



Using Cyclic Quadrilaterals to Design Cylindrical Developable Mechanisms

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Abstract. Developable mechanisms on regular cylindrical surfaces can be described using cyclic quadrilaterals. Mechanisms can exist in either an open or crossed configuration, and these configurations correspond to convex and crossed cyclic quadrilaterals. Using equations developed for both convex and crossed cyclic quadrilaterals, the geometry of the reference surface to which a four-bar mechanism can be mapped is found. Grashof mechanisms can be mapped to two surfaces in open or crossed configurations. The only way to map a non-Grashof mechanism to a cylindrical surface is in its open configuration. Extramobile and intramobile behavior can be achieved depending on selected pairs within a cyclic quadrilateral and its position within the circumcircle. Selecting different sets of links as the ground link changes the potential behavior of the mechanism. Different cases are tabulated to represent all possibilities. A non-Grashof developable mechanism can only exhibit extramobile or intramobile behavior if all of the joints lie on one half of the circumcircle.

1 Introduction

Developable surfaces (e.g. generalized cylinders, generalized cones, and tangent developed surfaces) are common in many engineering applications [1–5]. Developable mechanisms are mechanisms that conform to the shape of developable surfaces [6]. These mechanisms are advantageous because they can be incorporated into the surface of an object, allowing for compact storage and precise deployed motion.

While developable mechanisms provide unique advantages, they can be difficult to design. To conform to a developable surface, the joint axes must align with the ruling lines, and the links must be shaped to fit within the surface [6]. Methods have been proposed for designing developable mechanisms on regular cylindrical [7] and circular conical [8] surfaces using a circle to represent the reference surface.

Cyclic quadrilaterals are quadrilaterals whose vertices all lie on the circumference of a single circle [9, 10]. Using these quadrilaterals to represent a four-bar mechanism whose joints lie on a circle yields useful insights to developable mechanism design.

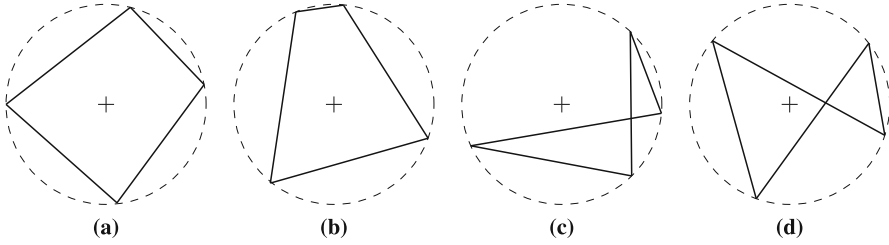


Fig. 1. (a) and (b) are examples of convex cyclic quadrilaterals, while (c) and (d) are examples of crossed cyclic quadrilaterals.

This paper introduces concepts and conclusions derived from applying the mathematical relationships of cyclic quadrilaterals to the design of developable mechanisms that conform to regular cylindrical surfaces. These resulting principles are presented and discussed to aid designers through the process of creating developable mechanisms capable of achieving desired behaviors.

1.1 Cyclic Quadrilaterals

A cyclic quadrilateral is a quadrilateral whose four vertices all lie on a single circle, often called the circumcircle [9,10]. Both convex and crossed cyclic quadrilaterals are possible [11], as shown in Fig. 1.

Paramesvara [12] showed that any set of four lines (a, b, c, d) can form a convex cyclic quadrilateral of radius R as long as the longest line is shorter than the sum of the other 3 lines, where R is

$$R = \frac{1}{4} \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}} \quad (1)$$

where

$$s = \frac{a + b + c + d}{2}$$

Josefsson [11] provided a similar equation for crossed cyclic quadrilaterals, using

$$R = \sqrt{\frac{(ab - cd)(ac - bd)(ad - bc)}{(-a + b + c - d)(a - b + c - d)(a + b - c - d)(a + b + c + d)}} \quad (2)$$

1.2 Developable Mechanisms

Developable mechanisms are contained within or conform to developable surfaces [6]. There are three types of non-trivial developable surfaces: generalized cylinders, generalized cones, and tangent developed surfaces. By aligning the

joints of a mechanism to the ruling lines of these surfaces, different mechanisms can be mapped to these different surfaces. A planar mechanism can be mapped to a cylindrical surface as all of the joint axes are parallel, a spherical mechanism can be mapped to a conical surface as all of the joint axes intersect at a single point, and spatial mechanisms can be mapped to tangent developed surfaces. The links can then be shaped so that the mechanism conforms to the surface.

