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Limit law for the cover time of a random walk on a binary tree

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Abstract. Let $\mathcal{T}_n$ denote the binary tree of depth n augmented by an extra edge connected to its root. Let $\mathcal{C}_n$ denote the cover time of $\mathcal{T}_n$ by simple random walk. We prove that the sequence of random variables $\sqrt{\mathcal{C}_n 2^{-(n+1)}} - m_n$, where m_n is an explicit constant, converges in distribution as $n \to \infty$, and identify the limit.

Résumé. Soit $\mathcal{T}_n$ l'arbre binaire de profondeur n, augmenté par une arête attachée à la racine. Soit $\mathcal{C}_n$ le temps de recouvrement de $\mathcal{T}_n$ par une marche aléatoire simple. Nous montrons que la suite de variables aléatoires $\sqrt{\mathcal{C}_n 2^{-(n+1)}} - m_n$, avec m_n une constante explicite, converge quand $n \to \infty$. Nous identifions la limite.

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1. Introduction

We introduce in this section notation and our main results, provide background, and give a road map for the rest of the paper.

1.1. *Notation and main result*

Let $\mathcal{T}_n$ denote the binary tree of depth n, whose only vertex of degree 2 is attached to an extra vertex ρ, called the ROOT. The COVER TIME $\mathcal{C}_n$ of $\mathcal{T}_n$ is the number of steps of a (discrete time) simple random walk started at ρ, till visiting all vertices of $\mathcal{T}_n$. We write SRW for such a random walk. The main result of this paper is the following theorem, which gives convergence in law of a (normalized) version of the cover time $\mathcal{C}_n$.

Theorem 1.1. *Let $\mathcal{C}'_n := 2^{-(n+1)}\mathcal{C}_n$, and set*

$$m_n := \rho_n n, \qquad \rho_n := c_* - \frac{\log n}{c_* n}, \quad c_* = \sqrt{2\log 2}. \tag{1.1}$$

There exist a random variable $X'_\infty > 0$ and $\alpha_* > 0$ finite, so that, for any fixed $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\big(\sqrt{\mathcal{C}'_n} - m_n \le y\big) = \mathbb{E}\big(\exp\{-\alpha_* X'_\infty \mathrm{e}^{-c_* y}\}\big) := \mathbb{P}\big(Y'_\infty \le y\big). \tag{1.2}$$

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That is, the normalized cover time $\sqrt{\mathcal{C}'_n} - m_n$ converges in distribution to Y'_∞, a standard Gumbel random variable shifted by $\log(\alpha_* X'_\infty)$ then scaled by $1/c_*$. See Lemma 1.4 for a description of the random variable X'_∞ in terms of the limit of a derivative martingale.

As is often the case, most of the work in proving a statement such as Theorem 1.1 involves the control of certain excursion counts. We now introduce notation in order to describe these. Let V_j denote the set of vertices of $\mathcal{T}_n$ at level j, with $V_{-1} = \{\rho\}$. For each $v \in V_n$ let $v(j)$ denote the ancestor of v at level $j \le n$, i.e. the unique vertex in V_j on the geodesic connecting v and ρ. In particular, $v(-1)$ is the root ρ.

Next, for each $v \in V_n$ let

$$\begin{aligned} T^s_{v,j} = \#\bigl\{&\text{excursions from } v(j-1) \text{ to } v(j) \text{ made by the SRW on } \mathcal{T}_n \\ &\text{prior to completing its first } [s] \text{ excursions from the root } \rho\bigr\}. \end{aligned} \tag{1.3}$$

Setting

$$t^*_{v,n} = \inf\bigl\{s \in \mathbb{Z}_+ : T^s_{v,n} \neq 0\bigr\}, \tag{1.4}$$

for the number of excursions from the root by the SRW till reaching v, we consider the corresponding excursion cover time of V_n,

$$t^*_n := \sup_{v \in V_n} \bigl\{t^*_{v,n}\bigr\}. \tag{1.5}$$

Our main tool in proving Theorem 1.1 is a generalization (see Proposition 2.2 below), of the following theorem concerning t^*_n.

Theorem 1.2. *With notation as in Theorem 1.1,*

$$\sqrt{2t^*_n} - m_n \overset{\text{dist}}{\Longrightarrow} Y_\infty \quad \text{as } n \to \infty, \tag{1.6}$$

where $Y_\infty := Y'_\infty - \bar{g}_\infty$ for a standard Gaussian random variable $\bar{g}_\infty$, independent of Y'_∞. Alternatively, for some random variable $X_\infty > 0$,

$$\mathbb{P}(Y_\infty \le y) := \mathbb{E}\bigl(\exp\bigl\{-\alpha_* X_\infty e^{-c_* y}\bigr\}\bigr). \tag{1.7}$$

The main challenge in proving Theorem 1.2 (or Proposition 2.2), is in obtaining the following key sharp right tail estimates for the excursion cover times.

Theorem 1.3. *There exists a finite $\alpha_* > 0$ such that*

$$\lim_{z\to\infty} \limsup_{n\to\infty} \bigl|z^{-1} e^{c_* z} \mathbb{P}\bigl(\sqrt{2t^*_n} - m_n > z\bigr) - \alpha_*\bigr| = 0. \tag{1.8}$$

Tail estimates at such fine resolution were established, for example, en-route to the convergence in law of the maximum of BRW (see [14]). However, what significantly complicates our analysis, is having here a more general branching process where the motion of individual particles follows a Markov chain (and not merely a sum of i.i.d.-s).

To describe X_∞ and X'_∞, let $\{g_u, u \in \mathcal{T}_\infty \backslash \{\rho\}\}$ be the standard Gaussian branching random walk (BRW), on the infinite binary tree $\mathcal{T}_\infty \backslash \{\rho\}$. That is, denoting by 0 the unique vertex at V_0 and placing i.i.d. standard normal weights on the edges of $\mathcal{T}_\infty$, we write g_u for the sum of the weights along the geodesic connecting 0 to u. We further consider the empirically centered $g'_u := g_u - \bar{g}_{|u|}$, where

$$\bar{g}_k = 2^{-k} \sum_{u' \in V_k} g_{u'}, \quad k \in \mathbb{Z}_+, \tag{1.9}$$

denotes the average of the BRW at level k, and set

$$X_k = \sum_{u \in V_k} (c_* k + g_u) e^{-c_*(c_* k + g_u)}, \qquad X'_k = \sum_{u \in V_k} \bigl(c_* k + g'_u\bigr) e^{-c_*(c_* k + g'_u)}. \tag{1.10}$$

It is not hard to verify that $\{X_k\}$ is a martingale, referred to as the DERIVATIVE MARTINGALE. We then have that

Lemma 1.4. *X_k, X'_k and $\bar{g}_k$ converge a.s. to positive, finite limits X_∞, X'_∞ and a standard Gaussian variable $\bar{g}_\infty$, independent of X'_∞, such that*

$$X_\infty = X'_\infty e^{-c_*\bar{g}_\infty}. \tag{1.11}$$

The convergence of X_k to X_∞ is well known, see e.g. [3,10] (and, for its first occurrence in terms of limits in branching processes, [25]), and building on it, we easily deduce the corresponding convergence for X'_k.

1.2. *Background and related results*

Theorem 1.1 is closely related to the recent paper [16], which deals with continuous time SRW, and we wish to acknowledge the priority of their work. The proofs however are different – while [16] builds heavily on the isomorphism theorem of [22] to relate directly the occupation time on the tree to the Gaussian free field on the tree which is nothing but the BRW described above Lemma 1.4, our proof is a refinement of [8, Theorem 1.3], where the tightness of the LHS of (1.6) is proved. Our proof, which is based on the strategy for proving convergence in law of the maximum of branching random walk described in [14], was obtained independently of [16], except that in proving that Theorem 1.2 implies Theorem 1.1, we do borrow some ideas from [16]. As motivation to our work, we note that estimates from [8] were instrumental in obtaining the tightness of the (centered) cover time of the two dimensional sphere by ϵ-blowup of Brownian motion, see [9]. Many of the ideas and computations we do in proving Theorem 1.3 play an important role in a forthcoming work where we plan to upgrade the tightness result of [9] to a convergence in law.

We next put our work in context. The study of the cover time of graphs by SRW has a long history. Early bounds appear in [27], and a general result showing that the cover time is concentrated as soon as it is much longer than the maximal hitting time appears in [4]. A modern general perspective linking the cover time of graphs to Gaussian processes appears in [20], and was refined to sharp concentration in [19] (for many graphs including trees) and [31] (for general graphs). See also [26] for a different perspective on [20]. For the cover time of trees, an exact first order asymptotic appears in [5]. The tightness of $\sqrt{C'_n}$ around an implicit constant was derived by analytic methods in [15], and, following the identification of the logarithmic correction in m_n [21], its $O(1)$ identification appears in [8].

We note that the evaluation of the cover time is but one of many natural questions concerning the process of points with a-typical (local) occupation time, and quite a bit of work has been devoted to this topic. We do not elaborate here and refer the reader to [2,18,29]. Particularly relevant to this paper is the recent [1].

It has been recognized for quite some time that the study of the cover time of two dimensional manifolds by Brownian motion (and of the cover time of two dimensional lattices by SRW) is related to a hierarchical structure similar to that appearing in the study of the cover time for trees, see e.g. [17] and, for a recent perspective, [28]. A similar hierarchical structure also appears in the study of extremes of the critical Gaussian free field, and in other logarithmically correlated fields appearing e.g. in the study of random matrices. We do not discuss that literature and refer instead to recent surveys offering different perspectives [6,11,12,24,30].

1.3. *Structure of the paper*

In contrast with [16] the key to our proof of Theorem 1.2 is the sharp right tail of Theorem 1.3. Indeed, after quickly dispensing of Lemma 1.4, in Section 2 we obtain Theorem 1.2 out of Theorem 1.3, by adapting the approach of [14] for the convergence in law of the maximum of BRW. In the short Section 3, which is the only part of this work that parallels the derivation of [16], we deduce Theorem 1.1 out of Theorem 1.2.

As mentioned before, the bulk of this paper is devoted to the proof of Theorem 1.3, which we establish in Section 4 by a refinement of the approach used in deriving [8, Theorem 1.3]. In doing so, we defer the a-priori bounds we need on certain barrier events, which might be of some independent interest, to Section 5, where we derive these bounds by refining estimates from [8]. The proof of the main contribution to the tail estimate of Theorem 1.3, as stated in Proposition 4.3, is further deferred to Section 6. There, utilizing the close relation between our Markov chain and the 0-dimensional Bessel process, we get sharper barrier estimates, now up to $(1 + o(1))$ factor of the relevant probabilities.

2. From tail to limit: Lemma 1.4 and Theorem 1.2

We start by proving the elementary Lemma 1.4, denoting throughout the last common ancestor of $u, u' \in \mathcal{T}_\infty$ by $w = u \wedge u'$. Namely, $w = u(|w|)$ for $|w| = \max\{j \geq 0, u(j) = u'(j)\}$.

Proof of Lemma 1.4. The BRW $\{g_u; u \in \mathcal{T}_k\}$ of Lemma 1.4 is the centered Gaussian random vector having

$$\mathrm{Cov}(g_u, g_{u'}) = \left|u \wedge u'\right|. \tag{2.1}$$

Further, the average of the BRW weights on the edges of $\mathcal{T}_\infty$ between levels $(k-1)$ and k, is precisely $\Delta g_k := \bar{g}_k - \bar{g}_{k-1}$ for $\bar{g}_k$ of (1.9). With $(\Delta g_k, k \geq 1)$ independent centered Gaussian random variables with $\mathrm{Var}(\Delta g_k) = 2^{-k}$, we have that $\bar{g}_k$ converges a.s. to the standard Gaussian $\bar{g}_\infty := \sum_k \Delta g_k$. Next, recall the existence of $w_k \to \infty$ such that, a.s.,

$$A_k := \left\{c_* k + g_u \in w_k + (0, 2c_* k), \forall u \in V_k\right\} \quad \text{occurs for all } k \text{ large}, \tag{2.2}$$

see e.g. [23, (1.8)]. For X_k of (1.10), it follows from [3] that $X_k \overset{\text{a.s.}}{\to} X_\infty \in (0, \infty)$, while

$$X_k > w_k \widetilde{X}_k \quad \text{on } A_k, \text{for } \widetilde{X}_k := \sum_{u \in V_k} e^{-c_*(c_* k + g_u)}. \tag{2.3}$$

Thus, $\widetilde{X}_k \overset{\text{a.s.}}{\to} 0$ as $k \to \infty$. From the two expressions in (1.10) we have that

$$X'_k = (X_k - \bar{g}_k \widetilde{X}_k) e^{c_* \bar{g}_k}$$

which thereby converges a.s. to $X'_\infty = X_\infty e^{c_* \bar{g}_\infty}$ as claimed in (1.11). Finally, from (2.1) we deduce that for any $u \in V_k$, $k \geq 0$,

$$\mathrm{Cov}(g_u, \bar{g}_k) = 2^{-k} \sum_{u' \in V_k} \left|u \wedge u'\right| = \sum_{j=1}^{k} (j-1) 2^{-j} + k 2^{-k} = 1 - 2^{-k}. \tag{2.4}$$

This covariance is constant over $u \in V_k$, hence $\mathrm{Cov}(g'_u, \bar{g}_{|u|}) = 0$ for $g'_u := g_u - \bar{g}_{|u|}$, implying the independence of $\bar{g}_k$ and $\{g'_u, u \in V_k\}$. The latter variables are further independent of the BRW edge weights outside $\mathcal{T}_k$, hence of $\bar{g}_\infty$. Thus, the random variable X'_∞, which is measurable on $\sigma(g'_u, u \in \mathcal{T}_\infty)$, must also be independent of $\bar{g}_\infty$. $\square$

We next normalize the counts $T^s_{u,j}$ of (1.3) and define

$$\widehat{T}_u(s) := \frac{T^s_{u,|u|} - s}{\sqrt{2s}}, \qquad \widehat{S}_k(s) := 2^{-k} \sum_{u \in V_k} \widehat{T}_u(s), \tag{2.5}$$

and get from the CLT for sums of i.i.d. the following relation with the BRW.

