ASYMPTOTIC GLUING OF SHEAR-FREE HYPERBOLOIDAL INITIAL DATA SETS

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Abstract. We present a procedure for asymptotic gluing of hyperboloidal initial data sets for the Einstein field equations that preserves the shear-free condition. Our construction is modeled on the gluing construction in [10], but with significant modifications that incorporate the shear-free condition. We rely on the special Hölder spaces, and the corresponding theory for elliptic operators on weakly asymptotically hyperbolic manifolds, introduced by the authors in [2] and applied to the Einstein constraint equations in [3].

1. Introduction

One of the most useful ways to mathematically define asymptotically flat spacetimes – solutions to the Einstein field equations that model isolated gravitational systems – is to require that they admit a conformal compactification; see [12], [9]. Such spacetimes can be foliated by spacelike leaves intersecting the conformal boundary along future null infinity. The intrinsic and extrinsic geometry induced on such a leaf comprises a solution to the Einstein constraint equations, commonly referred to as *hyperboloidal* in the literature.

In [3] the authors have constructed constant-mean-curvature hyperboloidal solutions to the Einstein constraint equations satisfying a boundary condition, known as the *shear free condition*, along the conformal boundary. Being shear-free is a necessary condition on the initial data set for any spacetime development of that data to admit a regular conformal structure at future null infinity; see [5].

In this paper we present an asymptotic gluing procedure for vacuum constant-mean-curvature shear-free hyperboloidal initial data as constructed in [3]. Previous gluing constructions for the solutions to the Einstein constraint equations with asymptotically hyperbolic geometry [8],[10] have not accounted for the shear-free condition.

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Topologically the gluing construction produces a connected sum of the conformal boundary. While our construction is independent of the topological type of the boundary, we note the important special case that the original data consists of two connected components, each having a spherical conformal boundary; see Figure 1. In this case, our construction yields "two-body" initial data sets having a spherical conformal boundary.

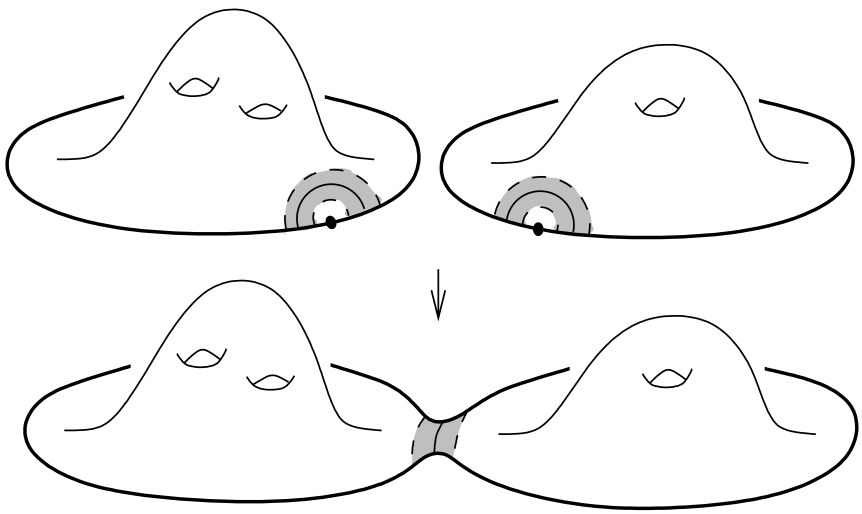

FIGURE 1. This diagram, adapted from [10], shows the boundary gluing construction in the case of two connected components.

Our construction produces a one-parameter family of shear-free hyperboloidal initial data sets. We are able to show, in the limit as the parameter tends to zero, that the geometry converges to that of the original data set. Furthermore, the geometry in the center of the gluing region converges to a portion of the hyperboloid inside the Minkowski spacetime.

1.1. **Constant-mean-curvature hyperboloidal data.** We now give a definition of the hyperboloidal data to which our result applies. This type of initial data has been discussed extensively in [3], which in turn relies heavily on [2].

First, we briefly review the definitions of the function spaces we work with; see [2], [11], and §2 below for more details. We assume that M is the interior of a compact 3-dimensional manifold $\overline{M}$ having boundary ∂M and let ρ be a smooth defining function on $\overline{M}$ (meaning ρ vanishes to first order on ∂M and is positive in M). Let $C^{k,\alpha}(M)$ be the intrinsic Hölder space of tensor fields on M and for $\delta \in \mathbb{R}$ let $C^{k,\alpha}_{\delta}(M) = \rho^{\delta} C^{k,\alpha}(M)$. A covariant 2-tensor field u is defined to be of class $\mathscr{C}^{k,\alpha;m}(M)$ if

$$\mathscr{L}_{X_1} \dots \mathscr{L}_{X_j} u \in C^{k-j,\alpha}_{2}(M)$$

for all $0 \le j \le m$ and for all smooth vector fields $X_1, \dots, X_j$ on $\overline{M}$; here $\mathscr{L}$ denotes the Lie derivative. For example, if $u \in C^{m,\alpha}(\overline{M})$ then $u \in \mathscr{C}^{m,\alpha;m}(M)$.

A complete metric g and a symmetric covariant 2-tensor K (representing the second fundamental form) form a ***constant-mean-curvature shear-free (CMCSF) hyperboloidal data set of class*** $\boldsymbol{\mathscr{C}^{k,\alpha;2}}$ on M if

(a) $g = \rho^{-2}\overline{g}$ for some $\overline{g} \in \mathscr{C}^{k,\alpha;2}(M)$ that extends to a metric on $\overline{M}$ and is such that $|d\rho|_{\overline{g}} = 1 + O(\rho)$;
(b) $K = \Sigma - g$ for some traceless tensor $\Sigma = \rho^{-1}\overline{\Sigma}$ with $\overline{\Sigma} \in \mathscr{C}^{k-1,\alpha;1}(M)$;
(c) the shear-free condition holds, meaning that

$$\overline{\Sigma}\Big|_{\rho=0} = \left[\operatorname{Hess}_{\overline{g}}\rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g}\right]_{\rho=0}; \tag{1.1}$$

and
(d) the vacuum constraint equations hold, meaning that

$$\mathrm{R}[g] + 6 - |\Sigma|_g^2 = 0 \quad \text{and} \quad \operatorname{div}_g \Sigma = 0, \tag{1.2}$$

where $\mathrm{R}[g]$ is the scalar curvature of g.

A metric g satisfying condition (a) is said to be ***weakly asymptotically hyperbolic of class*** $\boldsymbol{\mathscr{C}^{k,\alpha;2}}$.

An important example of CMCSF hyperboloidal data is the data induced on the unit hyperboloid in the Minkowski spacetime, given in the usual Cartesian coordinates by $\{(x^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + 1, x^0 > 0\}$. The induced metric for this example is the hyperbolic metric $\breve{g}$, while the second fundamental form is given by $K = -\breve{g}$; thus $\Sigma = 0$ for this data.

1.2. **Statement of the main result.** The asymptotic gluing procedure of [10] produces data which generically fails to be shear-free (cf. Proposition 3.2 of [5]). Here we present a modification of the gluing method of [10] within the category of shear-free initial data. We now give a precise statement of our result.

Theorem 1.1. *Suppose that a metric g and a tensor field Σ give rise to a CMCSF hyperboloidal data set of class $\mathscr{C}^{k,\alpha;2}$ on M for some $k \ge 3$ and $\alpha \in (0,1)$. Fixing $p_1, p_2 \in \partial M$, we define for each sufficiently small $\varepsilon > 0$ a manifold M_ε, which is the interior of a compact manifold $\overline{M}_\varepsilon$ whose boundary is obtained from a connected sum joining neighborhoods of p_1, p_2.*

For each sufficiently small $\varepsilon > 0$ there exist a metric g_ε and a tensor field Σ_ε that give rise to a CMCSF hyperboloidal data set of class $\mathscr{C}^{k,\alpha;2}$ on M_ε. As $\varepsilon \to 0$, the tensor fields $(g_\varepsilon, \Sigma_\varepsilon)$ converge to (g, Σ) in the following sense:

Convergence in the exterior region: *For each sufficiently small $c > 0$ we define an open set $E_c \subseteq M$, whose closure in $\overline{M}$ is disjoint from p_1, p_2. The sets E_c exhaust M in the sense that $\bigcup_{c>0} E_c = M$.*

For each $\varepsilon \ll c$ there exists an embedding $\iota_\varepsilon \colon E_c \to M_\varepsilon$. Our convergence result in the exterior region is that for fixed c we have

$$(\rho^2 \iota_\varepsilon^* g_\varepsilon, \rho \iota_\varepsilon^* \Sigma_\varepsilon) \to (\rho^2 g, \rho\Sigma) \tag{1.3}$$

in the $\mathscr{C}^{k,\alpha;2} \times \mathscr{C}^{k-1,\alpha;1}$ topology on E_c.

Convergence in the neck: *For each sufficiently small $c > 0$ we define a subset A_c of hyperbolic space $\mathbb{H}$ that in the half-space model corresponds to a semi-annular region; see* (2.1)*. The sets A_c exhaust hyperbolic space in the sense that $\bigcup_{c>0} A_c = \mathbb{H}$.*

For each $\varepsilon \ll c$ there exists an embedding $\Psi_\varepsilon \colon A_c \to M_\varepsilon$ such that $\Psi_\varepsilon(A_c) \cap \iota_\varepsilon(E_c) = \emptyset$. Our convergence result in the neck is that the data converges to the unit hyperboloid of Minkowski space, in the sense that for fixed c we have

$$(\breve{\rho}^2 \Psi_\varepsilon^* g_\varepsilon, \breve{\rho} \Psi_\varepsilon^* \Sigma_\varepsilon) \to (\breve{\rho}^2 \breve{g}, 0) \tag{1.4}$$

in the $\mathscr{C}^{k,\alpha;2} \times \mathscr{C}^{k-1,\alpha;1}$ topology on A_c. Here $\breve{\rho}$ is a fixed defining function for hyperbolic space; see (2.6)*.*

We emphasize that the above topology is the "right space" for convergence with regard to the shear free condition in view of the results of [3] that show that the shear-free condition is continuous in this topology, and the results of [1], which shows that shear-free data sets are dense with respect to the weaker $C^{k,\alpha}$ topology.

We note that the class of initial data sets considered here includes those with polyhomogeneous regularity along the conformal boundary; see [6],[3]. The observant reader will note that each step in our construction preserves polyhomogeneity, and thus the application of Theorem 1.1 to initial data that is both polyhomogeneous and shear-free yields polyhomogeneous data on M_ε. We refer the reader to [3], and the references therein, for additional details concerning polyhomogeneous data.

1.3. Overview of the construction.

We begin our construction in the same manner as in [10]. First, given (g, Σ) on M, and given the two gluing points $p_1, p_2 \in \partial M$, we use inversion with respect to half-spheres to construct a manifold M_ε, along with a defining function ρ_ε. We then use cutoff functions to construct a spliced metric λ_ε and spliced tensor field μ_ε on M_ε. Second, we apply the conformal method of [3] to $(\lambda_\varepsilon, \mu_\varepsilon)$ in order to obtain $(g_\varepsilon, \Sigma_\varepsilon)$ satisfying the constraint equations (1.2).

The spliced metric λ_ε is obtained from g using a cutoff function. To construct μ_ε we follow the approach of [3], and express the shear-free condition (1.1) using a tensor $\mathcal{H}_{\overline{g}}(\rho)$ that, for the metrics appearing here, agrees with the traceless Hessian of ρ along ∂M. The definition and properties of $\mathcal{H}_{\overline{g}}(\rho)$ are detailed in §3. In order to splice the second fundamental forms, we write $\Sigma = \rho^{-1}\mathcal{H}_{\overline{g}}(\rho) + \nu$ and then use a cutoff function to construct a tensor $\nu_\varepsilon^{\mathrm{ext}}$ that agrees with ν in the exterior region and vanishes in the neck.

We require that the metric λ_ε, together with the tensor μ_ε, form a good approximate solution to the constraint equations. In the middle of the neck we expect the solution to be very close to data corresponding to a hyperboloid in Minkowski space; for such data, $g = \breve{g}$ and $\Sigma = 0$. However, while $\nu_\varepsilon^{\mathrm{ext}} = 0$ in the neck, the tensor $\rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$ is not small there. Thus we must correct our approximate data by constructing a tensor $\nu_\varepsilon^{\mathrm{neck}}$ that counteracts the large terms in $\rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$. The result is a family of *spliced data sets*, each consisting of the metric λ_ε together with the tensor

$$\mu_\varepsilon = \rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon) + \nu_\varepsilon^{\mathrm{neck}} + \nu_\varepsilon^{\mathrm{ext}},$$

which approximately solve the constraint equations (1.2).

In order to obtain an exact solution to the constraint equations from each spliced data set $(\lambda_\varepsilon, \mu_\varepsilon)$, we make use of the conformal method; see [3] for a detailed description of the conformal method in this setting. The first step of this method is to prove the existence of a vector field W_ε such that

$$L_{\lambda_\varepsilon} W_\varepsilon = (\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon)^\sharp. \tag{1.5}$$

Here $L_{\lambda_\varepsilon} = \mathcal{D}^*_{\lambda_\varepsilon} \circ \mathcal{D}_{\lambda_\varepsilon}$ is the ***vector Laplace operator***, defined in terms of the ***conformal Killing operator*** $\mathcal{D}_{\lambda_\varepsilon}$, which acts on vector fields by

$$\mathcal{D}_{\lambda_\varepsilon} \colon W_\varepsilon \mapsto \frac{1}{2}\mathscr{L}_{W_\varepsilon}\lambda_\varepsilon - \frac{1}{3}(\operatorname{div}_{\lambda_\varepsilon} W_\varepsilon)\lambda_\varepsilon. \tag{1.6}$$

Note that the adjoint $\mathcal{D}^*_{\lambda_\varepsilon}$ acts on symmetric traceless covariant 2-tensors by

$$\mathcal{D}^*_{\lambda_\varepsilon} \colon \mu_\varepsilon \mapsto -(\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon)^\sharp$$

and thus $L_{\lambda_\varepsilon} = -\operatorname{div}_{\lambda_\varepsilon} \circ \mathcal{D}_{\lambda_\varepsilon}(\cdot)^\sharp$. That W_ε satisfies (1.5) ensures that the tensor

$$\begin{aligned} \sigma_\varepsilon &= \rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon) + \nu_\varepsilon^{\mathrm{neck}} + \nu_\varepsilon^{\mathrm{ext}} + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon \\ &= \mu_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon \end{aligned} \tag{1.7}$$

is divergence-free with respect to λ_ε. Furthermore, we solve for W_ε in a weighted function space that implies that the tensor $\overline{\sigma}_\varepsilon = \rho_\varepsilon\sigma_\varepsilon$ satisfies

$$\overline{\sigma}_\varepsilon\big|_{\rho_\varepsilon=0} = \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)\big|_{\rho_\varepsilon=0}.$$

This ensures that the resulting data set satisfies the shear-free condition.

Subsequently, we show the existence of a positive function ϕ_ε satisfying the Lichnerowicz equation

$$\Delta_{\lambda_\varepsilon}\phi_\varepsilon - \frac{1}{8}\,\mathrm{R}[\lambda_\varepsilon]\phi_\varepsilon + \frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}\phi_\varepsilon^{-7} - \frac{3}{4}\phi_\varepsilon^5 = 0 \tag{1.8}$$

and such that $\phi_\varepsilon \to 1$ as $\rho_\varepsilon \to 0$. Direct computation shows that the metric $g_\varepsilon = \phi_\varepsilon^4\lambda_\varepsilon$ and tensor $\Sigma_\varepsilon = \phi_\varepsilon^{-2}\sigma_\varepsilon$ satisfy the constraint equations (1.2), while a more delicate argument shows that in fact g_ε and Σ_ε have the necessary regularity to give rise to a CMCSF hyperboloidal data set as defined above.

In order to control the properties of g_ε and Σ_ε, and thus establish the main theorem, the above process must be carried out in such a way that we obtain uniform control (in ε) for each step of the process. Quantifying this uniform control is a somewhat delicate matter, and we make use of specially weighted function spaces in order to accomplish the task. Among other things, this requires uniform estimates on the mapping properties of various elliptic operators arising from λ_ε.

Our work is organized as follows. In §2 we define the regularity classes used, and recall from [2, 3, 11] their basic properties. Subsequently in §3 we recall from [2] the tensor $\mathcal{H}$, which is used in [3] to characterize the shear-free condition in a manner compatible with the conformal method. The proof of Theorem 1.1 begins in §4 with the construction of the spliced manifolds M_ε. The spliced metrics λ_ε are defined in §5, where their properties are established. In §6 we construct the spliced tensors μ_ε that give rise to the tensors σ_ε, for which we obtain a number of crucial estimates. We analyze the Lichnerowicz equation in §7 before assembling the final bits of the proof in §8. The uniform estimates for mapping properties of various elliptic operators arising from λ_ε involve a framework more general than our construction requires, and are placed in the appendix.

2. Function spaces

Since the gluing construction uses the fact that the asymptotic geometry of (M, g) is locally close to that of hyperbolic space, we first fix some notation involving hyperbolic space. Using this, we briefly recall from [2] the construction of various function spaces on M.

2.1. **Hyperbolic space.** Let $(\mathbb{H}, \breve{g})$ denote the upper half space model of 3-dimensional hyperbolic space; in coordinates $X = (x, y) \in \mathbb{R}^2 \times (0, \infty)$ we have

$$\breve{g} = \frac{(dx^1)^2 + (dx^2)^2 + dy^2}{y^2}.$$

For $r > 0$ we define the following subsets of $\mathbb{H}$:

$$\begin{aligned} \breve{B}_r &= \{(x, y) : d_{\breve{g}}((x, y), (0, 1)) < r\}, \\ Y_r &= \{(x, y) : |x| < r, y < r\}, \\ A_r &= \{(x, y) : r^2 < |x|^2 + y^2 < 1/r^2\}; \end{aligned} \tag{2.1}$$

here $|x|^2 = (x^1)^2 + (x^2)^2$. We note for later use that, since $e^2 < 8$ and since $\breve{B}_r$ is determined by the hyperbolic metric $\breve{g}$, we have

$$\breve{B}_2 \subseteq A_{1/8}. \tag{2.2}$$

We make use of the fact that the inversion map

$$\mathcal{I}\colon (x, y) \mapsto \left(\frac{x}{|x|^2 + y^2}, \frac{y}{|x|^2 + y^2} \right) \tag{2.3}$$

and the scaling maps

$$\mathcal{S}_\varepsilon \colon (x,y) \mapsto (\varepsilon x, \varepsilon y), \qquad \varepsilon > 0, \tag{2.4}$$

are isometries of $\mathbb{H}$. Note that $\mathcal{I}$ restricts to a map $A_r \to A_r$.

We identify $\overline{\mathbb{H}}$ with the half space $\{y \geq 0\} \subseteq \mathbb{R}^3$ and denote

$$\begin{aligned} B_r &= \{(x,y) : |x|^2 + y^2 < r^2\} \subseteq \overline{\mathbb{H}}, \\ \overline{B}_r &= \{(x,y) : |x|^2 + y^2 \leq r^2\} \subseteq \overline{\mathbb{H}}. \end{aligned} \tag{2.5}$$

We make use of the defining function

$$\breve{\rho} = \frac{2y}{|x|^2 + (1+y)^2}, \tag{2.6}$$

which is the pullback to the half-space model of the standard defining function $\frac{1}{2}(1 - |u|^2)$ for the ball model. On any fixed $\overline{B}_r$ we have

$$\frac{1}{C_r} y \leq \breve{\rho} \leq C_r y,$$

for some constant C_r depending only on r.

It is convenient to construct an inversion-invariant defining function on the annular region A_c. To accomplish this, we first recall the following.

Lemma 2.1 (Lemma 5 in [10])**.** *There exists a nonnegative and nondecreasing smooth cutoff function $\chi \colon \mathbb{R} \to \mathbb{R}$ that is identically 1 on $[2,\infty)$, is supported in $(\frac{1}{2},\infty)$, and satisfies the condition*

$$\chi(r) + \chi(1/r) \equiv 1. \tag{2.7}$$

We now define the function $F \colon (0,\infty) \to (0,\infty)$ by

$$F(r) = \chi(r) + \frac{1}{r^2}\chi(1/r).$$

The following is immediate from this definition.

Proposition 2.2. *The function $F \colon (0,\infty) \to (0,\infty)$ satisfies*

$$\begin{aligned} F(1/r) &= r^2 F(r), && r \in \mathbb{R}, \\ F(r) &= 1, && r \geq 2, \\ F(r) &= \frac{1}{r^2}, && r \leq 1/2. \end{aligned}$$

The functions χ and F give rise to functions on $\mathbb{H}$, which we denote by the same symbols, by taking r to be given by $r^2 = |x|^2 + y^2$. Using this, we see that yF is inversion-invariant; i.e., $\mathcal{I}^*(yF) = yF$; and that on each A_c we have

$$\frac{1}{C} y \leq \breve{\rho} \leq Cy \quad \text{and} \quad \frac{1}{C'} yF \leq \breve{\rho} \leq C' yF, \tag{2.8}$$

for constants C and C' depending only on c.

2.2. **Background coordinates and Möbius parametrizations.** The construction of function spaces on M given in [2] (see also [11]) relies on identifying coordinate neighborhoods of M with neighborhoods in hyperbolic space. Here we slightly modify that construction to be compatible with our gluing construction.

By choosing a collar neighborhood of $\partial M \subseteq \overline{M}$ and rescaling ρ by a constant if necessary, we hereafter identify a neighborhood of ∂M in $\overline{M}$ with $\partial M \times [0,1)$, and identify ρ with the coordinate on $[0,1)$.

For each $\hat{p} \in \partial M$ we choose smooth coordinates θ taking a neighborhood $U(\hat{p}) \subseteq \partial M$ to a ball of radius 1 in $\mathbb{R}^2$. We extend these coordinates to smooth coordinates

$$\Theta = (\theta, \rho)\colon Z(\hat{p}) \to Y_1, \tag{2.9}$$

where $Z(\hat{p}) = U(\hat{p}) \times [0,1)$ and Y_1 is given by (2.1). We may assume that Θ extends smoothly to $\overline{Z(\hat{p})}$. We fix a finite collection of points $\hat{p}$ such that the corresponding sets $\Theta^{-1}(Y_{1/8})$ cover $\partial M \times [0, 1/8)$. The finite collection of coordinates Θ we refer to as ***background coordinates***.

We may assume that the finite collection of points $\hat{p}$ contains the points p_i, $i = 1, 2$, of the main theorem. About these two points we may further assume that we have ***preferred background coordinates*** $\Theta_i = (\theta_i, \rho)$ centered at p_i satisfying the following conditions:

- $\Theta_i(p_i) = (0, 0)$,
- the coordinates Θ_i are defined on the set $Z(p_i)$ with
$$\Theta_i(Z(p_i)) = \{(\theta, \rho) : (\theta_i^1)^2 + (\theta_i^2)^2 \leq 1 \text{ and } \rho^2 \leq 1\},$$
and
- in coordinates Θ_i the metric $\overline{g} = \rho^2 g$ takes the form $\overline{g}\big|_{p_i} = \delta_{ab} d\Theta_i^a d\Theta_i^b$.

