(L)}, \hat{h}^{(L)})$ and their derivatives are bounded uniformly in $L \ge B$ (of Lemma 6.1) and over Δ_T.*

Proof With $|C^{(L)}(s,t)| \le \sqrt{K^{(L)}(s)K^{(L)}(t)}$ and $|q^{(L)}(s)| \le \sqrt{K^{(L)}(s)}$, the bound (6.2) on $K^{(L)}$ results for $L \ge B$ with $C^{(L)}, q^{(L)} \in [-2,2]$. Further, then $|\mu^{(L)}(s)| \le 2B + |\boldsymbol{\varphi}|2^{k-1}$ (see [24, proof of Lemma 2.3]). In view of (6.4),

$$\hat{h}^{(L)}(s) = 1 - 2K^{(L)}(s)\mu^{(L)}(s) + 2\beta\mathsf{A}^{(L)}(s,s), \tag{6.5}$$

yielding in turn the uniform boundedness of $\hat{h}^{(L)}(s)$.

Since (1.17) matches [24, (1.7)], it follows that for the function $H_L(s,t)$ of [24, (2.2)],

$$R^{(L)}(s,t) = \Lambda_L(s,t)H_L(s,t), \qquad \Lambda_L(s,t) = \exp(-\int_t^s \hat{\mu}^{(L)}(u)du), \qquad \forall (s,t) \in \Delta_T. \tag{6.6}$$

Recall that $\nu''(\cdot)$ is uniformly bounded on the compact $[-2,2]$, hence H_L of [24, (2.2)] is uniformly bounded over Δ_T and $L \ge B$, and thereby the same applies for $R^{(L)}$.

Upon replacing $f'_L(K^{(L)}(s))$ by $\mu^{(L)}(s)$ in (1.17)–(1.20), we deduce from our preceding statements the claimed uniform boundedness for $\partial_s q^{(L)}$, $\partial_s K^{(L)}$, $\partial_s C^{(L)}(s,t)$ and $\partial_s R^{(L)}(s,t)$, when $s \ge t$. Following [24, proof of Lemma 2.3], the same applies for $\partial_t H_L(s,t)$ and consequently for $\partial_t R^{(L)}(s,t)$. Further, from (3.23), such uniform boundedness applies to $\bar{D}^{(L)}(s,t)$ of (4.37), hence by (4.35) also to

$$\partial_t C^{(L)}(s,t) = \bar{D}^{(L)}(s,t) + R^{(L)}(s,t) + (q^{(L)}(s)/q_\star^2)\partial_t q^{(L)}(t).$$

Next, $\partial_s \hat{h}^{(L)}(s) = -4L\hat{h}^{(L)}(s) + \kappa_L(s)$ for

$$\kappa_L(s) := (g_L'(K^{(L)}(s)) + 4L)\hat{h}^{(L)}(s) + 2\beta\partial_s \mathsf{A}^{(L)}(s,s)\,.$$

In view of (6.3) we have that $|g_L'(r) + 4L| \le 4B + k|\boldsymbol{\varphi}|2^{2k}$ whenever $|r-1| \le B/(2L) \le 1/2$, while $|\partial_s A^{(L)}(s,s)|$ is bounded uniformly in $L \ge B$ and $s \le T$ (by (1.20) and the uniform boundedness of $(R^{(L)}, C^{(L)}, q^{(L)})$ and $\partial_s(R^{(L)}, C^{(L)}, q^{(L)})$). In particular, $\alpha(T) := \sup\{|\kappa_L(u)| : L \ge B, u \le T\}$ is finite. Next, recall (6.4) that $K^{(L)}(0) = 1$ and $g_L(1) = 1 - \boldsymbol{\varphi}$ (see (6.3)), resulting for our choice of $\boldsymbol{\varphi} = 1 + 2\beta q_o \mathsf{v}_\star'(q_o) = 1 + 2\beta \mathsf{A}^{(L)}(0,0)$ with $\hat{h}^{(L)}(0) = 0$. Thus,

$$\hat{h}^{(L)}(s) = \int_0^s e^{-4L(s-u)}\kappa_L(u)du$$

yielding that

$$\sup_{s\in[0,T]} |\hat{h}^{(L)}(s)| \le \frac{\alpha(T)}{4L}\,, \qquad \forall L \ge B\,, \tag{6.7}$$