Lemma 2.1. *For fixed k and the* BRW $\{g_u; u \in \mathcal{T}_k\}$ *of Lemma* 1.4, *we have*

$$\left\{\widehat{T}_u(s), u \in \mathcal{T}_k\right\} \underset{s \to \infty}{\overset{\text{dist}}{\Longrightarrow}} \{g_u, u \in \mathcal{T}_k\}, \tag{2.6}$$

$$\left\{\widehat{T}_u(s) - \widehat{S}_k(s), u \in V_k\right\} \underset{s \to \infty}{\overset{\text{dist}}{\Longrightarrow}} \left\{g'_u, u \in V_k\right\}. \tag{2.7}$$

Proof. The consecutive excursions from ρ by the SRW on $\mathcal{T}_n$ are i.i.d. Hence, $s \mapsto \{T^s_{u,|u|}, u \in \mathcal{T}_k\}$ is an $\mathbb{R}^d$-valued random walk (with d the finite size of $\mathcal{T}_k$). Further, projecting the SRW on $\mathcal{T}_n$ to the geodesic from u to ρ, yields a symmetric SRW on $\{-1, 0, \ldots, |u|\}$. Thus, denoting by T_j the number of excursions from $u(j-1)$ to $u \in V_j$ during a single excursion from ρ, we have that $\mathbb{P}(T_j \geq 1) = \mathsf{p}_j := 1/(j+1)$ (for reaching u before returning to ρ), and T_j conditional on $T_j \geq 1$, follows a geometric law of success probability $\mathbb{P}(T_j = 1 | T_j \geq 1) = \mathsf{p}_j$. Consequently, for any $j \in [0, k]$,

$$\mathbb{E}(T_j) = 1, \qquad \mathrm{Var}(T_j) = \mathbb{E}\left[T_j(T_j - 1)\right] = \frac{2(1 - \mathsf{p}_j)}{\mathsf{p}_j} = 2j. \tag{2.8}$$

Note that $T^1_{u,|u|}$ and $T^1_{u',|u'|}$ are independent, conditionally on $T^1_{w,|w|}$, for $w = u \wedge u'$, each having the conditional mean $T^1_{w,|w|}$. We thus see that for any $u, u' \in \mathcal{T}_k$, in view of (2.8),

$$\mathrm{Cov}\left(T^1_{u,|u|}, T^1_{u',|u'|}\right) = \mathrm{Var}(T_{|u \wedge u'|}) = 2\left|u \wedge u'\right|. \tag{2.9}$$

Comparing with (2.1), the i.i.d. increments of our $\mathbb{R}^d$-valued random walk have the mean vector **1** and covariance matrix which is twice that of the BRW, with (2.6) and (2.7) as immediate consequences of the multivariate CLT. □

Using throughout the notation

$$s_{n,y} := (m_n + y)^2/2 \tag{2.10}$$

for m_n of (1.1), we have that

$$\left\{\sqrt{2t_n^*} - m_n \le y\right\} = \left\{t_n^* \le s_{n,y}\right\}.$$

In view of Lemma 2.1, we thus see that Theorem 1.2 is an immediate consequence (for non-random $\tau_k(s) = s$), of (2.12) in the following proposition. (The additional statement employing (2.13) is utilized in the proof of Theorem 1.1.)

For any $k \ge 1$, let $\mathcal{F}_k$ denote the σ-algebra generated by the collection $\{T^\ell_{u,j}, \ell \ge 0, j \le k, u \in V_k\}$ for a SRW on $\mathcal{T}_n$. A moment's thought reveals that $\mathcal{F}_k$ does not depend on $n \ge k$.

Proposition 2.2. *Assuming Theorem 1.3, if $\mathcal{F}_k$-measurable $\{\tau_k(s), s \ge 0\}$ are such that*

$$\widehat{\tau}_k(s) := \left(\frac{\tau_k(s) - s}{\sqrt{2s}}\right) \xrightarrow[s\to\infty]{p} 0, \tag{2.11}$$

then for any fixed $y \in \mathbb{R}$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \left|\mathbb{P}\left(t_n^* \le \tau_k(s_{n,y})\right) - \mathbb{P}(Y_\infty \le y)\right| = 0. \tag{2.12}$$

Further, replacing (2.11) by

$$\widehat{\tau}_k(s) + \widehat{S}_k(s) \xrightarrow[s\to\infty]{p} 0, \tag{2.13}$$

leads to (2.12) holding with Y'_∞ of (1.2) instead of Y_∞.

Proof. For a possibly random, $\mathcal{F}_k$-measurable τ, we set

$$\widehat{T}_u(\tau; s) := \frac{T^\tau_{u,|u|} - s}{\sqrt{2s}}, \tag{2.14}$$

in analogy to $\widehat{T}_u(s; s) = \widehat{T}_u(s)$ of (2.5). With $(T^\tau_{u,|u|} - \tau)/\sqrt{2}$ the sum of τ i.i.d. standardized variables, by Donsker's invariance principle, as $s \to \infty$, the path $\theta \mapsto W^{(s)}(\theta) := \widehat{T}_u(\theta s; s) - ((\theta s - s)/\sqrt{2s})$, converges in law to a Brownian motion. Thus, if $\frac{\tau_k(s)}{s} \xrightarrow{p} 1$, then $W^{(s)}(\frac{\tau_k(s)}{s}) - W^{(s)}(1) \xrightarrow{p} 0$. That is,

$$\widehat{T}_u\left(\tau_k(s); s\right) - \widehat{T}_u(s; s) - \widehat{\tau}_k(s) \xrightarrow[s\to\infty]{p} 0, \quad \forall u \in \mathcal{T}_k. \tag{2.15}$$

Further,

$$f_s(x) := \sqrt{2(s + \sqrt{2s}x)} - \sqrt{2s} \xrightarrow[s\to\infty]{} x, \tag{2.16}$$

uniformly over bounded x. Hence, setting

$$\widetilde{T}_u(\tau; s) := \sqrt{2T^\tau_{u,|u|}} - \sqrt{2s} = f_s\left(\widehat{T}_u(\tau; s)\right), \tag{2.17}$$

upon combining Lemma 2.1 and (2.15), we deduce from (2.11) that

$$\left\{\widetilde{T}_u\left(\tau_k(s); s\right), u \in V_k\right\} \overset{\text{dist}}{\underset{s\to\infty}{\Longrightarrow}} \{g_u, u \in V_k\}, \tag{2.18}$$

whereas under (2.13) we merely replace g_u by g'_u on the RHS. Proceeding under the assumption (2.11), fix $y \in \mathbb{R}$ and an integer $k \ge 1$, setting

$$z_u^{(n)} := \sqrt{2T^{\tau_k(s_{n,y})}_{u,k}} - m_{n-k}, \qquad z_u^{(\infty)} := c_* k + g_u + y, \quad \forall u \in V_k, \tag{2.19}$$

with c_* as in (1.1). For fixed y and k, we have, using (2.10), that

$$\sqrt{2s_{n,y}} - m_{n-k} - (c_* k + y) = m_n - m_{n-k} - c_* k = \frac{1}{c_*}\log\left(1-\frac{k}{n}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence from (2.18) at $s = s_{n,y}$ it follows that

$$\{z_u^{(n)}, u \in V_k\} \underset{n\to\infty}{\overset{\text{dist}}{\Longrightarrow}} \{z_u^{(\infty)}, u \in V_k\}. \tag{2.20}$$

In particular, for X_k of (1.10) and $\widetilde{X}_k$ of (2.3) we have that

$$X_k^{(n,y)} := \sum_{u\in V_k} z_u^{(n)} e^{-c_* z_u^{(n)}} \underset{n\to\infty}{\overset{\text{dist}}{\Longrightarrow}} (X_k + y\widetilde{X}_k)e^{-c_* y}. \tag{2.21}$$

For any fixed $y \in \mathbb{R}$, we have by (2.2) and (2.20) that

$$\lim_{k\to\infty} \varliminf_{n\to\infty} \mathbb{P}(A_k^{(n)}) = 1, \quad A_k^{(n)} = \{z_u^{(n)} \in w_k + y + (0, 2c_* k), \forall u \in V_k\}. \tag{2.22}$$

Recalling that $\widetilde{X}_k \overset{\text{a.s.}}{\to} 0$ (see the line following (2.3)), and the definition of Y_∞ from (1.7), we have in view of (2.21) that for any $\alpha_k \to \alpha_*$

$$\begin{aligned}
\mathbb{P}(Y_\infty \le y) &= \lim_{k\to\infty} \mathbb{E}\big[\mathbf{1}_{A_k} \exp\{-\alpha_k (X_k + y\widetilde{X}_k)\mathrm{e}^{-c_* y}\}\big] \\
&= \lim_{k\to\infty} \overline{\underline{\lim}}_{n\to\infty} \mathbb{E}\big[\mathbf{1}_{A_k^{(n)}} \exp\{-\alpha_k X_k^{(n,y)}\}\big] \\
&= \lim_{k\to\infty} \overline{\underline{\lim}}_{n\to\infty} \mathbb{E}\Big[\mathbf{1}_{A_k^{(n)}} \prod_{u\in V_k} \big(1 - \alpha_k z_u^{(n)} e^{-c_* z_u^{(n)}}\big)\Big],
\end{aligned} \tag{2.23}$$

where $\overline{\underline{\lim}}_{n\to\infty} f(n)$ stands for bounds given by both $\limsup_{n\to\infty} f(n)$ and $\liminf_{n\to\infty} f(n)$, and in the last equality of (2.23) we relied on having

$$\delta_k := \sup_n \mathbf{1}_{A_k^{(n)}} \sup_{u\in V_k} \{\alpha_k z_u^{(n)} e^{-c_* z_u^{(n)}}\} \underset{k\to\infty}{\longrightarrow} 0,$$

as well as $e^{-a} \ge 1 - a \ge e^{-a(1+\delta)}$ for $a \in [0, \delta \wedge 1/2]$. For $u \in V_k$ let $V_n^u = \{v \in V_n : v(k) = u\}$ denote the leaves of the binary sub-tree of $\mathcal{T}_n$ of depth $n-k$, emanating from u, with $u(k-1)$ acting as its (extra) root. The event $\{t_n^* \le \tau\}$ of the SRW reaching all of V_n within its first τ excursions from ρ is the intersection over $u \in V_k$ of the events of reaching all of V_n^u within the first $T_{u,k}^\tau$ excursions of the SRW between $u(k-1)$ and u. By the Markov property, for $\mathcal{F}_k$-measurable τ, conditionally on $\mathcal{F}_k$ the latter events are mutually independent, of conditional probabilities $\bar{\gamma}_{n-k}(T_{u,k}^\tau)$ for $u \in V_k$ and $\bar{\gamma}_n(s) := \mathbb{P}(t_n^* \le s)$. Consequently, for $\tau = \tau_k(s_{n,y})$ we get that

$$\mathbb{P}(t_n^* \le \tau \mid \mathcal{F}_k) = \prod_{u\in V_k} \bar{\gamma}_{n-k}(T_{u,k}^\tau) = \prod_{u\in V_k} \big(1 - \gamma_{n-k}(z_u^{(n)})\big), \tag{2.24}$$

for $\gamma_n(z) := \mathbb{P}(\sqrt{2t_n^*} - m_n > z)$ and $z_u^{(n)}$ of (2.19). Theorem 1.3 and the monotonicity of $z \mapsto \gamma_n(z)$ yield that for some $n_k < \infty$ and $\alpha_k^{(\pm)} \to \alpha_*$,

$$\alpha_k^{(-)} z e^{-c_* z} \le \gamma_n(z) \le \alpha_k^{(+)} z e^{-c_* z} \quad \forall n \ge n_k, \forall z \in w_k + y + [0, 2c_* k]. \tag{2.25}$$

Under the event $A_k^{(n)}$, which is measurable on $\mathcal{F}_k$, the latter bounds apply for all $z = z_u^{(n)}$. Hence, we get from (2.24) that

$$\begin{aligned}
\varliminf_{n\to\infty} \mathbb{E}\Big[\mathbf{1}_{A_k^{(n)}} \prod_{u\in V_k} \big(1 - \alpha_k^{(+)} z_u^{(n)} e^{-c_* z_u^{(n)}}\big)\Big] &\le \varliminf_{n\to\infty} \mathbb{P}\big(t_n^* \le \tau_k(s_{n,y}); A_k^{(n)}\big) \le \varlimsup_{n\to\infty} \mathbb{P}\big(t_n^* \le \tau_k(s_{n,y}); A_k^{(n)}\big) \\
&\le \varlimsup_{n\to\infty} \mathbb{E}\Big[\mathbf{1}_{A_k^{(n)}} \prod_{u\in V_k} \big(1 - \alpha_k^{(-)} z_u^{(n)} e^{-c_* z_u^{(n)}}\big)\Big].
\end{aligned}$$

We now establish (2.12), by taking $k \to \infty$ while utilizing (2.22) and (2.23).

The same argument applies under (2.13), now replacing g_u by g'_u in (2.19), thereby changing X_k, $\widetilde{X}_k$ and Y_∞ in (2.21) and (2.23), to X'_k, $\widetilde{X}'_k$ and Y'_∞. □

3. Excursion counts to real time: From Proposition 2.2 to Theorem 1.1

Theorem 1.1 amounts to showing that for any fixed $y \in \mathbb{R}$ and $\epsilon > 0$,

$$\overline{\lim_{n\to\infty}}\, \mathbb{P}\big(\mathcal{C}'_n \le 2s_{n,y-2\epsilon}\big) \le \mathbb{P}\big(Y'_\infty \le y\big) \le \underline{\lim_{n\to\infty}}\, \mathbb{P}\big(\mathcal{C}'_n \le 2s_{n,y+2\epsilon}\big), \tag{3.1}$$

where throughout $s_{n,y} := (m_n + y)^2/2$, as in (2.10). To this end, let

$$R_n^s := 2^{-n} \sum_{u\in\mathcal{T}_n} T^s_{u,|u|}. \tag{3.2}$$

The SRW on $\mathcal{T}_n$ makes $2^{(n+1)} R_n^s$ steps during its first s excursions from the root to itself. Thus, $\{t_n^* \le \tau\} = \{\mathcal{C}'_n \le R_n^\tau\}$, so for any random t, τ,

$$\mathbb{P}\big(t_n^* \le \tau\big) - \mathbb{P}\big(R_n^\tau > 2t\big) \le \mathbb{P}\big(\mathcal{C}'_n \le 2t\big) \le \mathbb{P}\big(t_n^* \le \tau\big) + \mathbb{P}\big(R_n^\tau < 2t\big). \tag{3.3}$$

Considering (3.3) at $t = s_{n,y\pm 2\epsilon}$, the insufficient concentration of R_n^τ at the non-random $\tau = s_{n,y}$ rules out establishing (3.1) directly from Theorem 1.2. We thus follow the approach of [16, Section 9], in employing instead (3.3) for $\tau = \tau_k(s_{n,y})$ and the $\mathcal{F}_k$-measurable

$$\tau_k(s) := \inf\big\{\ell \in \mathbb{Z}_+ \mid S_k^\ell \ge s\big\}, \quad S_k^\ell := 2^{-k} \sum_{u\in V_k} T^\ell_{u,k}. \tag{3.4}$$

Recall that $\mathbb{E}[S_k^1] = 1$ (see (2.8)), while setting $\bar\sigma_k^2 := \mathrm{Var}(\bar g_k) = 1 - 2^{-k}$ (see (2.4)), and comparing (2.1) to (2.9), we arrive at $\mathrm{Var}(S_k^1) = 2\bar\sigma_k^2$. Hence, Donsker's invariance principle yields a coupling between the piece-wise linear interpolation $t \mapsto \widehat{S}_{s,k}(t)$ of $\{(S_k^t - t)/\sqrt{2s}; t \in \mathbb{Z}_+\}$, and a standard Brownian motion $\{W_\theta\}$, such that

$$\sup_{\theta\in[0,2]} \big|\widehat{S}_{s,k}(\theta s) - \bar\sigma_k W_\theta\big| \xrightarrow[s\to\infty]{p} 0. \tag{3.5}$$