Note that we can arrange the preferred background coordinates so that $Z(p_1) \cap Z(p_2) = \emptyset$. We also define, for use below, the coordinate half balls

$$\begin{aligned} U_{i,r} &= \{q \in Z(p_i)\colon (\theta_i^1(q))^2 + (\theta_i^2(q))^2 + \rho(q)^2 < r^2\}, \\ \overline{U}_{i,r} &= \{q \in Z(p_i)\colon (\theta_i^1(q))^2 + (\theta_i^2(q))^2 + \rho(q)^2 \leq r^2\}. \end{aligned} \tag{2.10}$$

We fix a smooth ***preferred background metric*** $\overline{h}$ on $\overline{M}$ that satisfies $\overline{h} = \delta_{ab} d\Theta_i^a d\Theta_i^b$ in each of the two preferred background coordinate charts. Let $\overline{\nabla}$ denote the Levi-Civita connection of $\overline{h}$ on $\overline{M}$.

For each $p \in M$ with $\rho(p) < 1/8$ we define a ***Möbius parametrization*** $\Phi_p\colon \breve{B}_2 \to M$ as follows. Let Θ be a background coordinate chart with $p \in \Theta^{-1}(Y_{1/8})$; denote $\Theta(p)$ by (θ_p, ρ_p). Define Φ_p by $(\Theta \circ \Phi_p)(x, y) = (\theta_p + \rho_p x, \rho_p y)$. The inclusion (2.2) ensures that Φ_p is well-defined. To this collection we append an additional finite number of smooth parametrizations $\Phi\colon \breve{B}_2 \to M \smallsetminus \{\rho < 1/16\}$ such that the sets $\Phi(\breve{B}_1)$ cover $M \smallsetminus \{\rho < 1/8\}$ and such that Φ extends smoothly to the closure of $\breve{B}_2$. We denote this extended collection by $\{\Phi\}$ and refer to them as Möbius parametrizations.

2.3. **Hölder spaces on M.** Hölder spaces of tensor fields on M are defined using the norms

$$\|u\|_{C^{k,\alpha}(M)} = \sup_{\Phi} \|\Phi^* u\|_{C^{k,\alpha}(\breve{B}_2)},$$

where the $C^{k,\alpha}(\breve{B}_2)$ norm is computed using the Euclidean metric. Uniformly equivalent norms are produced by replacing $\breve{B}_2$ by $\breve{B}_r$ for any $1 \leq r \leq 2$. If $U \subseteq M$ is open, the $C^{k,\alpha}(U)$ norms are defined by appropriately restricting the domains of the Möbius parametrizations.

In [2], weighted function spaces are defined using ρ as a weight function. Here we generalize that construction; see [7]. We say that a smooth function $w\colon M \to (0,\infty)$ satisfies the ***scaling hypotheses*** if there exist constants c_0 and c_k such that for every Möbius parametrization Φ_p we have

$$\begin{aligned} &c_0^{-1} w(p) \leq \Phi_p^* w \leq c_0 w(p) \quad \text{and} \\ &\|\Phi_p^* w\|_{C^k(\breve{B}_2)} \leq c_k w(p), \quad k \geq 1. \end{aligned} \tag{2.11}$$

It is straightforward to see that if functions w_1 and w_2 each satisfy the scaling hypotheses, then so do the functions $w_1 w_2$ and w_1/w_2.

Let $U \subseteq M$ be open. Two functions w_1 and w_2 satisfying the scaling hypotheses are said to be ***equivalent weight functions on U*** if there exists a constant C_U such that

$$\frac{1}{C_U} w_1(p) \leq w_2(p) \leq C_U w_1(p), \qquad p \in U. \tag{2.12}$$

For any function w satisfying the scaling hypotheses and for any $\delta \in \mathbb{R}$, we endow the weighted Hölder spaces $C^{k,\alpha}_\delta(M;w) = w^\delta C^{k,\alpha}(M)$ with the norm

$$\|u\|_{C^{k,\alpha}_\delta(M;w)} = \|w^{-\delta} u\|_{C^{k,\alpha}_\delta(M)}.$$

Lemma 2.3. *Let $U \subseteq M$ be open.*

(a) *Suppose w satisfies the scaling hypotheses. For each Möbius parametrization Φ set $w_\Phi = w(\Phi(0,1))$. For any $1 \leq r \leq 2$ we have the norm equivalence*

$$\frac{1}{C}\|u\|_{C^{k,\alpha}_\delta(U;w)} \leq \sup_{\Phi} w_\Phi^{-\delta} \|\Phi^* u\|_{C^{k,\alpha}(\breve{B}_r)} \leq C \|u\|_{C^{k,\alpha}_\delta(U;w)},$$

where the constant C depends only on k, δ, and where the constants $c_0, \dots, c_{k+1}$ appearing in the scaling hypotheses for w.

(b) *If w_1 and w_2 are equivalent on U then we have the norm equivalence*

$$\frac{1}{C'}\|u\|_{C^{k,\alpha}_\delta(U;w_1)} \leq \|u\|_{C^{k,\alpha}_\delta(U;w_2)} \leq C' \|u\|_{C^{k,\alpha}_\delta(U;w_1)},$$

where the constant C' depends only the constant C from part (a) *and the constant C_U in* (2.12).

Proof. The first claim is a straightforward generalization of [11, Lemma 3.5], while the second follows immediately from the first. □

It follows from [11, Lemma 3.3] that the defining function ρ satisfies the scaling hypotheses. We suppress explicit reference to the weight function if $w = \rho$, so that $C^{k,\alpha}_\delta(M) = C^{k,\alpha}_\delta(M;\rho)$.

It is straightforward to verify that for any fixed $0 < c < 1$ the functions $\breve{\rho}$, y, and yF satisfy the scaling hypotheses on $A_c \subseteq \mathbb{H}$. Furthermore, from (2.8) we see that these functions are equivalent weight functions on A_c. Unless otherwise specified, we use the weight function $\breve{\rho}$. Thus the norms

$$\|u\|_{C^{k,\alpha}_\delta(A_c)}, \qquad \|u\|_{C^{k,\alpha}_\delta(A_c;y)}, \qquad \|u\|_{C^{k,\alpha}_\delta(A_c;yF)}$$

are all equivalent.

The following lemma relates Hölder spaces on M and $\overline{M}$. We define the ***weight*** of a tensor field to be the covariant rank less the contravariant rank; thus a Riemannian metric has weight 2.

Lemma 2.4. [11, Lemma 3.7] *Suppose u is a tensor field of weight r.*

(a) *If $u \in C^{k,\alpha}(\overline{M})$ and $|u|_{\overline{h}} = O(\rho^s)$ then $u \in C^{k,\alpha}_{r+s}(M)$ with*

$$\|u\|_{C^{k,\alpha}_{r+s}(M)} \le C\|u\|_{C^{k,\alpha}(\overline{M})}$$

for some constant C.

(b) *If $u \in C^{k,\alpha}_{k+\alpha+r}(M)$ then $u \in C^{k,\alpha}(\overline{M})$ with*

$$\|u\|_{C^{k,\alpha}(\overline{M})} \le C\|u\|_{C^{k,\alpha}_{k+\alpha+r}(M)}$$

for some constant C.

We now introduce the spaces $\mathscr{C}^{k,\alpha;m}(M)$, intermediate between $C^{k,\alpha}(M)$ and $C^{k,\alpha}(\overline{M})$, first defined in [2]. For $0 \le m \le k$ and $\alpha \in (0,1)$ we say that a tensor field u having weight r is in $\mathscr{C}^{k,\alpha;m}(M)$ if

$$\mathscr{L}_{X_1} \dots \mathscr{L}_{X_j} u \in C^{k-j,\alpha}_r(M)$$

for all $0 \le j \le m$ and for all smooth vector fields $X_1, \dots, X_j$ on $\overline{M}$. By [2, Lemma 2.2] this is equivalent to requiring that the norm

$$\|u\|_{\mathscr{C}^{k,\alpha;m}(M)} = \sum_{j=0}^{m} \|\overline{\nabla}^j u\|_{C^{k-j,\alpha}_{r+j}(M)} \tag{2.13}$$

be finite; recall that $\overline{\nabla}$ is the connection associated to the preferred background metric $\bar{h}$. We also have occasion to use norms such as $\|\cdot\|_{\mathscr{C}^{k,\alpha;m}(M;w)}$, defined by replacing $C^{k-j,\alpha}_{r+j}(M)$ by $C^{k-j,\alpha}_{r+j}(M;w)$ in (2.13).

We also define similar norms on $\mathbb{H}$. The Hölder norms are defined as above using the half-space model and the Möbius parametrizations $\breve{\Phi}\colon \breve{B}_2 \to \mathbb{H}$ of the form

$$\breve{\Phi}\colon (x,y) \mapsto (x_* + y_* x, y_* y). \tag{2.14}$$

On $B_{1/c}$ for any $c < 1/4$, we define the $\mathscr{C}^{k,\alpha;m}$ norms using ${}^{\mathrm{E}}\nabla$, the connection associated to the Euclidean metric g_{E}, and the hyperbolic defining function $\breve{\rho}$.

The following proposition records several important properties of the function spaces described above.

Proposition 2.5 ([2, Lemma 2.3])**.**

(a) *The space of tensor fields on M of a specific type and of class $\mathscr{C}^{k,\alpha;m}$ forms a Banach space with the norm* (2.13)*. The space of all tensor fields of class $\mathscr{C}^{k,\alpha;m}$ forms a Banach algebra under the tensor product, and is invariant under contraction.*

(b) *If $u \in C^{k,\alpha}_{r+m}(M)$ is a tensor field of weight r then $u \in \mathscr{C}^{k,\alpha;m}(M)$ with*

$$\|u\|_{\mathscr{C}^{k,\alpha;m}(M)} \leq C\|u\|_{C^{k,\alpha}_{r+m}(M)}.$$

(c) *If $u \in \mathscr{C}^{k,\alpha;m}(M)$ is a tensor field of weight r and*

$$|\overline{\nabla}^j u|_{\overline{h}} \to 0 \text{ and } \rho \to 0, \quad 0 \leq j \leq m-1$$

then $u \in C^{k,\alpha}_{r+m}(M)$ with $\|u\|_{C^{k,\alpha}_{r+m}(M)} \leq C\|u\|_{\mathscr{C}^{k,\alpha;m}(M)}$.

We record a regularization result that follows from [2, Theorem 2.6].

Proposition 2.6. *Suppose $\tau \in \mathscr{C}^{k,\alpha;m}(M)$ is a tensor field of weight r. Then there exists a tensor field $\tilde{\tau}$ such that $\tilde{\tau} \in \mathscr{C}^{l,\beta;m}(M)$ for all l and β, and such that $\tilde{\tau} - \tau \in C^{k,\alpha}_{r+m}(M)$.*

Furthermore, for each l and β there exists a constant C such that $\|\tilde{\tau}\|_{\mathscr{C}^{l,\beta;m}(M)} \leq C\|\tau\|_{\mathscr{C}^{k,\alpha;m}(M)}$.

Finally, in the case that $M = \mathbb{H}$ and τ is supported in A_r then for any $0 < \tilde{r} < r$ it can be arranged that $\tilde{\tau}$ is supported in $A_{\tilde{r}}$.

We recall also the following version of Taylor's theorem.

Proposition 2.7 (Lemma 3.2 of [2])**.** *Suppose g is weakly asymptotically hyperbolic of class $\mathscr{C}^{k,\alpha;2}$. Then for any function $u \in \mathscr{C}^{k,\alpha;2}(M) \cap C^{k,\alpha}_1(M)$ we have $u - \rho\langle d\rho, du\rangle_{\overline{g}} \in C^{k-1,\alpha}_2(M)$ with*

$$\|u - \rho\langle d\rho, du\rangle_{\overline{g}}\|_{C^{k-1,\alpha}_2(M)} \leq C\|u\|_{\mathscr{C}^{k,\alpha;2}(M)},$$

where the constant C depends only on $\|\overline{g}\|_{\mathscr{C}^{k,\alpha;2}(M)}$.

We conclude this section by noting the effects of the scaling maps (2.4) on weighted Hölder norms in hyperbolic space. Direct computation using the Möbius parametrizations (2.14) shows that for a tensor field u of weight r we have

$$\begin{aligned} \|\mathcal{S}^*_\varepsilon u\|_{C^{k,\alpha}_\delta(B_{1/c};y)} &= \varepsilon^\delta \|u\|_{C^{k,\alpha}_\delta(B_{\varepsilon/c};y)} \\ \|({}^{\mathrm{E}}\nabla)^j(\mathcal{S}^*_\varepsilon u)\|_{C^{k-j,\alpha}_{r+j}(B_{1/c};y)} &= \varepsilon^{r+j}\|({}^{\mathrm{E}}\nabla)^j u\|_{C^{k-j,\alpha}_{r+j}(B_{\varepsilon/c};y)}. \end{aligned} \tag{2.15}$$

3. The shear-free condition and the tensor $\mathcal{H}$

We recall here the tensor $\mathcal{H}_{\overline{g}}(\omega)$ introduced in [2] and used in [3] to incorporate the shear-free condition into the conformal method. For any metric $\overline{g}$ and for any function ω we define the tensor $\mathcal{H}_{\overline{g}}(\omega)$ by

$$\mathcal{H}_{\overline{g}}(\omega) = |d\omega|^6_{\overline{g}}\,\mathcal{D}_{\overline{g}}\left(|d\omega|^{-2}_{\overline{g}}\operatorname{grad}_{\overline{g}}\omega\right) + A_{\overline{g}}(\omega)\left(d\omega\otimes d\omega - \frac{1}{3}|d\omega|^2_{\overline{g}}\overline{g}\right), \tag{3.1}$$

where

$$A_{\overline{g}}(\omega) = \frac{1}{2}|d\omega|_{\overline{g}}\operatorname{div}_{\overline{g}}\left(|d\omega|_{\overline{g}}\operatorname{grad}_{\overline{g}}\omega\right)$$

and where $\mathcal{D}_{\overline{g}}$ is the conformal-Killing operator defined in (1.6). The following properties of this tensor are established in §4 of [2].

Proposition 3.1. *For any C^1 metric $\overline{g}$ and C^2 function ω we have the following.*

(a) *$\mathcal{H}_{\overline{g}}(\omega)$ is symmetric and trace-free.*
(b) *$\mathcal{H}_{\overline{g}}(\omega)(\operatorname{grad}_{\overline{g}}\omega,\cdot) = 0$.*
(c) *$\mathcal{H}_{\overline{g}}(c\omega) = c^5\mathcal{H}_{\overline{g}}(\omega)$ for all constants c.*
(d) *For any strictly positive function θ, we have $\mathcal{H}_{\theta\overline{g}}(\omega) = \theta^{-2}\mathcal{H}_{\overline{g}}(\omega)$.*

If ρ is a smooth defining function, we furthermore have the following.

(e) *If $\overline{g}\in\mathscr{C}^{k,\alpha;2}(M)$, then $\mathcal{H}_{\overline{g}}(\rho)\in\mathscr{C}^{k-1,\alpha;1}(M)$ and $\operatorname{div}_{\overline{g}}\mathcal{H}_{\overline{g}}(\rho)\in C^{k-2,\alpha}_1(M)$.*
(f) *If $\overline{g}\in\mathscr{C}^{k,\alpha;2}(M)$ and $g=\rho^{-2}\overline{g}$ satisfies $\mathrm{R}[g]+6 = C^{k-2,\alpha}_2(M)$, then*

$$\mathcal{H}_{\overline{g}}(\rho) - \left(\operatorname{Hess}_{\overline{g}}\rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g}\right)\in C^{k-1,\alpha}_3(M). \tag{3.2}$$

Due to the property (3.2), the shear-free condition (1.1) is equivalent to requiring $\overline{\Sigma} = \mathcal{H}_{\overline{g}}(\rho)$ along ∂M.

4. The spliced manifold M_ε

We now begin the proof of the main theorem. We consider a CMCSF hyperboloidal data set (g,K) of class $\mathscr{C}^{k,\alpha;2}$ on M for fixed $k\geq 3$ and $\alpha\in(0,1)$. As outlined above, the first step of the proof is to construct the spliced manifold M_ε and the spliced defining function ρ_ε, as well as various function spaces on M_ε.

4.1. The splicing construction.

Recall the definition of $\overline{B}_r$ in (2.5) and let $\varepsilon>0$ be a small parameter. For each of the gluing points p_i, $i=1,2$, let the mapping $\alpha_{\varepsilon,i}\colon \overline{B}_{1/\varepsilon}\to\overline{M}$ be given in preferred background coordinates by

$$\Theta_i = (\theta_i,\rho) = \alpha_{\varepsilon,i}(x,y) = (\varepsilon x,\varepsilon y).$$

The mappings $\alpha_{\varepsilon,i}$ give us scaled parametrizations of neighborhoods of $p_i\in\overline{M}$. For future use we note that because $\alpha_{\varepsilon,i} = \Theta_i^{-1}\circ\mathcal{S}_\varepsilon$ and Θ_i is an isometry from $\overline{h}$ to g_E, it follows from (2.15) that for any tensor $u \in \mathscr{C}^{k,\alpha;m}(M)$ of weight r and any $0 \leq j \leq m$, we have

$$\begin{aligned} \|({}^E\nabla)^j \alpha^*_{\varepsilon,i} u\|_{C^{k-j}_{r+j}(B_{1/c};y)} &\leq C\varepsilon^{r+j} \|\overline{\nabla}^j u\|_{C^{k-j,\alpha}_{r+j}(M)} \\ &\leq C\varepsilon^{r+j} \|u\|_{\mathscr{C}^{k,\alpha;m}(M)}, \end{aligned} \tag{4.1}$$

where the constant C depends on c, but is independent of ε.

Recall the sets $\overline{U}_{i,r}$ defined in (2.10). Consider the equivalence relation $\sim$ on

$$\overline{M} \smallsetminus \left(\alpha_{\varepsilon,1}(\overline{B}_\varepsilon) \cup \alpha_{\varepsilon,2}(\overline{B}_\varepsilon)\right) = \overline{M} \smallsetminus \left(\overline{U}_{1,\varepsilon^2} \cup \overline{U}_{2,\varepsilon^2}\right)$$

generated by

$$\alpha_{\varepsilon,1}(x,y) \sim (\alpha_{\varepsilon,2} \circ \mathcal{I})\,(x,y),$$

where $\mathcal{I}$ is the inversion map defined in (2.3). Define the ***spliced manifold*** $\overline{M}_\varepsilon$ as the quotient manifold whose points are the equivalence classes of $\sim$. It is clear from the construction that $\overline{M}_\varepsilon$ is a family of smooth manifolds with boundary. In addition, define M_ε as the subset of $\overline{M}_\varepsilon$ consisting of points whose representatives are elements of M; thus M_ε is the interior of $\overline{M}_\varepsilon$. Denote the underlying quotient map by π_ε. The map

$$\Psi_\varepsilon = \pi_\varepsilon \circ \alpha_{\varepsilon,1} = \pi_\varepsilon \circ \alpha_{\varepsilon,2} \circ \mathcal{I} \colon A_\varepsilon \to M_\varepsilon$$

is used throughout this paper to parametrize a region of M_ε that we refer to as the ***neck***.

Recall the definition of $\overline{U}_{i,r}$ in (2.10). For each sufficiently small $c > 0$ we define the ***exterior region*** $E_c \subseteq M$ by

$$E_c = M \smallsetminus \left(\overline{U}_{1,c} \cup \overline{U}_{2,c}\right).$$

Note that $\overline{U}_{i,c} = \alpha_{\varepsilon,i}(\overline{B}_{c/\varepsilon})$ and thus

$$E_c = M \smallsetminus \left(\alpha_{\varepsilon,1}(\overline{B}_{c/\varepsilon}) \cup \alpha_{\varepsilon,2}(\overline{B}_{c/\varepsilon})\right).$$

Clearly $\bigcup_{c>0} E_c = M$.

For the rest of this paper we assume

$$0 < \varepsilon < c^2 < \frac{1}{64}. \tag{4.2}$$

In particular, this implies that the map $\iota_\varepsilon = \pi_\varepsilon\big|_{E_c} \colon E_c \to M_\varepsilon$ is an embedding. Note that $\Psi_\varepsilon^{-1}(E_c) = A_\varepsilon \smallsetminus \overline{A}_{\varepsilon/c}$.

In establishing the main theorem, it is important to obtain estimates that are uniform in ε; thus we adopt the following notational convention. For quantities X and Z, both depending on c and ε, we write $X \lesssim Z$ to mean that $X \leq CZ$ for some constant C that may depend on c, but is independent of ε satisfying (4.2). We write $X \approx Z$ when both $X \lesssim Z$ and $Z \lesssim X$.

4.2. **The defining function ρ_ε.** We now introduce a family of defining functions ρ_ε on M_ε. These functions agree with the original defining function ρ away from the neck $\Psi_\varepsilon(A_\varepsilon)$, while on $\Psi_\varepsilon(A_\varepsilon)$ they are determined by

$$\Psi_\varepsilon^* \rho_\varepsilon = \varepsilon y F. \tag{4.3}$$

Thus

$$\Psi_\varepsilon^* \rho_\varepsilon = \begin{cases} \alpha_{\varepsilon,1}^* \rho & \text{where } r > 2, \\ (\alpha_{\varepsilon,2} \circ \mathcal{I})^* \rho & \text{where } r < 1/2. \end{cases} \tag{4.4}$$

Furthermore, since yF satisfies the scaling hypotheses on A_ε, the functions ρ_ε satisfy the scaling hypotheses on M_ε.

Lemma 4.1. *Provided* (4.2) *holds we have the following.*

(a) *On E_c we have $\iota_\varepsilon^* \rho_\varepsilon = \rho$.*
(b) *On A_c the weight functions*

$$\Psi_\varepsilon^* \rho_\varepsilon = \varepsilon y F, \quad \varepsilon y, \quad \varepsilon \breve{\rho},$$

are all equivalent, uniformly in ε.

Proof. Part (a) is due to (4.4), while part (b) is a consequence of (2.8). □

4.3. **Function spaces on M_ε.** In order to define function spaces on M_ε, we first construct a collection of Möbius parametrizations for M_ε of two types:

- parametrizations of the form $\pi_\varepsilon \circ \Phi \colon \breve{B}_2 \to M_\varepsilon$, where Φ is a Möbius parametrization for M such that

$$\Phi(\breve{B}_2) \subseteq M \setminus \left(\overline{U}_{1,\varepsilon^2} \cup \overline{U}_{2,\varepsilon^2}\right),$$

and
- parametrizations of the form $\Psi_\varepsilon \circ \breve{\Phi} \colon \breve{B}_2 \to M_\varepsilon$, where $\breve{\Phi} \colon \breve{B}_2 \to \mathbb{H}$ is a Möbius parametrization of $\mathbb{H}$ such that $\breve{\Phi}(\breve{B}_2) \subseteq A_\varepsilon$.

The second type of parametrizations allow us to compare the geometry of the neck with that of hyperbolic space. It is easy to see that these parametrizations cover all of M_ε, and that this remains true if restricted to $\breve{B}_r$ for any $1 \le r \le 2$.

This collection of Möbius parametrizations is used to define the intrinsic Hölder spaces $C^{k,\alpha}(M_\varepsilon)$ with norms

$$\|u\|_{C^{k,\alpha}(M_\varepsilon)} = \sup \|\Phi_\varepsilon^* u\|_{C^{k,\alpha}(\breve{B}_2)};$$

as before, we obtain alternate norms, uniformly equivalent in ε, by replacing $\breve{B}_2$ with $\breve{B}_r$ for any $1 \le r \le 2$.