The design of developable mechanisms can be more complex than traditional mechanisms because they need to take into account the shape and position of the developable surface [6, 7]. The general principles for designing developable mechanisms have been described by [6].

1.3 Grashof Condition

The Grashof criterion classifies the relative motion of links in a four-bar mechanism. For a Grashof mechanism, at least one link of a four-bar mechanism can rotate completely with respect to the other links [13]. The traditional equations are shown below in Table 1, where the shortest and longest links are labeled as s and l , respectively and the other two links are p and q .

Table 1. Two analytical methods for determining the Grashof condition of four-bar mechanisms.

Traditional eq.	McCarthy's eq.	Grashof criteria
$s + l \leq p + q$	$T_1 T_2 T_3 \geq 0$	Grashof mechanism
$s + l > p + q$	$T_1 T_2 T_3 < 0$	non-Grashof mechanism
$s + l = p + q$	$T_1 T_2 T_3 = 0$	folding mechanism (special case Grashof)

McCarthy used three different parameters to determine the Grashof condition for planar mechanisms [14]. Given a mechanism with links (a, b, c, d) , the parameters T_1, T_2 , and T_3 are defined below. The product of these three parameters can determine the Grashof condition, as shown in Table 1.

$$T_1 = a + b - c - d \tag{3}$$

$$T_2 = a - b + c - d \tag{4}$$

$$T_3 = -a + b + c - d \tag{5}$$

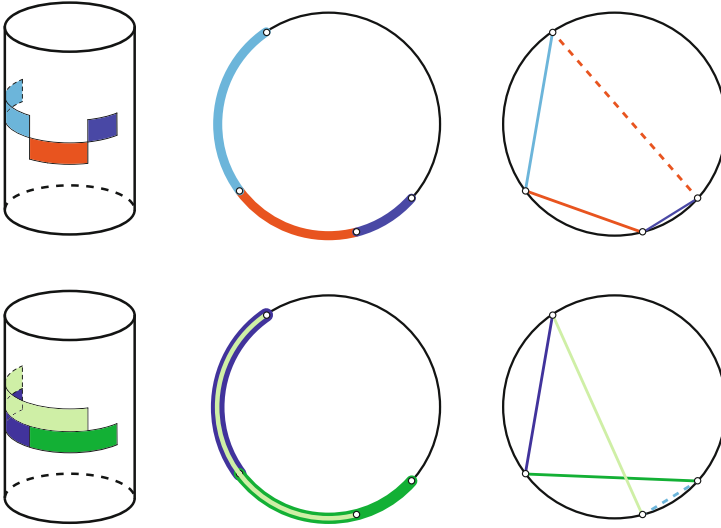


Fig. 2. Two cylindrical developable mechanisms (left) with a cross-section (middle) and skeleton diagram (right). The mechanism on the top is in its open configuration, and the bottom mechanism is in a crossed configuration. The dashed line indicates the grounded link.

2 Cyclic Quadrilaterals and Four-Bar Cylindrical Developable Mechanisms

In traditional mechanism analysis, a skeleton diagram is often used to describe mechanisms. While the links can have any arbitrary shape, kinematic analysis relies on the relative position of the joints [15]. Therefore, skeleton diagrams are made with straight lines connecting the joints. The links of a developable mechanism must conform to a reference surface, however, the kinematic behavior depends on the joint location. For a four-bar developable mechanism on a regular cylindrical surface, the skeleton diagram will be a cyclic quadrilateral. The sides of the quadrilateral are the links of the mechanism, the vertices are the joints, and the circumcircle is a cross-section of the cylindrical reference circle. Hence, four-bar cylindrical developable mechanisms can be represented by cyclic quadrilaterals, either convex or crossed. Figure 2 shows an example of this relationship.

This section will discuss cyclic quadrilaterals and their application to four-bar developable mechanisms.

The convex cyclic position represents the mechanism's open circuit. Parameshvara's equation (Eq. 1) shows that any set of 4 link lengths can form a convex cyclic quadrilateral (provided they can be assembled). This confirms what Bowman found, that all four-bar linkages have a possible convex cyclic position [16]. Interestingly, the order in which the links are arranged ($abcd$ or

$abdc$ or $adbc$) does not affect the radius of the reference circle [16], as long as the mechanisms are in the open circuit.

The crossed cyclic position represents the mechanism's crossed circuit. Crossed cyclic quadrilaterals are possible and Josefsson provided an equation for their circumcircle (Eq. 2). While any four-bar mechanism can be in the convex cyclic position, only Grashof mechanisms can be in the crossed cyclic position, as shown below.