From (3.4) we see that $S_k^{\tau_k(s)} - s \ge 0$ is at most 2^{-k} times the total number of excursions from V_{k-1} to V_k made by the SRW started at some $v \in V_k$, before hitting the root, plus 1. The latter has exactly the law of S_k^1 given $S_k^1 > 0$. Thus,

$$c_k := \sup_{s\ge 0}\big\{\mathbb{E}\big[S_k^{\tau_k(s)} - s\big]\big\} \le \frac{\mathbb{E}[S_k^1]}{\mathbb{P}(S_k^1 > 0)} < \infty, \tag{3.6}$$

and for $\widehat{\tau}_k(s)$ defined as in (2.11), one has when $s \to \infty$, that

$$\widehat{S}_{s,k}\big(\tau_k(s)\big) + \widehat{\tau}_k(s) = \frac{S_k^{\tau_k(s)} - s}{\sqrt{2s}} \xrightarrow{p} 0, \qquad \theta_s := \frac{\tau_k(s)}{s} \xrightarrow{p} 1. \tag{3.7}$$

In particular, considering (3.5) at θ_s, by the continuity of $\theta \mapsto W_\theta$

$$\widehat{S}_{s,k}(\theta_s s) = \bar\sigma_k W_{\theta_s} + o_p(1) = \bar\sigma_k W_1 + o_p(1) = \widehat{S}_{s,k}(s) + o_p(1).$$

Since $\widehat{S}_{s,k}(s) = \widehat{S}_k(s)$ of (2.5), we conclude that $\{\tau_k(s), s \ge 0\}$ of (3.4) satisfy (2.13), and with $|2s_{n,y\pm 2\epsilon} - 2s_{n,y}| \ge 4\epsilon\sqrt{s_{n,y}}$ for n large enough, we finish the proof of Theorem 1.1 upon showing that for $s = s_{n,y}$ and any fixed $\epsilon > 0$,

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\big(\big|R_n^{\tau_k(s)} - 2s\big| \ge 4\epsilon\sqrt{s}\big) = 0. \tag{3.8}$$

To this end, recall that in view of (2.8) and (3.2)

$$r_j := \mathbb{E}\big(R_j^1\big) = 2^{-j}|\mathcal{T}_j| = 2 - 2^{-j}, \tag{3.9}$$

and similarly, by (2.9) and (3.2) we get that

$$\sigma_n^2 := \operatorname{Var}(R_n^1) = 2 \sum_{u,u' \in \mathcal{T}_n} 2^{-2n} |u \wedge u'| \le 4. \tag{3.10}$$

Next, writing in short $\tau = \tau_k(s)$, we have for any $n \ge k$, the representation

$$R_n^\tau = 2^{k-n} R_k^\tau + r_{n-k} S_k^\tau + 2^{-k} \Delta_{k,n}(2^k S_k^\tau),$$

where the random variable $\Delta_{k,n}(\ell)$ is the centered and scaled total time spent by the SRW on $\mathcal{T}_n$ below level k during the first ℓ excursions from V_{k-1} to V_k. For fixed ℓ, $\Delta_{k,n}(\ell) \overset{(d)}{=} R_{n-k}^\ell - \mathbb{E}(R_{n-k}^\ell)$ and has variance $\sigma_{n-k}^2 \ell \le 4\ell$. Further, $\Delta_{k,n}(2^k S_k^\tau)$ conditioned on $\mathcal{F}_k$ is distributed as $R_{n-k}^\ell - \mathbb{E}(R_{n-k}^\ell)$ with $\ell = 2^k S_k^\tau$. Recalling that $R_{k-1}^\ell = 2(R_k^\ell - S_k^\ell)$ and utilizing (3.6), we thus get by Markov's inequality (conditional on $\mathcal{F}_k$), that

$$\delta_{k,n} := \mathbb{P}\big(\big|R_n^\tau - 2^{k-n-1} R_{k-1}^\tau - 2S_k^\tau\big| \ge \epsilon\sqrt{s}\big) \le \frac{4 2^k \mathbb{E}(S_k^\tau)}{(2^k \epsilon \sqrt{s})^2} = \frac{4 2^{-k}}{\epsilon^2}\left(1 + \frac{c_k}{s}\right),$$

goes to zero for $s = s_{n,y} \to \infty$ followed by $k \to \infty$. Also, by the union bound,

$$\mathbb{P}\big(|R_n^\tau - 2s| \ge 4\epsilon\sqrt{s}\big) \le \delta_{k,n} + \mathbb{P}\big(S_k^\tau - s \ge \epsilon\sqrt{s}\big) + \mathbb{P}(\tau \ge 2s) + \mathbb{P}\big(2^{k-n-1} R_{k-1}^{2s} \ge \epsilon\sqrt{s}\big).$$

Next, employing Markov's inequality, we deduce that the last term goes to zero, since $2^{k-n} s r_{k-1}/(\epsilon\sqrt{s}) \to 0$ for $s = s_{n,y}$ and $n \to \infty$. By (3.6) and having WHP $\tau = \tau_k(s) < 2s$ (see (3.7)), we thus arrive at (3.8) and thereby conclude the proof of Theorem 1.1.

4. Sharp right tail: Auxiliary lemmas and proof of Theorem 1.3

Hereafter we denote by $\mathbb{P}_s$ probabilities of events occurring up to the completion of the first $[s]$ excursions at the root and let $\eta_v(j) := \sqrt{2T_{v,j}^s}$ for $v \in V_k$, $j \le k$ and $T_{v,j}^s$ of (1.3), with the value of s implicit. For $u \in V_{n'}$ where $n' := n - \ell$, and $V_n^u := \{v \in V_n : v(n') = u\}$, let

$$\eta_\ell^\sharp(u) := \min_{v \in V_n^u} \{\eta_v(n)\}, \tag{4.1}$$

denote the minimal (normalized) occupation time of edges entering leaves of the sub-tree of depth ℓ rooted at u (during the first $[s]$ excursions from the root ρ), abbreviating $\eta_n^\sharp$ for $\eta_n^\sharp(0)$. Since

$$\{t_n^* > s\} = \Big\{\min_{v \in V_n} \{T_{v,n}^s\} = 0\Big\},$$

Theorem 1.3 amounts to the claim

$$\alpha_* = \lim_{z \to \infty} z^{-1} e^{c_* z} \varliminf_{n \to \infty} \mathbb{P}_{s_{n,z}}\big(\eta_n^\sharp = 0\big), \tag{4.2}$$

for $s_{n,z}$ and c_* of (2.10) and (1.1), respectively. Our proof of (4.2) is based on a refinement of the probability estimates of [8, Section 5], intersecting here the event $\{\eta_n^\sharp = 0\}$ with barrier events involving the (normalized) edge occupation times $\{j \mapsto \eta_v(j), v \in \mathcal{T}_n\}$. More precisely, we adapt the strategy of [14, Section 3], by essentially bounding $\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)$ between the expectations of counts $\Lambda_{n,\ell} \le \Gamma_{n,\ell}$ for two barrier type events, which are equivalent at the claimed scale of asymptotic growth in z (see Lemma 4.2). Our curved barrier event for $\Gamma_{n,\ell}$ is relaxed enough to deduce that the event $\{\Gamma_{n,\ell} \ge 1\}$ is for large n, ℓ, about the same as having $\{\eta_n^\sharp = 0\}$ (see Lemma 4.1). The straight barrier event for $\Lambda_{n,\ell}$ is strict enough to yield a negligible variance (see Lemma 4.4), so its expectation serves to lower bound $\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)$. Our claim (4.2) then follows from such a limit for $\mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}]$ (which is a consequence of Proposition 4.3). Specifically, for $s = s_{n,z}$ consider the excess edge occupation times, over the barrier

$$\bar{\varphi}_n(j) := \rho_n(n - j), \quad j \in [0, n'], n' = n - \ell. \tag{4.3}$$

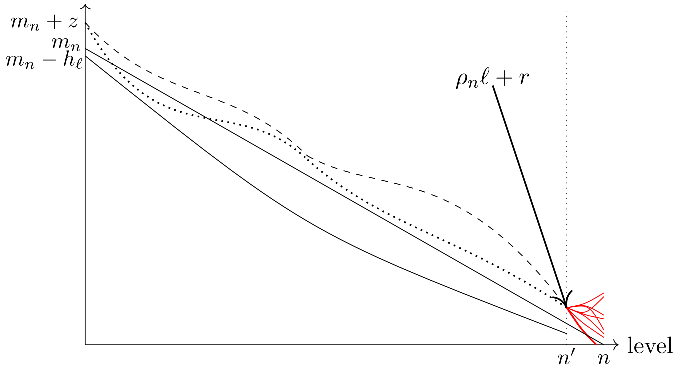

Fig. 1. Depiction of the events $E_{n,\ell}(u)$ (dashed line) and $F_{n,\ell}(u)$ (dotted line) for some $u \in V_{n'}$. In either case, the red paths emanating from level $n' = n - \ell$ denote excursion counts corresponding to different children of u. Note the curved vs. straight barrier and the excursion count that reaches 0.

In the sequel we show that the main contribution to $\{\eta_n^\sharp = 0\}$ is due to not covering a sub-tree rooted at some $u \in V_{n'}$ while the edge occupation times along the geodesic to u exceed the barrier $\bar{\varphi}_n(\cdot)$ of (4.3), with the excess at the edge into u further restricted to

$$I_\ell := \sqrt{\ell}\big[r_\ell^{-1}, r_\ell\big], \quad r_\ell := \sqrt{\log \ell}. \tag{4.4}$$

To this end, let

$$\hat{\eta}_v(j) := \eta_v(j) - \bar{\varphi}_n(j), \tag{4.5}$$

considering for $u \in V_{n'}$ the events

$$E_{n,\ell}(u) := \bigcap_{0 \le j \le n'} \big\{\hat{\eta}_u(j) > 0\big\} \cap \big\{\hat{\eta}_u\big(n'\big) \in I_\ell, \eta_\ell^\sharp(u) = 0\big\}, \tag{4.6}$$

and the corresponding counts

$$\Lambda_{n,\ell} := \sum_{u \in V_{n'}} \mathbf{1}_{E_{n,\ell}(u)}. \tag{4.7}$$

See Figure 1 for a pictorial illustration of the event $E_{n,\ell}(u)$. As explained before, aiming first to upper bound $\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)$, we fix $\delta \in (0, \frac{1}{2})$ and for $k \in [1, n]$, $h \in [0, n-k]$, consider the curved, relaxed barriers

$$\varphi_{n,k,h}(j) := \bar{\varphi}_n(j) - \psi_{k,h}(j), \quad j \in [0, k], \tag{4.8}$$

using hereafter the notations

$$\psi_{k,h}(j) := h + j_k^\delta, \quad j_k := j \wedge (k - j), j \in [0, k]. \tag{4.9}$$

We further use the abbreviated notation

$$\psi_\ell(\cdot) := \psi_{n',h_\ell}(\cdot), \quad \text{where } h_\ell := \frac{1}{2}\log \ell, \tag{4.10}$$

with $n' = n - \ell \ge 1$. Replacing the barriers of (4.3) by those of (4.8), we then form the larger counts

$$\Gamma_{n,\ell} = \sum_{u \in V_{n'}} \mathbf{1}_{F_{n,\ell}(u)}, \tag{4.11}$$

where in terms of (4.1), (4.5) and (4.8), we define for each $u \in V_{n'}$

$$F_{n,\ell}(u) := \bigcap_{0 \le j \le n'} \big\{\hat{\eta}_u(j) + \psi_\ell(j) > 0\big\} \cap \big\{\eta_\ell^\sharp(u) = 0\big\}. \tag{4.12}$$

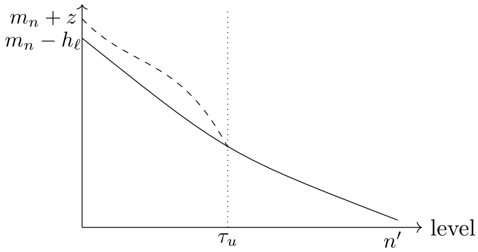

Fig. 2. Depiction of an event from $G_{n,\ell}$ (dashed line) corresponding to some $u \in V_{n'}$, $n' = n - \ell$. Note the curved barrier.

See Figure 1 for a pictorial illustration of the event $F_{n,\ell}(u)$. If $\eta_n^\sharp = 0$, then necessarily $\eta_\ell^\sharp(u) = 0$ for some $u \in V_{n'}$ and either $F_{n,\ell}(u)$ occurs (so $\Gamma_{n,\ell} \geq 1$), or else the event $G_{n,\ell} := G_{n,n'}(h_\ell)$ must occur, where, see Figure 2,

$$G_{n,k'}(h) := \bigcup_{u \in V_{k'}} \bigcup_{0 \leq j \leq k'} \{\hat{\eta}_u(j) \leq -\psi_{k',h}(j)\}. \tag{4.13}$$

Hence, for any ℓ,

$$\mathbb{E}_{s_{n,z}}[\Gamma_{n,\ell}] \geq \mathbb{P}_{s_{n,z}}(\Gamma_{n,\ell} \geq 1) \geq \mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0) - \mathbb{P}_{s_{n,z}}(G_{n,\ell}). \tag{4.14}$$

Recall that by [8, proof of Corollary 5.4], for some $c' > 0$ and all $z \geq 1$,

$$\varliminf_{n \to \infty} \mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0) \geq c' z \mathrm{e}^{-c_* z}, \tag{4.15}$$

so our next lemma, which is an immediate consequence of Lemma 5.1 below, shows that the right-most term in (4.14) is negligible.

Lemma 4.1. *We have that*

$$\lim_{\ell \to \infty} \sup_{z \geq 1} \Big\{ z^{-1} e^{c_* z} \varlimsup_{n \to \infty} \mathbb{P}_{s_{n,z}}(G_{n,\ell}) \Big\} = 0. \tag{4.16}$$

Combining (4.14)–(4.16), we arrive at

$$\varlimsup_{\ell \to \infty} \varlimsup_{z \to \infty} \varlimsup_{n \to \infty} \frac{\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)}{\mathbb{E}_{s_{n,z}}[\Gamma_{n,\ell}]} \leq 1. \tag{4.17}$$

Restricting hereafter to $\delta \in (0, \frac{1}{6})$ allows us to further show in Section 5.2 the following equivalence of first moments (cf. (5.41) for why we take δ small).

Lemma 4.2. *For any $\delta \in (0, \frac{1}{6})$ we have that*

$$\lim_{\ell \to \infty} \varlimsup_{z \to \infty} \Big\{ z^{-1} e^{c_* z} \varlimsup_{n \to \infty} \mathbb{E}_{s_{n,z}}[\Gamma_{n,\ell} - \Lambda_{n,\ell}] \Big\} = 0. \tag{4.18}$$

Now, from (4.17) and (4.18), we have the upper bound

$$\varlimsup_{\ell \to \infty} \varlimsup_{z \to \infty} \varlimsup_{n \to \infty} \frac{\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)}{\mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}]} \leq 1. \tag{4.19}$$

For such expected counts with straight barriers, we establish in Section 6.1, using the connection to the 0-Bessel process, the following large n and z asymptotic.