Suppose that $\Phi_p \colon \breve{B}_2 \to M$ is a Möbius parametrization arising from one of the preferred background coordinate charts Θ_i such that $\Phi_p(\breve{B}_2) \cap B_{\varepsilon^2}(p_i) = \emptyset$. Then the corresponding Möbius parametrization $\pi_\varepsilon \circ \Phi_p \colon \breve{B}_2 \to M_\varepsilon$ coincides with the parametrization $\Psi_\varepsilon \circ \breve{\Phi} \colon \breve{B}_2 \to M_\varepsilon$ centered at X_* with $\alpha_{\varepsilon,i}(X_*) = p$. Such parametrizations, which can be viewed as arising either from hyperbolic space or from the manifold M, cover the the transitional region of M_ε between the exterior region $\iota_\varepsilon(E_c)$ and the neck region $\Psi_\varepsilon(A_c)$.

The following is immediate.

Lemma 4.2. *For any l and β we have*

(a) $\|u\|_{C^{l,\beta}(\iota_\varepsilon(E_c))} = \|\iota_\varepsilon^* u\|_{C^{l,\beta}(E_c)}$,
(b) $\|u\|_{C^{l,\beta}(\Psi_\varepsilon(A_c))} = \|\Psi_\varepsilon^* u\|_{C^{l,\beta}(A_c)}$.

Notice that $\Psi_\varepsilon^*\rho_\varepsilon \approx \varepsilon y$ becomes degenerate as $\varepsilon \to 0$. In order to avoid difficulties associated with this degeneracy, we define weighted Hölder spaces and intermediate spaces on M_ε using an alternate defining function $\tilde\rho_\varepsilon$ that scales better in the neck as $\varepsilon \to 0$. In order to accomplish this, let $\psi\colon (0,\infty) \to (0,1]$ be a smooth, nondecreasing function such that

$$\psi(x) = \begin{cases} 2x & \text{if } 0 < x \le \frac{1}{4} \\ 1 & \text{if } x \ge \frac{3}{4}. \end{cases}$$

We subsequently define a smooth function ω_ε on $\overline{M}_\varepsilon$ by setting $\omega_\varepsilon = 1$ outside $\Psi_\varepsilon(A_\varepsilon)$ and requiring that

$$\Psi_\varepsilon^*\omega_\varepsilon = \psi(\varepsilon r + \varepsilon/r) \tag{4.5}$$

on A_ε. Note that

$$\Psi_\varepsilon^*\omega_\varepsilon = \begin{cases} 2\left(\varepsilon r + \frac{\varepsilon}{r}\right) & \text{on } A_{8\varepsilon}, \\ 1 & \text{on } A_\varepsilon \setminus A_{4\varepsilon/3}. \end{cases} \tag{4.6}$$

Note also that in preferred background coordinates $\Theta_i = (\theta,\rho)$ we have $\pi_\varepsilon^*\omega_\varepsilon = \psi(|(\theta,\rho)| + \varepsilon^2|(\theta,\rho)|^{-1})$, while outside the domain of those coordinates we have $\pi_\varepsilon^*\omega_\varepsilon = 1$. Thus on E_c we have $\iota_\varepsilon^*\omega_\varepsilon = \widetilde{\omega}_\varepsilon \approx 1$.

With the function ω_ε in hand we define

$$\tilde\rho_\varepsilon = \rho_\varepsilon/\omega_\varepsilon. \tag{4.7}$$

Direct computation shows that both ρ_ε and ω_ε, and hence $\tilde\rho_\varepsilon$, satisfy the scaling hypotheses (2.11). Furthermore, for each fixed c, on A_c we have

$$\Psi_\varepsilon^*\tilde\rho_\varepsilon = \frac{yF}{2\left(r + \frac{1}{r}\right)} \approx yF \approx y \approx \breve\rho, \tag{4.8}$$

together with analogous uniform estimates for all derivatives of $\Psi_\varepsilon^*\tilde\rho_\varepsilon$, while on E_c we have $\iota_\varepsilon^*\tilde\rho_\varepsilon \approx \rho$. Combining this with Lemma 4.2 yields the following.

Lemma 4.3. *For any l, β, and δ we have*

(a) $\|u\|_{C^{l,\beta}_\delta(\iota_\varepsilon(E_c);\tilde\rho_\varepsilon)} \approx \|\iota_\varepsilon^* u\|_{C^{l,\beta}_\delta(E_c)}$,
(b) $\|u\|_{C^{l,\beta}_\delta(\Psi_\varepsilon(A_c);\tilde\rho_\varepsilon)} \approx \|\Psi_\varepsilon^* u\|_{C^{l,\beta}_\delta(A_c)}$.

For any region $U \subseteq M_\varepsilon$ we note that $C^{l,\beta}_\delta(U;\rho_\varepsilon)$ and $C^{l,\beta}_\delta(U;\tilde\rho_\varepsilon)$ coincide as sets; thus we only indicate the weight function $\tilde\rho_\varepsilon$ if we are referring to the corresponding norms.

In order to define the intermediate spaces $\mathscr{C}^{k,\alpha;m}$ on M_ε, we construct a family of smooth background metrics $\overline{h}_\varepsilon$. Recall the preferred background metric $\overline{h}$ defined in §2.2 and note that $h = \rho^{-2}\overline{h}$ satisfies

$$\alpha_{\varepsilon,1}^* h = \breve{g} = (\alpha_{\varepsilon,2} \circ \mathcal{I})^* h.$$

Thus h descends to a metric h_ε on M_ε under the quotient map π_ε. We set $\overline{h}_\varepsilon = \tilde{\rho}_\varepsilon^2 h_\varepsilon$. Note that on A_c we have

$$\Psi_\varepsilon^* \overline{h}_\varepsilon = F^2 g_{\mathrm{E}} \approx g_{\mathrm{E}}.$$

Using $\overline{h}_\varepsilon$ and $\tilde{\rho}_\varepsilon$, we define the intermediate norm $\mathscr{C}^{k,\alpha;m}$ of a tensor field u having weight r as follows:

$$\|u\|_{\mathscr{C}^{k,\alpha;m}(M_\varepsilon;\tilde{\rho}_\varepsilon)} = \sum_{j=0}^{m} \|{}^\varepsilon\overline{\nabla}^j u\|_{C^{k-j,\alpha}_{r+j}(M_\varepsilon,\tilde{\rho}_\varepsilon)},$$

where ${}^\varepsilon\overline{\nabla}$ is the connection associated to $\overline{h}_\varepsilon$.

The following lemma follows directly from the various definitions involved.

Lemma 4.4. *For any l, β, and m we have*

(a) $\|u\|_{\mathscr{C}^{l,\beta;m}(\iota_\varepsilon(E_c);\tilde{\rho}_\varepsilon)} \approx \|\iota_\varepsilon^* u\|_{\mathscr{C}^{l,\beta;m}(E_c)}$,
(b) $\|u\|_{\mathscr{C}^{l,\beta;m}(\Psi_\varepsilon(A_c);\tilde{\rho}_\varepsilon)} \approx \|\Psi_\varepsilon^* u\|_{\mathscr{C}^{l,\beta;m}(A_c)}$.

5. The spliced metrics

For each $0 < \varepsilon < 1/64$, we define the ***spliced metric*** λ_ε to be the metric on M_ε that agrees with $(\pi_\varepsilon)_* g$ away from the neck $\Psi_\varepsilon(A_\varepsilon)$, while on $\Psi_\varepsilon(A_\varepsilon)$ it satisfies

$$\Psi_\varepsilon^* \lambda_\varepsilon^{-1} = \chi(r)[(\alpha_{\varepsilon,1})^* g]^{-1} + \chi(1/r)\,[(\alpha_{\varepsilon,2} \circ \mathcal{I})^* g]^{-1}; \tag{5.1}$$

here χ is the cutoff in (2.7). The reason for splicing cometrics rather than metrics is that it is easier to verify that the asymptotic hyperbolicity property holds if we work with cometrics; see Proposition 5.6 below. We set $\overline{\lambda}_\varepsilon = \rho_\varepsilon^2 \lambda_\varepsilon$. Note that

$$\Psi_\varepsilon^* \lambda_\varepsilon = \begin{cases} \alpha_{\varepsilon,1}^* g & \text{where } r > 2, \\ (\alpha_{\varepsilon,2} \circ \mathcal{I})^* g & \text{where } r < 1/2. \end{cases} \tag{5.2}$$

In order to establish estimates for the spliced metric λ_ε, we first analyze the pullback metrics $\alpha_{\varepsilon,i}^* g$. Following [10], we write

$$\alpha_{\varepsilon,i}^* g = y^{-2}(g_{\mathrm{E}} + m_{\varepsilon,i})$$

for tensors $m_{\varepsilon,i}$; here g_{E} represents the Euclidean metric on the half space. We furthermore define the contravariant 2-tensor fields $j_{\varepsilon,i}$ by

$$(g_{\mathrm{E}} + m_{\varepsilon,i})^{-1} = g_{\mathrm{E}}^{-1} + j_{\varepsilon,i}.$$

Recall that throughout we let c be a fixed constant less than 1/8 and assume that ε satisfies (4.2).

Proposition 5.1. *The tensors $m_{\varepsilon,i}$ and $j_{\varepsilon,i}$ are in $\mathscr{C}^{k,\alpha;2}(A_c)$ and satisfy*

(a) $\|m_{\varepsilon,i}\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon$,
(b) $\|j_{\varepsilon,i}\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon$.

Proof. Note that $\overline{g} = \rho^2 g$ satisfies

$$\alpha^*_{\varepsilon,i}\overline{g} = \varepsilon^2(g_{\mathrm{E}} + m_{\varepsilon,i}). \tag{5.3}$$

Since $\overline{g} \in \mathscr{C}^{k,\alpha;2}(M)$ and $g_{\mathrm{E}} \in \mathscr{C}^{k,\alpha;2}(B_{1/c})$, we immediately have $m_{\varepsilon,i} \in \mathscr{C}^{k,\alpha;2}(B_{1/c})$. This inclusion, however, does not come with an estimate uniform in ε.

The preferred background coordinates are constructed so that $m_{\varepsilon,i} = 0$ at $(x, y) = (0, 0)$. Thus the mean value theorem implies that

$$\|m_{\varepsilon,i}\|_{C^0_2(B_{1/c})} \lesssim \|{}^{\mathrm{E}}\nabla m_{\varepsilon,i}\|_{C^0_2(B_{1/c})}.$$

As a consequence

$$\|m_{\varepsilon,i}\|_{C^{k,\alpha}_2(B_{1/c})} \lesssim \|{}^{\mathrm{E}}\nabla m_{\varepsilon,i}\|_{C^{k-1,\alpha}_2(B_{1/c})}.$$

Using this, (5.3), and (4.1) we find

$$\begin{aligned}
&\|m_{\varepsilon,i}\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \\
&\quad\lesssim \|{}^{\mathrm{E}}\nabla m_{\varepsilon,i}\|_{C^{k-1,\alpha}_2(B_{1/c})} \\
&\qquad + \|{}^{\mathrm{E}}\nabla m_{\varepsilon,i}\|_{C^{k-1,\alpha}_3(B_{1/c})} + \|({}^{\mathrm{E}}\nabla)^2 m_{\varepsilon,i}\|_{C^{k-2,\alpha}_4(B_{1/c})} \\
&\quad\lesssim \|{}^{\mathrm{E}}\nabla m_{\varepsilon,i}\|_{C^{k-1,\alpha}_3(B_{1/c};y)} + \|({}^{\mathrm{E}}\nabla)^2 m_{\varepsilon,i}\|_{C^{k-2,\alpha}_4(B_{1/c};y)} \\
&\quad= \varepsilon^{-2}\|{}^{\mathrm{E}}\nabla \alpha^*_{\varepsilon,i}\overline{g}\|_{C^{k-1,\alpha}_3(B_{1/c};y)} + \varepsilon^{-2}\|({}^{\mathrm{E}}\nabla)^2 \alpha^*_{\varepsilon,i}\overline{g}\|_{C^{k-2,\alpha}_4(B_{1/c};y)} \\
&\quad\lesssim \varepsilon\|\overline{g}\|_{\mathscr{C}^{k,\alpha;2}(M)},
\end{aligned}$$

which establishes the first claim. The second claim follows from similar reasoning, together with the fact that $\overline{g} \in C^{1,1}(\overline{M})$ (see [2, Lemma 2.2]), from which we easily obtain a uniform invertibility bound. □

Proposition 5.2. *Along $y = 0$ we have*

$$j_{\varepsilon,i}(dy, dy) = 0 \quad \textit{and} \quad (\mathcal{I}^* j_{\varepsilon,i})(dy, dy) = 0.$$

Proof. In view of (5.3) we have

$$\alpha^*_{\varepsilon,i}(|d\rho|^2_{\overline{g}}) = |\varepsilon dy|^2_{\varepsilon^2(g_{\mathrm{E}}+m_{\varepsilon,i})} = |dy|^2_{g_{\mathrm{E}}+m_{\varepsilon,i}} = (g_{\mathrm{E}}^{-1} + j_{\varepsilon,i})(dy, dy).$$

Thus the assumption that $|d\rho|_{\overline{g}} = 1$ along ∂M implies that $j_{\varepsilon,i}(dy, dy) = 0$ where $y = 0$. The second claim follows from the first and the fact that $\mathcal{I}_* dy = r^{-2} dy + O(y)$. □

We now obtain uniform bounds on the metric λ_ε; these give rise to uniform bounds for geometric differential operators on M_ε. We subsequently obtain stronger estimates in the neck region $\Psi_\varepsilon(A_c)$.

Proposition 5.3.
(a) $\|\lambda_\varepsilon^{-1}\|_{C^{k,\alpha}(M_\varepsilon)} \lesssim 1$.

(b) $\|\lambda_\varepsilon\|_{C^{k,\alpha}(M_\varepsilon)} \lesssim 1$.

Proof. Due to (5.2) it suffices to establish the estimates on $\Psi_\varepsilon(A_{1/2})$. Since $A_{1/2} \subseteq A_c$ by (4.2), we use Lemma 4.2 and (5.1) to estimate

$$\begin{aligned}\|\lambda_\varepsilon^{-1}\|_{C^{k,\alpha}(\Psi_\varepsilon(A_{1/2}))} &\le \|\Psi_\varepsilon^*\lambda_\varepsilon^{-1}\|_{C^{k,\alpha}(A_c)} \\ &\lesssim \|\alpha_{1,\varepsilon}^* g^{-1}\|_{C^{k,\alpha}(A_c)} \\ &\quad + \|(\alpha_{2,\varepsilon}\circ \mathcal{I})^* g^{-1}\|_{C^{k,\alpha}(A_c)} \\ &\lesssim \|g^{-1}\|_{C^{k,\alpha}(M)},\end{aligned}$$

which is finite due to the fact that $\overline{g} = \rho^2 g \in \mathscr{C}^{k,\alpha;2}(M)$, and therefore $\overline{g}^{-1} \in \mathscr{C}^{k,\alpha;2}(M)$.

Proposition 5.1 implies that $\Psi_\varepsilon^*\lambda_\varepsilon$ is uniformly invertible on A_c, and thus the desired bound on λ_ε follows from the corresponding bound on λ_ε^{-1}. □

The bounds in the previous proposition imply estimates for differential operators arising from λ_ε. We say that a differential operator $P = P[g]$ is a ***geometric operator of order l*** determined by the metric g if in any coordinate frame the components of Pu are linear functions of u and its partial derivatives, whose coefficients are universal polynomials in the components of g, their partial derivatives, and $(\det g_{ij})^{-1/2}$. We furthermore require that the coefficients of the j^{th} derivatives of u involve no more than $l-j$ derivatives of the metric. Examples of geometric operators include the scalar Laplacian Δ_g, the divergence operator, and the conformal Killing operator $\mathcal{D}_g$. The mapping properties of geometric operators arising from asymptotically hyperbolic metrics are studied systematically in [11] (see also [4]); the extension of that work to the weakly asymptotically hyperbolic setting appears in [3]. From these works we deduce the following.

Proposition 5.4. *Let P be a geometric operator of order l and suppose that $l \le j \le k$ and $\delta \in \mathbb{R}$.*

(a) *If $u \in C^{j,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ then*

$$\|P[\lambda_\varepsilon]u\|_{C^{j-l,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \lesssim \|u\|_{C^{j,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}.$$

(b) *Furthermore, if P is an elliptic operator and $u \in C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ with $P[\lambda_\varepsilon]u \in C^{j-l,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ then $u \in C^{j,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ with*

$$\|u\|_{C^{j,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \lesssim \|P[\lambda_\varepsilon]u\|_{C^{j-l,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} + \|u\|_{C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}.$$

Proof. The statements follow directly from [11, Lemmas 4.6 and 4.8], making use of the uniform bounds established in Proposition 5.3. □

The previous proposition immediately implies the following.

Proposition 5.5.

(a) *The conformal Killing operator $\mathcal{D}_{\lambda_\varepsilon}$ defined in* (1.6) *satisfies*

$$\|\mathcal{D}_{\lambda_\varepsilon} W\|_{C_2^{k-1,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} \lesssim \|W\|_{C_2^{k,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)}$$

for any vector field $W \in C_2^{k,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)$.

(b) *The divergence operator satisfies*

$$\|\operatorname{div}_{\lambda_\varepsilon} T\|_{C_2^{k-2,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} \lesssim \|T\|_{C_2^{k-1,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)}$$

for any covariant 2-tensor field $T \in C_2^{k-1,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)$.

(c) *Let L_{λ_ε} be the vector Laplace operator defined in* (1.5). *For any vector field $W \in C_2^0(M_\varepsilon,\tilde\rho_\varepsilon)$ with $L_{\lambda_\varepsilon} W \in C_2^{k-2,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)$ we have $W \in C_2^{k,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)$ with*

$$\|W\|_{C_2^{k,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} \lesssim \|L_\varepsilon W\|_{C_2^{k-2,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} + \|W\|_{C_2^0(M_\varepsilon,\tilde\rho_\varepsilon)}.$$

(d) *Suppose the functions $f_\varepsilon \in C^{k-2,\alpha}(M_\varepsilon)$ satisfy $\|f_\varepsilon\|_{C^{k-2,\alpha}(M_\varepsilon)} \le K$. Let $\mathcal{P}_\varepsilon = \Delta_{\lambda_\varepsilon} + f_\varepsilon$. For any function $u \in C_2^0(M_\varepsilon,\tilde\rho_\varepsilon)$ with $\mathcal{P}_\varepsilon u \in C_2^{k-2,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)$ we have*

$$\|u\|_{C_2^{k,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} \lesssim \|\mathcal{P}_\varepsilon u\|_{C_2^{k-2,\alpha}(M_\varepsilon,\tilde\rho_\varepsilon)} + \|u\|_{C_2^0(M_\varepsilon,\tilde\rho_\varepsilon)},$$

where the implicit constant depends on K.

We now use Proposition 5.1 to obtain additional estimates for λ_ε in the neck region. To accomplish this, we write

$$\Psi_\varepsilon^*\lambda_\varepsilon = y^{-2}(g_{\mathrm{E}} + m_\varepsilon) \quad \text{and} \quad (\Psi_\varepsilon^*\lambda_\varepsilon)^{-1} = y^2(g_{\mathrm{E}}^{-1} + j_\varepsilon). \tag{5.4}$$

Since $\mathcal{I}^* y = y/r^2$, the tensor j_ε takes the form

$$j_\varepsilon = \chi(r) j_{\varepsilon,1} + \frac{\chi(1/r)}{r^4}\mathcal{I}^* j_{\varepsilon,2}. \tag{5.5}$$

The following proposition uses (5.1) and (5.4) to show that λ_ε satisfies the regularity and boundary conditions necessary to be part of a CMCSF hyperboloidal data set of class $\mathscr{C}^{k,\alpha;2}$ on M_ε. In particular, λ_ε satisfies the hypotheses of the conformal method in [3].

Proposition 5.6. *For each ε satisfying* (4.2) *the metric λ_ε satisfies*

(a) $\overline{\lambda}_\varepsilon = \rho_\varepsilon^2 \lambda_\varepsilon \in \mathscr{C}^{k,\alpha;2}(M_\varepsilon)$ *and*

(b) $|d\rho_\varepsilon|^2_{\lambda_\varepsilon} = 1$ *along ∂M_ε.*

Furthermore,

(c) *in the exterior region E_c we have $\iota_\varepsilon^*\lambda_\varepsilon = g$, while*

(d) *in the neck region we have*

$$\|y^2\left(\Psi_\varepsilon^*\lambda_\varepsilon - \breve g\right)\|_{\mathscr{C}^{k,\alpha;2}(A_c)} = \|m_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon$$

and thus

$$\|y^2\Psi_\varepsilon^*\lambda_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim 1 \quad \text{and} \quad \|y^{-2}(\Psi_\varepsilon^*\lambda_\varepsilon)^{-1}\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim 1.$$

Proof. Note that the cutoff functions in (5.1) extend smoothly to $\overline{M}$. Thus Proposition 5.1 implies that $\overline{\lambda}_\varepsilon \in \mathscr{C}^{k,\alpha;2}(M_\varepsilon)$.

Using Proposition 5.2 and (5.5) we have that

$$|dy|^2_{g_{\mathrm{E}}+m_\varepsilon} = 1 \quad \text{where} \quad y = 0. \tag{5.6}$$

This, together with the computation

$$\begin{aligned}\Psi^*_\varepsilon(|d\rho_\varepsilon|^2_{\overline{\lambda}_\varepsilon}) &= |d(yF)|^2_{F^2(g_{\mathrm{E}}+m_\varepsilon)}\\ &= |dy|^2_{g_{\mathrm{E}}+m_\varepsilon} + 2\frac{F'}{F}y\langle dy, dr\rangle_{g_{\mathrm{E}}+m_\varepsilon} + y^2\left(\frac{F'}{F}\right)^2 |dr|^2_{g_{\mathrm{E}}+m_\varepsilon},\end{aligned}$$

shows that $|d\rho_\varepsilon|^2_{\overline{\lambda}_\varepsilon} = 1$ where $\rho_\varepsilon = 0$.

The identity in the exterior region is immediate from the construction. The estimate in the neck region follows from Proposition 5.1 and the fact that the coordinate expression for $\mathcal{I}^* j_{\varepsilon,2}$ is

$$(\mathcal{I}^* j_{\varepsilon,2})^{ab} = Q^{ab}_{cd}(j_{\varepsilon,2})^{cd}$$

for some rational functions Q^{ab}_{cd} that are uniformly bounded on A_c; see equations (36)–(37) of [10]. □

Finally, we obtain the following estimates in the neck region.

Proposition 5.7.

(a) $\|1 - |dy|^2_{g_E+m_\varepsilon}\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon$,
(b) $\|y\Delta_{g_E+m_\varepsilon} y\|_{\mathscr{C}^{k-1,\alpha;2}(A_c)} \lesssim \varepsilon$,
(c) $\|\,\mathrm{R}[g_E + m_\varepsilon]\|_{C^{k-2,\alpha}(A_c)} \lesssim \varepsilon$,
(d) $\|1 - |dy|^2_{g_E+m_\varepsilon}\|_{C^{k,\alpha}_1(A_c)} \lesssim \varepsilon$,
(e) $\|y\Delta_{g_E+m_\varepsilon} y\|_{C^{k-1,\alpha}_1(A_c)} \lesssim \varepsilon$.

Proof. The first three claims follow directly from Proposition 5.6(d), while the latter two also make use of part (c) of Proposition 2.5. □

6. The second fundamental form

In this section we obtain, for each $\varepsilon > 0$ satisfying (4.2), a symmetric covariant 2-tensor σ_ε such that the pair $(\lambda_\varepsilon, \sigma_\varepsilon)$ satisfies the shear-free condition (1.1), approximately solves the constraint equations (1.2), and satisfies estimates compatible with the convergence statements in the main theorem. Our construction differs significantly from that used in [10], as the procedure there does not account for the shear-free condition.