from which the uniform boundedness of $|\partial_s \hat{h}^{(L)}|$ follows. Finally, by definition, for our choice of $f_L(\cdot)$,

$$\partial_s \mu^{(L)}(s) = f_L''(K^{(L)}(s))\hat{h}^{(L)}(s) = \left[2L + \frac{(2k-1)\boldsymbol{\varphi}}{2}K^{(L)}(s)^{2k-2}\right]\hat{h}^{(L)}(s)\,,$$

which by (6.7) provides the uniform boundedness of $|\partial_s \mu^{(L)}|$. □

Proof of Proposition 1.6 Recall Lemma 6.2 that $(R^{(L)}, C^{(L)}, q^{(L)}, K^{(L)}, \mu^{(L)}, \hat{h}^{(L)})$, $L \ge B$ are equi-continuous and uniformly bounded on Δ_T. Hence, by the Arzela-Ascoli theorem, this collection has a limit point $(C, R, q, K, \mu, \hat{h})$ with respect to uniform convergence on Δ_T.

By Lemma 6.1 we know that the limit $K(s) \equiv 1$ on $[0,T]$, whereas by (6.7) we have that $\hat{h}(s) \equiv 0$ on $[0,T]$. Considering $L_n \to \infty$ for which $(R^{(L_n)}, C^{(L_n)}, q^{(L_n)}, K^{(L_n)}, \mu^{(L_n)}, \hat{h}^{(L_n)})$ converges to $(R, C, q, K, \mu, \hat{h})$ we find that the latter must satisfy (1.33). Further, since $R^{(L)}(t,t) = 1$, $C^{(L)}(t,t) = K^{(L)}(t)$ and $q^{(L)}(0) = q_o$, integrating (1.17)–(1.19) we see that $R^{(L)}(s,t) = 1 + \int_t^s \mathsf{A}_R^{(L)}(\theta,t)d\theta$, $C^{(L)}(s,t) = K^{(L)}(t) + \int_t^s \mathsf{A}_C^{(L)}(\theta,t)d\theta$ and $q^{(L)}(s) = q_o + \int_0^s \mathbf{A}_q^{(L)}(\theta)d\theta$, where

$$\begin{aligned}
&\mathsf{A}_R^{(L)}(\theta,t) := -\mu^{(L)}(\theta)R^{(L)}(\theta,t) + \beta^2\int_t^\theta R^{(L)}(u,t)R^{(L)}(\theta,u)\nu''(C^{(L)}(\theta,u))du,\\
&\mathsf{A}_C^{(L)}(\theta,t) := -\mu^{(L)}(\theta)C^{(L)}(\theta,t) + \beta\mathsf{A}^{(L)}(\theta,t)\,,\\
&\mathsf{A}_q^{(L)}(\theta) := -\mu^{(L)}(\theta)q^{(L)}(\theta)\\
&+\beta^2\int_0^\theta R^{(L)}(\theta,u)\Big[q^{(L)}(u)\nu''(C^{(L)}(\theta,u)) - \frac{q_\star^2\nu'(q^{(L)}(u))\nu''(q^{(L)}(\theta))}{\nu'(q_\star^2)}\Big]du + \beta q_\star^2\mathsf{v}_\star'(q^{(L)}(\theta))\,.
\end{aligned}$$

Note that $\mu^{(L_n)}(s) \to \mu(s)$, while $\mathsf{A}_R^{(L_n)}(s,t)$, $\mathsf{A}_C^{(L_n)}(s,t)$ and $\mathsf{A}_q^{(L_n)}(s,t)$ converge, uniformly on Δ_T, to the right-hand-sides of (1.30)–(1.32), respectively. We thus deduce that for each limit point (C, R, q, μ), the functions $C(s,t)$, $R(s,t)$ and $q(s)$ are differentiable in s on Δ_T and all limit points satisfy (1.30)–(1.33). Further, $C^{(L)}(s,t)$ are non-negative definite kernels with $C^{(L)}(t,t) \to 1$ as $L \to \infty$. Consequently, each of their limit points corresponds to a $[-1,1]$-valued non-negative kernel on $[0,T]^2$. Similarly, as $R^{(L)}(t,t) = 1$ and $R^{(L)}(s,t)$

satisfy (6.1), both constraints apply for any limit point $R(s,t)$. We further extend $R(\cdot,\cdot)$ to a function on $[0,T]^2$ by setting $R(s,t) = R^{(L)}(s,t) = 0$ whenever $s < t$.