Proposition 4.3. *There exists $\alpha_\ell > 0$ such that*

$$\lim_{\ell\to\infty} \alpha_\ell^{-1}\Big\{ \varlimsup_{z\to\infty} z^{-1} \mathrm{e}^{c_* z} \varlimsup_{n\to\infty} \mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}]\Big\} = 1, \tag{4.20}$$

where by (4.15) *and* (4.19), $\liminf\{\alpha_\ell\}$ *is strictly positive.*

As shown in Section 5.3, the barrier event we have added in the definition (4.6) of $E_{n,\ell}(u)$ yields the following tight control on the second moment of $\Lambda_{n,\ell}$.

Lemma 4.4. *We have that*

$$\lim_{\ell\to\infty} \varlimsup_{z\to\infty} \Big\{ z^{-1} e^{c_* z} \varlimsup_{n\to\infty} \mathbb{E}_{s_{n,z}}\big[\Lambda_{n,\ell}(\Lambda_{n,\ell}-1)\big]\Big\} = 0. \tag{4.21}$$

Note that $\Lambda_{n,\ell} \ge 1$ implies having $\eta_v(n) = 0$ for some $v \in V_n$, that is, having $\eta_n^\sharp = 0$. Hence, with $\Lambda_{n,\ell}$ integer valued, for any choice of ℓ,

$$\mathbb{P}_{s_{n,z}}\big(\eta_n^\sharp = 0\big) \ge \mathbb{P}_{s_{n,z}}(\Lambda_{n,\ell} \ge 1) \ge \mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}] - \mathbb{E}_{s_{n,z}}\big[\Lambda_{n,\ell}(\Lambda_{n,\ell}-1)\big]. \tag{4.22}$$

Having a positive $\liminf\{\alpha_\ell\}$, the latter bound, together with (4.20) and (4.21), imply that

$$\varliminf_{\ell\to\infty} \varliminf_{z\to\infty} \varliminf_{n\to\infty} \frac{\mathbb{P}_{s_{n,z}}(\eta_n^\sharp = 0)}{\mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}]} \ge 1. \tag{4.23}$$

Proof of Theorem 1.3. Comparing first (4.19) to (4.23) and then with (4.20), we conclude that

$$\lim_{\ell\to\infty} \alpha_\ell^{-1}\Big\{ \varlimsup_{n\to\infty} z^{-1} \mathrm{e}^{c_* z} \varlimsup_{n\to\infty} \mathbb{P}_{s_{n,z}}\big(\eta_n^\sharp = 0\big)\Big\} = 1. \tag{4.24}$$

Necessarily $\alpha_\ell \to \alpha_*$ for which (4.2) holds (with $\alpha_* > 0$ in view of (4.15) and $\alpha_* < \infty$ by [8, Proposition 5.2]). $\square$

5. Barrier bounds for excursion counts

We keep the barrier sequences of (4.8) and all other related notation from Section 4. Further, with $\rho_n \to c_* > 1.1$, see (1.1), WLOG we restrict to $n \ge n_\star \ge 64$ with $\rho_n \ge \rho_* =: 1.1$, starting at the following a-priori bound on the events $G_{n,k'}(h)$ from (4.13). Recall the notation $s_{n,z}$ of (2.10).

Lemma 5.1. *For some $c < \infty$, any $n \ge n_\star$, $z, k' \ge 1$ and $h \in [0, n-k']$,*

$$\mathbb{P}_{s_{n,z}}\big(G_{n,k'}(h)\big) \le c(z+h)e^{-c_*(z+h)}e^{-(z+h)^2/(8n)}. \tag{5.1}$$

Proof of Lemma 4.1. Setting $h = h_\ell = \frac{1}{2}\log\ell$ in (5.1), results with

$$\mathbb{P}_{s_{n,z}}(G_{n,\ell}) \le c(z+\log\ell)\ell^{-c_*/2}e^{-c_* z}e^{-z^2/(8n)}, \tag{5.2}$$

so taking $\ell \to \infty$ establishes (4.16). $\square$

Before embarking on the proof of Lemma 5.1, we deduce from it certain useful a-priori tail bounds on the non-covering events $\{\eta_\ell^\sharp = 0\}$.

Corollary 5.2. *For some $c < \infty$ and all $n \ge n_\star$, $z \ge 1$,*

$$\mathbb{P}_{s_{n,z}}\big(\eta_n^\sharp = 0\big) \le cze^{-c_* z}e^{-z^2/(8n)}. \tag{5.3}$$

Further, for some $\hat{\ell}$ finite, any $\hat{\ell} \le \ell \le n/\log n$ and $r \ge -h_\ell$,

$$\gamma_{n,\ell}(r) := \mathbb{P}_{[(\rho_n\ell+r)^2/2]}\big(\eta_\ell^\sharp = 0\big) \le c\ell^{-1}(r+\log\ell)e^{-c_* r}e^{-r^2/(8\ell)}. \tag{5.4}$$

Proof. The event $\eta_n^\sharp = 0$ amounts to $\eta_v(n) = 0$ for some $v \in V_n$. With $\varphi_{n,n,0}(n) = 0$ (see (4.8)–(4.9)), this implies that $\eta_v(n) \le \varphi_{n,n,0}(n)$ and consequently that $G_{n,n}(0)$ also occurs (take $j = k' = n$ in (4.13)). That is $\{\eta_n^\sharp = 0\} \subseteq G_{n,n}(0)$, so the bound (5.3) follows from (5.1). Proceeding to prove (5.4), setting $\hat{\ell} := n_\star \vee \exp(2(c_* + 1)/(2 - c_*))$, one easily checks that $z = r + (\log \ell - 1)/c_* \ge 1$ whenever $r \ge -\frac{1}{2}\log \ell$ and $\ell \ge \hat{\ell}$. If further $1 \le \ell \le n/\log n$, then

$$\rho_n \ell + r = c_* \ell - \frac{\ell \log n}{c_* n} + r \ge m_\ell + z,$$

so (5.3) at $n = \ell$ and such z yields the bound (5.4), possibly with $c \mapsto ec$. □

We recall (4.5), (4.9) and take throughout

$$H_x := [x, x+1]. \tag{5.5}$$

The key to this section are the following a-priori barrier estimates adapted from [8, Section 5] (though it is advised to skip the proofs at first reading).

Lemma 5.3. *Let $g_m(i) := i \exp(c_* i + i^2/m)$, $\beta_{n,k} := \frac{k}{n}\log n - \log k$ and $z_h := z + h$. For some $c_1 \ge 1$, all $z \ge 1$, $k \ge 0$, $h \in [0, n-k)$ and $i \in \mathbb{Z}_+$*

$$q_{n,k,z}(i; h) := \mathbb{P}_{s_{n,z}}\Big(\min_{j \le k}\{\hat{\eta}_v(j) + \psi_{k,h}(j)\} \ge 0, \hat{\eta}_v(k) \in H_{i-h}\Big) \tag{5.6}$$

$$\le c_1 2^{-k}\bigg(\frac{z_h}{\sqrt{k_n}} \wedge k\bigg) e^{\beta_{n,k}} e^{-c_* z_h} e^{-z_h^2/(4m)} g_m(i+1), \quad \forall m \in \big[2k, n^2\big] \tag{5.7}$$

(*replacing for $k = 0$ the ill-defined factor $(\frac{z_h}{\sqrt{k_n}} \wedge k)e^{\beta_{n,k}}$ by 1*).

Likewise, for $i, k' \in \mathbb{Z}_+$, $n' = k' + k \in (k', n)$, $m \in [2k, (n-k')^2]$ and $z \ge 0$,

$$p_{n,k,z}(i) := \mathbb{P}\Big(\min_{j \in (k', n']}\{\hat{\eta}_v(j)\} \ge 0, \hat{\eta}_v(n') \in H_i \;\Big|\; \hat{\eta}_v(k') = z\Big)$$

$$\le c_1 \frac{2^{-k} e^{\beta_{n,k}}}{\sqrt{k_{n-k'}}}(z \vee 1) e^{-c_* z} e^{-(z \vee 1)^2/(4m)} g_m(i+1). \tag{5.8}$$

The bound (5.8) applies also to $z \in [-\rho_n k, 0]$, now with $m = -4k$.

Proof. In case $k \ge 1$, setting $a = \rho_n n - h$ and $b = \rho_n(n-k) - h$, using [8, Lemma 5.1], the event considered in (5.6) corresponds to [8, (1.1)] for $L = k$, $C = 1$, $\varepsilon = \frac{1}{2} - \delta$ and the line $f_{a,b}(j; k)$ between $(0, a)$ and (k, b), taking there $y = b + i$ and $x = \sqrt{2[s_{n,z}]} \ge a$. Having $z \ge 1$ and $m_n \ge 1$ yields that $x \ge \sqrt{2}$. Further, with $h \in [0, n-k)$ and $\rho_n \ge \rho_*$,

$$y \ge b = \rho_n(n-k) - h \ge 1 + \frac{1}{10}(n-k) \tag{5.9}$$

(the restriction to $y \ge \sqrt{2}$ in [8, (1.1)] clearly can be replaced with $y > 1$ there, since for $y \in [1, \sqrt{2})$ one has $H_y^2/2 \cap \mathbb{Z} = (H_{\sqrt{2}})^2/2 \cap \mathbb{Z}$). Here x/L and y/L are not uniformly bounded above but following [8, proof of (4.2)] and utilizing [8, Remark 2.6] to suitably modify [8, (4.16)], we nevertheless arrive at the bound

$$q_{n,k,z}(i; h) \le c \frac{(1 + x - a)(1 + i)}{k} \sqrt{\frac{x}{ky}} \sup_{w \in H_y}\big\{e^{-(x-w)^2/(2k)}\big\}. \tag{5.10}$$

In addition, whenever $x \ge \sqrt{2}$, $y, k \ge 1$, we have that

$$q_{n,k,z}(i; h) \le \mathbb{P}_{s_{n,z}}\big(\hat{\eta}_v(k) \in H_{i-h}\big) \le c \sup_{w \in H_y}\big\{e^{-(x-w)^2/(2k)}\big\} \tag{5.11}$$

(see [9, Lemma 3.6]). Next, since $x \le c_* n + z$, we deduce from (5.9) that

$$\frac{x}{ky} \le \frac{20}{k_n}\bigg(c_* + \frac{z}{n}\bigg) \le \frac{c_o}{k_n}\exp\bigg(\frac{z^2}{12n^2}\bigg), \tag{5.12}$$

for some constant $c_o < \infty$. Further, setting $\delta_{n,z} := a + z_h - x$ and

$$\bar{\varepsilon}_{n,k} = \varepsilon_{n,k} + \delta_{n,z}, \quad \varepsilon_{n,k} := \frac{k \log n}{c_* n}, \tag{5.13}$$

we have from (1.1) that $x - b = c_* k + z_h - \bar{\varepsilon}_{n,k}$. With $\frac{c_*^2}{2} = \log 2$, it follows that for any real $\widetilde{w}$,

$$\frac{1}{2k}(x - b - \widetilde{w})^2 = k \log 2 + c_*(z_h - \bar{\varepsilon}_{n,k} - \widetilde{w}) + \frac{1}{2k}(z_h - \bar{\varepsilon}_{n,k} - \widetilde{w})^2. \tag{5.14}$$

By an elementary inequality, for any $m \geq 2k$,

$$(z_h - \widetilde{w} - \bar{\varepsilon}_{n,k})^2 \geq \frac{2k}{3m}\big(z_h^2 - 3\widetilde{w}^2 - 3\bar{\varepsilon}_{n,k}^2\big), \tag{5.15}$$

so that by (5.14)

$$\frac{1}{2k}(x - b - \widetilde{w})^2 \geq k \log 2 + c_*(z_h - \bar{\varepsilon}_{n,k} - \widetilde{w}) + \frac{z_h^2}{3m} - \frac{\widetilde{w}^2}{m} - \frac{\bar{\varepsilon}_{n,k}^2}{m}. \tag{5.16}$$

With $c_* \varepsilon_{n,k} - \log k = \beta_{n,k} \leq 1$ for $k \in [1, n]$ (while $\delta_{n,z}$ and $\bar{\varepsilon}_{n,k}^2/k$ are uniformly bounded), we plug into the smaller among (5.10) and (5.11) the bounds (5.12) and (5.16) (for $w = b + \widetilde{w}$ and $\widetilde{w} \in H_i$), to arrive at (5.7). Further, $q_{n,0,z}(i; h) = 0$ for any $i < z_h - 1.1$, whereas for $i \geq z_h - 1.1$, having

$$\inf_{m, z_h \geq 1} \big\{ g_m(z_h - 0.1) e^{-c_* z_h - z_h^2/(4m)} \big\} > 0,$$

yields that (5.7) holds also for $k = 0$ (under our convention).

Turning to the proof of (5.8), we consider first $z > 0$, proceeding as in the proof of (5.7) with the line $f_{a,b}(j; k)$ of length k and same slope as in the preceding, now connecting (k', a) to (n', b), where $a = \rho_n(n - k')$ and $b = \rho_n(n - n')$. For $x = a + z$ and $y = b + i$ we have $x \geq a > b$ and $y \geq b > 0$, thanks to our assumption that $n' \in (k', n)$. By the Markov property of $j \mapsto \eta_v(j)$, for such values of (a, b, x, y) the RHS of (5.10) with $h = 0$ necessarily bounds the probability $p_{n,k,z}(i)$, and thereafter one merely follows the derivation of (5.7), now with $h = \delta_{n,z} = 0$ and $n - k'$ replacing n when bounding x/y. The latter modification results in having $\sqrt{k_{n-k'}}$ in (5.8), instead of $\sqrt{k_n}$. Next, for $z \leq 0$ we simply lower the barrier line $f_{a,b}(j; k)$ to start at $a = x = \bar{\varphi}_n(k') + z$, where our assumption that $z \geq -\rho_n k$ guarantees having $x \geq \bar{\varphi}_n(n') > 0$ (thereby $x \geq \sqrt{2}$, with $p_{n,k,z}(i)$ bounded by the RHS of (5.10)). Here $x/(ky) \leq c_o/k_{n-k'}$, and $z_h = z \leq 0$ allows us to replace the RHS of (5.15) for $\widetilde{w} \geq i$ by $\frac{(i+1)^2}{2} - 1$, yielding the stated form of (5.8). □

We conclude this sub-section by adapting the bounds of Lemma 5.3 to the form needed when proving Lemma 4.4 and Lemma 4.2.