6.1. The spliced second fundamental form.

Recall that we express the second fundamental form of the original initial data as $K = \Sigma - g$. We decompose the traceless part Σ as

$$\Sigma = \rho^{-1}\mathcal{H}_{\overline{g}}(\rho) + \nu.$$

By hypothesis, $\overline{\Sigma} = \rho\Sigma \in \mathscr{C}^{k-1,\alpha;1}(M)$, and by Proposition 3.1 $\mathcal{H}_{\overline{g}}(\rho) \in \mathscr{C}^{k-1,\alpha;1}(M)$. Thus $\rho\nu \in \mathscr{C}^{k-1,\alpha;1}(M)$. The shear-free condition implies that $\rho\nu$ vanishes on ∂M; thus by part (c) of Proposition 2.5 we have $\nu \in C^{k-1,\alpha}_2(M)$.

We splice the tensor ν using a cutoff function to construct a tensor $\nu^{\mathrm{ext}}_\varepsilon$ that agrees with ν in the exterior region and vanishes inside the neck so as to ensure that $\nu^{\mathrm{ext}}_\varepsilon$ is trace-free with respect to λ_ε. More precisely, we set $\nu^{\mathrm{ext}}_\varepsilon = (\pi_\varepsilon)_*\nu$ outside of $\Psi_\varepsilon(A_\varepsilon)$, and on $\Psi_\varepsilon(A_\varepsilon)$ require

$$\Psi^*_\varepsilon \nu^{\mathrm{ext}}_\varepsilon = \chi\big(\tfrac{r^2}{8}\big)(\alpha_{\varepsilon,1})^*\nu + \chi\big(\tfrac{1}{8r^2}\big)(\alpha_{\varepsilon,2}\circ\mathcal{I})^*\nu. \tag{6.1}$$

Note that

$$\Psi^*_\varepsilon \nu^{\mathrm{ext}}_\varepsilon = \begin{cases} (\alpha_{\varepsilon,1})^*\nu & \text{where } r > 4,\\ 0 & \text{where } \frac{1}{2} < r < 2,\\ (\alpha_{\varepsilon,2}\circ\mathcal{I})^*\nu & \text{where } r < \frac{1}{4}. \end{cases} \tag{6.2}$$

Furthermore, we have the following.

Lemma 6.1. *The tensor ν^{ext}_ε is trace-free with respect to λ_ε and is an element of $C^{k-1,\alpha}_2(M_\varepsilon)$ with $\|\nu^{ext}_\varepsilon\|_{C^{k-1,\alpha}_2(M_\varepsilon)} \lesssim 1$.*

Proof. We may view $\nu^{\mathrm{ext}}_\varepsilon$ as the pushforward under the projection π_ε of the tensor field $\chi_\varepsilon\nu$ on M, where the function χ_ε is identically equal to 1 except in the vicinity of the gluing points where, for $i = 1, 2$, we have

$$\alpha^*_{\varepsilon,i}\chi_\varepsilon = \chi(r^2/8). \tag{6.3}$$

In preferred background coordinates Θ_i on M we have

$$\chi_\varepsilon(\Theta_i) = \chi\left(\frac{|\Theta_i|^2}{8\varepsilon^2}\right) \quad \text{where} \quad \frac{1}{2} \le \frac{|\Theta|^2}{8\varepsilon^2} \le 2;$$

otherwise χ_ε is constant. From this it follows that

$$\|\chi_\varepsilon\|_{C^k(M)} \lesssim 1. \tag{6.4}$$

Thus

$$\|\chi_\varepsilon\nu\|_{C^{k-1,\alpha}_2(M)} \le \|\chi_\varepsilon\|_{C^k(M)}\|\nu\|_{C^{k-1,\alpha}_2(M)} \lesssim 1.$$

As $\nu^{\mathrm{ext}}_\varepsilon = (\pi_\varepsilon)_*(\chi_\varepsilon\nu)$, this implies the desired estimate.

Note that the support of $\nu^{\mathrm{ext}}_\varepsilon$ is contained in the region where $\lambda_\varepsilon = (\pi_\varepsilon)_*g$. Since ν is trace-free with respect to g, it follows that $\nu^{\mathrm{ext}}_\varepsilon$ is trace-free with respect to λ_ε. □

Since $\nu^{\mathrm{ext}}_\varepsilon \in C^{k-1,\alpha}_2(M_\varepsilon)$, one could for each sufficiently small $\varepsilon > 0$ apply the results of [3] to the pair $(\lambda_\varepsilon, \nu^{\mathrm{ext}}_\varepsilon)$ and obtain a shear-free initial data set via the conformal method. However, the resulting solutions to the constraint equations might not satisfy the convergence statements of the main theorem. This is because the term $\rho^{-1}_\varepsilon \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$ is generally of significant size in the neck, and thus corrections arising from the conformal method are generally not small. For this reason we add a correction term, supported in $\Psi_\varepsilon(A_\varepsilon)$, to the tensor $\nu^{\mathrm{ext}}_\varepsilon$.

It follows from (5.4) that $\Psi_\varepsilon^*\overline{\lambda}_\varepsilon = \varepsilon^2 F^2(g_{\mathrm{E}} + m_\varepsilon)$. Direct computation using parts (c) and (d) of Proposition 3.1 shows

$$\Psi_\varepsilon^*(\rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)) = (yF)^{-1}\mathcal{H}_{F^2(g_{\mathrm{E}}+m_\varepsilon)}(yF). \tag{6.5}$$

Thus our plan is to approximate $\Psi_\varepsilon^*(\rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon))$ by $(yF)^{-1}\mathcal{H}_{F^2 g_{\mathrm{E}}}(yF)$. We note the following properties of this approximating tensor.

Lemma 6.2.

(a) $\mathcal{H}_{F^2 g_E}(yF)$ *is supported on* $A_{1/2}$*,*
(b) $\mathcal{H}_{F^2 g_E}(yF)$ *vanishes at* $y = 0$*, and*
(c) $(yF)^{-1}\mathcal{H}_{F^2 g_E}(yF) \in C_2^{k-1,\alpha}(A_{1/2})$.

Proof. Note that $\mathcal{H}_{F^2 g_{\mathrm{E}}}(yF) = F^{-4}\mathcal{H}_{g_{\mathrm{E}}}(yF)$. Thus to establish the first property, it suffices to consider $\mathcal{H}_{g_{\mathrm{E}}}(yF)$. Where $r \geq 2$ we have $\mathcal{H}_{g_{\mathrm{E}}}(yF) = \mathcal{H}_{g_{\mathrm{E}}}(y) = 0$, while for $r < 1/2$ we have

$$\mathcal{H}_{g_{\mathrm{E}}}(yF) = r^{-8}\mathcal{H}_{r^{-4}g_{\mathrm{E}}}(y/r^2) = r^{-8}\mathcal{H}_{\mathcal{I}^* g_{\mathrm{E}}}(\mathcal{I}^* y) = r^{-8}\mathcal{I}^*\mathcal{H}_{g_{\mathrm{E}}}(y) = 0.$$

This establishes the first claim.

For the second claim, we compute and estimate the various terms appearing in (3.1), restricting to the domain $A_{1/2}$, where $\mathcal{H}_{g_{\mathrm{E}}}(yF)$ is supported. First note that

$$d(yF) = F dy + y dF \quad \text{and} \quad dF = \frac{F'(r)}{r}\left(x^1 dx^2 + x^2 dx^2 + y dy\right).$$

Thus

$$g_{\mathrm{E}}(dy, dF) = O(y) \quad \text{and} \quad |d(yF)|^2_{g_{\mathrm{E}}} = F^2 + O(y^2).$$

Direct computation shows that

$$^E\nabla\left(|d(yF)|^{-2}_{g_{\mathrm{E}}} d(yF)\right) = F^{-2}\left(dy \otimes dF - dF \otimes dy\right) + O(y). \tag{6.6}$$

Since $\mathcal{D}_{g_{\mathrm{E}}}\left(|d(yF)|^{-2}_{g_{\mathrm{E}}}\,\mathrm{grad}_{g_{\mathrm{E}}}(yF)\right)$ is the symmetric and traceless part of (6.6), we conclude that $\mathcal{D}_{g_{\mathrm{E}}}\left(|d(yF)|^{-2}_{g_{\mathrm{E}}}\,\mathrm{grad}_{g_{\mathrm{E}}}(yF)\right) = O(y)$. We furthermore compute

$$|d(yF)|_{g_{\mathrm{E}}} A_{g_{\mathrm{E}}}(yF) = O(y).$$

Thus we establish the second claim.

Since $\mathcal{H}_{g_{\mathrm{E}}}(yF)$ is a smooth covariant 2 tensor on $\overline{M}_\varepsilon$ and vanishes at the boundary we have $\mathcal{H}_{g_{\mathrm{E}}}(yF) \in C_3^{k-1,\alpha}(M_\varepsilon)$ by Lemma (2.4). This establishes the third claim. ☐

Lemma 6.2 implies that we may obtain a well-defined $C_2^{k-1,\alpha}$ tensor field $\nu_\varepsilon^{\mathrm{approx}}$ on M_ε by requiring

$$\Psi_\varepsilon^*\nu_\varepsilon^{\mathrm{approx}} = -(yF)^{-1}\mathcal{H}_{F^2 g_{\mathrm{E}}}(yF),$$

and by setting $\nu_\varepsilon^{\mathrm{approx}} = 0$ outside $\Psi_\varepsilon(A_\varepsilon)$. However, the tensor $\nu_\varepsilon^{\mathrm{approx}}$ is not trace-free with respect to λ_ε. Thus we define the correction term $\nu_\varepsilon^{\mathrm{neck}}$ by

$$\nu_\varepsilon^{\mathrm{neck}} = \nu_\varepsilon^{\mathrm{approx}} - \frac{1}{3}\left(\mathrm{tr}_{\lambda_\varepsilon}\nu_\varepsilon^{\mathrm{approx}}\right)\lambda_\varepsilon. \tag{6.7}$$

We note the following basic properties of $\nu_\varepsilon^{\mathrm{neck}}$.

Lemma 6.3. *We have $\nu_\varepsilon^{neck} \in C_2^{k-1,\alpha}(M_\varepsilon)$ with $\|\nu_\varepsilon^{neck}\|_{C_2^{k-1,\alpha}(M_\varepsilon)} \lesssim 1$. Furthermore, ν_ε^{neck} is supported in $\Psi_\varepsilon(A_{1/2})$, where we have*

$$\Psi_\varepsilon^* \nu_\varepsilon^{neck} = -(yF)^{-1} \left(\mathcal{H}_{F^2 g_E}(yF) - \frac{1}{3}(j_\varepsilon^{ab} \mathcal{H}_{F^2 g_E}(yF)_{ab})(g_E + m_\varepsilon) \right).$$

Proof. Lemma 6.2 yields an ε-independent estimate for $\nu_\varepsilon^{\mathrm{approx}}$. Together with Proposition 5.1, this implies the $C_2^{k-1,\alpha}$ estimate. The remaining claims follow directly from the definition and from the fact that $\operatorname{tr}_{g_{\mathrm{E}}} \mathcal{H}_{g_{\mathrm{E}}}(yF) = 0$. □

We now define, as discussed in the introduction, the spliced second fundamental form μ_ε by

$$\mu_\varepsilon = \rho_\varepsilon^{-1} \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon) + \nu_\varepsilon^{\mathrm{neck}} + \nu_\varepsilon^{\mathrm{ext}}. \tag{6.8}$$

It is important to note that $\mu_\varepsilon = (\pi_\varepsilon)_* \Sigma$ outside of $\Psi_\varepsilon(A_{1/4})$.

6.2. **Estimates for μ_ε.** Lemmas 6.1 and 6.3 indicate that $\nu_\varepsilon^{\mathrm{neck}} + \nu_\varepsilon^{\mathrm{ext}}$ has the regularity required for using the results of [3] in order to obtain a shear-free initial data set from each pair $\lambda_\varepsilon, \mu_\varepsilon$ according to the procedure outlined in the introduction.

We now establish estimates on μ_ε needed for the convergence results of the main theorem. Recall that we are assuming (4.2) throughout.

Proposition 6.4. *For all ε satisfying (4.2) we have*

(a) $\|\mu_\varepsilon\|_{C^{k-1,\alpha}(M_\varepsilon)} \lesssim 1$.

Furthermore, in the neck and exterior regions we have

(b) $\|y\Psi_\varepsilon^* \mu_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)} \lesssim \varepsilon$,

(c) $\iota_\varepsilon^* \mu_\varepsilon = \Sigma$.

Proof. Due to Lemmas 6.1 and 6.3, the first claim is established once we have estimated $\rho_\varepsilon^{-1} \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$. From (4.4) and (5.2), we see that both ρ_ε and λ_ε agree with the projections of ρ and g, respectively, outside of $\Psi_\varepsilon(A_{1/2}) \subseteq \Psi_\varepsilon(A_c)$. Thus it suffices to obtain an estimate in the neck region. Noting that each term in $\rho_\varepsilon^{-1} \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$ is a contraction of $\rho_\varepsilon^{-1}(\otimes^3 \overline{\lambda}_\varepsilon{}^{-1}) \otimes (\otimes^4 d\rho_\varepsilon) \otimes \operatorname{Hess}_{\overline{\lambda}_\varepsilon} \rho_\varepsilon$, we see that the last estimate in Proposition 5.6 implies the desired uniform bound.

We now establish the estimate in the neck region. First we consider $y\Psi_\varepsilon^* \nu_\varepsilon^{\mathrm{ext}}$. Since $\chi(r^2/8)$, $\chi(1/8r^2)$ are smooth and uniformly bounded on A_c, it suffices to estimate $y\alpha_{\varepsilon,1}^* \nu$ and $y(\alpha_{\varepsilon,2} \circ \mathcal{I})^* \nu$. Note that

$$y\alpha_{\varepsilon,1}^* \nu = \frac{1}{\varepsilon} \alpha_{\varepsilon,1}^*(\rho\nu).$$

Since $\rho\nu \in \mathscr{C}^{k-1,\alpha;1}(M)$, it follows from (4.1) that

$$\|y\alpha_{\varepsilon,1}^* \nu\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)} \lesssim \varepsilon.$$

A similar computation, relying on uniform estimates on A_c for the inversion map, yields a corresponding estimate for $y(\alpha_{\varepsilon,2}\circ\mathcal{I})^*\nu$. This gives the desired estimate for $y\Psi_\varepsilon^*\nu_\varepsilon^{\mathrm{ext}}$.

In order to estimate $y\Psi_\varepsilon^*(\rho_\varepsilon^{-1}\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)+\nu_\varepsilon^{\mathrm{neck}})$ we make use of [3, Lemma 7.5(b)], which implies that $\overline{\lambda}\mapsto\mathcal{H}_{\overline{\lambda}}(\rho)$ is a locally Lipschitz mapping $\mathscr{C}^{k,\alpha;2}\to\mathscr{C}^{k-1,\alpha;1}$. Thus the boundedness of F and its derivatives on A_c implies that

$$\|\mathcal{H}_{F^2(g_{\mathrm{E}}+m_\varepsilon)}(yF)-\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)}\lesssim\|m_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)}.$$

Furthermore, we have

$$\|(j_\varepsilon^{ab}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)_{ab})(g_{\mathrm{E}}+m_\varepsilon)\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)}\lesssim\|j_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)}.$$

Thus Proposition 5.1 implies the desired estimate. □

We now show that μ_ε is close to being divergence-free as measured by the weight function $\tilde{\rho}_\varepsilon$; see (4.7) and (4.8).

Proposition 6.5. $\|\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon\|_{C_2^{k-2,\alpha}(M_\varepsilon,\tilde{\rho}_\varepsilon)}\lesssim\varepsilon$

Proof. If restricted to the complement of $\Psi_\varepsilon(A_{1/4})$, we have $\pi_\varepsilon^*\lambda_\varepsilon=g$ and $\pi_\varepsilon^*\mu_\varepsilon=\Sigma$. Thus, as $\operatorname{div}_g\Sigma=0$ by hypothesis, $\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon$ is supported on $\Psi_\varepsilon(A_{1/4})\subseteq\Psi_\varepsilon(A_c)$. From Lemma 4.3 we have

$$\|\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon\|_{C_2^{k-2,\alpha}(\Psi_\varepsilon(A_c);\tilde{\rho}_\varepsilon)}\approx\|\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon)\|_{C_2^{k-2,\alpha}(A_c)}.$$

Using (5.4), (6.5), and (6.7), and adding and subtracting a term, we write

$$\begin{aligned}\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon)=&\operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}\big((yF)^{-1}\mathcal{H}_{F^2(g_{\mathrm{E}}+m_\varepsilon)}(yF)\big)\\&-\operatorname{div}_{y^{-2}g_{\mathrm{E}}}\big((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\big)\\&+\operatorname{div}_{y^{-2}g_{\mathrm{E}}}\big((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\big)\\&-\operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}\big((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\big)\\&+\frac{1}{3}\operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}\Big((yF)^{-1}(j_\varepsilon^{ab}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)_{ab})(g_{\mathrm{E}}+m_\varepsilon)\Big)\\&+\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon}\nu_\varepsilon^{\mathrm{ext}}).\end{aligned}$$

We now invoke [3, Lemma 7.5(c)], which implies that $\overline{\lambda}\mapsto\operatorname{div}_{\rho^{-2}\overline{\lambda}}(\rho^{-1}\mathcal{H}_{\overline{\lambda}}(\rho))$ is a locally Lipschitz mapping $\mathscr{C}^{k,\alpha;2}\to C_2^{k-2,\alpha}$. Thus

$$\begin{aligned}\|\operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}&\big((yF)^{-1}\mathcal{H}_{F^2(g_{\mathrm{E}}+m_\varepsilon)}(yF)\big)\\&-\operatorname{div}_{y^{-2}g_{\mathrm{E}}}\big((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\big)\|_{C_2^{k-2,\alpha}(A_c)}\\&\qquad\lesssim\|m_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\lesssim\varepsilon.\end{aligned}$$

Next, we apply [3, Proposition 7.9] to the divergence operator, concluding that

$$\begin{aligned}\| \operatorname{div}_{y^{-2}g_{\mathrm{E}}} &\left((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\right) \\ &- \operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)} \left((yF)^{-1}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)\right) \|_{C^{k-2,\alpha}_2(A_c)} \\ &\lesssim \|m_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon.\end{aligned}$$

Since $\operatorname{div}_{\lambda_\varepsilon}(u\lambda_\varepsilon) = du$ for any function u,

$$\begin{aligned}\operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)} &\Big((yF)^{-1}(j_\varepsilon^{ab}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)_{ab})(g_{\mathrm{E}}+m_\varepsilon)\Big) \\ &= d\left(y^2(yF)^{-1}(j_\varepsilon^{ab}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)_{ab})\right).\end{aligned}$$

Using this, together with Lemma 6.2(3), we easily see that

$$\begin{aligned}\| \operatorname{div}_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)} &\Big((yF)^{-1}(j_\varepsilon^{ab}\mathcal{H}_{F^2g_{\mathrm{E}}}(yF)_{ab})(g_{\mathrm{E}}+m_\varepsilon)\Big) \|_{C^{k-2,\alpha}_2(A_c)} \\ &\lesssim \|j_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon.\end{aligned}$$

Thus it remains only to estimate

$$\|\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon} \nu_\varepsilon^{\mathrm{ext}})\|_{C^{k-2,\alpha}_2(A_{1/4})}.$$

By inversion symmetry, it suffices to consider the set where $r > 2$. On this set we have

$$\begin{aligned}\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon} \nu_\varepsilon^{\mathrm{ext}}) &= \operatorname{div}_{\Psi_\varepsilon^*\lambda_\varepsilon}(\chi(r^2/8)\alpha^*_{\varepsilon,1}\nu) \\ &= \chi(r^2/8)\alpha^*_{\varepsilon,1}(\operatorname{div}_g \nu) + \alpha^*_{\varepsilon,1}\nu(\operatorname{grad}_{\Psi_\varepsilon^*\lambda_\varepsilon} \chi(r^2/8), \cdot)\end{aligned}$$

due to (5.2). As the cutoff function is smooth and uniformly bounded, we may use (4.1) with the indices m and j set to zero, together with Proposition 5.5, to conclude that

$$\begin{aligned}\|\Psi_\varepsilon^*(\operatorname{div}_{\lambda_\varepsilon} \nu_\varepsilon^{\mathrm{ext}})\|_{C^{k-2,\alpha}_2(A_{1/4})} &\lesssim \|\alpha^*_{1,\varepsilon}\nu\|_{C^{k-1,\alpha}_2(A_{1/4})} + \|\alpha^*_{2,\varepsilon}\nu\|_{C^{k-1,\alpha}_2(A_{1/4})} \\ &\lesssim \varepsilon^2\|\nu\|_{C^{k-1,\alpha}_2(M)} \\ &\lesssim \varepsilon^2.\end{aligned}$$

□

6.3. **The tensor σ_ε.** The metric λ_ε and tensor μ_ε satisfy regularity and boundary conditions suitable to apply the conformal method in order to obtain CMCSF hyperboloidal solutions to the constraint equations as outlined in the introduction. The first step in that procedure is to solve

$$L_{\lambda_\varepsilon} W_\varepsilon = (\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon)^\sharp \tag{6.9}$$

for a vector field W_ε and subsequently define a tensor field σ_ε by (1.7). We now establish that this process can be accomplished with appropriate uniform estimates in ε.

Lemma 6.6. *For each* ε *satisfying* (4.2) *there exists a unique vector field* $W_\varepsilon \in C^{k,\alpha}_2(M_\varepsilon; \tilde{\rho}_\varepsilon)$ *satisfying* (6.9). *Furthermore we have the uniform estimates*

$$\|W_\varepsilon\|_{C^{k,\alpha}_2(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon,$$
$$\|\mathcal{D}_{\lambda_\varepsilon} W_\varepsilon\|_{C^{k-1,\alpha}_2(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon,$$
$$\|\tilde{\rho}_\varepsilon \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon.$$

Proof. Since $\overline{\lambda}_\varepsilon \in \mathscr{C}^{k,\alpha;2}(M_\varepsilon)$ and $\nu^{\text{neck}}_\varepsilon + \nu^{\text{ext}}_\varepsilon \in C^{k-1,\alpha}_2(M_\varepsilon)$ we may apply Proposition 6.3 of [3] to conclude that there exists a unique vector field $W_\varepsilon \in C^{k,\alpha}_2(M_\varepsilon)$ satisfying (6.9).