With $\widehat{H}(\cdot)$ a continuous functional of (R, C, q), it remains only to verify that the system of equations (1.30)–(1.33) with $q(0) = q_o$, $C(s,t) = C(t,s)$, $C(t,t) = R(t,t) = 1$ and $R(s,t) = 0$ for $s < t$, admits at most one bounded solution (R, C, q) on $[0,T]^2$. To this end consider the difference between the integrated form of (1.30)–(1.32) for two such solutions (C, R, q) and $(\bar{C}, \bar{R}, \bar{q})$. Since ν'', ν', $\mathsf{v}'_\star$ are locally Lipschitz, we get as in [24, proof of Prop. 1.1], that $\Delta R = |R - \bar{R}|$, and $\Delta C = |C - \bar{C}| + |q(s) - \bar{q}(s)| + |q(t) - \bar{q}(t)|$ satisfy on Δ_T

$$\Delta R(s,t) \le \kappa_1 \Big\{ \int_t^s [\Delta R(v,t) + \Delta C(v,t)] dv + \int_t^s h(v) dv \Big\},$$

$$\Delta C(s,t) \le \kappa_1 \Big[\int_t^s \Delta C(v,t) dv + h(t) + \int_t^s h(v) dv \Big],$$

where $h(v) := \int_0^v [\Delta R(v,u) + \Delta C(v,u)] du$ and $\kappa_1 < \infty$ depends on T, β, $\nu(\cdot)$, $\mathsf{v}'_\star(\cdot)$ and the maximum of $|R|$, $|C|$, $|q|$, $|\bar{R}|$, $|\bar{C}|$ and $|\bar{q}|$ on $[0,T]^2$. Integrating these inequalities over $t \in [0,s]$, since $\Delta R(v,u) = 0$ for $u \ge v$ and $\Delta C(v,u) = \Delta C(u,v)$, we find similarly to [24, Page 860], that

$$0 \le h(s) \le 2\kappa_2 \int_0^s h(v) dv, \qquad h(0) = 0,$$

for some finite constant κ_2 (of the same type of dependence as κ_1). By Gronwall's lemma we deduce that $h \equiv 0$ on $[0,T]$, hence $\Delta R(s,t) = \Delta C(s,t) = 0$ for a.e. $(s,t) \in \Delta_T$. By the continuity and symmetry of these functions, the same applies for all $(s,t) \in [0,T]^2$, yielding the stated uniqueness and thereby completing the proof. □

7 Proof of Proposition 2.1

Consider the convex set $\mathcal{A}^+$ of *bounded* continuous functions $(R, C, q) \in \mathcal{C}_b(\Delta_\infty) \times \mathcal{C}_b(\mathbb{R}_+^2) \times \mathcal{C}_b(\mathbb{R}_+)$ such that $C(s,t) = C(t,s)$, $R(s,s) = C(s,s) = 1$ and $q(0) = q_o$, equipped with the norm

$$\|(R, C, q)\| = \sup_{(s,t)\in\Delta_\infty} |R(s,t)| + \sup_{(s,t)\in\Delta_\infty} |C(s,t)| + \sup_{s\ge 0} |q(s)|. \tag{7.1}$$

Analogously to [24, (4.1)–(4.3)], we recall from Proposition 1.6 that (R, C, q) of (1.30)–(1.33) is the unique fixed point of the mapping $\Psi : (R, C, q) \mapsto (\widetilde{R}, \widetilde{C}, \widetilde{q})$ on $\mathcal{A}^+$ such that for any $(s,t) \in \Delta_\infty$

$$\partial_s \widetilde{R}(s,t) = -\mu(s)\widetilde{R}(s,t) + \beta^2 \int_t^s \widetilde{R}(u,t)\widetilde{R}(s,u)\nu''(C(s,u)) du, \tag{7.2}$$