Lemma 5.4. *There exists a constant $c_2 < \infty$ satisfying the following. Fix $\hat{\ell}$ as in Corollary 5.2 and I_ℓ as in (4.4). For any $z \geq 0$, $\ell \in [\hat{\ell}, \frac{n}{\log n}]$ and $k \geq 1$, setting $k' = n' - k$ with n' as in (4.3),*

$$\begin{aligned}\theta_{n,k,\ell}(z) &:= \mathbb{P}\Big(\min_{j \in (k', n']}\big\{\hat{\eta}_v(j)\big\} \geq 0, \hat{\eta}_v\big(n'\big) \in I_\ell, \eta_\ell^\sharp(v) = 0 \,\big|\, \hat{\eta}_v\big(k'\big) = z\Big) \\ &\leq c_2 2^{-k} e^{\beta_{n,k}}\big(1 \vee \sqrt{\ell/k}\big)(1 + z) e^{-c_* z} e^{-z^2/(8(k \vee 8\ell))}.\end{aligned} \tag{5.17}$$

If in addition $z \geq 4h_\ell$, $n \geq 3\ell$, then for any $r \geq 0$,

$$\begin{aligned}\mathbf{p}_{n,k,z}(r) &:= \mathbb{P}_{s_{n,z}}\Big(\min_{j \leq n'}\big\{\hat{\eta}_v(j) + \psi_\ell(j)\mathbf{1}_{j \leq k}\big\} > 0, \hat{\eta}_v(k) \leq 0, \hat{\eta}_v\big(n'\big) \in H_r\Big) \\ &\leq c_2 2^{-n'} \frac{\psi_\ell(k)^3}{k_{n'}^{3/2} \ell^{1/2}} e^{-z^2/(16k)} z(r+1) e^{-c_*(z-r)} e^{-\frac{r^2}{4(n'-k)}}.\end{aligned} \tag{5.18}$$

Proof. Starting with (5.17), by the Markov property of $\eta_v(j)$ at $j = n'$ and monotonicity of $y \mapsto \gamma_{n,\ell}(y)$ of (5.4), we have in terms of $p_{n,k,z}(\cdot)$ of (5.8)

$$\theta_{n,k,\ell}(z) \le \sum_{r \in I_\ell} p_{n,k,z}(r)\gamma_{n,\ell}(r).$$

Plugging the bounds of (5.4) and (5.8) (at $m = 2(k \vee 8\ell)$), yields for c_1' finite

$$\theta_{n,k,\ell}(z) \le c_1' \frac{2^{-k} e^{\beta_{n,k}} \sqrt{\ell}}{\sqrt{k_{n-k'}}}(1+z)e^{-c_* z} e^{-z^2/(8(k\vee 8\ell))} \sum_{r\in I_\ell} \frac{r^2}{\ell^{3/2}} e^{-r^2/(16\ell)}.$$

With the latter sum uniformly bounded and $k_{n-k'} = k \wedge \ell$, we arrive at (5.17).

Next, turning to establish (5.18), note first that

$$j_{k'}^\beta \le k_{k'}^\beta + j_k^\beta, \quad \forall 0 \le j \le k < k' \tag{5.19}$$

when $\beta = 1$ and consequently also for all $\beta \in [0, 1]$. For $\beta = \delta$, it results with

$$\psi_{k',h}(j) \le \psi_{k,h'}(j), \quad \text{for } h' = \psi_{k',h}(k),\ j \in [0, k],$$

with equality at $j = k$. In particular, recalling (4.10) and considering $k' = n'$, we have that

$$\psi_\ell(j) \le \psi_{k,h}(j) \quad \text{for } h = \psi_\ell(k),\ j \in [0, k]. \tag{5.20}$$

Employing (5.20) to enlarge the event whose probability is $\mathbf{p}_{n,k,z}(i)$, we get by the Markov property of $\eta_v(j)$ at $j = k$, in terms of $q_{n,k,z}(\cdot;\cdot)$, $p_{n,k,z}(\cdot)$ and $h = \psi_\ell(k)$, that

$$\mathbf{p}_{n,k,z}(r) \le \sum_{i=1}^{h} q_{n,k,z}(h-i; h) \sup_{z' \in H_{-i}} \{p_{n,k',z'}(r)\}. \tag{5.21}$$

Substituting first our bound (5.8) at $m = -4k'$, and then (5.7) at $m = 2k$, yields that for some c_1' finite,

$$\begin{aligned}\mathbf{p}_{n,k,z}(r) &\le c_1 \sum_{i=1}^{h} q_{n,k,z}(h-i; h) 2^{-k'} e^{\beta_{n,k'}} \left(k'_{k'+\ell}\right)^{-1/2} e^{c_*(r+i)}(r+1)e^{-\frac{r^2}{4k'}} \\ &\le c_1' h 2^{-n'} \frac{e^{\beta_{n,k}+\beta_{n,k'}}}{\sqrt{k'_{k'+\ell} k_n}} z_h e^{-c_* z_h} e^{-z^2/(8k)} g_{2k}(h)(r+1)e^{c_* r} e^{-r^2/(4k')}.\end{aligned} \tag{5.22}$$

With $\delta \le \frac{1}{2}$ and $z \ge 4h_\ell$, it follows that

$$\frac{h^2}{2k} \le \frac{h_\ell^2 + k_{n'}^{2\delta}}{k} \le \frac{z^2}{16k} + 1.$$

Further, recall that $k + k' = n'$ and $\beta_{n,n'} \le 1$, hence

$$e^{\beta_{n,k}+\beta_{n,k'}} = e^{\beta_{n,n'}} \frac{n'}{kk'} \le \frac{2e}{k_{n'}}. \tag{5.23}$$

Our assumption $n \ge 3\ell$ results with $k \vee k' \ge \ell$ and thereby $k'_{k'+\ell} k_n \ge \ell k_{n'}$. Applying the preceding within (5.22), we arrive at (5.18). □

5.1. *Negligible crossings*: *Proof of Lemma* 5.1

Fixing n, k', h as in Lemma 5.1, consider for $u \in V_{k'}$, the first time

$$\tau_u := \min\{j \ge 0 : \eta_u(j) \le \varphi_{n,k',h}(j)\}$$

that the process $j \mapsto \eta_u(j)$ reaches the relevant barrier of (4.8), see Figure 2. For $z, m_n \geq 1$ we have that $\tau_u \geq 1$, since $\eta_u(0) = \sqrt{2[s_{n,z}]} > m_n \geq \varphi_{n,k',h}(0)$ under $\mathbb{P}_{s_{n,z}}$. Decomposing $G_{n,k'}(h)$ according to the possible values of $\{\tau_u\}$, results with

$$\mathbb{P}_{s_{n,z}}\big(G_{n,k'}(h)\big) \leq \sum_{k=1}^{k'} \underbrace{\mathbb{P}_{s_{n,z}}(\exists u \in V_{k'} \text{ such that } \tau_u = k)}_{(\mathrm{I}_k)}. \tag{5.24}$$

The event $\{\tau_u = k\}$ depends only on the value of $u(k) \in V_k$. Hence, by the union bound we have that for any fixed $u \in V_{k'}$

$$(\mathrm{I}_k) \leq 2^k \mathbb{P}_{s_{n,z}}(\tau_u = k). \tag{5.25}$$

With $\rho_n > 1 > \delta$, it is easy to verify that $j \mapsto \varphi_{n,k',h}(j)$ is strictly decreasing with $\varphi_{n,k',h}(k') = \rho_n(n-k') - h \geq 0$. Further, for $b := \varphi_{n,k',h}(k)$, $\widetilde{b} := \varphi_{n,k',h}(k+1)$, we get upon conditioning on $\eta_u(k) = y$, that

$$\begin{aligned} &\mathbb{P}_{s_{n,z}}(\tau_u = k+1) \\ &\quad \leq \sum_{i=0}^{\infty} \mathbb{P}_{s_{n,z}}\big(\tau_u > k, \eta_u(k) \in H_{b+i}\big) \sup_{y \in H_{b+i}} \mathbb{P}_{\frac{y^2}{2}}\big(\eta_u(1) \leq \widetilde{b}\big). \end{aligned} \tag{5.26}$$

Applying [7, Lemma 4.6] at $p = q = 1/2$ and $\theta = \widetilde{b}^2/2 \leq y^2/2$

$$\sup_{y \in H_{b+i}} \mathbb{P}_{\frac{y^2}{2}}\big(\eta_u(1) \leq \widetilde{b}\big) \leq e^{-i^2/4}. \tag{5.27}$$

Setting $h' = \psi_{k',h}(k)$, we proceed to bound the first probability on the RHS of (5.26). To this end, recall (5.19), yielding that $\varphi_{n,k,h'}(j) \leq \varphi_{n,k',h}(j)$ for $j \in [0,k]$, with equality at $j = k$ (see (4.8)–(4.9)). Consequently,

$$\mathbb{P}_{s_{n,z}}\big(\tau_u > k, \eta_u(k) \in H_{b+i}\big) \leq q_{n,k,z}\big(i; h'\big), \tag{5.28}$$

for $q_{n,k,z}(\cdot;\cdot)$ of Lemma 5.3. Since $n - k' \geq h$ and $\delta < 1$, for any $k < k'$,

$$n - k - h' \geq k' - k - \big(k' - k\big)^{\delta} > 0,$$

in which case, by (5.7) we have that for any $i \in \mathbb{Z}_+$

$$q_{n,k,z}\big(i; h'\big) \leq c_1 2^{-k} z_{h'} e^{-c_* z_{h'}} e^{-z_{h'}^2/(8n)} g_{2n}(i+1). \tag{5.29}$$

Noting that $\sup_{n\geq 3}\{g_{2n}(i+1)\}e^{-i^2/4}$ is summable (and $z_{h'} \geq z_h$), we find upon combining (5.25)–(5.29), that for some c_3 finite and any $1 \leq k < k'$,

$$\begin{aligned} (\mathrm{I}_{k+1}) &\leq 2^{k+1} \sum_{i=0}^{\infty} q_{n,k,z}\big(i; h'\big) e^{-i^2/4} \\ &\leq c_3\big(z_h + k_{k'}^{\delta}\big) e^{-c_*(z_h + k_{k'}^{\delta})} e^{-z_h^2/(8n)}. \end{aligned} \tag{5.30}$$

Further, with $\varphi_{n,k',h}(1) \leq m_n - h$ we have similarly to (5.25)–(5.27) that

$$(\mathrm{I}_1) \leq 2\mathbb{P}_{s_{n,z}}(\tau_u = 1) \leq 2\mathbb{P}_{s_{n,z}}\big(\eta_u(1) \leq m_n - h\big) \leq 2e^{-z_h^2/4},$$

which is further bounded for $z_h \geq z \geq 1$ by the RHS of (5.30) at $k = 0$ (possibly increasing the universal constant c_3). Summing over $k \leq k'$ it follows from (5.24) and (5.30) that for some universal $c_4 < \infty$,

$$\begin{aligned} \mathbb{P}_{s_{n,z}}\big(G_{n,k'}(h)\big) &\leq c_3 e^{-c_* z_h} e^{-z_h^2/(8n)} \sum_{k=0}^{k'} \big(z_h + k_{k'}^{\delta}\big) e^{-c_* k_{k'}^{\delta}} \\ &\leq c_4 z_h e^{-c_* z_h} e^{-z_h^2/(8n)}, \end{aligned}$$

as claimed in (5.1).

5.2. *Comparing barriers*: *Proof of Lemma* 4.2

Hereafter, let $\nu_{n,k,z}(\cdot)$ denote the finite measure on $[0,\infty)$ such that

$$\nu_{n,k,z}(A) := 2^k \mathbb{P}_{s_{n,z}}\Big(\min_{j\le k}\{\hat{\eta}_v(j)\} > 0, \hat{\eta}_v(k) \in A\Big), \tag{5.31}$$

using the abbreviation $\nu_{n,z} = \nu_{n,n',z}$ (where $n' = n - \ell$). In view of (4.6) and the Markov property of $\{\eta_v(j)\}$ at $j = n'$ we have that

$$\mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}] = 2^{n'}\mathbb{P}_{s_{n,z}}\big(E_{n,\ell}(v)\big) = \int_{I_\ell} \gamma_{n,\ell}(y)\nu_{n,z}(dy), \tag{5.32}$$

for $\gamma_{n,\ell}(\cdot)$ of (5.4). Similarly, setting the finite measure on $[-h_\ell,\infty)$

$$\mu_{n,z}(A) := 2^{n'}\mathbb{P}_{s_{n,z}}\Big(\min_{j\le n'}\{\hat{\eta}_v(j) + \psi_\ell(j)\} > 0, \hat{\eta}_v\big(n'\big) \in A\Big), \tag{5.33}$$

we have by the Markov property and (4.12) that

$$\mathbb{E}_{s_{n,z}}[\Gamma_{n,\ell}] = 2^{n'}\mathbb{P}_{s_{n,z}}\big(F_{n,\ell}(v)\big) = \int_{-h_\ell}^{\infty} \gamma_{n,\ell}(y)\mu_{n,z}(dy). \tag{5.34}$$

For $r \ge 1$, recalling $H_r = [r, r+1]$, we decompose $\mu_{n,z}(H_r) - \nu_{n,z}(H_r)$ according to the possible values of $\tau := \max\{j < n' : \hat{\eta}_v(j) \le 0\}$, to arrive at

$$\mu_{n,z}(H_r) - \nu_{n,z}(H_r) \le 2^{n'} \sum_{k=1}^{n'-1} \mathbf{p}_{n,k,z}(r), \tag{5.35}$$

for $\mathbf{p}_{n,k,z}(r)$ of Lemma 5.4. By (5.32), (5.34), (5.35) and the monotonicity of $y \mapsto \gamma_{n,\ell}(y)$ we have that

$$\begin{aligned}\mathbb{E}_{s_{n,z}}[\Gamma_{n,\ell} - \Lambda_{n,\ell}] &\le \sum_{r\notin I_\ell} \gamma_{n,\ell}(r)\mu_{n,z}(H_r) + \sum_{r\in I_\ell} \gamma_{n,\ell}(r) \sum_{k=1}^{n'-1} 2^{n'}\mathbf{p}_{n,k,z}(r) \\ &:= \mathsf{I}_n(z,\ell) + \mathsf{II}_n(z,\ell).\end{aligned} \tag{5.36}$$

Dealing first with $\mathsf{I}_n(z,\ell)$ of (5.36), note that $\mu_{n,z}(H_r) = 2^{n'} q_{n,n'}(r+h;h)$ for $q_{n,k}(i;h)$ of Lemma 5.3 and $h = h_\ell$. Combining (5.4) with (5.7) at $k = n'$ (where $k_n = \ell$), and having $z + h_\ell \le 2z$ (as $z \to \infty$ before $\ell \to \infty$), yields for some c_5 finite, any $\ell \ge \hat{\ell}$, large n and all $r \ge -h_\ell$

$$\gamma_{n,\ell}(r)\mu_{n,z}(H_r) \le c_5 z e^{-c_* z} \frac{(r+2h_\ell)^2}{\ell^{3/2}} e^{-r^2/(8\ell)} e^{(r+h_\ell)^2/n}. \tag{5.37}$$

Substituting (5.37) in (5.36) and taking $n \to \infty$ results with

$$\varlimsup_{z\to\infty}\Big\{z^{-1}e^{c_* z}\varlimsup_{n\to\infty}\mathsf{I}_n(z,\ell)\Big\} \le \varepsilon_{\mathsf{I}}(\ell), \tag{5.38}$$

where by our choice (4.4) of I_ℓ, for any $\ell \to \infty$

$$\varepsilon_{\mathsf{I}}(\ell) := c_5 \sum_{r\notin I_\ell} \frac{(r+2h_\ell)^2}{\ell^{3/2}} e^{-r^2/(8\ell)} \longrightarrow 0.$$

In view of (5.4), it suffices to show that for some $\varepsilon_{\mathsf{II}}(\ell) \to 0$ and all $r \in I_\ell$,

$$\varlimsup_{z\to\infty}\Big\{z^{-1}e^{c_* z}\varlimsup_{n\to\infty}\sum_{k=1}^{n'-1} 2^{n'}\mathbf{p}_{n,k,z}(r)\Big\} \le \frac{\varepsilon_{\mathsf{II}}(\ell)}{\sqrt{\ell}}(r+1)e^{c_* r}, \tag{5.39}$$

in order to get the analog of (5.38) for $\mathsf{II}_n(z,\ell)$ and thereby complete the proof of the lemma. In view of (5.18) we get (5.39) upon showing that

$$\lim_{\ell\to\infty}\overline{\lim_{z\to\infty}}\sup_{r\in I_\ell}\overline{\lim_{n\to\infty}}\sum_{k=1}^{n'-1}\psi_\ell(k)^3 k_{n'}^{-3/2}\exp\left(-\frac{z^2}{16k}-\frac{r^2}{4(n'-k)}\right)=0. \tag{5.40}$$

Even without the exponential factor, since $\delta<\frac{1}{6}$ the sum in (5.40) over $\{k: k_{n'}^\delta\ge h_\ell\}$, where $\psi_\ell(k)\le 2k_{n'}^\delta$ is bounded above by

$$4\sum_{k\ge h_\ell^{1/\delta}}k^{3\delta-3/2}\le ch_\ell^{(3-\frac{1}{2\delta})}\underset{\ell\to\infty}{\longrightarrow}0. \tag{5.41}$$

Further, the sum in (5.40) over $\{k: k_{n'}^\delta<h_\ell\}$ has $2h_\ell^{1/\delta}$ terms, which are uniformly bounded by $(2h_\ell)^3\exp(-b_\ell/(4h_\ell^{1/\delta}))$, where having

$$b_\ell:=\frac{z^2}{4}\wedge\inf_{r\in I_\ell}\{r^2\}=\frac{\ell}{\log\ell}, \tag{5.42}$$

makes that sum also negligible, as claimed in (5.40).