To obtain the uniform estimates we first use Proposition A.1 to obtain

$$\begin{aligned}\|W_\varepsilon\|_{C^{k,\alpha}_2(M_\varepsilon,\tilde{\rho}_\varepsilon)} &\lesssim \|L_{\lambda_\varepsilon} W_\varepsilon\|_{C^{k-2,\alpha}_2(M_\varepsilon,\tilde{\rho}_\varepsilon)} \\ &= \|(\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon)^\sharp\|_{C^{k-2,\alpha}_2(M_\varepsilon,\tilde{\rho}_\varepsilon)}.\end{aligned}$$

Using Propositions 5.3 and 6.5 we have

$$\|(\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon)^\sharp\|_{C^{k-2,\alpha}_2(M_\varepsilon,\tilde{\rho}_\varepsilon)} \le \|\lambda_\varepsilon^{-1}\|_{C^{k,\alpha}(M_\varepsilon)} \|\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon\|_{C^{k-2,\alpha}_2(M_\varepsilon,\tilde{\rho}_\varepsilon)} \lesssim \varepsilon,$$

which proves the first estimate. The second estimate then follows from Proposition 5.5, and the third from Proposition 2.5(b) . □

Using W_ε, we define the symmetric, trace-free tensor σ_ε by

$$\begin{aligned}\sigma_\varepsilon &= \rho_\varepsilon^{-1} \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon) + \nu^{\text{neck}}_\varepsilon + \nu^{\text{ext}}_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon \\ &= \mu_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon.\end{aligned}$$

Proposition 6.7. *We have*

(a) $\overline{\sigma}_\varepsilon = \rho_\varepsilon \sigma_\varepsilon \in \mathscr{C}^{k-1,\alpha;1}(M_\varepsilon)$,
(b) $\operatorname{div}_{\lambda_\varepsilon} \sigma_\varepsilon = 0$, *and*
(c) $\overline{\sigma}_\varepsilon = \mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho_\varepsilon)$ *along* ∂M_ε.

Furthermore we have the global estimate

(d) $\|\sigma_\varepsilon\|_{C^{k-1,\alpha}(M_\varepsilon)} \lesssim 1$.

Finally, in the neck and exterior regions we have the uniform estimates

(e) $\|y\Psi^*_\varepsilon \sigma_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)} \lesssim \varepsilon$, *and*
(f) $\|\rho(\iota^*_\varepsilon \sigma_\varepsilon - \Sigma)\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)} \lesssim \varepsilon$

Proof. The first three claims are direct consequences of the construction and regularity involved; see also [3, Theorem 8.2]. The remaining claims follow from Proposition 6.4 and Lemma 6.6. □

The following additional estimates on $|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ are required in our analysis of the Lichnerowicz equation.

Proposition 6.8. *We have*

$$\| |\sigma_\varepsilon|^2_{\lambda_\varepsilon} \|_{C^{k-1,\alpha}(M_\varepsilon)} \lesssim 1, \tag{6.10}$$

$$\| \iota_\varepsilon^* |\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\Sigma|^2_g \|_{C_2^{k-1,\alpha}(E_c)} \lesssim \varepsilon, \tag{6.11}$$

$$\| \Psi_\varepsilon^* |\sigma_\varepsilon|^2_{\lambda_\varepsilon} \|_{C_2^{k-1,\alpha}(A_c)} \lesssim \varepsilon, \tag{6.12}$$

$$\| |\sigma_\varepsilon|^2 - |\mu_\varepsilon|^2 \|_{C_2^{k-1,\alpha}(M_\varepsilon;\tilde\rho_\varepsilon)} \lesssim \varepsilon. \tag{6.13}$$

Proof. Note that $|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ is a contraction of

$$\lambda_\varepsilon^{-1} \otimes \sigma_\varepsilon \otimes \sigma_\varepsilon.$$

This, together with Propositions 5.3 and 6.7, yields (6.10). For (6.12), we view $\Psi_\varepsilon^* |\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ as a contraction of

$$y^{-2}(\Psi_\varepsilon^* \lambda_\varepsilon)^{-1} \otimes y \Psi_\varepsilon^* \sigma_\varepsilon \otimes y \Psi_\varepsilon^* \sigma_\varepsilon$$

and use Propositions 5.6 and 6.7. Analogous reasoning yields (6.11). For the final estimate we note that $|\sigma_\varepsilon|^2 - |\mu_\varepsilon|^2$ is a contraction of

$$\lambda_\varepsilon^{-1} \otimes \lambda_\varepsilon^{-1} \otimes (\sigma_\varepsilon + \mu_\varepsilon) \otimes (\sigma_\varepsilon - \mu_\varepsilon) = \lambda_\varepsilon^{-1} \otimes \lambda_\varepsilon^{-1} \otimes (\sigma_\varepsilon + \mu_\varepsilon) \otimes \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon.$$

Using Propositions 5.3, 6.4, and 6.7, together with Lemma 6.6, we have

$$\begin{aligned} &\| |\sigma_\varepsilon|^2 - |\mu_\varepsilon|^2 \|_{C_2^{k-1,\alpha}(M_\varepsilon;\tilde\rho_\varepsilon)} \\ &\quad \le \| \lambda_\varepsilon^{-1} \|^2_{C^{k,\alpha}(M_\varepsilon)} \| \sigma_\varepsilon + \mu_\varepsilon \|_{C^{k-1,\alpha}(M_\varepsilon)} \| \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon \|_{C_2^{k-1,\alpha}(M_\varepsilon;\tilde\rho_\varepsilon)} \\ &\quad \lesssim \varepsilon, \end{aligned}$$

which concludes the proof. ☐

7. Analysis of the Lichnerowicz equation

As discussed in §1.3, the results of [3] (see also [2]) imply that for each λ_ε and σ_ε there exists a positive solution ϕ_ε to the Lichnerowicz equation (1.8); the solution ϕ_ε is the unique such function satisfying $\phi_\varepsilon - 1 \in C_1^{k,\alpha}(M_\varepsilon)$. In this section we establish the following estimates for ϕ_ε in the exterior region E_c and gluing region A_c.

Proposition 7.1. *Suppose that* (4.2) *holds. There exists* $\varepsilon_* > 0$*, depending on* c*, such that for* $0 < \varepsilon < \varepsilon_*$ *we have*

$$\begin{aligned} &\| \iota_\varepsilon^* \phi_\varepsilon - 1 \|_{\mathscr{C}^{k,\alpha;2}(E_c)} \lesssim \varepsilon, \\ &\| \Psi_\varepsilon^* \phi_\varepsilon - 1 \|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon. \end{aligned} \tag{7.1}$$

The proof of Proposition 7.1 appears in §7.3 below.

The proof of Theorem 1.1 makes use of the following, which is an immediate consequence of $\iota_\varepsilon^* \phi_\varepsilon$ and $\Psi_\varepsilon^* \phi_\varepsilon$ being uniformly bounded close to 1, and thus away from zero.

Corollary 7.2. *For any integer p we have*

$$\begin{aligned} \|\iota_\varepsilon^*(\phi_\varepsilon^p - 1)\|_{\mathscr{C}^{k,\alpha;2}(E_c)} &\lesssim \varepsilon, \\ \|\Psi_\varepsilon^*(\phi_\varepsilon^p - 1)\|_{\mathscr{C}^{k,\alpha;2}(A_c)} &\lesssim \varepsilon. \end{aligned} \tag{7.2}$$

The proof of Proposition 7.1, which is completed at the end of §7.3 below, follows the arguments in [10] and [1], and makes use of a function θ_ε that approximately solves the Lichnerowicz equation (1.8). We then use the linearization of (1.8) at θ_ε to estimate the difference $\phi_\varepsilon - \theta_\varepsilon$ by means of a fixed-point argument, and thus establish (7.1).

7.1. **The approximate solution θ_ε.** For ε satisfying (4.2) we define the Lichnerowicz operator $\mathcal{L}_\varepsilon$ to act on a function θ by

$$\mathcal{L}_\varepsilon(\theta) = \Delta_{\lambda_\varepsilon}\theta - \frac{1}{8}\,\mathrm{R}[\lambda_\varepsilon]\theta + \frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}\theta^{-7} - \frac{3}{4}\theta^5,$$

so that the Lichnerowicz equation (1.8) can be written $\mathcal{L}_\varepsilon(\phi_\varepsilon) = 0$. In this subsection we establish the following.

Proposition 7.3. *For each ε there exists a positive function θ_ε with $\theta_\varepsilon - 1 \in C^{k,\alpha}_1(M_\varepsilon)$ such that*

$$\|\theta_\varepsilon - 1\|_{C^{k,\alpha}_1(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon, \tag{7.3}$$

$$\|\mathcal{L}_\varepsilon(\theta_\varepsilon)\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde{\rho}_\varepsilon)} \leq \varepsilon C_\delta, \quad \delta \in \{0,2\}, \tag{7.4}$$

for constants C_δ independent of ε satisfying (4.2)*. Furthermore we have improved regularity in the exterior and neck regions:*

$$\|\iota_\varepsilon^*\theta_\varepsilon - 1\|_{\mathscr{C}^{k,\alpha;2}(E_c)} \lesssim \varepsilon, \tag{7.5}$$

$$\|\Psi_\varepsilon^*\theta_\varepsilon - 1\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon. \tag{7.6}$$

To prove Proposition 7.3 we need to establish a number of lemmas. We first show that in the exterior region, it suffices to take the constant function 1 as the approximate solution θ_ε.

Lemma 7.4. *We have*

$$\|\mathcal{L}_\varepsilon(1)\|_{C^{k,\alpha}_2(M_\varepsilon \setminus \Psi_\varepsilon(A_{1/4});\tilde{\rho}_\varepsilon)} \lesssim \varepsilon.$$

Proof. On $M_\varepsilon \setminus \Psi_\varepsilon(A_{1/4})$ we have $\lambda_\varepsilon = (\pi_\varepsilon)_* g$ and $\mu_\varepsilon = (\pi_\varepsilon)_*\Sigma$. Since g and Σ satisfy the constraint equations (1.2), we have

$$\mathrm{R}[\lambda_\varepsilon] = -6 + |\mu_\varepsilon|^2_{\lambda_\varepsilon}.$$

Thus on $M_\varepsilon \setminus \Psi_\varepsilon(A_{1/4})$ we have

$$\mathcal{L}_\varepsilon(1) = \frac{1}{8}\left(|\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\mu_\varepsilon|^2_{\lambda_\varepsilon}\right).$$

The desired estimate now follows from Proposition 6.8. □

In the neck region we cannot simply set θ_ε equal to 1 because the scalar curvature of the spliced metric λ_ε need not be close to -6 in $C^{k-2,\alpha}_2(M_\varepsilon;\tilde\rho_\varepsilon)$. Rather, we seek an approximate solution θ_ε that is a perturbation of the constant function 1, with the perturbation supported in the neck region. Before giving a careful definition of θ_ε, we establish some preliminary results.

Observe that for any function v we have

$$\mathcal{L}_\varepsilon(1+v)=\Delta_{\lambda_\varepsilon}v-3v-\frac{1}{8}(\mathrm{R}[\lambda_\varepsilon]+6)+\mathscr{R}_\varepsilon(v), \tag{7.7}$$

where the remainder term $\mathscr{R}_\varepsilon(v)$ is given by

$$\mathscr{R}_\varepsilon(v)=-\frac{1}{8}(\mathrm{R}[\lambda_\varepsilon]+6)v+\frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}(1+v)^{-7}-\frac{3}{4}\left((1+v)^5-1-5v\right).$$

Using $\Psi_\varepsilon^*\lambda_\varepsilon=y^{-2}(g_{\mathrm{E}}+m_\varepsilon)$, the formula for how the scalar curvature changes under a conformal change of the metric yields

$$\Psi_\varepsilon^*\,\mathrm{R}[\lambda_\varepsilon]+6=6v_\varepsilon+y^2\,\mathrm{R}[g_{\mathrm{E}}+m_\varepsilon], \tag{7.8}$$

where

$$v_\varepsilon=1-|dy|^2_{g_{\mathrm{E}}+m_\varepsilon}+\frac{2}{3}y\Delta_{g_{\mathrm{E}}+m_\varepsilon}y. \tag{7.9}$$

Note that Proposition 5.3 implies that $\mathrm{R}[g_{\mathrm{E}}+m_\varepsilon]\in C^{k-2,\alpha}(A_c)$ with

$$\|y^2\,\mathrm{R}[g_{\mathrm{E}}+m_\varepsilon]\|_{C^{k-2,\alpha}_2(A_c)}\approx\|\,\mathrm{R}[g_{\mathrm{E}}+m_\varepsilon]\|_{C^{k-2,\alpha}(A_c)}\lesssim\varepsilon. \tag{7.10}$$

Lemma 7.5. *We have that*

(a) v_ε *vanishes where* $y=0$*,*
(b) $v_\varepsilon\in\mathscr{C}^{k-1,\alpha;2}(A_c)$ *with* $\|v_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;2}(A_c)}\lesssim\varepsilon$*,*
(c) $v_\varepsilon\in C^{k-1,\alpha}_1(A_c)$ *with* $\|v_\varepsilon\|_{C^{k-1,\alpha}_1(A_c)}\lesssim\varepsilon$*,*
(d) $v_\varepsilon\in C^{k-2,\alpha}_2(A_c\smallsetminus A_{1/2})$ *with* $\|v_\varepsilon\|_{C^{k-2,\alpha}_2(A_c\smallsetminus A_{1/2})}\lesssim\varepsilon$*.*

Proof. From Proposition 5.7 we have $v_\varepsilon\in C^{k-1,\alpha}_1(A_c)$, which implies that v_ε vanishes where $y=0$. The second and third claims also follow directly from Proposition 5.7.

To establish the final claim we first note that by assumption we have $\rho\Sigma\in C^{k-1,\alpha}_2(M)$ and $g^{-1}\in C^{k,\alpha}(M)$. Thus (1.2) implies that

$$\mathrm{R}[g]+6=|\Sigma|^2_g\in C^{k-1,\alpha}_2(M).$$

Since $\lambda_\varepsilon=(\pi_\varepsilon)_*g$ on $\Psi_\varepsilon(A_c\smallsetminus A_{1/2})$, this implies that

$$\Psi_\varepsilon^*\,\mathrm{R}[\lambda_\varepsilon]+6\in C^{k-1,\alpha}_2(A_c\smallsetminus A_{1/2}).$$

Taken together with (7.10) and (7.8), this implies that $v_\varepsilon\in C^{k-2,\alpha}_2(A_c\smallsetminus A_{1/2})$. Thus $v_\varepsilon\to0$ and $|{}^{\mathrm{E}}\nabla v_\varepsilon|_{g_{\mathrm{E}}}\to0$ as $y\to0$, which, in view of Proposition 2.5(c), yields

$$\|v_\varepsilon\|_{C^{k-2,\alpha}_2(A_c\smallsetminus A_{1/2})}\le C\|v_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;2}(A_c)}.$$

The final claim now follows from the second. $\square$

We use Lemma 7.5 to estimate the remainder term $\mathscr{R}_\varepsilon(v)$ in (7.7).

Lemma 7.6. *If* $v \in C_1^{k,\alpha}(M_\varepsilon)$ *and if* v *satisfies*

$$|v| < \frac{1}{2} \quad \text{and} \quad \|v\|_{C_1^{k,\alpha}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon$$

then

$$\|\Psi_\varepsilon^* \mathscr{R}_\varepsilon(v)\|_{C_2^{k-2,\alpha}(A_c)} \lesssim \varepsilon.$$

Proof. It is easy to see that

$$\|(1+v)^5 - 1 - 5v\|_{C_2^{k-2,\alpha}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \leq 26\|v\|^2_{C_1^{k,\alpha}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon.$$

Since $|v| < \frac{1}{2}$ we have

$$\|(1+v)^{-7}\|_{C^{k-2,\alpha}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim 1.$$

Thus (6.12) implies that

$$\||\sigma_\varepsilon|^2_{\lambda_\varepsilon}(1+v)^{-7}\|_{C_2^{k-2,\alpha}(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \varepsilon.$$

To estimate the scalar curvature term we use (7.8) to write

$$(\mathrm{R}[\lambda_\varepsilon] + 6)v = 6v_\varepsilon v + y^2\,\mathrm{R}[g_{\mathrm{E}} + m_\varepsilon]v.$$

Thus Lemma 7.5 and (7.10) imply that

$$\begin{aligned}
&\|(\mathrm{R}[\lambda_\varepsilon] + 6)v\|_{C_2^{k-2,\alpha}(A_c)} \\
&\qquad \lesssim \|v_\varepsilon\|_{C_1^{k-2,\alpha}(A_c)}\|v\|_{C_1^{k-2,\alpha}(A_c)} \\
&\qquad\qquad + \|y^2\,\mathrm{R}[g_{\mathrm{E}} + m_\varepsilon]\|_{C_2^{k-2,\alpha}(A_c)}\|v\|_{C^{k-2,\alpha}(A_c)} \\
&\qquad \lesssim \varepsilon.
\end{aligned}$$

□

We construct a regularization of v_ε that is supported in A_c. Let η be a smooth cutoff function on A_c that is supported on $A_{1/6}$ and such that $\eta = 1$ on $A_{1/4}$. We now apply Proposition 2.6 in order to obtain a function $\tilde{\tau}_\varepsilon$ that approximates $\eta\Delta_{g_{\mathrm{E}}+m_\varepsilon}y$ in the following sense:

Lemma 7.7. *There exists* $\tilde{\tau}_\varepsilon \in \mathscr{C}^{k,\alpha;1}(\mathbb{H})$ *such that*

(a) $\tilde{\tau}_\varepsilon$ *is supported in* $A_{1/8}$,
(b) $\|\tilde{\tau}_\varepsilon\|_{\mathscr{C}^{k,\alpha;1}(\mathbb{H})} \lesssim \varepsilon$,
(c) $\|\tilde{\tau}_\varepsilon - \eta\Delta_{g_E+m_\varepsilon}y\|_{C_1^{k-1,\alpha}(\mathbb{H})} \lesssim \varepsilon$.

Proof. It follows from the definition of η and from Proposition 5.6 that $\eta\Delta_{g_{\mathrm{E}}+m_\varepsilon}y \in \mathscr{C}^{k-1,\alpha;1}(\mathbb{H})$ is supported in $A_{1/6}$ and satisfies

$$\|\eta\Delta_{g_{\mathrm{E}}+m_\varepsilon}y\|_{\mathscr{C}^{k-1,\alpha;1}(\mathbb{H})} \lesssim \varepsilon.$$

We obtain $\tilde{\tau}_\varepsilon$ by applying Proposition 2.6 to the function $\eta\Delta_{g_{\mathrm{E}}+m_\varepsilon}y$. Proposition 2.6 immediately implies the first and second claims, and also implies that $\tilde{\tau}_\varepsilon - \eta\Delta_{g_{\mathrm{E}}+m_\varepsilon}y \in C_1^{k-1,\alpha}(\mathbb{H};\breve{\rho})$. In view of part (c) of Proposition 2.5, this latter fact, together with the second claim, implies the third claim. □

We now define $\tilde{v}_\varepsilon$, the regularization of ηv_ε, by

$$\tilde{v}_\varepsilon = \eta(1 - |dy|^2_{g_{\mathrm{E}}+m_\varepsilon}) + \frac{2}{3} y \tilde{\tau}_\varepsilon.$$

Lemma 7.8. *The function $\tilde{v}_\varepsilon$ satisfies*

(a) *$\tilde{v}_\varepsilon$ vanishes where $y = 0$,*
(b) *$\tilde{v}_\varepsilon \in \mathscr{C}^{k,\alpha;2}(\mathbb{H})$ with $\|\tilde{v}_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(\mathbb{H})} \lesssim \varepsilon$,*
(c) *$\tilde{v}_\varepsilon \in C^{k,\alpha}_1(\mathbb{H})$ with $\|\tilde{v}_\varepsilon\|_{C^{k,\alpha}_1(\mathbb{H})} \lesssim \varepsilon$,*
(d) *$\eta v_\varepsilon - \tilde{v}_\varepsilon \in C^{k-1,\alpha}_2(\mathbb{H})$ with $\|\eta v_\varepsilon - \tilde{v}_\varepsilon\|_{C^{k-1,\alpha}_2(\mathbb{H})} \lesssim \varepsilon$,*
(e) *$\tilde{v}_\varepsilon \in C^{k-1,\alpha}_2(A_c \setminus A_{1/2})$ with $\|\tilde{v}_\varepsilon\|_{C^{k-1,\alpha}_2(A_c \setminus A_{1/2})} \lesssim \varepsilon$.*

Proof. The first claim follows from (5.6) and the boundedness of $\tilde{\tau}_\varepsilon$. The second and third claims follow from analogous estimates in Proposition 5.7 and Lemma 7.7. For the fourth claim we note that

$$\eta v_\varepsilon - \tilde{v}_\varepsilon = \frac{2}{3} y(\eta \Delta_{g_{\mathrm{E}}+m_\varepsilon} y - \tilde{\tau}_\varepsilon);$$

thus the desired estimate follows from Lemma 7.7. For the last claim we write $\tilde{v}_\varepsilon = (\tilde{v}_\varepsilon - \eta v_\varepsilon) + \eta v_\varepsilon$ and apply the final statements of Lemmas 7.7 and 7.5. $\square$

We now define the approximate solution θ_ε by requiring that $\theta_\varepsilon = 1$ outside $\Psi_\varepsilon(A_{1/8})$ and that $\Psi^*_\varepsilon \theta_\varepsilon = 1 - \frac{3}{16}\tilde{v}_\varepsilon$; note that θ_ε is well-defined since $\tilde{v}_\varepsilon$ is supported on $A_{1/8}$. With this definition in hand, we may use Lemmas 7.4 through 7.8 to prove Proposition 7.3, showing that θ_ε is in fact an approximate solution to the Lichnerowicz equation.

Proof of Proposition 7.3. The estimate (7.3) follows from Lemmas 7.8 and 4.3, and the fact that $\tilde{v}_\varepsilon$ is supported in $A_{1/8}$.

We next establish (7.4). In view of Lemma 7.4, it suffices to estimate $\|\Psi^*_\varepsilon \mathcal{L}_\varepsilon(\theta_\varepsilon)\|_{C^{k-2,\alpha}_2(A_c)}$. Using (7.7) and (7.8) we have

$$\Psi^*_\varepsilon \mathcal{L}_\varepsilon(\theta_\varepsilon) = -\frac{3}{16}\left(\Delta_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)} \tilde{v}_\varepsilon + \tilde{v}_\varepsilon\right) + \frac{3}{4}\left(\tilde{v}_\varepsilon - v_\varepsilon\right) - \frac{1}{8} y^2 \,\mathrm{R}[g_{\mathrm{E}} + m_\varepsilon] + \Psi^*_\varepsilon \mathscr{R}_\varepsilon\left(-\frac{3}{16}\tilde{v}_\varepsilon\right). \tag{7.11}$$

The final two terms in (7.11) are easily estimated in $C^{k-2,\alpha}_2(A_c)$ using (7.10) and Lemmas 7.6 and 7.8. Since $\eta = 1$ on $A_{1/4}$ we can estimate the second term in (7.11) by

$$\|\tilde{v}_\varepsilon - v_\varepsilon\|_{C^{k-2,\alpha}_2(A_c)} \le \|\tilde{v}_\varepsilon\|_{C^{k-2,\alpha}_2(A_c \setminus A_{1/2})} + \|v_\varepsilon\|_{C^{k-2,\alpha}_2(A_c \setminus A_{1/2})} + \|\eta v_\varepsilon - \tilde{v}_\varepsilon\|_{C^{k-2,\alpha}_2(A_c)},$$

which in turn is controlled by Lemmas 7.5 and 7.8.