$$\partial_s \widetilde{C}(s,t) = -\mu(s)\widetilde{C}(s,t) + \beta^2 I_1(s,t) + \beta^2 I_2(s,t), \tag{7.3}$$

$$\partial_s \widetilde{q}(s) = -\mu(s)\widetilde{q}(s) + \beta^2 I_3(s), \tag{7.4}$$

with $\mu(s) = \mu_{(R,C,q)}(s) = \frac{1}{2} + \beta^2 I_0(s)$ of (1.33) and

$$I_0(t) := \int_{-t}^0 R(t, t+u) \Big[\psi(C(t, t+u))$$

$$
\begin{aligned}
&\quad - \frac{\psi(q(t))\nu'(q(t+u))}{\nu'(q_\star^2)}\Big] du \\
&\quad + \beta^{-1} q(t) \mathsf{v}_\star'(q(t)) , \\
I_1(t+v,t) &:= \int_{-t}^{v} R(t+v,t+u) \\
&\quad \Big[\nu''(C(t+v,t+u))C(t+u,t) \\
&\quad - \frac{q(t)\nu'(q(t+u))\nu''(q(t+v))}{\nu'(q_\star^2)}\Big] du , \\
I_2(t+v,t) &:= \int_{-t}^{0} R(t,t+u)\Big[\nu'(C(t+v,t+u)) \\
&\quad - \frac{\nu'(q(t+v))\nu'(q(t+u))}{\nu'(q_\star^2)}\Big] du \\
&\quad + \beta^{-1} q(t) \mathsf{v}_\star'(q(t+v)) , \\
I_3(t) &:= \int_{-t}^{0} R(t,t+u)\Big[q(t+u)\nu''(C(t,t+u)) \\
&\quad - \frac{q_\star^2 \nu'(q(t+u))\nu''(q(t))}{\nu'(q_\star^2)}\Big] du \\
&\quad + \beta^{-1} q_\star^2 \mathsf{v}_\star'(q(t)) .
\end{aligned}
$$

We next characterize the possible limits $(R_{\rm fdt}, C_{\rm fdt})$ in (2.1) in case we have for $\beta > 0$, $|q_o| \le q_\star$ that:

(H1). There exists a closed set $\mathcal{A} \subset \{(R,C,q) \in \mathcal{A}^+ : \|(R,C,q)\| \le \rho\}$, where the functions $\{R(t+\cdot,t), t \ge T_0\}$ are uniformly integrable WRT Lebesgue measure on $\mathbb{R}$ and

$$
\liminf_{v\to-\infty} \inf_{t\ge -v} \Big\{ \frac{1}{|v|} \int_v^0 \mu_{(R,C,q)}(t+u) du \Big\} > 0 . \tag{7.5}
$$

(H2). Ψ is a contraction on $(\mathcal{A}, \|\cdot\|)$ and the subset $\mathcal{S}$ of $\mathcal{A}$ with property (2.1) for some $|\alpha| \le 1$, is non-empty.

Proposition 7.1 *Assuming* **(H1)–(H2)**, *the solution* (R,C,q) *of* (1.30)–(1.33) *is the unique fixed point of* Ψ *in* $\mathcal{S}$ *and* $(R_{\rm fdt}, C_{\rm fdt})$ *of* (2.1) *are a solution in* $\widetilde{\mathcal{B}} := \{(R,C) \in \mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{R}) : C(0) = R(0) = 1, C(\tau) = C(-\tau)\}$ *of [24, (4.15)–(4.16)], with* μ *as in [24, (4.17)], but now for* $(\mathbf{I}, \alpha)$ *satisfying* (2.4) *and* (2.5).

Proof We first verify that for the given β and q_o, any $S = (R,C,q) \in \mathcal{S}$ results with $\Psi(S) \in \mathcal{S}$. To this end, proceeding similarly to [24, proof of (4.7)], we have for $(R,C,q) \in \mathcal{S}$ that as $t \to \infty$ the bounded integrands in the formulas for $I_i(\cdot,\cdot)$, $i = 0,1,2,3$, converge pointwise (per fixed $u = v - \theta$), to the corresponding expression for $(R_{\rm fdt}, C_{\rm fdt}, \alpha q_\star)$. Further, thanks to the uniform integrability of the collection $\{R(t+\cdot,t), t \ge T_0\}$ (when $(R,C,q) \in \mathcal{A}$, see **(H1)**), the contributions of the integrals over $[-t,-m]$ decay to zero as $m \to \infty$, uniformly in t. Thus, applying the bounded convergence theorem for the integrals over $[-m,v]$, then taking $m \to \infty$, we deduce that for each fixed $v \ge 0$, in analogy with [24, (4.11)–(4.12)],