Josefsson [11] derived an equation for the area of a crossed cyclic quadrilateral given by

$$K = \frac{1}{4} \sqrt{(P_1)(P_2)(P_3)(P_4)} \quad (6)$$

where

$$P_1 = -a + b + c - d$$

$$P_2 = a - b + c - d$$

$$P_3 = a + b - c - d$$

$$P_4 = a + b + c + d$$

The parameters P_1, P_2 , and P_3 are equivalent to the parameters T_3, T_2 , and T_1 used by McCarthy [14] to determine the Grashof condition (see Sect. 1.3). Links cannot have a negative length, so parameter P_4 is always positive.

Assuming that a non-Grashof mechanism ($P_1 P_2 P_3 < 0$) can be in the crossed cyclic position, the product $P_1 P_2 P_3 P_4$ must be negative. When applied to Eq. 6, the resulting area is a complex number, which is impossible to physically make. Therefore, a non-Grashof mechanism does not have a crossed cyclic configuration.

2.1 Special Case: Folding Mechanism

A folding, or change point mechanism ($T_1 T_2 T_3 = 0$) is a mechanism where all links can be simultaneously co-linear. It is a special-case Grashof mechanism and can therefore have positions in both the open and crossed circuits without disassembly.

Equation 1 can be used to find the radius of the circumcircle for the set of links when in an open circuit. However, when the mechanism is in its crossed configuration, Eq. 6, shows that $P_1 P_2 P_3 = 0$, resulting in an area of zero. This suggests that folding mechanisms in the crossed configuration can only conform to a circumcircle of infinite radius (i.e. all the links lie in a single line).

There are only two known cases where the crossed configuration of a folding mechanism has a non-infinite radius. The first case is a parallelogram folding mechanism where opposite links have equal length. Parallelogram mechanisms in the crossed configuration can exist on a wide range of circumcircle radii. The range is from half the length of the longest link, l , to infinity ($l/2 \leq R < \infty$).

An example of a parallelogram mechanism in several crossed cyclic configurations is shown in Fig. 3. The second case is a kite mechanism with two sets of adjacent links with equal length. The crossed configuration of a kite mechanism is a special case where the equal adjacent links are stacked. The resulting skeleton diagram appears as two line segments because the links with equal length are coincident. This crossed or stacked configuration has the same range of radii as the parallelogram mechanism. The open and stacked configurations of a kite mechanism are shown in Fig. 4.

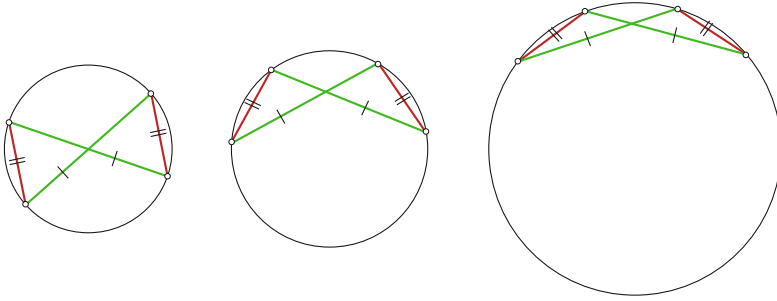


Fig. 3. A folding mechanism with the green links equal and the red links equal. As a parallelogram mechanism, the same set of links can be circumscribed by circles of increasing radius as the mechanism approaches its folded position.

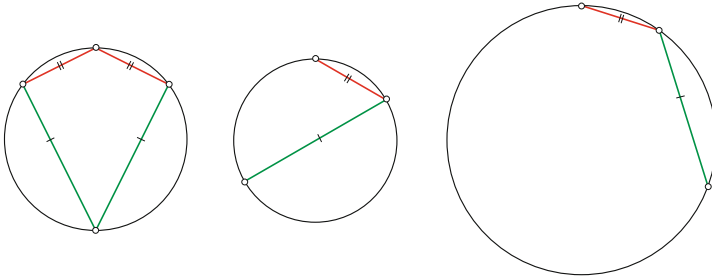


Fig. 4. A kite mechanism with the green links equal and the red links equal. The mechanism is shown in its open configuration (left) and two examples of its stacked configuration.