5.3. *Second moment*: *Proof of Lemma* 4.4

In view of (4.7) we have that

$$\mathbb{E}_s\big[\Lambda_{n,\ell}(\Lambda_{n,\ell}-1)\big]=\sum_{\substack{u,v\in V_{n'}\\ u\ne v}}\mathbb{P}_s\big(E_{n,\ell}(u)\cap E_{n,\ell}(v)\big).$$

We recall the definition (4.6) of $E_{n,\ell}(\cdot)$ and split the preceding sum according to the values of $k'=|u\wedge v|<n'$ and $\hat\eta_v(k')>0$. Specifically, having $2^{n'+k-1}$ such ordered pairs (for $k=n'-k'$), yields the bound

$$\mathbb{E}_{s_{n,z}}\big[\Lambda_{n,\ell}(\Lambda_{n,\ell}-1)\big]\le\sum_{k=1}^{n'}2^{2k}\int_0^\infty\theta_{n,k,\ell}^2(y)\nu_{n,k',z}(dy)=:\sum_{k=1}^{n'}\mathsf{J}_\mathsf{k} \tag{5.43}$$

in terms of $\theta_{n,k,\ell}(\cdot)$ and $\nu_{n,k',z}(\cdot)$ of (5.17) and (5.31), respectively. Further, the Markov property of $\eta_v(j)$ at $j=k'$ yields in terms of $q_{n,k',z}(\cdot;0)$ of (5.6)

$$\mathsf{J}_\mathsf{k}\le\sum_{i=0}^\infty 2^{k'}q_{n,k',z}(i;0)\Big[2^k\sup_{y\in H_i}\theta_{n,k,\ell}(y)\Big]^2.$$

Plugging in the preceding the bounds (5.17) and (5.7) (at $m=64n\le n^2$), we find that for some $c_6<\infty$, any $k\in(0,n')$, $z\ge1$ and $n\ge n_0(\ell)$ as in Lemma 5.4,

$$\mathsf{J}_\mathsf{k}\le c_6e^{\beta_{n,k'}+2\beta_{n,k}}(1\vee\ell/k)\left(\frac{z}{\sqrt{k_n'}}\wedge k'\right)e^{-c_*z}\sum_{i=0}^\infty(i+1)^3e^{-c_*i}. \tag{5.44}$$

Using (5.23), $\beta_{n,k}\le1$ and $k_n'\ge k_{n'}$ we get from (5.44) that for some $c_7<\infty$,

$$\mathsf{J}_\mathsf{k}\le\begin{cases}c_7ze^{-c_*z}k_{n'}^{-3/2}, & k_{n'}\ge\ell,\\ c_7e^{-c_*z}, & k'<\ell,\\ c_7ze^{-c_*z}\sqrt{\ell}k^{-3}, & k<\ell,\end{cases} \tag{5.45}$$

where for $k<\ell$ we used the alternative bounds $\beta_{n,k}\le1-\log k$ and $k_n'\ge\ell$. Now, (5.45) implies that for all n,

$$z^{-1}e^{c_*z}\sum_{k=\ell^{1/3}}^{n'}\mathsf{J}_k\le c_7\left[\sqrt{\ell}\sum_{k=\ell^{1/3}}^{\ell-1}k^{-3}+\sum_{k=\ell}^{n'-\ell}k_{n'}^{-3/2}+\sum_{k=n'-\ell}^{n'}z^{-1}\right]\le\delta_{\ell,z}$$

for some $\delta_{\ell,z} \to 0$, when $z \to \infty$ followed by $\ell \to \infty$. Turning to control the remaining sum of J_k over $k < \ell^{1/3}$, note that by (5.42), upon comparing (5.4) and (5.17) we find that for some $\varepsilon(\ell) \to 0$ as $\ell \to \infty$ and any $n \ge n_0(\ell)$,

$$\sum_{k=1}^{\ell^{1/3}} 2^k \sup_{y \ge 0} \{\theta_{n,k,\ell}(y)\} \le 4^{\ell^{1/3}} \sup_{r \in I_\ell} \{\gamma_{n,\ell}(r)\} \le \varepsilon(\ell).$$

To complete the proof of Lemma 4.4, recall (4.6) and (5.31), that for $k' = n' - k$

$$2^k \int_0^\infty \theta_{n,k,\ell}(y) \nu_{n,k',z}(dy) = \mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}]. \tag{5.46}$$

Hence, we have on the RHS of (5.43) that

$$\overline{\lim_{z\to\infty}} \left\{ z^{-1} e^{c_* z} \overline{\lim_{n\to\infty}} \sum_{k=1}^{\ell^{1/3}} \mathsf{J}_k \right\} \le \varepsilon(\ell) \overline{\lim_{z\to\infty}} \left\{ z^{-1} e^{c_* z} \overline{\lim_{n\to\infty}} \mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}] \right\},$$

which together with (4.22) and (5.3) imply that the RHS is for all ℓ large enough at most $2c\varepsilon(\ell)$ (i.e. negligible, as claimed).

6. The Bessel process: Proof of Proposition 4.3

Hereafter, set $\lambda_\ell(y) := \frac{1}{2}(c_*\ell + y)^2$, $y \ge -c^*\ell$, with

$$\begin{aligned} \widehat{\gamma}_\ell(y) &:= \mathbb{P}_{[\lambda_\ell(y)]}\big(\eta_\ell^\sharp = 0, \eta(0) \in c_*\ell + I_\ell\big), \\ \widetilde{\gamma}_\ell(y) &:= \mathbb{E}^\xi \big\{ \mathbb{P}_{\xi(\lambda_\ell(y))}\big(\eta_\ell^\sharp = 0, \eta(0) \in c_*\ell + I_\ell\big) \big\}, \end{aligned} \tag{6.1}$$

which are $\gamma_{\infty,\ell}(\cdot)$ from (5.4), restricted to I_ℓ of (4.4), and its regularization by an expectation, denoted $\mathbb{E}^\xi$, over the independent Poisson(λ) variable $\xi(\lambda)$ at $\lambda = \lambda_\ell(y)$. We emphasize that the law of ξ depends on ℓ but we suppress this from the notation. We follow this convention of suppressing dependence in ℓ, n in many places throughout this section, e.g. in the definitions (6.5), (6.10), (6.17), (6.24) and (6.28) below. Our goal here is to prove Proposition 4.3, with

$$\alpha_\ell := \frac{1}{\sqrt{\pi \ell}} \int_0^\infty y e^{c_* y} \widetilde{\gamma}_\ell(y)\, dy. \tag{6.2}$$

In particular $\alpha_\ell < \infty$, since by standard Poisson tail estimates for some $c < \infty$,

$$\widetilde{\gamma}_\ell(y) \le \mathbb{P}\Big(\sqrt{2\xi\big(\lambda_\ell(y)\big)} - c_*\ell \in I_\ell\Big) \le \exp\big\{-c \operatorname{dist}(y, I_\ell)^2\big\} \tag{6.3}$$

(see [8, (3.8)]). Omitting hereafter from the notation the (irrelevant) specific choice $v \in V_{n'}$, we recall from (4.5), (5.31) and (5.32) that

$$2^{-n'} \mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}] = \mathbb{E}_{s_{n,z}}\Big(\widehat{\gamma}_\ell\big(\eta(n') - c_*\ell\big); \min_{j<n'}\{\hat{\eta}(j)\} > 0\Big). \tag{6.4}$$

The first step towards Proposition 4.3 is our next lemma, utilizing the Markov structure from [8, Lemma 3.1] to estimate the barrier probabilities on the RHS of (6.4) via the law $\mathbb{P}_x^Y$ of a 0-dimensional Bessel process $\{Y_t\}$, starting at $Y_1 = x$. To this end, define for $\kappa \in \mathbb{R}$ the events

$$B_\kappa := \bigcap_{j=1}^{n'} \big\{ Y_j > \bar{\varphi}_n(j) + \kappa \psi_\ell(j) \big\}, \tag{6.5}$$

in terms of the barrier notations (4.3), (4.9), (4.10), and associate to each $[0, 1]$-valued $g(\cdot)$, the function

$$\widetilde{g}(w) := \mathbb{E}^\xi\big[g\big(\sqrt{2\xi\big(w^2/2\big)}\big)\big], \tag{6.6}$$

so in particular $g(w) = \widehat{\gamma}_\ell(w - c_*\ell)$ yields $\widetilde{g}(w) = \widetilde{\gamma}_\ell(w - c_*\ell)$.

Lemma 6.1. *There exist $U_s \stackrel{\text{dist}}{\Longrightarrow} U_\infty$, a centered Gaussian of variance $1/2$, and $\varepsilon_\ell \to 0$ as $\ell \to \infty$, such that for $s = s_{n,z}$, any $[0,1]$-valued $g(\cdot)$ supported on $[\bar{\varphi}_n(n'), \infty)$ and $z \ge \ell$,*

$$(1-\varepsilon_\ell)^{-1}\mathbb{E}^{Y,s}\big(\widetilde{g}(Y_{n'}); B_{-1}\big) \ge \mathbb{E}_s\Big(g\big(\eta\big(n'\big)\big); \min_{j<n'}\big\{\widehat{\eta}(j)\big\} > 0\Big) \tag{6.7}$$

$$\ge (1-\varepsilon_\ell)\mathbb{E}^{Y,s}\big(\widetilde{g}(Y_{n'}); B_2\big), \tag{6.8}$$

where $\mathbb{E}^{Y,s}$ denotes expectation with respect to a 0-dimensional Bessel process starting at $Y_1 = U_s + \sqrt{2s}$. Further, for some $\delta > 0$,

$$\sup_s\big\{\mathbb{E}\big[e^{\delta U_s^2}\big]\big\} < \infty. \tag{6.9}$$

Proof. Recall from [8, Lemma 3.1] the time in-homogeneous Markov chain

$$\big(\eta(0) = \sqrt{2[s]}, Y_1, \eta(1), \dots, Y_{n'}, \eta\big(n'\big), \dots\big),$$

of law $\mathbb{Q}_1^s = \mathbb{Q}_1^{[s]}$. From [8, Lemma 3.1(a)] we have that $\mathbb{Q}_1^s[g(\eta(j))|Y_j] = \widetilde{g}(Y_j)$ of (6.6), and that $Y_1 = \sqrt{2\mathcal{L}_1(s)}$ for a $\Gamma([s],1)$-random variable $\mathcal{L}_1(s)$. Set $U_s := \sqrt{2\mathcal{L}_1(s)} - \sqrt{2s}$ and note that by [8, Lemma 3.1(d,e)], the random variables $\{\eta(j), j \ge 0\}$ and $\{Y_j, j \ge 1\}$ have respectively, the marginal laws $\mathbb{P}_s$ and $\mathbb{P}^{Y,s}$.

Standard large deviations for Gamma variables yield (6.9) with $\delta < 1/2$ (cf. [8, (3.13)]). Recall that $(\mathcal{L}_1(s) - s)/\sqrt{2s} \stackrel{\text{dist}}{\Longrightarrow} U_\infty$ when $s \to \infty$ (by the CLT), hence the same convergence applies for $U_s = f_s((\mathcal{L}_1(s) - s)/\sqrt{2s})$ and $f_s(\cdot)$ of (2.16). In addition, setting for $k \in \mathbb{N}$ the events

$$A_k := \bigcap_{j=0}^{k}\big\{\eta(j) > \bar{\varphi}_n(j)\big\}, \tag{6.10}$$

we have by the preceding and (6.5) that the bound (6.7) follows from

$$\mathbb{Q}_1^s\big(g\big(\eta\big(n'\big)\big); A_{n'} \cap B_{-1}\big) \ge (1-\varepsilon_\ell)\mathbb{E}_s\big(g\big(\eta\big(n'\big)\big); A_{n'}\big) \tag{6.11}$$

(taking $k = n'$ due to the assumed support of $g(\cdot)$ and including $j = 0$ at no loss of generality since $z > 0$). Now, recall from [8, Lemma 3.1(b)] that

$$\mathbb{Q}_1^s\big(g\big(\eta\big(n'\big)\big); A_{n'} \cap B_{-1}\big) = \mathbb{E}_s\Bigg(g\big(\eta\big(n'\big)\big)\prod_{j=1}^{n'} F_j^\eta; A_{n'}\Bigg), \tag{6.12}$$

where if

$$\sqrt{\eta^2(j-1)/2 + \eta^2(j)/2} > \bar{\varphi}_n(j), \tag{6.13}$$

then by [8, (3.14)],

$$F_j^\eta := \mathbb{Q}_1^s\big(Y_j > \bar{\varphi}_n(j) - \psi_\ell(j) \mid \eta(j-1), \eta(j)\big) \ge 1 - ce^{-c\psi_\ell^2(j)}.$$

Since (6.13) holds on the event $A_{n'}$ for $j = 1, \dots, n'$ recalling (4.10) that $\psi_\ell(j) = \psi_\ell(n'-j)$ and splitting the product on the RHS of (6.12) to $j > n/2$ and $j \le n/2$, yields the inequality (6.11), and thereby (6.7), with

$$\varepsilon_\ell = 1 - \prod_{j=0}^{\infty}\big[1 - ce^{-c(j^\delta + h_\ell)^2}\big]^2, \tag{6.14}$$

which converges to zero when $\ell \to \infty$.