$$
\widehat{I}_0 := \lim_{t\to\infty} I_0(t) = \int_0^\infty R_{\rm fdt}(\theta)\Big[\psi(C_{\rm fdt}(\theta)) - \frac{\psi(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}\Big] d\theta
$$

$$+\beta^{-1}\alpha q_\star \mathsf{v}'_\star(\alpha q_\star)\,, \tag{7.6}$$

$$\widehat{I}_1(v) := \lim_{t\to\infty} I_1(t+v,t) = \int_0^\infty R_{\rm fdt}(\theta)\Big[\nu''(C_{\rm fdt}(\theta))C_{\rm fdt}(v-\theta) - \frac{\alpha q_\star \nu''(\alpha q_\star)\nu'(\alpha q_\star)}{\nu'(q_\star^2)}\Big]d\theta\,, \tag{7.7}$$

$$\widehat{I}_2(v) := \lim_{t\to\infty} I_2(t+v,t) = \int_v^\infty R_{\rm fdt}(\theta-v)\Big[\nu'(C_{\rm fdt}(\theta)) - \frac{\nu'(\alpha q_\star)^2}{\nu'(q_\star^2)}\Big]d\theta + \beta^{-1}\alpha q_\star \mathsf{v}'_\star(\alpha q_\star)\,, \tag{7.8}$$

$$\widehat{I}_3 := \lim_{t\to\infty} I_3(t) = \int_0^\infty R_{\rm fdt}(\theta)\Big[\alpha q_\star \nu''(C_{\rm fdt}(\theta)) - \frac{q_\star^2\nu'(\alpha q_\star)\nu''(\alpha q_\star)}{\nu'(q_\star^2)}\Big]d\theta + \beta^{-1}q_\star^2 \mathsf{v}'_\star(\alpha q_\star)\,. \tag{7.9}$$

Using the notation $I_i(\cdot,t) := I_i(t)$ for $i = 0,3$, we further know by the preceding that $\sup_{t,v\ge 0}\{|I_i(t+v,t)|\} < \infty$ for $0 \le i \le 3$, yielding in particular the finiteness of $\sup_{t\ge 0}\sup_{v\in[0,\tau]}\{\Lambda(t+\tau,t+v)\}$ for $\Lambda(\cdot,\cdot)$ of [24, (4.8)]. Recall from (7.2) that $\widetilde{R}(s,t) = \Lambda(s,t)\widetilde{H}(s,t)$ for $\widetilde{H}(\cdot,\cdot)$ of [24, (4.9)], hence by bounded convergence (as in [24]), we have for any $\tau \ge 0$,

$$\widehat{\Lambda}(\tau - v) := \lim_{t\to\infty} \Lambda(t+\tau,t+v) = e^{-(\tau-v)\mu}\,, \qquad \forall v\in[0,\tau], \tag{7.10}$$

$$\widetilde{C}_{\rm fdt}(\tau) := \lim_{t\to\infty} \widetilde{C}(t+\tau,t) = \widehat{\Lambda}(\tau) + \beta^2\int_0^\tau \widehat{\Lambda}(\tau-v)\widehat{I}_1(v)dv + \beta^2\int_0^\tau \widehat{\Lambda}(\tau-v)\widehat{I}_2(v)dv\,, \tag{7.11}$$

$$\widetilde{H}_{\rm fdt}(\tau) := \lim_{t\to\infty} \widetilde{H}(t+\tau,t) = 1 + \sum_{n\ge 1}\beta^{2n}\sum_{\sigma\in {\rm NC}_n}\int_{0\le\theta_1\le\cdots\le\theta_{2n}\le\tau} \prod_{i\in{\rm cr}(\sigma)} \nu''(C_{\rm fdt}(\theta_i-\theta_{\sigma(i)}))\prod_{j=1}^{2n}d\theta_j\,, \tag{7.12}$$

$$\widetilde{R}_{\rm fdt}(\tau) := \lim_{t\to\infty} \widetilde{R}(t+\tau,t) = \widehat{\Lambda}(\tau)\widetilde{H}_{\rm fdt}(\tau)\,. \tag{7.13}$$