In order to estimate $\|\Delta_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}\tilde v_\varepsilon+\tilde v_\varepsilon\|_{C_2^{k-2,\alpha}(A_c)}$ we use the identity

$$\Delta_{y^{-2}(g_{\mathrm{E}}+m_\varepsilon)}\tilde v_\varepsilon+\tilde v_\varepsilon = y^2\Delta_{g_{\mathrm{E}}+m_\varepsilon}\tilde v_\varepsilon+\tilde v_\varepsilon - y\langle d\tilde v_\varepsilon, dy\rangle_{g_{\mathrm{E}}+m_\varepsilon}. \tag{7.12}$$

Using Lemma 7.8 and Proposition 5.6 we estimate the first term in (7.12) as follows

$$\begin{aligned}\|y^2\Delta_{g_{\mathrm{E}}+m_\varepsilon}\tilde v_\varepsilon\|_{C_2^{k-2,\alpha}(A_c)} &\approx \|\Delta_{g_{\mathrm{E}}+m_\varepsilon}\tilde v_\varepsilon\|_{C^{k-2,\alpha}(A_c)}\\ &\lesssim \|\tilde v_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\\ &\lesssim \varepsilon.\end{aligned}$$

Note that Lemma 7.8 implies that the expression

$$\tilde v_\varepsilon - y\langle d\tilde v_\varepsilon, dy\rangle_{g_{\mathrm{E}}+m_\varepsilon}$$

satisfies the hypotheses of Proposition 2.7 and thus

$$\|\tilde v_\varepsilon - y\langle d\tilde v_\varepsilon, dy\rangle_{g_{\mathrm{E}}+m_\varepsilon}\|_{C_2^{k-2,\alpha}(A_c)} \lesssim \|\tilde v_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)} \lesssim \varepsilon.$$

Thus (7.4) is established for $\delta=2$.

Note that for any function u we have

$$\|u\|_{C^{k-2,\alpha}(A_c)} = \|\breve\rho^2 u\|_{C_2^{k-2,\alpha}(A_c)} \lesssim \|\breve\rho^2\|_{C^{k-2,\alpha}(A_c)}\|u\|_{C_2^{k-2,\alpha}(A_c)}$$

and thus the $\delta=0$ estimate in (7.4) follows from the estimate with $\delta=2$.

Finally, (7.6) follows from the second claim of Lemma 7.8, while (7.5) holds trivially due to our definition that $\theta_\varepsilon=1$ outside $\Psi_\varepsilon(A_{1/8})$. □

7.2. **Linearization of the Lichnerowicz equation.** Let $\mathcal{P}_\varepsilon[\theta]$ denote the linearization of the Lichnerowicz operator $\mathcal{L}_\varepsilon$ at a function θ. We have

$$\mathcal{P}_\varepsilon[\theta]u = \Delta_{\lambda_\varepsilon}u - \frac{1}{8}\left(\mathrm{R}[\lambda_\varepsilon] + 7|\sigma_\varepsilon|^2_{\lambda_\varepsilon}\theta^{-8} + 30\theta^4\right)u. \tag{7.13}$$

Proposition 7.9. *Suppose $-1<\delta<3$ and let θ_ε be the function given by Proposition 7.3. Then there exists $\varepsilon_*>0$ such that if $0<\varepsilon<\varepsilon_*$, then the operator*

$$\mathcal{P}_\varepsilon[\theta_\varepsilon]\colon C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)\to C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$$

is invertible, and there exists a constant K_δ independent of ε such that the operator norm of $\mathcal{P}_\varepsilon[\theta_\varepsilon]^{-1}\colon C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)\to C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ satisfies

$$\|\mathcal{P}_\varepsilon[\theta_\varepsilon]^{-1}\|_\delta \le K_\delta. \tag{7.14}$$

Proof. From Proposition A.1 we know that $\mathcal{P}_\varepsilon[1]$ is uniformly invertible. Thus it remains to show that

$$\|\mathcal{P}_\varepsilon[1]u - \mathcal{P}_\varepsilon[\theta_\varepsilon]u\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \lesssim \varepsilon\|u\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \tag{7.15}$$

for all $u\in C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$. We have

$$\mathcal{P}_\varepsilon[1]u - \mathcal{P}_\varepsilon[\theta_\varepsilon]u = \left(\frac{7}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}(1-\theta_\varepsilon^{-8}) + \frac{15}{4}(1-\theta_\varepsilon^4)\right)u.$$

Recall that from (6.10) we have $\| |\sigma_\varepsilon|^2_{\lambda_\varepsilon} \|_{C^{k-2,\alpha}(M_\varepsilon)} \lesssim 1$. From (7.3) we can choose ε_* small enough that

$$\|1 - \theta_\varepsilon^{-8}\|_{C^{k-2,\alpha}(M_\varepsilon)} \lesssim \varepsilon \quad \text{and} \quad \|1 - \theta_\varepsilon^{4}\|_{C^{k-2,\alpha}(M_\varepsilon)} \lesssim \varepsilon,$$

from which (7.15) follows. □

We define the error term $\mathcal{Q}_\varepsilon(u)$ by

$$\mathcal{L}_\varepsilon(\theta_\varepsilon + u) = \mathcal{L}_\varepsilon(\theta_\varepsilon) + \mathcal{P}_\varepsilon[\theta_\varepsilon](u) + \mathcal{Q}_\varepsilon(u). \tag{7.16}$$

In order to describe the mapping properties of $\mathcal{Q}_\varepsilon$, we use the following.

Proposition 7.10. *There exist $r_* > 0$, $\varepsilon_* > 0$, and D_* such that for each $\delta \geq 0$ and $0 < \varepsilon < \varepsilon_*$ we have*

$$\begin{aligned} &\|\mathcal{Q}_\varepsilon(u_1) - \mathcal{Q}_\varepsilon(u_2)\|_{C^{k-1,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \\ &\qquad \leq D_* \|u_1 - u_2\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \Big(\|u_1\|_{C^{k,\alpha}(M_\varepsilon)} + \|u_2\|_{C^{k,\alpha}(M_\varepsilon)} \Big) \end{aligned} \tag{7.17}$$

for all functions $u_1, u_2 \in C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)$ with $\|u_i\|_{C^{k,\alpha}(M_\varepsilon)} \leq r_*$.

Proof. Note that

$$\mathcal{Q}_\varepsilon(u) = f(\theta_\varepsilon + u) - f(\theta_\varepsilon) - f'(\theta_\varepsilon)u, \tag{7.18}$$

where

$$f(x) = \frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon} x^{-7} - \frac{3}{4}x^5.$$

We now make use of the integral form of Taylor's remainder formula

$$f(b) - f(a) = (b-a)f'(a) + (b-a)^2 \int_0^1 (1-t) f''\big(a + t(b-a)\big)\,dt. \tag{7.19}$$

First consider (7.19) with $a = \theta_\varepsilon + u_1$ and $b = \theta_\varepsilon + u_2$, and then consider (7.19) with $a = \theta_\varepsilon + u_2$ and $b = \theta_\varepsilon + u_1$. Taking the difference of (7.19) with these two choices of a and b and then using (7.18) we find that

$$\begin{aligned} Q_\varepsilon(u_1) - Q_\varepsilon(u_2) = {} & \frac{1}{2}(u_1 - u_2)\big(f'(\theta_\varepsilon + u_1) - 2f'(\theta_\varepsilon) + f'(\theta_\varepsilon + u_2)\big) \\ & + \frac{1}{2}(u_1 - u_2)^2 \int_0^1 (1-t)\Big(f''(\theta_\varepsilon + u_2 - t(u_2 - u_1)) \\ & \qquad\qquad - f''(\theta_\varepsilon + u_2 - (1-t)(u_2 - u_1))\Big)dt. \end{aligned}$$

Using the fundamental theorem of calculus we write this expression as

$$\begin{aligned}
&Q_\varepsilon(u_1) - Q_\varepsilon(u_2) \\
&\quad =(u_1-u_2)u_1\frac{1}{2}\int_0^1 f''(\theta_\varepsilon + tu_1)dt \\
&\quad - (u_1-u_2)u_2\frac{1}{2}\int_0^1 f''(\theta_\varepsilon + tu_2)dt \\
&\quad + (u_1-u_2)^2\frac{1}{2}\int_0^1 (1-t)f''(\theta_\varepsilon + u_2 - t(u_2-u_1))\,dt \\
&\quad - (u_1-u_2)^2\frac{1}{2}\int_0^1 (1-t)f''(\theta_\varepsilon + u_2 - (1-t)(u_2-u_1))dt.
\end{aligned}$$

From Proposition 6.8 we have that $|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ is bounded in $C^{k-1,\alpha}(M_\varepsilon)$. Using (7.3) we can choose ε_* to ensure that $|\theta_\varepsilon|$ is uniformly bounded away from zero. Thus we can choose r_* sufficiently small that each of the four integrals above is bounded in $C^{k-1,\alpha}(M_\varepsilon)$, which concludes the proof. □

7.3. **Proof of Proposition 7.1.** In order to prove Proposition 7.1 we first establish an estimate for the difference between the solution ϕ_ε to the Lichnerowicz equation and the approximate solution θ_ε defined in §7.1. Our strategy is to use a contraction-mapping argument. For each $r > 0$ let

$$\overline{B}^{k,\alpha}_\delta(r) = \{u \in C^{k,\alpha}_\delta(M_\varepsilon)\colon \|u\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \leq r\}.$$

For $\varepsilon > 0$ we define

$$X_\varepsilon = \overline{B}^{k,\alpha}_2(2K_2C_2\varepsilon) \cap \overline{B}^{k,\alpha}_0(2K_0C_0\varepsilon),$$

where C_2, C_0 are the constants appearing in (7.4), and K_2, K_0 are those appearing in (7.14). Choosing the metric

$$d(u,v) = \|u-v\|_{C^{k,\alpha}_2(M_\varepsilon;\tilde\rho_\varepsilon)} + \|u-v\|_{C^{k,\alpha}(M_\varepsilon)}$$

we find that X_ε is a complete metric space.

From (7.16) we have that $\theta_\varepsilon + u$ is a solution to the Lichnerowicz equation if

$$\mathcal{L}_\varepsilon[\theta_\varepsilon]u = -\left(\mathcal{L}_\varepsilon(\theta_\varepsilon) + \mathcal{Q}_\varepsilon(u)\right).$$

This holds provided u is a fixed point of the map

$$\mathcal{G}_\varepsilon\colon u \mapsto -\mathcal{P}_\varepsilon[\theta_\varepsilon]^{-1}\left(\mathcal{L}_\varepsilon(\theta_\varepsilon) + \mathcal{Q}_\varepsilon(u)\right).$$

Lemma 7.11. *We may choose ε_* such that for $0 < \varepsilon < \varepsilon_*$ the map $\mathcal{G}_\varepsilon$ is a contraction mapping on X_ε.*

Proof. Start by taking ε_* to be smaller than the choices made for this constant in Propositions 7.9 and 7.10. We first show that $\mathcal{G}_\varepsilon$ maps X_ε to itself. If $u \in X_\varepsilon$, then taking $u_2 = 0$ in (7.17) implies that for $\delta \in \{0, 2\}$ we have

$$\|\mathcal{Q}_\varepsilon(u)\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \leq 4D_*K_\delta K_0C_\delta C_0\varepsilon^2.$$

Thus from (7.14) we have

$$\|\mathcal{P}_\varepsilon[\theta_\varepsilon]^{-1}\mathcal{Q}_\varepsilon(u)\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le 4D_* K_\delta^2 K_0 C_\delta C_0 \varepsilon^2.$$

From (7.14) and (7.4) we have

$$\|\mathcal{P}_\varepsilon[\theta_\varepsilon]^{-1}\mathcal{L}_\varepsilon(\theta_\varepsilon)\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le K_\delta C_\delta \varepsilon.$$

Thus by choosing ε_* small enough, we can guarantee that $\mathcal{G}_\varepsilon(u) \in X_\varepsilon$.

To see that $\mathcal{G}_\varepsilon$ is a contraction, suppose that $u_1, u_2 \in X_\varepsilon$. Using (7.14) and (7.17) we have

$$\begin{aligned}\|\mathcal{G}_\varepsilon(u_1) - \mathcal{G}_\varepsilon(u_2)\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} &\le K_\delta \|\mathcal{Q}_\varepsilon(u_1) - \mathcal{Q}_\varepsilon(u_2)\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \\ &\le K_\delta D_* 4K_0 C_0 \varepsilon \|u_1 - u_2\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}.\end{aligned}$$

Thus $\mathcal{G}_\varepsilon$ is a contraction so long as ε is sufficiently small. □

Proposition 7.12. *We may choose ε_* sufficiently small that if $0 < \varepsilon < \varepsilon_*$ then $\phi_\varepsilon - \theta_\varepsilon \in X_\varepsilon$. In particular, we have*

$$\|\phi_\varepsilon - \theta_\varepsilon\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \lesssim \varepsilon, \qquad \delta \in \{0, 2\}. \tag{7.20}$$

Proof. Lemma 7.11 shows that if ε_* is sufficiently small then for $0 < \varepsilon < \varepsilon_*$ the map $\mathcal{G}_\varepsilon$ has a unique fixed point $u_\varepsilon \in X_\varepsilon \subseteq C^{k,\alpha}_2(M_\varepsilon)$. Since $\mathcal{G}_\varepsilon(u_\varepsilon) = u_\varepsilon$ we see from (7.16) that $\mathcal{L}_\varepsilon(\theta_\varepsilon + u_\varepsilon) = 0$ and thus $\theta_\varepsilon + u_\varepsilon$ is a solution to the Lichnerowicz equation (1.8). By Proposition 7.3 we have $\theta_\varepsilon - 1 \in C^{k,\alpha}_1(M_\varepsilon)$. Thus, since $u_\varepsilon \in X_\varepsilon \subseteq C^{k,\alpha}_1(M_\varepsilon)$, we have $(\theta_\varepsilon + u_\varepsilon) - 1 \in C^{k,\alpha}_1(M_\varepsilon)$. Furthermore, for sufficiently small ε we have $\theta_\varepsilon + u_\varepsilon > 0$. But from [2, Proposition 6.4] we have that ϕ_ε is the unique positive solution to (1.8) such that $\phi_\varepsilon - 1 \in C^{k,\alpha}_1(M_\varepsilon)$. Thus we have $\phi_\varepsilon = \theta_\varepsilon + u_\varepsilon$. In particular, $\phi_\varepsilon - \theta_\varepsilon = u_\varepsilon \in X_\varepsilon$, which immediately implies (7.20). □

Proof of Proposition 7.1. We establish the estimates (7.1) by writing

$$\phi_\varepsilon - 1 = (\theta_\varepsilon - 1) + (\phi_\varepsilon - \theta_\varepsilon).$$

We estimate $\theta_\varepsilon - 1$ using (7.5) and (7.6). We estimate $\phi_\varepsilon - \theta_\varepsilon$ using (7.20), together with Proposition 2.5(b). □

8. Proof of Theorem 1.1

We now complete the proof of our main theorem. We assume (4.2), and that $\varepsilon < \varepsilon_*$ as in Proposition 7.1. From Proposition 5.6 we have $\overline{\lambda}_\varepsilon = \rho_\varepsilon^2 \lambda_\varepsilon \in \mathscr{C}^{k,\alpha;2}(M_\varepsilon)$ with $|d\rho_\varepsilon|^2_{\overline{\lambda}_\varepsilon} = 1$ along ∂M_ε, and from Lemmas 6.1 and 6.3 we have

$$\nu_\varepsilon^{\text{ext}} + \nu_\varepsilon^{\text{neck}} \in C^{k-1,\alpha}_2(M_\varepsilon).$$

Thus the results of [3] imply that

$$g_\varepsilon = \phi_\varepsilon^4 \lambda_\varepsilon \quad \text{and} \quad \Sigma_\varepsilon = \phi_\varepsilon^{-2} \sigma_\varepsilon$$

constitutes an appropriate seed data set so that the conformal method produces a CMCSF hyperboloidal data set on M_ε. It remains to verify the convergence statements (1.3) and (1.4).

We first consider the metric g_ε. In the exterior region E_c we have

$$\begin{aligned}\rho^2(\iota_\varepsilon^* g_\varepsilon - g) &= \iota_\varepsilon^*\left(\rho_\varepsilon^2(g_\varepsilon - \lambda_\varepsilon)\right)\\ &= \iota_\varepsilon^*\left((\phi_\varepsilon^4 - 1)\rho_\varepsilon^2\lambda_\varepsilon\right)\\ &= \iota_\varepsilon^*(\phi_\varepsilon^4 - 1)\rho^2 g.\end{aligned}$$

By hypothesis we have $\overline{g} = \rho^2 g \in \mathscr{C}^{k,\alpha;2}(M)$. Thus (7.2) implies that

$$\begin{aligned}\|\rho^2(\iota_\varepsilon^* g_\varepsilon - g)\|_{\mathscr{C}^{k,\alpha;2}(E_c)} &\lesssim \|\iota_\varepsilon^*(\phi_\varepsilon^4 - 1)\|_{\mathscr{C}^{k,\alpha;2}(E_c)}\|\rho^2 g\|_{\mathscr{C}^{k,\alpha;2}(M)}\\ &\lesssim \varepsilon.\end{aligned}$$

In the neck region A_c we write

$$\breve{\rho}^2(\Psi_\varepsilon^* g_\varepsilon - \breve{g}) = \Psi_\varepsilon^*(\phi_\varepsilon^4 - 1)\breve{\rho}^2\Psi_\varepsilon^*\lambda_\varepsilon + \breve{\rho}^2\left(\Psi_\varepsilon^*\lambda_\varepsilon - \breve{g}\right).$$

Applying (7.2) and Proposition 5.6, and using (4.8), we have

$$\begin{aligned}\|\breve{\rho}^2(\Psi_\varepsilon^* g_\varepsilon - \breve{g})\|_{\mathscr{C}^{k,\alpha;2}(A_c)} &\lesssim \|\Psi_\varepsilon^*(\phi_\varepsilon^4 - 1)\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\|y^2\Psi_\varepsilon^*\lambda_\varepsilon\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\\ &\qquad + \|y^2(\Psi_\varepsilon^*\lambda_\varepsilon - \breve{g})\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\\ &\lesssim \varepsilon.\end{aligned}$$

We now turn attention to the tensor Σ_ε. In the exterior region E_c we have

$$\rho(\iota_\varepsilon^*\Sigma_\varepsilon - \Sigma) = \iota_\varepsilon^*(\phi_\varepsilon^{-2} - 1)\rho\iota_\varepsilon^*\sigma_\varepsilon + \rho\left(\iota_\varepsilon^*\sigma_\varepsilon - \Sigma\right).$$

Using Proposition 6.7 we have

$$\|\rho\iota_\varepsilon^*\sigma_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)} \le \|\rho(\iota_\varepsilon^*\sigma_\varepsilon - \Sigma)\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)} + \|\rho\Sigma\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)} \lesssim 1.$$

Thus applying (7.2) and Proposition 6.7 we have

$$\begin{aligned}\|\rho(\iota_\varepsilon^*\Sigma_\varepsilon - \Sigma)\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)} &\lesssim \|\iota_\varepsilon^*(\phi_\varepsilon^{-2} - 1)\|_{\mathscr{C}^{k,\alpha;2}(E_c)}\|\rho\iota_\varepsilon^*\sigma_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)}\\ &\qquad + \|\rho(\iota_\varepsilon^*\sigma_\varepsilon - \Sigma)\|_{\mathscr{C}^{k-1,\alpha;1}(E_c)}\\ &\lesssim \varepsilon.\end{aligned}$$

In the neck region A_c, the analogous decomposition yields

$$\begin{aligned}\|\breve{\rho}(\Psi_\varepsilon^*\Sigma_\varepsilon - 0)\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)} &\lesssim \|\Psi_\varepsilon^*(\phi_\varepsilon^{-2} - 1)\|_{\mathscr{C}^{k,\alpha;2}(A_c)}\|\rho\Psi_\varepsilon^*\sigma_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)}\\ &\qquad + \|\rho\Psi_\varepsilon^*\sigma_\varepsilon\|_{\mathscr{C}^{k-1,\alpha;1}(A_c)}\\ &\lesssim \varepsilon.\end{aligned}$$

This concludes the proof of the main theorem. □

Appendix A. Uniform invertibility for elliptic operators

In this appendix we study the vector Laplace operator L_{λ_ε} defined in (1.5) and the linearized Lichnerowicz operator

$$\mathcal{P}_\varepsilon[1] = \Delta_{\lambda_\varepsilon} - \frac{1}{8}\left(\mathrm{R}[\lambda_\varepsilon] + 7|\sigma_\varepsilon|^2_{\lambda_\varepsilon} + 30\right) \tag{A.1}$$

given by (7.13) in the special case $\theta = 1$. We obtain uniform invertibility of these operators in the following sense.

Proposition A.1. *Let λ_ε be the metrics constructed in* (5.1)*. For each $\delta \in [0,3)$ there exists a constant C_δ, independent of ε, such that:*

(a) $L_{\lambda_\varepsilon}\colon C^{k,\alpha}_\delta(M_\varepsilon) \to C^{k-2,\alpha}_\delta(M_\varepsilon)$ *is invertible with*

$$\|X\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le C_\delta \|L_{\lambda_\varepsilon} X\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}$$

for all vector fields $X \in C^{k,\alpha}_\delta(M_\varepsilon)$,

(b) $\mathcal{P}_\varepsilon[1]\colon C^{k,\alpha}_\delta(M_\varepsilon) \to C^{k-2,\alpha}_\delta(M_\varepsilon)$ *is invertible with*

$$\|u\|_{C^{k,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le C_\delta \|\mathcal{P}_\varepsilon[1] u\|_{C^{k-2,\alpha}_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}$$

for all functions $u \in C^{k,\alpha}_\delta(M_\varepsilon)$.

Theorem 1.6 of [2] implies that L_{λ_ε} and $\mathcal{P}_\varepsilon[1]$ are Fredholm of index zero; see also [11]. Thus Proposition A.1 is an immediate consequence of the elliptic regularity estimates in Proposition 5.5 and of the following lemma, which controls the kernels of the operators, and is proven in section A.3 below.

Lemma A.2. *For each $\delta \in [0,3)$ there exists C_δ, independent of ε, such that:*

(a) $\|X\|_{C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le C_\delta \|L_{\lambda_\varepsilon} X\|_{C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}$ *for all $X \in C^{2,\alpha}_\delta(M_\varepsilon)$,*

(b) $\|u\|_{C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)} \le C_\delta \|\mathcal{P}_\varepsilon[1] u\|_{C^0_\delta(M_\varepsilon;\tilde\rho_\varepsilon)}$ *for all $u \in C^{2,\alpha}_\delta(M_\varepsilon)$.*

Prior to proving Lemma A.2, we introduce a general framework for blowup analysis and establish some results concerning the kernels of model operators.

A.1. Exhaustions of weighted Riemannian manifolds.

Let (M_*, g_*) be a Riemannian manifold. We say that a sequence of Riemannian manifolds ***(M_j, g_j) forms an exhaustion of (M_*, g_*)*** if

- M_j are non-empty precompact open subsets of M_*,
- $M_1 \subseteq \overline{M}_1 \subseteq M_2 \subseteq \overline{M}_2 \subseteq M_3 \subseteq \cdots$,
- $\bigcup_{j=1}^\infty M_j = M_*$, and
- $\|g_j - g_*\|_{C^2(K,g_*)} \to 0$ on each precompact set $K \subseteq M_*$.

If in addition we have continuous functions $w_j\colon M_j \to (0,\infty)$ and $w_*\colon M_* \to (0,\infty)$ such that $\|w_j - w_*\|_{C^0(K)} \to 0$ on each precompact set $K \subseteq M_*$, then we say that ***(M_j, g_j, w_j) forms an exhaustion of (M_*, g_*, w_*)***.