Recall from [8, (3.4)] the notation $\mathbb{Q}_2^{x^2/2}$ for the law of the Markov chain $(Y_1, \eta(1), \dots)$ started at $Y_1 = x$. To see (6.8), we will show

$$\mathbb{Q}_2^{x^2/2}\big(\widetilde{g}(Y_{n'}); A_{n'-1} \cap B_2\big) \ge (1-\varepsilon_\ell)\mathbb{P}_x^Y\big(\widetilde{g}(Y_{n'}); B_2\big). \tag{6.15}$$

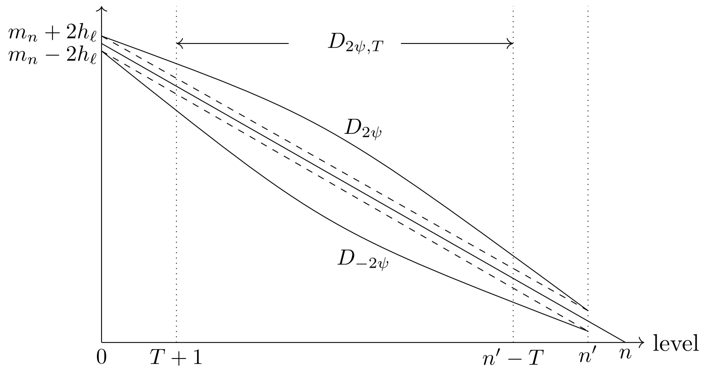

Fig. 3. The curves in the events $D_{\pm 2h_\ell}$ (dashed lines) and $D_{\pm 2\psi}$ (curved, solid lines). The event $D_{2\psi,T}$ involves the curves between $T+1$ and $n'-T$.

Taking the expectation over x with respect to the law of Y_1 under $\mathbb{Q}_1^s$ and arguing as in the proof of (6.7) will then give (6.8). Turning to establishing (6.15), recall from [8, Lemma 3.1(c)] that

$$\mathbb{Q}_2^{x^2/2}\big(\widetilde{g}(Y_{n'}); A_{n'-1} \cap B_2\big) = \mathbb{P}_x^Y\left(\widetilde{g}(Y_{n'}) \prod_{j=1}^{n'-1} F_j^Y; B_2\right),$$

where by [8, (3.16)],

$$F_j^Y := \mathbb{Q}_2^{x^2/2}\big(\eta(j) > \bar{\varphi}_n(j) \mid Y_j, Y_{j+1}\big) \ge 1 - ce^{-c\psi_\ell^2(j)},$$

provided that

$$\sqrt{Y_j Y_{j+1}} \ge \bar{\varphi}_n(j) + \psi_\ell(j). \tag{6.16}$$

On B_2, we have that for any $j < n'$, and all ℓ larger than some fixed universal constant,

$$\sqrt{Y_j Y_{j+1}} > \bar{\varphi}_n(j+1) + 2\psi_\ell(j+1) \ge \bar{\varphi}_n(j) + \psi_\ell(j).$$

In particular, with (6.16) holding on B_2 for any $j \in \{1, \ldots, n'-1\}$, by the same reasoning as before, this yields the inequality (6.15), and hence also (6.8). □

We next estimate the barrier probabilities for $\{Y_j\}$ in terms of the law $\mathbb{P}_x^W$ of a Brownian motion $\{W_t\}$, starting at $W_1 = Y_1 = x$. For $0 \le T < T' \le n'$, introduce the events

$$D_{\pm 2\psi,T,T'} := \big\{W_t > \bar{\varphi}_n(t) \pm 2\psi_\ell(t), \forall t \in \big[T+1, T'\big]\big\}, \tag{6.17}$$

using hereafter $D_{\kappa h_\ell,T,T'}$ if $\pm 2\psi_\ell(t)$ in (6.17) is replaced by the constant function κh_ℓ, with abbreviated notation $D_{\pm 2\psi,T}$ when $T' = n' - T$ and $D_{\pm 2\psi}$ for $D_{\pm 2\psi,0}$. See Figure 3 for a pictorial description. Recall the sets B_κ, see (6.5).

Lemma 6.2. *For $\widehat{g}(w) := \widetilde{g}(w)/\sqrt{w}$, some $\varepsilon_\ell \to 0$, any $\widetilde{g}(\cdot)$, n, ℓ as in Lemma 6.1 and all $x > 0$,*

$$\mathbb{E}_x^Y\big(\widetilde{g}(Y_{n'}); B_2\big) \ge (1-\varepsilon_\ell)\sqrt{x}\,\mathbb{E}_x^W\big(\widehat{g}(W_{n'}); D_{2\psi}\big), \tag{6.18}$$

$$\mathbb{E}_x^Y\big(\widetilde{g}(Y_{n'}); B_{-1}\big) \le (1-\varepsilon_\ell)^{-1}\sqrt{x}\,\mathbb{E}_x^W\big(\widehat{g}(W_{n'}); D_{-2\psi}\big). \tag{6.19}$$

Proof. Recall that up to the absorption time $\tau_* := \inf\{t > 1 : Y_t = 0\}$, the 0-dimensional Bessel process satisfies the SDE

$$Y_t = W_t - \int_1^t \frac{1}{2Y_s}\,ds, \qquad Y_1 = W_1 = x,$$

with $\{W_t\}$ having the Brownian law $\mathbb{P}^W_x$. Further, the event $\{Y_{n'} > 0\}$ implies that $\{\tau_* > n'\}$, in which case by Girsanov's theorem and monotone convergence, we have that for any bounded $\mathcal{F}_{n'}$-measurable Z,

$$\mathbb{E}^Y_x(Z; Y_{n'} > 0) = \mathbb{E}^W_x\left(Z\sqrt{\frac{x}{W_{n'}}}e^{-\frac{3}{8}\int_1^{n'}(W_s)^{-2}\,ds}; \inf_{t\in[1,n']}\{W_t\} > 0\right). \tag{6.20}$$

With B_2 containing the event corresponding to $D_{2\psi}$ for the process Y_t, we get (6.18) by considering (6.20) for $Z = \widetilde{g}(W_{n'})\mathbf{1}_{D_{2\psi}}$. Indeed, the event $D_{2\psi}$ implies that $\inf_{t\le n'}\{W_t - \bar{\varphi}_n(t)\} \ge 0$, hence

$$e^{-\frac{3}{8}\int_1^{n'}(W_s)^{-2}\,ds} \ge e^{-\frac{3}{8}\int_1^{n'}(n-s)^{-2}\,ds} \ge e^{-\frac{3}{8\ell}} := 1 - \varepsilon_\ell$$

with $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Next, for all ℓ larger than some universal constant,

$$\inf_{t\in[1,n']}\bigl\{\bar{\varphi}_n(t) - 2\psi_\ell(t)\bigr\} = \bar{\varphi}_n\bigl(n'\bigr) - 2h_\ell > 0,$$

and with $B' := \{W_j > \bar{\varphi}_n(j) - \psi_\ell(j),\ j = 1, 2, \dots, n'\}$, it suffices for (6.19) to show that

$$\mathbb{E}^W_x\bigl(\widehat{g}(W_{n'}); D_{-2\psi} \cap B'\bigr) \ge (1 - \varepsilon_\ell)\mathbb{E}^W_x\bigl(\widehat{g}(W_{n'}); B'\bigr). \tag{6.21}$$

To this end, since $\phi_t := \bar{\varphi}_n(t) - 2\psi_\ell(t)$ is a convex function, we get upon conditioning on $\{W_1, W_2, \dots, W_{n'}\}$, that

$$\mathbb{E}^W_x\bigl(\widehat{g}(W_{n'}); D_{-2\psi} \cap B'\bigr) \ge \mathbb{E}^W_x\left(\widehat{g}(W_{n'})\prod_{j=1}^{n'-1}F^W_j; B'\right),$$

where by the reflection principle (see [8, (2.1)] or [13, Lemma 2.2]),

$$\begin{aligned} F^W_j &:= \mathbb{P}^W\Bigl(\min_{u\in[0,1]}\bigl\{W_{j+u} - f_{\phi_j,\phi_{j+1}}(u; 1)\bigr\} > 0 \mid W_j, W_{j+1}\Bigr) \\ &= 1 - \exp\bigl(-2(W_j - \phi_j)(W_{j+1} - \phi_{j+1})\bigr), \end{aligned} \tag{6.22}$$

with $f_{a,b}(\cdot; 1)$ denoting the line segment between $(0, a)$ and $(1, b)$. On the event B' we thus have that $F^W_j \ge 1 - \exp(-2\psi_\ell(j)\psi_\ell(j+1))$ for all $j \in \{1, 2, \dots, n'-1\}$, thereby in analogy with (6.14), establishing (6.21) for

$$\varepsilon_\ell = 1 - \prod_{j=0}^{\infty}\bigl[1 - e^{-2(j^\delta+h_\ell)((j+1)^\delta+h_\ell)}\bigr]^2,$$

which converges to zero as $\ell \to \infty$. □

6.1. *Proof of Proposition* 4.3

Taking $s = s_{n,z}$ yields that $W_1 = m_n + z + U_s$. For such W_1 let

$$\alpha^{(\pm)}_{n,\ell,z} := z^{-1}e^{c_*z}2^{n'}\mathbb{E}\bigl[\sqrt{W_1/W_{n'}}\widetilde{\gamma}_\ell(W_{n'} - c_*\ell)\mathsf{q}^{(\pm)}_{\widetilde{n}}(W_1, W_{n'})\bigr], \tag{6.23}$$

with $\widetilde{n} := n' - 1$ denoting our barrier length and $\mathsf{q}^{(\pm)}_{\widetilde{n}}(x, w) := \mathsf{q}^{(\pm)}_{\widetilde{n},0}(x, w)$ for the corresponding non-crossing probabilities

$$\mathsf{q}^{(\pm)}_{\widetilde{n},T}(x, w) := \mathbb{P}^W_x(D_{\pm 2\psi,T} \mid W_{n'} = w). \tag{6.24}$$

Combining (6.4) with Lemmas 6.1 and 6.2 for $g(\cdot) = \widehat{\gamma}_\ell(\cdot - c_*\ell)$ and $\widetilde{g}(\cdot) = \widetilde{\gamma}_\ell(\cdot - c_*\ell)$, respectively, we have that

$$(1 - \varepsilon_\ell)^{-2}\alpha^{(-)}_{n,\ell,z} \ge z^{-1}e^{c_*z}\mathbb{E}_{s_{n,z}}[\Lambda_{n,\ell}] \ge (1 - \varepsilon_\ell)^2\alpha^{(+)}_{n,\ell,z}.$$

The proof of Proposition 4.3 thus amounts to showing that for any $\epsilon > 0$ and all large enough ℓ,

$$(1 + \epsilon)^3\alpha_\ell \ge \varlimsup_{z\to\infty}\varlimsup_{n\to\infty}\bigl\{\alpha^{(-)}_{n,\ell,z}\bigr\} \ge \varliminf_{z\to\infty}\varliminf_{n\to\infty}\bigl\{\alpha^{(+)}_{n,\ell,z}\bigr\} \ge (1 - \epsilon)^3\alpha_\ell. \tag{6.25}$$

To this end, setting $z' = z + U_s$ and $W_{n'} = c_*\ell + y$, we write (6.23) explicitly as

$$\alpha^{(\pm)}_{n,\ell,z} = \frac{e^{c_* z} 2^{n'}}{z\sqrt{2\pi\widetilde{n}}} \mathbb{E}\left[\int \widetilde{\gamma}_\ell(y)\, dy \frac{\sqrt{m_n + z'}}{\sqrt{c_*\ell + y}} \mathsf{q}^{(\pm)}_{\widetilde{n}}\big(m_n + z', c_*\ell + y\big) e^{-(m_n - c_*\ell + z' - y)^2/2\widetilde{n}}\right],$$

with the expectation over z'. Hereafter $\ell \le n/\log n$ so $|c_* - \rho_n|\ell \le 1$ and $D_{\pm 2\psi}$ imposes heights $a^{(\pm)} = \rho_n\widetilde{n} + b^{(\pm)}$, $b^{(\pm)} = \rho_n\ell \pm 2h_\ell$ at barrier end points. Thus, in the preceding formula one needs only consider z', $y \ge \pm 3h_\ell$. Recall that $m_n - c_*\ell = c_* n' - \varepsilon_{n,n}$ for $\varepsilon_{n,n}$ of (5.13). Thus, upon setting $\Delta_n := \frac{1}{2\widetilde{n}}(z' - y + c_* - \varepsilon_{n,n})^2$, we then get similarly to (5.14) that

$$\frac{1}{2\widetilde{n}}\big(c_* n' + z' - y - \varepsilon_{n,n}\big)^2 = \big(n' + 1\big)\log 2 + c_*\big(z' - y - \varepsilon_{n,n}\big) + \Delta_n.$$

Since $c_*\varepsilon_{n,n} = \log n$, this simplifies our formula for $\alpha^{(\pm)}_{n,\ell,z}$ to

$$\begin{aligned} \alpha^{(\pm)}_{n,\ell,z} &= \frac{1}{\sqrt{\pi\ell}} \int_{\pm 3h_\ell}^{\infty} e^{c_* y} \widetilde{\gamma}_\ell(y) \frac{\widehat{f}^{(\pm)}_{n,\ell,z}(y)}{\sqrt{1 + y/(c_*\ell)}}\, dy, \\ \widehat{f}^{(\pm)}_{n,\ell,z}(y) &:= \frac{n}{2\sqrt{2}z} \mathbb{E}\left[\frac{\sqrt{m_n + z'}}{\sqrt{c_*\widetilde{n}}} \mathsf{q}^{(\pm)}_{\widetilde{n}}\big(m_n + z', c_*\ell + y\big) e^{-c_* U_s} e^{-\Delta_n}\right]. \end{aligned} \tag{6.26}$$

By our uniform tail estimate (6.9) for U_s and the tail bound (6.3) on $\widetilde{\gamma}_\ell(y)$, up to an error $\varepsilon_n \to 0$ as $n \to \infty$, we can restrict the evaluation of $\widehat{f}^{(\pm)}_{n,\ell,z}(y)$ to $|z'| + y \le C\sqrt{\log n}$. This forces $m_n + z' = c_*\widetilde{n}(1 + \varepsilon_n)$ and eliminates Δ_n, thereby allowing us to replace $\widehat{f}^{(\pm)}_{n,\ell,z}(y)$ in (6.26) by

$$f^{(\pm)}_{n,\ell,z}(y) = \mathbb{E}\left[\frac{e^{-c_* U_s}}{\sqrt{2}z} \frac{\widetilde{n}}{2} \mathsf{q}^{(\pm)}_{\widetilde{n}}\big(m_n + z', c_*\ell + y\big)\right]. \tag{6.27}$$

Recalling the events $D_{\kappa h_\ell, T}$, see below (6.17), we further consider the barrier probabilities

$$\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n},T}(x, w) := \mathbb{P}^W_x(D_{\pm 2h_\ell, T} \mid W_{n'} = w), \tag{6.28}$$

using the abbreviated notation $\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n}}(x, w) = \widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n},0}(x, w)$. Let $\mathbb{P}^{[t_1,t_2]}_{x\to w}(A_{m(t)})$ denote the probability that the Brownian bridge, taking the value x at t_1 and w at t_2 remains above the barrier $m(t)$ on the interval $[t_1, t_2]$. Recall from [13, Lemma 2.2] that for a linear barrier $m(t)$,

$$\mathbb{P}^{[t_1,t_2]}_{x\to w}(A_{m(t)}) = 1 - e^{-2(x - m(t_1))_+(w - m(t_2))_+/(t_2 - t_1)}. \tag{6.29}$$

It follows that

$$\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n}}(x, w) = \mathbb{P}^{[1,n']}_{x\to w}(A_{\bar{\varphi}_n(t)\pm 2h_\ell}) = 1 - e^{-2(x - a^{(\pm)})_+(w - b^{(\pm)})_+/\widetilde{n}}. \tag{6.30}$$

Fixing ℓ, with $\varepsilon_{n,\ell} \to 0$ and ρ_n bounded, taking $x - a^{(\pm)} = z' + \rho_n \mp 2h_\ell$ and $w - b^{(\pm)} = y - \varepsilon_{n,\ell} \mp 2h_\ell$ (which are both $O(\sqrt{\log n})$), we have for some $\varepsilon_n \to 0$,

$$\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n}}\big(m_n + z', c_*\ell + y\big) = \frac{2 + \varepsilon_n}{\widetilde{n}}\big(z' + c_* \mp 2h_\ell\big)(y \mp 2h_\ell). \tag{6.31}$$

Note further that

$$\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n},T}(x, w) = \mathbb{E}^W_x\big[\mathbb{P}^{[T+1,n'-T]}_{W_{T+1}\to W_{n'-T}}(A_{\bar{\varphi}_n(t)\pm 2h_\ell}) \mid W_{n'} = w\big]. \tag{6.32}$$

The next lemma paraphrases [13, Proposition 6.1] (with the proof given there also yielding the claimed uniformity).