Unlike [24], here in principle $I_i(s,t)$ might take negative values. However, thanks to (7.5),

$$\mu := \lim_{t\to\infty}\{\mu(t)\} = \frac{1}{2} + \beta^2\widehat{I}_0 > 0\,. \tag{7.14}$$

With $\Lambda(t,x) = \Lambda(t,0)\Lambda(0,x)$, also

$$\widetilde{q}(t) = \Lambda(t,0)q_0 + \beta^2\int_{-t}^0 \Lambda(t,t+v)I_3(t+v)dv\,,$$

where by (7.5) we have that $\Lambda(t,0)\to 0$ and the integral over $[-t,-m]$ decays to zero as $m\to\infty$, uniformly in t. Applying bounded convergence for the integral over $[-m,0]$, then taking $m\to\infty$, we see that

$$\widetilde{\alpha}q_\star := \lim_{t\to\infty}\widetilde{q}(t) = \beta^2\widehat{I}_3\int_0^\infty \widehat{\Lambda}(v)dv = \frac{\beta^2}{\mu}\widehat{I}_3\,. \tag{7.15}$$

Thus, $\Psi(\mathcal{S}) \subset \mathcal{S}$, with Ψ inducing on $\mathcal{S}$ the mapping $\Psi_{\rm fdt} : (R_{\rm fdt}, C_{\rm fdt}, \alpha) \to (\widetilde{R}_{\rm fdt}, \widetilde{C}_{\rm fdt}, \widetilde{\alpha})$ given by (7.10)–(7.15), for $\widehat{I}_i$, $i = 0, 1, 2, 3$ as in the RHS of (7.6)–(7.9). In particular, $\widetilde{R}_{\rm fdt}$ and $\widetilde{C}_{\rm fdt}$ are differentiable on $\mathbb{R}_+$ and satisfy [24, (4.23)–(4.24)] for $\widetilde{R}_{\rm fdt}(0) = \widetilde{C}_{\rm fdt}(0) = 1$ and the preceding values of $\widehat{I}_i$, $i = 0, 1, 2$.

Next, recall **(H2)** that Ψ is a contraction on $(\mathcal{A}, \|\cdot\|)$, hence also on its non-empty subset $\mathcal{S}$. Thus, starting at any $S^{(0)} = (R^{(0)}, C^{(0)}, q^{(0)}) \in \mathcal{S}$ yields a Cauchy sequence $S^{(k)} = \Psi(S^{(k-1)}) \in \mathcal{S}$, $k = 1, \ldots$ for the norm $\|\cdot\|$ of (7.1), with $S^{(k)} \to S^{(\infty)}$ in the closed subset $\mathcal{A}$ of $(\mathcal{A}^+, \|\cdot\|)$. Further, fixing $\tau \geq 0$, with $|(x, y, z)| := |x| + |y| + |z|$, since $S^{(k)} \in \mathcal{S}$ we have that

$$\lim_{T\to\infty} \sup_{t,t'\geq T} |S^{(\infty)}(t+\tau, t) - S^{(\infty)}(t'+\tau, t')| \leq 2\|S^{(\infty)} - S^{(k)}\| .$$

Taking $k \to \infty$ we deduce that $\{t \mapsto S^{(\infty)}(t+\tau, t)\}$ is a Cauchy mapping from $\mathbb{R}_+$ to $|(x, y, z)| \leq \rho$, hence $S^{(\infty)}(t+\tau, t)$ converges as $t \to \infty$. This applies for any $\tau \geq 0$, hence $S^{(\infty)} \in \mathcal{S}$ is the unique fixed point of the contraction Ψ on $(\mathcal{S}, \|\cdot\|)$. In particular, as shown in (7.6) this implies also that $\mu(t) \to \mu$ of (7.14). Recall that any fixed point of Ψ must satisfy (1.30)–(1.33), hence the unique solution of the latter equations in $\mathcal{A}^+$ must coincide with $S^{(\infty)}$ and in particular be in $\mathcal{S}$. As noted before, this yields the existence of $(R_{\rm fdt}, C_{\rm fdt}) \in \widetilde{\mathcal{B}}$ which for a suitable choice of α forms a fixed point of $\Psi_{\rm fdt}$. Considering (7.14) and [24, (4.24)] for $\widehat{I}_i(\cdot)$, $i = 0, 1, 2$, of (7.6)–(7.8) we arrive at [24, (4.15)–(4.17)], now with the possibly non-zero $\mathbf{I}$ as given in (2.5). Finally, in view of (7.15) and (7.9), our constraint (2.4) on α is merely the fixed point condition $\widetilde{\alpha} = \alpha$. □