We now give a definition of convergence for linear differential operators. Let (M_j, g_j) be an exhaustion of (M_*, g_*). Consider a second order linear differential operator P_* acting on sections of some tensor bundle over M_*, and operators P_j acting on the restriction of that bundle to M_j. We write $P_j = A_j\nabla^2 + B_j\nabla + C_j$ where A_j, B_j, C_j are appropriate bundle homomorphisms and where ∇ is the connection associated to g_j. Similarly, we write $P_* = A_*\nabla^2 + B_*\nabla + C_*$. We say that ***$P_j$ converges to P_****, and write $P_j \to P_*$, if

$$\|A_j - A_*\|_{C^2(K,g_*)} + \|B_j - B_*\|_{C^1(K,g_*)} + \|C_j - C_*\|_{C^0(K,g_*)} \to 0$$

on each precompact $K \subseteq M_*$.

Clearly, if $P_j \to P_*$ then for each precompact K and each smooth tensor field η we have

$$\|P_j\eta - P_*\eta\|_{C^0(K,g_*)} \to 0 \quad \text{and} \quad \|P_j^\dagger\eta - P_*^\dagger\eta\|_{C^0(K,g_*)} \to 0,$$

where $P_j^\dagger$ and $P_*^\dagger$ denote the formal adjoints of P_j and P_*, respectively. If in addition the operators P_j and P_* are elliptic then the constants in the interior elliptic regularity estimates can be chosen independently of (sufficiently large) j. Finally, the reader should notice that if (M_j, g_j) is an exhaustion of (M_*, g_*) then any family of second order geometric operators $P_j = P[g_j]$ and $P_* = P[g_*]$ satisfies $P_j \to P_*$.

Proposition A.3. *Let (M_j, g_j, w_j) be an exhaustion of (M_*, g_*, w_*), and let P_j and P_* be second order elliptic linear differential operators on (M_j, g_j) and (M_*, g_*) with $P_j \to P_*$. Suppose also that there exists points $q_j \in M_1$ converging to $q_* \in M_1$ with respect to g_*, a sequence of tensor fields $u_j \in C^2(M_j)$, and constants $c, C > 0$ such that:*

(a) *for all j we have $\left(w_j^{-1}|u_j|_{g_j}\right)\Big|_{q_j} \geq c$;*

(b) *for all j we have $\sup\limits_{M_j} w_j^{-1}|u_j|_{g_j} \leq C$;*

(c) *we have $\sup\limits_{M_j} w_j^{-1}|P_j u_j|_{g_j} \to 0$ as $j \to \infty$.*

Then there is a non-zero tensor field $u_* \in C^0(M_*, g_*)$ and a subsequence $\{u_{j_n}\}$ such that

- *$u_{j_n} \to u_*$ uniformly on compact sets;*
- *$\sup\limits_M w_*^{-1}|u_*|_{g_*} < \infty$;*
- *$P_* u_* = 0$ in the weak sense.*

Proof. Fix $p > \dim(M_*)$ so that the Sobolev space $H^{1,p}(M_j, g_*)$ embeds continuously into $C^0(M_j, g_*)$ for each j.

We now describe a process for extracting a subsequence of $\{u_j\}$ that we use iteratively in order to produce the desired subsequence via a diagonal argument. Given the sequence $\{u_j\}$ and the sets $M_1 \subseteq \overline{M}_1 \subseteq M_2$, we extract a subsequence $u_{j_n,1}$ as follows. Our assumptions imply that for sufficiently large j we have

$$|u_j|_{g_j} \leq 2Cw_*, \quad |P_j u_j|_{g_j} \leq Cw_* \quad \text{on } M_2.$$

As the volumes $\mathrm{vol}_{g_j}(M_2)$ are uniformly bounded for $j > 2$, we have that the Sobolev norms $\|u_j\|_{H^{0,p}(M_2,g_j)}$ and $\|P_j u_j\|_{H^{0,p}(M_2,g_j)}$ are bounded uniformly. Since the assumption $P_j \to P_*$ implies

$$\|u_j\|_{H^{2,p}(M_1,g_j)} \leq C' \left(\|P_j u_j\|_{H^{0,p}(M_2,g_j)} + \|u_j\|_{H^{0,p}(M_2,g_j)} \right)$$

for some constant C' independent of j, we have that $\|u_j\|_{H^{2,p}(M_1,g_j)}$ are bounded, and thus so are $\|u_j\|_{H^{2,p}(M_1,g_*)}$. Applying Rellich's lemma yields a subsequence $\{u_{j_n,1}\}$ that converges in $H^{1,p}(M_1, g_*)$ to some function u_1. Since p has been chosen such that $H^{1,p}(M_1, g_*) \subseteq C^0(M_1)$, it follows that we have uniform pointwise convergence

$$u_{j_n,1} \to u_1 \quad \text{in } C^0(M_1, g_*).$$

Furthermore, assumptions (a) and (b) imply that

$$|u_1(q_*)|_{g_*} \geq \frac{c}{2}, \quad |u_1|_{g_*} \leq 2Cw_* \quad \text{on } M_1.$$

The process that produces the subsequence $\{u_{j_n,1}\}$ from the sequence $\{u_j\}$ and the sets $M_1 \subseteq \overline{M}_1 \subseteq M_2$ is now applied iteratively. For example, applying this process to the sequence $\{u_{j_n,1}\}$ and the sets $M_2 \subseteq \overline{M}_2 \subseteq M_3$ gives rise to the subsequence $\{u_{j_n,2}\}$ of $\{u_{j_n,1}\}$ that converges in $C^0(M_2, g_*)$ to some limit u_2. Since $u_{j_n,1} \to u_1$ in $C^0(M_1, g_*)$, we see that the function u_2 is a continuous extension of u_1 to the domain M_2. Furthermore, we have that

$$|u_2(q_*)|_{g_*} \geq \frac{c}{2}, \quad |u_2|_{g_*} \leq 2Cw_* \quad \text{on } M_2.$$

Repeating this process inductively we obtain subsequences $\{u_{j_n,l}\}$ of $\{u_j\}$ and limiting functions $u_l \in C^0(M_l)$ such that

$$u_{j_n,l} \to u_l \quad \text{in } C^0(M_l, g_*)$$

as $n \to \infty$. Consequently, the diagonal sequence $\{u_{j_n,n}\}$ is uniformly convergent on every compact subset of M_* to a limit $u_* \in C^0(M_*, g_*)$. Furthermore, we have

$$|u_*(q_*)|_{g_*} \geq \frac{c}{2}, \quad \text{and} \quad |u_*|_{g_*} \leq 2Cw_* \quad \text{on } M_*.$$

For the remainder of the proof we denote the subsequence $\{u_{j_n,n}\}$ by $\{u_{j_n}\}$.

We now show that $P_* u_* = 0$ weakly. Consider a smooth tensor field η supported on some $\Omega \subseteq \overline{\Omega} \subseteq M_*$, where $\overline{\Omega}$ is compact. Since $P_{j_n} \to P_*$ and

$g_{j_n} \to g_*$ we have

$$\begin{aligned}\left|\int_{M_*} \langle P_*^\dagger \eta, u_* \rangle_{g_*} dV_{g_*}\right| &= \lim_{n\to\infty} \left|\int_\Omega \langle P_{j_n}^\dagger \eta, u_{j_n} \rangle_{g_{j_n}} dV_{g_{j_n}}\right| \\ &\le \lim_{n\to\infty} \int_\Omega \left|\langle \eta, P_{j_n} u_{j_n} \rangle_{g_{j_n}}\right| dV_{g_{j_n}} \\ &\le \|\eta\|_{C^0(\Omega,g_*)} \mathrm{Vol}_{g_*}(\Omega) \cdot \lim_{n\to\infty} \|P_{j_n} u_{j_n}\|_{C^0(\Omega,g_{j_n})}.\end{aligned}$$

It follows from our assumptions that $\sup_\Omega w_{j_n}^{-1} |P_{j_n} u_{j_n}|_{g_{j_n}} \to 0$. As the functions w_{j_n} converge uniformly to the positive function w_* on the precompact set Ω, they are uniformly bounded from above and below on Ω. Thus $\|P_{j_n} u_{j_n}\|_{C^0(\Omega,g_{j_n})} \to 0$ and hence

$$\int_{M_*} \langle P_*^\dagger \eta, u_* \rangle_{g_*} dV_{g_*} = 0.$$

Therefore $P_* u_* = 0$ weakly. □

A.2. **Invertibility of model operators.** Our blowup analysis uses the mapping properties of elliptic geometric operators defined using one of two model CMCSF hyperboloidal initial data sets: the data assumed in the main theorem, given by (g, Σ) on M, which serves as a model away from the gluing region, and the data given by $(\breve{g}, 0)$ on $\mathbb{H}^3$, which serves as a model in the gluing region. In the first case, our aim is to establish the injectivity of the vector Laplace operator L_g and of the operator $\mathcal{P}_0$ given by

$$\begin{aligned}\mathcal{P}_0 u &= \Delta_g u - \frac{1}{8}\left(\mathrm{R}[g] + 7|\Sigma|_g^2 + 30\right) u \\ &= \Delta_g u - (3 + |\Sigma|_g^2),\end{aligned}$$

where we have used (1.2). The operator $\mathcal{P}_0$ serves as a model for the linearization $\mathcal{P}_\varepsilon[1]$ of the Lichnerowicz operator about the function 1. In the second case, we establish injectivity of the analogous operators defined by $(\breve{g}, 0)$.

First we consider the case of the data assumed in the main theorem.

Proposition A.4. *Let (g, Σ) be initial data on M as in Theorem 1.1 and suppose $|1 - \delta| < 2$.*

(a) *If a continuous vector field X on M satisfies $|X|_g \le C\rho^\delta$ for some constant C and if $L_g X = 0$, then $X = 0$.*
(b) *If a continuous function u on M satisfies $|u| \le C\rho^\delta$ for some constant C and if $\mathcal{P}_0 u = 0$, then $u = 0$.*

Proof. For the first claim, we note that L_g is an elliptic geometric operator. Thus from the elliptic regularity results in [2, Lemma 5.1] we have $X \in C_\delta^{k,\alpha}(M)$. From Proposition 6.3 of [3] we have that

$$L_g \colon C_\delta^{k,\alpha}(M) \to C_\delta^{k-2,\alpha}(M)$$

is invertible, and thus $X = 0$.

For the second claim, we note that $\Delta_g - 3$ is an elliptic geometric operator. Since $|\Sigma|_g^2 \in C_2^{k-1,\alpha}(M)$, adding $-|\Sigma|_g^2 u$ to the lower order term does not affect the arguments leading to elliptic regularity results for $\mathcal{P}_0$; see [11, Lemma 4.8] and [2, Lemma 5.1]. (Note that the sign convention for the Laplacian Δ_g in [11] is the opposite of the one used here.) Thus $u \in C_\delta^{k,\alpha}(M)$. Since Proposition 6.5 of [3] implies that

$$\mathcal{P}_0 \colon C_\delta^{k,\alpha}(M) \to C_\delta^{k-2,\alpha}(M)$$

is invertible, we conclude that $u = 0$. □

We now turn to the model of hyperbolic space. As in section 2.1 we use the coordinates (x, y) on the half-space model of hyperbolic space and write $r^2 = |x|^2 + y^2$. Recall also the function $\breve{\rho}$ defined in (2.6) and the function F described in Proposition 2.2. It is established in [11, Theorem 5.9] that any self-adjoint elliptic geometric operator $\breve{P}$ on hyperbolic space is an isomorphism

$$\breve{P} \colon C_\delta^{k,\alpha}(\mathbb{H}^3) \to C_\delta^{k-2,\alpha}(\mathbb{H}^3)$$

provided $|\delta - 1| < R$, where R is the indicial radius of the operator $\breve{P}$. In particular, this applies to the vector Laplace operator $L_{\breve{g}}$ and to the operator $\Delta_{\breve{g}} - 3$ for $|\delta - 1| < 2$.

The isomorphism property of $\breve{P}$, together with interior elliptic regularity, implies that any continuous tensor field $v \in \ker \breve{P}$ with $|v|_{\breve{g}} \le C\breve{\rho}^\delta$ must in fact vanish. In our blowup analysis we require a slight strengthening of this statement that makes use of the functions y and yF on the half-space model of hyperbolic space. The argument we present is a generalization of the proof of Proposition 13 and Corollary 14 in [10].

Proposition A.5. *Let $\breve{P}$ be a self-adjoint elliptic geometric operator on $\mathbb{H}$ with indicial radius $R > 0$. Suppose that for some δ satisfying $|1 - \delta| < R$ there exists $v \in \ker \breve{P}$ satisfying either $|v|_{\breve{g}} \le C(yF)^\delta$ or $|v|_{\breve{g}} \le Cy^\delta$. Then v is identically zero.*

Proof. We argue by contradiction and consider first the case that there exists a nonzero tensor field $v \in \ker \breve{P}$ and constants $C, \delta \in \mathbb{R}$ such that $|\delta - 1| < R$ and $|v|_{\breve{g}} \le C(yF)^\delta$. Let $r_0 > 0$ be such that v does not vanish identically on the set where $r < r_0$. From Proposition 2.2 we have

$$|v|_{\breve{g}} \le Cy^\delta \quad \text{where } r \ge r_0, \tag{A.2}$$

$$|v|_{\breve{g}} \le C\frac{y^\delta}{r^{2\delta}} \quad \text{where } r \le 3r_0, \tag{A.3}$$

where here, and in the following, the value of C may vary from line to line.

Let $\varphi_0 \colon \mathbb{R}^2 \to [0, 1]$ be a smooth cutoff function supported on $|x| \le 2r_0$, where $x = (x^1, x^2)$ are Cartesian coordinates on $\mathbb{R}^2$, and define $\tilde{v}$ on $\mathbb{H}$ by

$$\tilde{v}(x, y) = \int_{\mathbb{R}^2} v(x - \xi, y)\varphi_0(\xi)\, d\xi.$$

Since v does not vanish on $r < 2r_0$ by assumption, one can always choose φ_0 so that $\tilde{v}$ is not identically zero. Differentiation under the integral sign shows that $\breve{P}\tilde{v} = 0$.

We claim that

$$|\tilde{v}|_{\breve{g}} \leq C\left(y^{2-\delta} + y^{\delta}\right). \tag{A.4}$$

On the region where $r \geq 3r_0$, the estimate (A.2) implies (A.4) and thus we focus attention on the region where $r \leq 3r_0$. There, (A.3) implies that

$$|\tilde{v}|_{\breve{g}} \leq C \int_{|\xi|\leq 2r_0} \frac{y^{\delta}}{(y^2 + |x-\xi|^2)^{\delta}} d\xi.$$

We now use the change of variables $\xi = x - y\zeta$ and observe that $r \leq 3r_0$ and $|\xi| \leq 2r_0$ implies $|\zeta| \leq \frac{5r_0}{y}$. Thus using polar coordinates yields

$$\begin{aligned} |\tilde{v}|_{\breve{g}} &\leq Cy^{2-\delta} \int_{|x-y\zeta|\leq 2r_0} \frac{d\zeta}{(1+|\zeta|^2)^{\delta}} \\ &\leq Cy^{2-\delta} \int_0^{5r_0/y} \frac{t}{(1+t^2)^{\delta}}\, dt. \end{aligned}$$

It follows from

$$\frac{1}{(1+t^2)^{\delta}} \leq \begin{cases} C & \text{for } \; t \leq 1, \\ Ct^{-2\delta} & \text{for } \; t \geq 1 \end{cases}$$

that

$$\int_0^{5r_0/y} \frac{t}{(1+t^2)^{\delta}}\, dt \leq C(1 + y^{2\delta-2}).$$

This completes the proof of (A.4).

We now define $u = \mathcal{I}^*\tilde{v}$, where $\mathcal{I}$ is the inversion operator defined in (2.3). Note that $\breve{P}u = 0$ due to $\mathcal{I}$-invariance of $\breve{g}$, and consequently the $\mathcal{I}$-invariance of $\breve{P}$. Choose $r_1 > 0$ so that $u \neq 0$ on $r \leq r_1$. Choose also a smooth function $\varphi_1 \colon \mathbb{R}^2 \to [0,1]$ supported on $|x| < 2r_1$ such that the tensor field

$$\tilde{u}(x,y) = \int_{\mathbb{R}^2} u(x-\xi, y)\varphi_1(\xi)\, d\xi$$

is non-zero. Differentiation under the integral sign shows that $\breve{P}\tilde{u} = 0$.

We claim that

$$|\widetilde{u}|_{\breve{g}} \lesssim \breve{\rho}^{2-\delta} + \breve{\rho}^{\delta}. \tag{A.5}$$

To this end, observe that (A.4) implies

$$|u|_{\breve{g}} \lesssim \left(\frac{y}{r^2}\right)^{2-\delta} + \left(\frac{y}{r^2}\right)^{\delta}.$$

The same change of variables argument involved in the proof of (A.4) shows that

$$|\widetilde{u}|_{\breve{g}} \leq Cy^{2-\delta} + y^{\delta} \quad \text{on} \quad r \leq 3r_1.$$

In the region where $r \leq 3r_1$, we have $C^{-1}y \leq \breve{\rho} \leq Cy$, which implies the estimate (A.5) in that region.

In the region where $r \geq 3r_1$ and $|\xi| \leq 2r_1$ we have

$$C^{-1}r^2 \leq y^2 + |x-\xi|^2 Cr^2.$$

The estimate (A.5) now follows from the fact that $C^{-1}\breve{\rho} \leq \frac{y}{r^2} \leq C\breve{\rho}$ where $r \geq 3r_1$.

Since $\breve{\rho} \leq C$, the estimate (A.5) implies that $|\tilde{u}|_{\breve{g}} \leq C\breve{\rho}^{\nu}$ where $\nu = \min(2-\delta, \delta)$. Thus $\tilde{u} \in \ker \breve{P}$ and $\tilde{u} \in C^0_\nu(\mathbb{H}^3)$, where $|1-\nu| < R$. The isomorphism property of $\breve{P}$ implies that $\tilde{u} = 0$, which is the desired contradiction.

Suppose now that $\breve{P}v = 0$ and that $|v|_{\breve{g}} \leq Cy^\delta$. If $\delta < 0$, then the fact that $y \geq C\breve{\rho}$ implies $|v|_{\breve{g}} \leq C\breve{\rho}^\delta$ and hence the isomorphism property of $\breve{P}$ implies $v = 0$. If $\delta \geq 0$, then the fact that $F^\delta \geq C$ implies that $|v|_{\breve{g}} \leq C(yF)^\delta$ and thus $v = 0$ by the previous argument. □

A.3. **Proof of Lemma A.2.** We now establish Lemma A.2. We present the argument for the estimate

$$\|u\|_{C^0_\delta(M_\varepsilon;\tilde{\rho}_\varepsilon)} \lesssim \|\mathcal{P}_\varepsilon[1]u\|_{C^0_\delta(M_\varepsilon;\tilde{\rho}_\varepsilon)}; \tag{A.6}$$

the estimate for the vector Laplace operator follows from analogous reasoning.

We argue by contradiction and assume that (A.6) does not hold. Thus there exists $\delta \in [0,3)$ and a sequence $\varepsilon_j \to 0$, together with functions $u_j \in C^{2,\alpha}_\delta(M_{\varepsilon_j})$, such that

$$\|u_j\|_{C^0_\delta(M_{\varepsilon_j};\tilde{\rho}_{\varepsilon_j})} = 1 \tag{A.7}$$

and

$$\|\mathcal{P}_{\varepsilon_j}[1]u_j\|_{C^0_\delta(M_{\varepsilon_j};\tilde{\rho}_{\varepsilon_j})} \to 0. \tag{A.8}$$

Hence there exist points $q_j \in M \setminus (U_{1,\varepsilon_j} \cup U_{2,\varepsilon_j})$, where we recall (2.10), such that at the point $\pi_{\varepsilon_j}(q_j) \in M_{\varepsilon_j}$ we have

$$\left(\tilde{\rho}_{\varepsilon_j}^{-\delta}|u_j|\right)\Big|_{\pi_{\varepsilon_j}(q_j)} \geq \frac{1}{2}. \tag{A.9}$$

Passing to a subsequence if necessary, we may assume that $q_j \to q \in \overline{M}$. We now consider several cases, depending on the location of $q \in \overline{M}$, obtaining a contradiction in each case.

Case 1: $q \in M$. In this case we define (M_j, g_j, w_j) by setting

$$M_j = \{p \in M \setminus (U_{1,\varepsilon_j} \cup U_{2,\varepsilon_j}) \colon \rho_{\varepsilon_j}(\pi_{\varepsilon_j}(p)) > \varepsilon_j\},$$

$g_j = \pi^*_{\varepsilon_j}\lambda_{\varepsilon_j}$, and $w_j = \pi^*_{\varepsilon_j}\tilde{\rho}^\delta_{\varepsilon_j} = \pi^*_{\varepsilon_j}(\rho_{\varepsilon_j}/\omega_{\varepsilon_j})^\delta$; see (4.7).

Let $\psi\colon (0,\infty) \to (0,1]$ be the smooth cutoff function used in (4.5), and define the function $\omega_*\colon \overline{M} \to (0,\infty)$ by setting $\omega = 1$ outside the domain of the preferred background coordinates $\Theta_i = (\theta, \rho)$ and by requiring that

$\omega_* = \psi(|(\theta,\rho)|)$ in each background coordinate chart. Set $w_* = (\rho/\omega_*)^\delta$ on M.

We claim that (M_j, g_j, w_j) forms an exhaustion of (M, g, w_*). The convergence of the metrics is immediate from the fact that $\iota_\varepsilon^*\lambda_\varepsilon = g$ on E_c; see Proposition 5.6. To see the convergence of the weight functions, we recall from §4.3 that in preferred background coordinates $\Theta_i = (\theta, \rho)$ we have $\pi_\varepsilon^*\omega_\varepsilon = \psi(|(\theta,\rho)| + \varepsilon^2|(\theta,\rho)|^{-1})$. Thus $\pi_{\varepsilon_j}^*\omega_{\varepsilon_j} \to \omega$ uniformly on every precompact subset of M. As $\pi_\varepsilon^*\rho_\varepsilon = \rho$ on E_c we see that $w_j \to w_*$ uniformly on precompact sets as well.

Let $v_j = \pi_{\varepsilon_j}^* u_j$. As $q_* \in M_j$ for sufficiently large j, we may pass to a subsequence to ensure that $q_j \in M_1$ for all j, and hence that $q_* \in M_1$. From (A.9) and (A.7) we have

$$\left(w_j^{-1}|v_j|\right)\Big|_{q_j} \geq \frac{1}{2} \quad \text{and} \quad \sup_{M_j} w_j^{-1}|u_j| \leq 1.$$

Furthermore, setting $P_j = \pi_{\varepsilon_j}^*\mathcal{P}_{\varepsilon_j}[1]$, we have $\sup_{M_j} w_j^{-1}|P_j v_j| \to 0$.

The convergence of $\iota_\varepsilon^*|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ to $|\Sigma|^2_g$ in the exterior region given by (6.11) implies that $\mathcal{P}_{\varepsilon_j}[1] \to \mathcal{P}_0$ as described in §A.1. Thus from Proposition A.3 there exists a nonzero function $v_* \in C^0(M)$ such that $|v_*| \leq C(\rho/\omega_*)^\delta$ and $\mathcal{P}_0 v_* = 0$. Note that $\rho \leq C\omega_* \leq C$. Thus, since $\delta \geq 0$, we have $|v_*| \leq C$. But Proposition A.4 implies that the only continuous and bounded function in the kernel of $\mathcal{P}_0$ is the zero function, contradicting that v_* is nonzero.