2.2 Generalized Equation for the Radius of the Circumcircle

Equations 1 and 2 are very similar, with the only changes dealing with the sign changes for d [11]. Hence, these equations can be combined to design four-bar cylindrical developable mechanisms in both open and crossed configurations. This combination introduces a variable (μ) that takes into account this sign

change and corresponds to the circuit of the mechanism. The equation for the radius of the cylindrical reference surface R is

$$R = \frac{1}{4} \sqrt{\frac{(ab + \mu cd)(ac + \mu bd)(\mu ad + bc)}{(p - a)(p - b)(p - c)(p - \mu d)}} \quad (7)$$

where

$$p = \frac{a + b + c + \mu d}{2}$$

$$\mu = \begin{cases} 1 & \text{for an open configuration} \\ -1 & \text{for a crossed configuration} \end{cases}$$

This equation is valid for any Grashof mechanism, regardless of its circuit. However, as discussed above, it is not valid for non-Grashof mechanisms in the crossed circuit.

3 Intramobility and Extramobility with Cyclic Quadrilaterals

The use of cyclic quadrilaterals in the design of developable mechanisms can be of particular use when modeling linkages that provide desired behaviors. Greenwood [7] proposed three types of behaviors that developable mechanisms can exhibit as they move from their conformed position: extramobile (all parts of the linkage move outside the reference surface), intramobile (all parts of the linkage move inside the reference surface), and transmobile (the linkage has parts that move both inside and outside the reference surface). Extramobile and intramobile behaviors are of particular interest as they present motion that allows for mechanisms that move purely into or away from the surface. Figure 5 shows an example of a cylindrical developable mechanism that exhibits both extramobile and intramobile behavior.

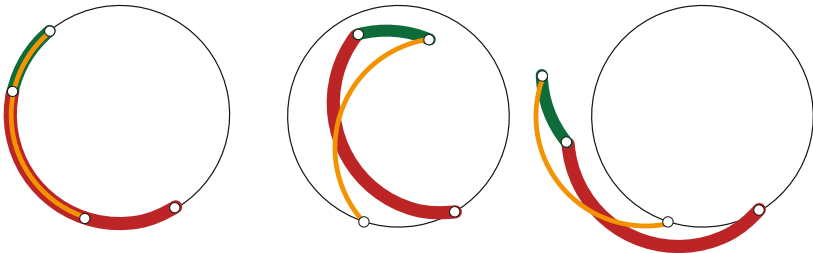


Fig. 5. An example of a cylindrical developable mechanism that show both intramobile and extramobile behavior. The left shows the mechanisms in their conformed position, the center shows all of the links moving within the surface (intramobile), and the right shows all of the links moving outside the surface (extramobile).

Greenwood proposed a graphical method, called the shadow method, to determine if a cylindrical developable mechanism is capable of exhibiting extramobility and intramobility. The shadow method is applied by shading the region bounded by vectors of the two side links (b and d in a traditional 4-bar mechanism) and contains the ground link and coupler. For a mechanism to exhibit purely extra and intramobile behavior (i.e. no transmobile behavior), the shaded region may not cover the center of the circle.

A four-bar linkage can be created by closing a loop of four bars. With four specified link locations on a circle, there are three possible combinations of cyclic quadrilaterals that result in a valid four-bar loop. The two cases shown on the left of Table 2 demonstrate three pairs of lines, indicated in blue, orange, and green. Any combination of two sets of lines will result in a valid four-bar linkage. Within one of these sets, any of the four bars may be selected as the ground link, changing the behavior of the mechanism. Extra and intramobility, however, are dependent upon which pair of links is selected as the ground and coupler links.

Table 2 demonstrates the general cases for how the outer loop of possible cyclic quadrilaterals may be oriented within the circle. Case 1 does not contain the center of the circle while Case 2 does contain the center of the circle. It can be seen that a cyclic quadrilateral following the condition of Case 1 is capable of producing three different mechanisms that can potentially demonstrate extra and intramobile behavior (Open A, Crossed 1A, Crossed 2B). However, a mechanism that follows the conditions of Case 2 may only produce one extra/intramobile mechanism (Crossed 1A).

Special conditions exist in the event that a link within the cyclic quadrilateral intersects the center of the circle. If one link extends halfway across the circle, the mechanism cannot exhibit intramobile behavior [7]. These special conditions are discussed in the following subsection.