Proof of Proposition 2.1 We start with our second claim, where we allow for arbitrary $\beta > 0$, but assume that the unique fixed point (R, C, q) of Ψ in $\mathcal{A}^+$ satisfies (2.1) as well as the properties in **(H1)**. While proving Proposition 7.1 we have showed that it results with (7.6)–(7.9), and thereby with $\mu(t) \to \mu$ for $(R_{\rm fdt}, C_{\rm fdt}, \mu)$ a solution of [24, (4.15)–(4.17)] on $\widetilde{\mathcal{B}}$ with $(\mathbf{I}, \alpha)$ satisfying (2.4)–(2.5). To complete our claim, note that (2.7) amounts to [24, (1.21)] holding for $\phi(\cdot)$ of (2.2) and $b = 1/2$, so by [24, Proposition 5.1] we have that $(R_{\rm fdt}, C_{\rm fdt}, \mu) = (-2D', D, \phi(1))$ satisfies [24, (4.15)–(4.17)] for $\mathbf{I}$ of (2.6) and the unique $D(\cdot)$ of (2.3).

Turning to our first claim, note that $\alpha = 0$ satisfies (2.4) for any value of β. Further, from [24, (4.17)] and (2.5) we see that $\mu \to \frac{1}{2}$ when $\beta \downarrow 0$ and since the finite polynomials $\nu'(x)$ and $\mathsf{v}'_\star(x)$ are both zero at $x = 0$, it is easy to check that $\alpha = 0$ is the *only* solution of (2.4) for small $\beta > 0$. In case $q_o = 0$ it is also shown in [24, Section 4] that for small β our assumptions **(H1)–(H2)** hold for $\mathcal{A}$ consisting of $e^{\delta|s-t|}(R, C, q)(s, t) \in [0, \rho(r|s-t| + 1)^{-3/2}] \times [0, c] \times \{0\}$ and suitably chosen parameters δ, r, ρ, c. Leaving the details to the reader, such analysis can be extended to yield **(H1)–(H2)** for any $|q_0| \leq q_\star$ and $\beta \in [0, \beta_1)$, again with $\alpha = 0$, but now for

$$\mathcal{A} := \Big\{(R, C, q) \in \mathcal{A}^+ : |R(t+\tau, t)| \leq \rho(r\tau + 1)^{-3/2} e^{-\delta\tau}, \\ |C(t+\tau, t)| \leq c e^{-\delta\tau}, |q(\tau)| \leq \kappa e^{-\eta\tau}\Big\}$$

and certain positive $\delta, r, \rho, c, \kappa, \eta$ (that may depend on β and q_o). The unique fixed point of Ψ in $\mathcal{S}$ one gets from Proposition 7.1 must then have $\mathbf{I} = \alpha = 0$, with $(R_{\rm fdt}, C_{\rm fdt}, \mu)$ the unique solution of [24, (4.15)–(4.17)] within a subset of $\widetilde{\mathcal{B}}$ analogous to $\mathcal{B}(\delta, r, \rho, c)$ of [24, Proposition 4.2], except for allowing here possible negative values of $R_{\rm fdt}$ or $C_{\rm fdt}$. Recall that for all β up to β_c of [24, (1.23)] both (2.6) and (2.7) hold for $\gamma = 1/2$ and $\mathbf{I} = D_\infty = 0$.

Thus, as we have seen before, for such β the unique solution of [24, (4.15)–(4.17)] alluded to above corresponds to $C_{\rm fdt}(\cdot) = D(\cdot)$ for the [0, 1]-valued solution of (2.3). □

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