Lemma 6.3. *For each $\epsilon > 0$ there exist T_ϵ, n_ϵ finite so that, for any $\ell \ge 0$, $T \in [T_\epsilon, \frac{1}{2}\widetilde{n}]$, $x - a^{(\pm)}$, $w - b^{(\pm)} \in [0, \log\widetilde{n}]$ and all $\widetilde{n} > n_\epsilon$*

$$(1 - \epsilon)\widetilde{\mathsf{q}}^{(+)}_{\widetilde{n},T}(x, w) \le \mathsf{q}^{(+)}_{\widetilde{n},T}(x, w) \le \mathsf{q}^{(-)}_{\widetilde{n},T}(x, w) \le (1 + \epsilon)\widetilde{\mathsf{q}}^{(-)}_{\widetilde{n},T}(x, w). \tag{6.33}$$

Fixing $\epsilon > 0$, we bound separately $f^{(\pm)}_{n,\ell,z}(y)$. Starting with $f^{(-)}_{n,\ell,z}(y)$, we have from (6.27), using the fact that $\mathsf{q}^{(-)}_{\widetilde{n}} \le \mathsf{q}^{(-)}_{\widetilde{n},T_\epsilon}$ and the RHS of (6.33), that

$$\begin{aligned} f^{(-)}_{n,\ell,z}(y) &\le \mathbb{E}\biggl[\frac{e^{-c_* U_s}}{\sqrt{2}z}\frac{\widetilde{n}}{2}\mathsf{q}^{(-)}_{\widetilde{n},T_\epsilon}\bigl(m_n+z', c_*\ell+y\bigr)\biggr] \\ &\le (1+\epsilon)\mathbb{E}\biggl[\frac{e^{-c_* U_s}}{\sqrt{2}z}\frac{\widetilde{n}}{2}\widetilde{\mathsf{q}}^{(-)}_{\widetilde{n},T_\epsilon}\bigl(m_n+z', c_*\ell+y\bigr)\biggr]. \end{aligned} \tag{6.34}$$

Turning to evaluate $\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n},T}(m_n+z', c_*\ell+y)$, we get from (6.32) and (6.29) that

$$\widetilde{\mathsf{q}}^{(\pm)}_{\widetilde{n},T}\bigl(m_n+z', c_*\ell+y\bigr) \le \frac{2}{\widetilde{n}-2T}\mathbb{E}\bigl[(\bar{Z}\mp 2h_\ell)_+(\bar{Y}\mp 2h_\ell)_+\bigr], \tag{6.35}$$

where $(\bar{Z}, \bar{Y})$ follow the joint Gaussian distribution of

$$\bigl(W_{T+1}-\bar{\varphi}_n(T+1), W_{n'-T}-\bar{\varphi}_n\bigl(n'-T\bigr)\bigr)$$

given $W_1 = m_n + z'$ and $W_{n'} = c_*\ell + y$. It is further easy to verify that

$$\operatorname{Cov}(\bar{Z}, \bar{Y}) = T\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{T^2}{\widetilde{n}}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}, \tag{6.36}$$

and that

$$\mathbb{E}\bigl[(\bar{Z}, \bar{Y})\bigr] - \biggl[z' + c_* + \frac{T}{\widetilde{n}}\bigl(y-z'\bigr), y + \frac{T}{\widetilde{n}}\bigl(z'-y\bigr)\biggr] = o_{\widetilde{n}}(1)$$

independently of (z', y), decaying to zero when $\widetilde{n} \to \infty$ with ℓ, T kept fixed. From this we get, in view of (6.34) and (6.35), that

$$\varlimsup_{z\to\infty}\varlimsup_{n\to\infty}\bigl\{f^{(-)}_{n,\ell,z}(y)\bigr\} \le (1+\epsilon)(y+3h_\ell)\lim_{z,s\to\infty}\mathbb{E}\biggl[\frac{(z'+3h_\ell)_+}{\sqrt{2}z}e^{-c_* U_s}\biggr]$$

provided $h_\ell \ge c_* + \sqrt{T_\epsilon/(2\pi)}$. In addition, with $\mathcal{B} = \{|U_s| < z/2\}$, or without such restriction, we get thanks to (6.9), via dominated convergence that

$$\lim_{z,s\to\infty}\mathbb{E}\biggl[\mathbf{1}_{\mathcal{B}}\frac{(z'\pm 3h_\ell)_+}{\sqrt{2}z}e^{-c_* U_s}\biggr] = \frac{1}{\sqrt{2}}\mathbb{E}\bigl(e^{-c_* U_\infty}\bigr) = \frac{1}{\sqrt{2}}e^{c_*^2/4} = 1. \tag{6.37}$$

(Recall Lemma 6.1 that $U_\infty \sim N(0, 1/2)$.) Combined with the previous display, we obtain

$$\varlimsup_{z\to\infty}\varlimsup_{n\to\infty}\bigl\{f^{(-)}_{n,\ell,z}(y)\bigr\} \le (1+\epsilon)(y+3h_\ell). \tag{6.38}$$

Note that for $\ell \to \infty$

$$\frac{1}{\sqrt{\pi}\ell}\int_{-3h_\ell}^{\infty} e^{c_* y}\widetilde{\gamma}_\ell(y)\frac{y+3h_\ell}{\sqrt{1+y/(c_*\ell)}}\,dy \le \alpha_\ell(1+\epsilon). \tag{6.39}$$

(Due to (6.3) the contribution to α_ℓ outside $[\sqrt{\ell}/(2r_\ell), 2r_\ell\sqrt{\ell}]$ is negligible, whereas within that interval $y/\ell \to 0$ and $h_\ell/y \to 0$.) Combining (6.26), (6.38) and (6.39) yields the LHS of (6.25), thereby completing the proof of the upper bound in Proposition 4.3.

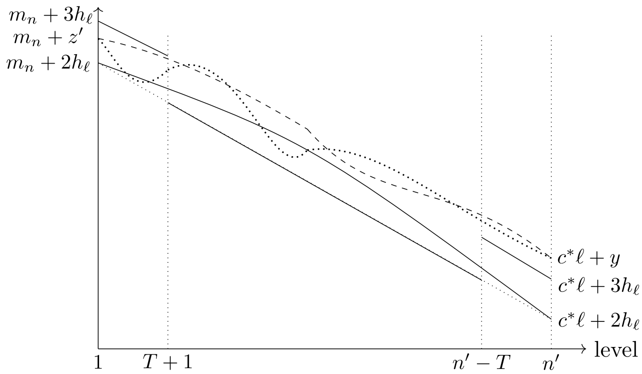

Fig. 4. Depiction of the events $D_{2\psi}$ (dashed curve) and $D^c_{3h_\ell,0,T+1} \cap D_{2h_\ell,T}$ (dotted line).

Turning next to the lower bound on $f^{(+)}_{n,\ell,z}(y)$, we first truncate to $y \in [\sqrt{\ell}/(2r_\ell), \sqrt{\ell}2r_\ell]$ and restrict to $z' \in [\frac{z}{2}, \frac{3z}{2}]$ via the event $\mathcal{B}$. Then, taking $h_\ell \ge 2(T+1)^\delta$ for $T = T_\epsilon$ of Lemma 6.3, guarantees that

$$3h_\ell \ge \sup_{t\in[1,T+1]\cup[n'-T,n']} \{2\psi_\ell(t)\},$$

for $\psi_\ell(\cdot) \ge h_\ell$ of (4.10). This in turn implies that

$$D_{2\psi,T} \subset D_{2\psi} \cup \big(D^c_{3h_\ell,0,T+1} \cap D_{2h_\ell,T}\big) \cup \big(D_{2h_\ell,T} \cap D^c_{3h_\ell,\widetilde{n}-T,n'}\big), \tag{6.40}$$

where $D^c_{3h_\ell,T,T'}$ denotes the event of the Brownian motion crossing below the linear barrier in the definition of $D_{3h_\ell,T,T'}$, see below (6.17). See Figure 4 for an illustration of these events. From (6.40) and the LHS of (6.33) we deduce by the union bound, that at $x = m_n + z'$, $w = c_*\ell + y$, for all $\widetilde{n}$ large enough

$$\mathsf{q}^{(+)}_{\widetilde{n}}(x,w) \ge (1-\epsilon)\widetilde{\mathsf{q}}^{(+)}_{\widetilde{n},T}(x,w) - \widetilde{\mathsf{q}}^{(\downarrow 3h_\ell,+)}_{\widetilde{n},T}(x,w) - \widetilde{\mathsf{q}}^{(+,\downarrow 3h_\ell)}_{\widetilde{n},T}(x,w), \tag{6.41}$$

where $\widetilde{\mathsf{q}}^{(\downarrow 3h_\ell,+)}_{\widetilde{n},T}(x,w)$ and $\widetilde{\mathsf{q}}^{(+,\downarrow 3h_\ell)}_{\widetilde{n},T}(x,w)$ are the probabilities of the events $D^c_{3h_\ell,0,T+1} \cap D_{2h_\ell,T}$ and $D_{2h_\ell,T} \cap D^c_{3h_\ell,\widetilde{n}-T,n'}$ under $\mathbb{P}^W_x(\cdot \mid W_{n'} = w)$.

Proceeding to evaluating the latter terms, note that conditional on $(\bar{Z},\bar{Y})$ and the given values of $W_1 = x$, $W_{n'} = w$, the events $D_{3h_\ell,0,T+1}$, $D_{2h_\ell,T}$ and $D_{3h_\ell,\widetilde{n}-T,n'}$ are mutually independent. Thus, setting $y' = w - \rho_n\ell \in [y, y+1]$ and assuming WLOG that $(\sqrt{\ell}/r_\ell) \wedge z \ge 8h_\ell$, we have from (6.29), that

$$\mathbb{P}\big(D^c_{3h_\ell,0,T+1} \mid \bar{Z},\bar{Y}\big) = 1 - \mathbb{P}^{[1,T+1]}_{x\to W_{T+1}}(A_{\bar{\varphi}_n(t)+3h_\ell}) = e^{-2(z'+\rho_n-3h_\ell)(\bar{Z}-3h_\ell)_+/T},$$
$$\mathbb{P}\big(D^c_{3h_\ell,\widetilde{n}-T,n'} \mid \bar{Z},\bar{Y}\big) = 1 - \mathbb{P}^{[n'-T,n']}_{W_{n'-T}\to w}(A_{\bar{\varphi}_n(t)+3h_\ell}) = e^{-2(\bar{Y}-3h_\ell)_+(y'-3h_\ell)/T}.$$

Combining these identities with (6.32) and the inequality (6.41), we arrive at

$$\begin{aligned}&\mathsf{q}^{(+)}_{\widetilde{n}}(x,w)\\ &\quad\ge \mathbb{E}\big[\big(1-\epsilon-e^{-2(z'+\rho_n-3h_\ell)(\bar{Z}-3h_\ell)_+/T} - e^{-2(y'-3h_\ell)(\bar{Y}-3h_\ell)_+/T}\big)\mathbb{P}^{[T+1,n'-T]}_{\bar{Z}+\bar{\varphi}_n(T+1)\to\bar{Y}+\bar{\varphi}_n(n'-T)}(A_{\bar{\varphi}_n(t)+2h_\ell})\big].\end{aligned}$$

The first factor on the RHS is at least -2 and for all ℓ larger than some universal $\ell_0(\epsilon)$ it exceeds $(1-\epsilon)^2$ on the event $A := \{\bar{Z} \wedge \bar{Y} \ge 4h_\ell\}$. Setting $V := (\bar{Z}-2h_\ell)_+(\bar{Y}-2h_\ell)_+$, we combine for the second term on the RHS the analog of identity (6.30) with the bound $1 - e^{-a} \in [a - a^2/2, a]$ on $\mathbb{R}_+$ to arrive at

$$(\widetilde{n}-2T)\mathsf{q}^{(+)}_{\widetilde{n}}(x,w) \ge 2\mathbb{E}\left[\left\{(1-\epsilon)^2 - 3\mathbf{1}_{A^c} - \frac{2V}{\widetilde{n}-2T}\right\}V\right].$$

Utilizing (6.27), the uniform tail bounds one has on $(\bar{Z}-z', \bar{Y}-y)$ when $\widetilde{n} \to \infty$, for our truncated range of z' and y, followed by (6.37), we conclude that

$$\begin{aligned}\varliminf_{z\to\infty}\varliminf_{n\to\infty}\left\{f_{n,\ell,z}^{(+)}(y)\right\} &\geq \varliminf_{z\to\infty}\varliminf_{n\to\infty}\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}\frac{e^{-c_*U_s}}{\sqrt{2}z}\frac{\widetilde{n}}{2}\mathsf{q}_{\widetilde{n}}^{(+)}\left(m_n+z', c_*\ell+y\right)\right] \\ &\geq (1-\epsilon)^2(y-3h_\ell)\varliminf_{z,s\to\infty}\mathbb{E}\left[\mathbf{1}_{\mathcal{B}}\frac{(z'-3h_\ell)_+}{\sqrt{2}z}e^{-c_*U_s}\right] \geq (1-\epsilon)^2(y-3h_\ell).\end{aligned}$$

Plugging this into (6.26) and noting that for $\ell \to \infty$

$$\frac{1}{\sqrt{\pi\ell}}\int_{\sqrt{\ell}/(2r_\ell)}^{2r_\ell\sqrt{\ell}} e^{c_*y}\widetilde{\gamma}_\ell(y)\frac{y-3h_\ell}{\sqrt{1+y/(c_*\ell)}}\,dy \geq \alpha_\ell(1-\epsilon)$$

we arrive at the RHS of (6.25), thereby completing the proof of Proposition 4.3.

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