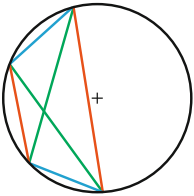
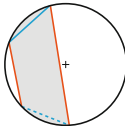
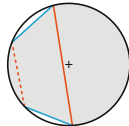
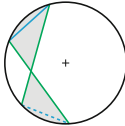
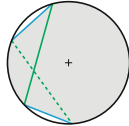
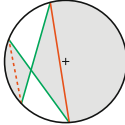
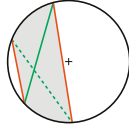
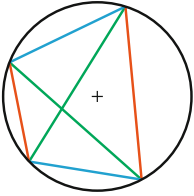
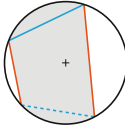
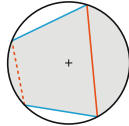
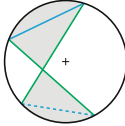
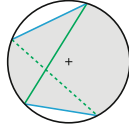
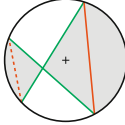
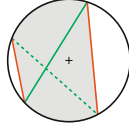
3.1 Special Conditions

The only condition for which Case 1 can have a link intersect the center of the circle is when this is caused by one of the outer links. In this scenario, both configurations Open A and Crossed 2B are now incapable of intramobile behavior (but may still exhibit extramobile behavior). There are no other changes for this special condition. Note that configuration Crossed 1A can still exhibit both intramobile behavior and extramobile behavior, even if one of the outer links intersects the center of the circle.

There are two special conditions that exist for Case 2. The first is if one of the crossed links intersects the center of the circle. If this occurs, configuration Crossed 1A cannot exhibit intramobile behavior. However, it is important to note that now configuration Crossed 2A, which previously could not exhibit intramobile or extramobile behavior, can now exhibit extramobile behavior (in this special condition). There are no other changes for this first special condition.

The second special condition exists when both crossed links intersect the center of the circle. (This can only happen if the mechanism is a crossed parallelogram mechanism.) Similar to above, in this condition, configurations Crossed 1A

Table 2. General cases of cyclic quadrilaterals and their capacity to generate extra and intramobile mechanisms (shown with the shadow method). Mechanisms without the shaded region overlapping the center of the circle can possibly be extra and intramobile. Dashed lines indicate the ground link.

Case		A	B
 Case 1	Open		
	Crossed 1		
	Crossed 2		
 Case 2	Open		
	Crossed 1		
	Crossed 2		

and 2A can exhibit extramobile behavior but cannot exhibit intramobile behavior. It should be noted that these configurations may have extramobile motion in only one of the two directions of motion. For a link to extend through the middle of the circle, its actual link shape must extend at an arc length πR around the circle. While either direction may be used (link curving either direction around the circle), only one direction can be selected for extramobile motion. In other

words, a link that extends halfway around the circle cannot be used to provide extramobile motion in both directions.

4 Discussion and Conclusion

This paper shows how cyclic quadrilaterals can be used to design cylindrical developable mechanisms. The generalized equation for the radius of the circumcircle (Eq. 7) provides a versatile way to determine the geometry of a developable mechanism. For example, the link lengths of a planar four-bar mechanism can be used to determine the geometry of the developable surface to which it can be mapped. Or a combination of the developable surface geometry and several link constraints can be used to find the other missing variables.

Planar four-bar mechanisms can be mapped to regular cylindrical surfaces, regardless of their Grashof condition. However, only Grashof mechanisms can be mapped to these surfaces in a crossed configuration. Therefore, the only way to map a non-Grashof mechanism to a regular cylindrical or circular conical surface is in its open configuration. Forming any four-bar mechanism from a crossed cyclic quadrilateral forces the mechanism to be a Grashof mechanism, regardless of any other geometrical considerations. Any cylindrical developable mechanism in a crossed configuration can be identified as a Grashof mechanism without any calculations.

It can be observed from Table 2 that the only way to create an open-loop mechanism that achieves extra or intramobile behavior is through Case 1, Open A. Because the center of the circle is not within the shaded portion, according to the shadow method, this type of mechanism can exhibit extramobile or intramobile behavior. All other cases of convex quadrilaterals result in exclusively transmobile behavior. This is significant because it has been established that non-Grashof mechanisms can only have convex cyclic quadrilateral configurations. Therefore, the only way to have a non-Grashof mechanism that can exhibit extramobile or intramobile behavior must belong to Case 1, Open A. Therefore, for non-Grashof mechanisms desiring intramobile and extramobile behavior, all vertices must lie on one half of the circumcircle. Any other case of mechanism that can exhibit extramobile or intramobile behavior is necessarily a Grashof mechanism.

This paper also demonstrates that depending on the position of the vertices of a cyclic quadrilateral, there are several cases that determine whether certain behaviors are possible from different developable mechanisms. For a given cyclic quadrilateral, it may be possible to achieve extramobile or intramobile behavior if the correct set of links is chosen to include the grounded link.

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