Case 2: $q \in \partial\overline{M} \smallsetminus \{p_1, p_2\}$. Let $\Theta = (\theta, \rho)$ be background coordinates on M centered at q as introduced in §2.2. After an affine change of coordinates we can arrange that at q we have $\bar{g}_{ij}d\Theta^i d\Theta^j = \delta_{ij}d\Theta^i d\Theta^j$. For j sufficiently large, q_j is contained in the domain $Z(q)$ of Θ; denote $\Theta(q_j)$ by $(\hat{\theta}_j, \hat{\rho}_j)$. Let $r_* > 0$ be such that neither p_1 nor p_2 is contained in that part of $Z(q)$ where $|(\theta,\rho)| \leq r_*$. Without loss of generality, we may assume that $|\hat{\theta}_j| < r_*/2$.

Set $M_j = \{(x,y) \in \mathbb{H} \colon |(x,y)| < r_*/4\hat{\rho}_j, y > \hat{\rho}_j/2\}$ and use the background coordinates (θ,ρ) about q to define $\Phi_j \colon M_j \to M$ by

$$\Phi_j \colon (x,y) \mapsto (\theta,\rho) = (\hat{\theta}_j + \hat{\rho}_j x, \hat{\rho}_j y).$$

Note that $\Phi_j(0,1) = q_j$ and that for sufficiently small c (and hence, in view of (4.2), sufficiently small ε_j) the image of Φ_j is contained in the exterior region E_c. Thus we may define $T_j \colon M_j \to M_{\varepsilon_j}$ by $T_j = \iota_{\varepsilon_j} \circ \Phi_j$.

Set $g_j = T_j^*\lambda_{\varepsilon_j}$. Let $\hat{\omega}_j = \omega_{\varepsilon_j}(\iota_{\varepsilon_j}(q_j))$ and, recalling from (4.7) that $\tilde{\rho}_\varepsilon = \rho_\varepsilon/\omega_\varepsilon$, define

$$w_j = \tilde{\rho}_{\varepsilon_j}(\iota_{\varepsilon_j}(q_j))^{-\delta} T_j^* \tilde{\rho}_{\varepsilon_j}^\delta = \left(\frac{\hat{\rho}_j}{\hat{\omega}_j}\right)^{-\delta} T_j^* \tilde{\rho}_{\varepsilon_j}^\delta. \tag{A.10}$$

We claim that (M_j, g_j, w_j) forms an exhaustion of $(\mathbb{H}, \breve{g}, y^\delta)$. Since $\hat{\rho}_j \to 0$ we have $\bigcup_j M_j = \mathbb{H}$. To see that $g_j \to \breve{g}$, recall from Proposition 5.6 that $\iota_\varepsilon^*\lambda_\varepsilon = g$ on E_c, and thus g_j is simply the pullback of g by Φ_j. In coordinates $\Theta = (\theta,\rho)$, we write $g = \rho^{-2}\overline{g}_{ij}(\theta,\rho)d\Theta^i d\Theta^j$. Thus in coordinates $X = (x,y)$

we have $g_j = \Phi_j^* g = y^{-2}\overline{g}_{ij}(\hat{\theta}_j + \hat{\rho}_j x, \hat{\rho}_j y) dX^i dX^j$. Thus on any precompact set $K \subseteq \mathbb{H}$ we have $g_j \to y^{-2}\overline{g}_{ij}(q) dX^i dX^j = y^{-2} g_{\mathrm{E}} = \breve{g}$ uniformly. Finally, to see the convergence of the weight function, note that $\hat{\rho}_j^{-1} T_j^* \rho_{\varepsilon_j} = y$ and that on any precompact $K \subseteq \mathbb{H}$ we have $\hat{\omega}_j^{-1} T_j^* \omega_{\varepsilon_j} \to 1$ uniformly. Thus the claim is verified.

Define the functions v_j on M_j by $v_j = (\hat{\rho}_j/\hat{\omega}_j)^{-\delta} T_j^* u_j$. Using (A.10) we see that $w_j^{-1}|v_j| = T_j^*(\tilde{\rho}_{\varepsilon_j}^{-\delta}|u_j|)$. Thus from (A.9) and (A.7) we have

$$\left(w_j^{-1}|v_j|\right)\Big|_{(0,1)} \geq \frac{1}{2} \quad \text{and} \quad \sup_{M_j} w_j^{-1}|v_j| \leq 1;$$

hence assumptions (a) and (b) of Proposition A.3 hold.

Define the differential operator P_j on M_j by $P_j = T_j^* \mathcal{P}_{\varepsilon_j}[1]$. We claim that $P_j \to \Delta_{\breve{g}} - 3$. To see this, note first that since $\iota_\varepsilon^* \lambda_\varepsilon = g$ in the exterior region E_c, applying the constraint equations (1.2) to (A.1) yields

$$\begin{aligned} \iota_\varepsilon^* \mathcal{P}_{\varepsilon_j}[1] &= \Delta_g - 3 - \frac{1}{8}|\Sigma|_g^2 - \frac{7}{8}\iota_\varepsilon^*|\sigma_\varepsilon|_{\lambda_\varepsilon}^2 \\ &= \Delta_g - 3 - |\Sigma|_g^2 - \frac{7}{8}\left(\iota_{\varepsilon_j}^*|\sigma_{\varepsilon_j}|_{\lambda_{\varepsilon_j}}^2 - |\Sigma|_g^2\right). \end{aligned} \tag{A.11}$$

By the hypotheses of Theorem 1.1 we have that $|\overline{\Sigma}|_{\overline{g}}^2$ is bounded, and thus for some constant C we have $\Phi_j^*|\Sigma|_g^2 = \Phi_j^*(\rho^2|\overline{\Sigma}|_{\overline{g}}^2) \leq C(\hat{\rho}_j y)^2$, which tends to zero uniformly on any precompact set. Furthermore, from (6.11) we have

$$\Phi_j^*\left|\iota_{\varepsilon_j}^*|\sigma_{\varepsilon_j}|_{\lambda_{\varepsilon_j}}^2 - |\Sigma|_g^2\right| \leq C(\hat{\rho}_j y)^2 \varepsilon_j,$$

which also tends to zero. Finally, since $g_j = \Phi_j^* g \to \breve{g}$ we have $\Delta_g \to \Delta_{\breve{g}}$, which establishes the claim.

From (A.8) we have $w_j^{-1}|P_j v_j| \to 0$. Applying Proposition A.3 we obtain a continuous, nonzero function v_* on $\mathbb{H}$ such that $|v_*| \leq C y^\delta$ and $\Delta_{\breve{g}} v_* - 3v_* = 0$. This, however, contradicts Proposition A.5.

Case 3: $q \in \{p_1, p_2\}$. In this case we may assume, without loss of generality, that $q = p_1$ and, by passing to a subsequence if necessary, that the points q_j are contained in the domain $Z(p_1)$ of the preferred background coordinates (θ, ρ) centered about p_1. Let $(\hat{\theta}_j, \hat{\rho}_j)$ be the background coordinates of q_j. Since $q_j \in M \smallsetminus (U_{1,\varepsilon_j} \cup U_{2,\varepsilon_j})$ we have $|(\hat{\theta}_j, \hat{\rho}_j)| \geq \varepsilon_j$. Since $|(\hat{\theta}_j, \hat{\rho}_j)| \to 0$ we may assume that $|\hat{\theta}_j| + \hat{\rho}_j < 1/8$.

Below, we consider three sub-cases, depending on the nature of the convergence $(\hat{\theta}_j, \hat{\rho}_j) \to (0,0)$. In each case we define nested precompact subsets $M_j \subseteq \mathbb{H}$ and maps $T_j \colon M_j \to M_{\varepsilon_j}$. We arrange T_j so that the preferred background coordinate expression for $T_j(x,y)$ satisfies

$$8\varepsilon_j^2 < |T_j(x,y)| < \frac{1}{8}, \tag{A.12}$$

which ensures that $g_j = T_j^*\lambda_{\varepsilon_j}$ is well defined. We then show that (M_j, g_j) forms an exhaustion of $(\mathbb{H}, \breve{g})$, and that $P_j = T_j^* P_{\varepsilon_j}[1] \to \Delta_{\breve{g}} - 3$. Finally, in each case we construct a sequence of functions v_j and weights w_j satisfying the hypotheses of Proposition A.3. We thus obtain a nonzero limiting function v_*, from which we obtain a contradiction via Proposition A.5.

Case 3(a): Both $|\hat{\theta}_j|/\hat{\rho}_j$ and $|(\hat{\theta}_j, \hat{\rho}_j)|/\varepsilon_j$ are bounded above. Thus there exists $C > 1$ such that $|\hat{\theta}_j| \leq C\hat{\rho}_j$ and $|(\hat{\theta}_j, \hat{\rho}_j)| \leq C\varepsilon_j$ for all j. Thus $\hat{\rho}_j \leq |(\hat{\theta}_j, \hat{\rho}_j)| \leq |\hat{\theta}_j| + \hat{\rho}_j \leq 2C\hat{\rho}_j$. Furthermore, since $\varepsilon_j \leq |(\hat{\theta}_j, \hat{\rho}_j)|$ we have $\varepsilon_j \leq 2C\hat{\rho}_j$ and $\hat{\rho}_j \leq |(\hat{\theta}_j, \hat{\rho}_j)| \leq C\varepsilon_j$. Combining these yields

$$\frac{1}{2C}\hat{\rho}_j \leq |(\hat{\theta}_j, \hat{\rho}_j)| \leq 2C\hat{\rho}_j \quad \text{and} \quad \frac{1}{2C}\varepsilon_j \leq \hat{\rho}_j \leq 2C\varepsilon_j. \tag{A.13}$$

Let

$$M_j = \left\{(x, y) \in \mathbb{H} \colon |(x, y)| < \frac{1}{8\varepsilon_j}, y > 8\varepsilon_j\right\}$$

and define $T_j \colon M_j \to M_{\varepsilon_j}$ by setting $T_j(x, y) = \Psi_{\varepsilon_j}(x, y) = (\varepsilon_j x, \varepsilon_j y)$. For each $(x, y) \in M_j$ we have, for sufficiently large j, that

$$8\varepsilon_j^2 < \varepsilon_j y \leq |(\varepsilon_j x, \varepsilon_j y)| < \frac{1}{8}$$

and thus (A.12) holds. Let

$$w = \left(\frac{yF}{2(r + 1/r)}\right)^{\delta}.$$

Thus from (4.3) and (4.6) we have $w = T_j^* \tilde{\rho}_{\varepsilon_j}^{\delta}$. From Proposition 5.6 we have $g_j = \Psi_{\varepsilon_j}^* \lambda_{\varepsilon_j} \to \breve{g}$ on precompact sets of $\mathbb{H}$, and thus (M_j, g_j, w) is an exhaustion of $(\mathbb{H}, \breve{g}, w)$. Furthermore, it follows from (6.12) that $T_j^*|\sigma_{\varepsilon_j}|^2_{\lambda_{\varepsilon_j}} \to 0$ uniformly on precompact subsets of $\mathbb{H}$; hence $P_j = T_j^* \mathcal{P}_{\varepsilon_j}[1] \to \Delta_{\breve{g}} - 3$.

Let $(\hat{x}_j, \hat{y}_j) = (\hat{\theta}_j/\varepsilon_j, \hat{\rho}_j/\varepsilon_j) \in M_j$ so that $T_j(\hat{x}_j, \hat{y}_j) = (\hat{\theta}_j, \hat{\rho}_j)$. From (A.13) the sequence $(\hat{x}_j, \hat{y}_j)$ is bounded, and $\hat{y}_j$ is bounded away from zero. Thus by passing to a subsequence we have $(\hat{x}_j, \hat{y}_j) \to (\hat{x}_*, \hat{y}_*)$ with $\hat{y}_* > 0$.

Set $v_j = T_j^* u_j$. By assumption we have

$$\left(w^{-1}|v_j|\right)\Big|_{(\hat{x}_j, \hat{y}_j)} = \left(\tilde{\rho}_{\varepsilon_j}^{-\delta}|u_j|\right)\Big|_{\pi_{\varepsilon_j}(q_j)} \geq \frac{1}{2}$$

and

$$\sup_{M_j} w^{-1}|v_j| = \sup_{M_j} T_j^*\left(\tilde{\rho}_{\varepsilon_j}^{-\delta}|u_j|\right) \leq 1.$$

Furthermore

$$\sup_{M_j} w^{-1}|P_j v_j| = \sup_{M_j} T_j^*\left(\tilde{\rho}_{\varepsilon_j}^{-\delta}|P_{\varepsilon_j}[1]u_j|\right) \leq \|\mathcal{P}_{\varepsilon_j}[1]u_j\|_{C^0_\delta(M_{\varepsilon_j}; \tilde{\rho}_{\varepsilon_j})} \to 0.$$

Thus the hypotheses of Proposition A.3 are satisfied and there exists a nonzero function v_* on $\mathbb{H}$ with $\Delta_{\breve{g}} v_* - 3v_* = 0$ and

$$|v_*| \leq w = \left(\frac{yF}{2(r+1/r)}\right)^{\delta}.$$

As we are assuming $\delta \geq 0$ this implies $|v_*| \leq (yF)^{\delta}$ and the desired contradiction is obtained from Proposition A.5.

Case 3(b): $|\hat{\theta}_j|/\hat{\rho}_j$ is bounded above, but $|(\hat{\theta}_j, \hat{\rho}_j)|/\varepsilon_j$ is not bounded above. In this case we may, after passing to a subsequence if necessary, suppose that $|\hat{\theta}_j| \leq C\hat{\rho}_j$ for some constant $C > 1$ and that $\hat{r}_j = |(\hat{\theta}_j, \hat{\rho}_j)|/\varepsilon_j \to \infty$. Thus

$$\hat{\rho}_j \leq \varepsilon_j \hat{r}_j = |(\hat{\theta}_j, \hat{\rho}_j)| \leq 2C\hat{\rho}_j \quad \text{and} \quad \frac{\hat{\rho}_j}{\varepsilon_j} \to \infty. \tag{A.14}$$

Let

$$M_j = \left\{(x,y) \in \mathbb{H} \colon |(x,y)| < \frac{1}{16C\hat{\rho}_j}, y > \frac{8\varepsilon_j}{\hat{\rho}_j}\right\}.$$

For sufficiently large j we have $\varepsilon_j/\hat{\rho}_j < 8C$ and thus $M_j \subseteq A_{\varepsilon_j}$. Hence we may define $T_j \colon M_j \to M_{\varepsilon_j}$ by $T_j(x,y) = \Psi_{\varepsilon_j}(\hat{r}_j x, \hat{r}_j y)$. In preferred background coordinates (θ, ρ) about p_1 we have $T_j(x,y) = (\varepsilon_j \hat{r}_j x, \varepsilon_j \hat{r}_j y)$ and thus from (A.14) we see that

$$8\varepsilon_j^2 < 8\varepsilon_j < \hat{\rho}_j y \leq \varepsilon_j \hat{r}_j y \leq |T_j(x,y)| \leq \varepsilon_j \hat{r}_j |(x,y)| < \frac{1}{8};$$

thus (A.12) holds.

Let $(\hat{x}_j, \hat{y}_j) = (\varepsilon_j \hat{r}_j)^{-1}(\hat{\theta}_j, \hat{\rho}_j)$ so that $T_j(\hat{x}_j, \hat{y}_j) = \pi_{\varepsilon_j}(q_j)$. By construction we have $|(\hat{x}_j, \hat{y}_j)| = 1$ and it follows from (A.14) that $\hat{y}_j \geq 1/2C$. Thus we may pass to a subsequence such that $(\hat{x}_j, \hat{y}_j) \to (\hat{x}_*, \hat{y}_*)$ with $|(\hat{x}_*, \hat{y}_*)| = 1$ and $y_* > 0$.

Setting $g_j = T_j^* \lambda_{\varepsilon_j}$ and $w_j = T_j^* \tilde{\rho}_{\varepsilon_j}^{\delta}$, we claim that (M_j, g_j, w_j) forms an exhaustion of $(\mathbb{H}, \breve{g}, (y/2r)^{\delta})$, where as usual we write $r = |(x,y)|$. To see this, first note that Proposition 5.6 implies that $\Psi_{\varepsilon_j}^* \lambda_{\varepsilon_j} \to \breve{g}$ uniformly on precompact sets. Dilation by $\hat{r}_j$ is an isometry of hyperbolic space that preserves unweighted norms; see (2.15). Thus $g_j \to \breve{g}$ uniformly on precompact sets. Next observe from (4.7), while using (4.3) and (4.6), that

$$T_j^* \tilde{\rho}_{\varepsilon_j} = \frac{yF(\hat{r}_j r)}{2\left(r + 1/\hat{r}_j^2 r\right)}.$$

The hypotheses that define this case include $\hat{r}_j \to \infty$, while Proposition 2.2 states that $F(\hat{r}_j r) = 1$ if $\hat{r}_j r > 2$. Thus we see that $T_j^* \tilde{\rho}_{\varepsilon_j} \to y/2r$ uniformly on precompact subsets of $\mathbb{H}$ and the claim is established.

Set $v_j = T_j^* u_j$ and $P_j = T_j^* \mathcal{P}_{\varepsilon_j}[1]$. We now verify that the hypotheses of Proposition A.3 are satisfied. The assumption (A.9) implies that

$$\left(w_j^{-1}|v_j|\right)\Big|_{(\hat{x}_j,\hat{y}_j)} \geq \frac{1}{2}$$

and thus hypothesis (a) holds. As the assumptions (A.7) and (A.8) imply that hypotheses (b) and (c) hold, it remains to establish the convergence of the operators P_j. Proposition 6.8 implies that $T_j^*|\sigma_{\varepsilon_j}|^2_{\lambda_{\varepsilon_j}} \to 0$ uniformly on precompact subsets of $\mathbb{H}$. Thus the convergence $g_j \to \breve{g}$ implies that $P_j \to \Delta_{\breve{g}} - 3$.

We now invoke Proposition A.3 to conclude that there exists a nonzero continuous function v_* on $\mathbb{H}$ such that $|v_*| \leq C(y/r)^\delta \leq Cy^0$ and $\Delta_{\breve{g}} v_* - 3v_* = 0$. Consequently, Proposition A.5 yields a contradiction.

Case 3(c): $|\hat{\theta}_j|/\hat{\rho}_j$ is not bounded above. Passing to a subsequence we may assume that $|\hat{\theta}_j|/\hat{\rho}_j \to \infty$. We may further assume that $\hat{\rho}_j/|\hat{\theta}_j| < 1/2$; when combined with the fact that $\varepsilon_j \leq |(\hat{\theta}_j, \hat{\rho}_j)|$ we find that $|\hat{\theta}_j| \geq \varepsilon_j/2$.

Let $(\hat{x}_j, \hat{y}_j) = \varepsilon_j^{-1}(\hat{\theta}_j, \hat{\rho}_j)$ so that $\Psi_{\varepsilon_j}(\hat{x}_j, \hat{y}_j) = \pi_{\varepsilon_j}(q_j)$. Set

$$M_j = \left\{ (x, y) \in \mathbb{H} \colon |(x, y)| < \frac{|\hat{\theta}_j|}{2\hat{\rho}_j}, y > \varepsilon_j \right\}.$$

For $(x, y) \in M_j$ we may use $|\hat{\theta}_j| < 1/8$ to conclude that

$$|(\hat{x}_j + \hat{y}_j x, \hat{y}_j y)| \leq \frac{|\hat{\theta}_j|}{\varepsilon_j} + \frac{\hat{\rho}_j}{\varepsilon_j}|(x, y)| < 2\frac{|\hat{\theta}_j|}{\varepsilon_j} < \frac{1}{8\varepsilon_j} \tag{A.15}$$

and use $\hat{\rho}_j|x| < |\hat{\theta}_j|/2$ to obtain

$$|(\hat{x}_j + \hat{y}_j x, \hat{y}_j y)| \geq |\hat{x}_j + \hat{y}_j x| = \frac{1}{\varepsilon_j}|\hat{\theta}_j + \hat{\rho}_j x| \geq \frac{|\hat{\theta}_j|}{2\varepsilon_j} > \frac{1}{4} > 8\varepsilon_j. \tag{A.16}$$

Thus the map $\Phi_j \colon M_j \to A_{8\varepsilon_j}$ given by $\Phi_j(x, y) = (\hat{x}_j + \hat{y}_j x, \hat{y}_j y)$ is well defined, and the map $T_j = \Psi_{\varepsilon_j} \circ \Phi_j \colon M_j \to M_{\varepsilon_j}$ satisfies (A.12). In preferred background coordinates (θ, ρ) about p_1 we have $T_j(x, y) = (\varepsilon_j \hat{x}_j + \varepsilon_j \hat{y}_j x, \varepsilon_j \hat{y}_j y) = (\hat{\theta}_j + \hat{\rho}_j x, \hat{\rho}_j y)$ and thus $T_j(0, 1) = \pi_{\varepsilon_j}(q_j)$.

We now estimate $T_j^* \tilde{\rho}_{\varepsilon_j}$ using (4.8), which implies that

$$T_j^* \tilde{\rho}_{\varepsilon_j} = \frac{\hat{y}_j y F(|\Phi_j(x, y)|)}{2\left(|\Phi_j(x, y)| + \frac{1}{|\Phi_j(x,y)|}\right)}.$$

The estimate (A.16) implies that $|\Phi_j(x, y)| > 1/4$ and thus from Proposition 2.2 we have $F(|\Phi_j(x, y)|)$ uniformly bounded above and below. The lower bound $|\Phi_j(x, y)| > 1/4$ furthermore implies that

$$|\Phi_j(x, y)| \leq |\Phi_j(x, y)| + \frac{1}{|\Phi_j(x, y)|} \leq 17|\Phi_j(x, y)|.$$

As (A.15) and (A.16) imply that

$$\frac{|\hat{\theta}_j|}{2\varepsilon_j} \leq |\Phi_j(x,y)| \leq 2\frac{|\hat{\theta}_j|}{\varepsilon_j},$$

and as $\hat{y}_j = \hat{\rho}_j/\varepsilon_j$, we find that

$$\frac{1}{C}\frac{\hat{\rho}_j}{|\hat{\theta}_j|}y \leq T_j^*\tilde{\rho}_{\varepsilon_j} \leq C\frac{\hat{\rho}_j}{|\hat{\theta}_j|}y \tag{A.17}$$

for some constant C.

Let $g_j = T_j^*\lambda_{\varepsilon_j}$ and set $w_j = y^\delta$. From Proposition 5.6 we see that $g_j \to \breve{g}$ on precompact sets and thus (M_j, g_j, w_j) forms an exhaustion of $(\mathbb{H}, \breve{g}, y^\delta)$. We seek to apply Proposition A.3 to the functions

$$v_j = \left(\frac{\hat{\rho}_j}{|\hat{\theta}_j|}\right)^{-\delta} T_j^* u_j$$

and operators $P_j = T_j^*\mathcal{P}_{\varepsilon_j}[1]$. Proposition 6.8 implies that $T_j^*|\sigma_{\varepsilon_j}|^2_{\lambda_{\varepsilon_j}} \to 0$ uniformly on precompact subsets of $\mathbb{H}$. Thus the convergence $g_j \to \breve{g}$ implies that $P_j \to \Delta_{\breve{g}} - 3$. Thus by applying (A.17) to (A.7), (A.8), and (A.9) we have that the hypotheses of Proposition A.3 are satisfied. The result is a nonzero function v_* satisfying both $|v_*| \leq Cy^\delta$ and $\Delta_{\breve{g}}v - 3v = 0$. This, however, is in contradiction to Proposition A.5.

With all cases exhausted, the proof of Lemma A.2 is complete.

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