Bourgain discretization using Lebesgue-Bochner spaces

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This paper is dedicated to the memory of our friend Joe Diestel (1943–2017). Lebesgue-Bochner spaces were one of the main passions of Joe. He started to work in this direction in his Ph.D. thesis [Die68], and devoted to Lebesgue-Bochner spaces a large part of his most popular, classical, Dunford-Schwartz-style monograph [DU77], joint with Jerry Uhl.

Abstract

We study the Lebesgue-Bochner discretization property of Banach spaces Y, which ensures that the Bourgain discretization modulus for Y has a good lower estimate. We prove that there exist spaces that do not have the Lebesgue-Bochner discretization property, and we give a class of examples of spaces that enjoy this property.

1 Introduction

We denote by $c_Y(X)$ the greatest lower bound of distortions of bilipschitz embeddings of a metric space (X, d_X) into a metric space (Y, d_Y), that is, the greatest lower bound of the numbers C for which there are a map $f : X \to Y$ and a real number $r > 0$ such that

$$\forall u, v \in X \quad r d_X(u, v) \le d_Y(f(u), f(v)) \le rC d_X(u, v).$$

See [Mat02], [Nao18], and [Ost13] for background on this notion. Let X be a finite-dimensional Banach space and Y be an infinite-dimensional Banach space.

Definition 1.1. For $\varepsilon \in (0, 1)$ let $\delta_{X \hookrightarrow Y}(\varepsilon)$ be the supremum of those $\delta \in (0, 1)$ for which every δ-net $\mathcal{N}_\delta$ in B_X satisfies $c_Y(\mathcal{N}_\delta) \ge (1-\varepsilon) c_Y(X)$. The function $\delta_{X \hookrightarrow Y}(\varepsilon)$ is called the *Bourgain discretization modulus for embeddings of X into Y.*

It is not immediate that the discretization modulus is defined for any $\varepsilon \in (0, 1)$, but this can be derived using the argument of [Rib76] and [HM82] (see [GNS12, Introduction]). Giving a new proof of the Ribe theorem [Rib76], Bourgain proved the following remarkable result [Bou87] (we state it in a stronger form which was proved in [GNS12]):

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Theorem 1.2 (Bourgain's discretization theorem)**.** *There exists $C \in (0,\infty)$ such that for every two Banach spaces X, Y with $\dim X = n < \infty$ and $\dim Y = \infty$, and every $\varepsilon \in (0,1)$, we have*

$$\delta_{X \hookrightarrow Y}(\varepsilon) \geq e^{-(c_Y(X)/\varepsilon)^{Cn}}. \tag{1}$$

Bourgain's discretization theorem and the described below result of [GNS12] on improved estimates in the case of L_p spaces have important consequences for quantitative estimates of L_1-distortion of the metric space consisting of finite subsets (of equal cardinality) in the plane with the minimum weight matching distance, see [NS07, Theorem 1.2].

The proof of Bourgain's discretization theorem was clarified and simplified in [Beg99] and [GNS12] (see also its presentation in [Ost13, Section 9.2]). Different approaches to proving Bourgain's discretization theorem in special cases were found in [LN13], [HLN16], and [HN16+]. However these approaches do not improve the order of estimates for the discretization modulus.

On the other hand the paper [GNS12] contains a proof with much better estimates in the case where $Y = L_p$. The approach of [GNS12] is based on the following result (whose proof uses methods of [JMS09]; origins of this approach can be found in [GK03]).

Theorem 1.3 ([GNS12, Theorem 1.3])**.** *There exists a universal constant $\kappa \in (0,\infty)$ with the following property. Assume that $\delta, \varepsilon \in (0,1)$ and $D \in [1,\infty)$ satisfy $\delta \leq \kappa\varepsilon^2/(n^2 D)$. Let X, Y be Banach spaces with $\dim X = n < \infty$, and let $\mathcal{N}_\delta$ be a δ-net in B_X. Assume that $c_Y(\mathcal{N}_\delta) \leq D$. Then there exists a separable probability space (Ω, ν), a finite dimensional linear subspace $Z \subseteq Y$, and a linear operator $T : X \to L_\infty(\nu, Z)$ satisfying*

$$\forall x \in X, \quad \frac{1-\varepsilon}{D}\|x\|_X \leq \|Tx\|_{L_1(\nu,Z)} \leq \|Tx\|_{L_\infty(\nu,Z)} \leq (1+\varepsilon)\|x\|_X. \tag{2}$$

Remark 1.4. The measure ν in the proof of [GNS12, Theorem 1.3] is atomless, it is the normalized Lebesgue measure on the unit ball of X.

As is noted in [GNS12], since (Ω, ν) is a probability measure, we have

$$\|\cdot\|_{L_1(\nu,Z)} \leq \|\cdot\|_{L_p(\nu,Z)} \leq \|\cdot\|_{L_\infty(\nu,Z)}$$

for every $p \in [1,\infty]$. Therefore (2) implies that X admits an embedding into $L_p(\nu, Z)$ with distortion $\leq \dfrac{D(1+\varepsilon)}{1-\varepsilon}$. Since, by the well-known Carathéodory theorem, $L_p(\nu, L_p)$ is isometric to L_p (see [Lac74, §14]), we get that if Z is a subspace of L_p, then $L_p(\nu, Z)$ is also a subspace of L_p. As explained in [GNS12], it follows that the Bourgain's discretization modulus for the case of $Y = L_p$ satisfies a much better estimate

$$\delta_{X \hookrightarrow L_p}(\varepsilon) \geq \frac{\kappa\varepsilon^2}{n^{5/2}}$$

(since for all spaces X, Y and all $\delta > 0$, $c_Y(\mathcal{N}_\delta) \leq \sqrt{n}$, see [GNS12]).

To generalize this approach to a wider class of spaces, it is natural to introduce the following definition.

Definition 1.5. We say that a Banach space Y has *the Lebesgue-Bochner discretization property* if there exists a function $f : [1, \infty) \to [1, \infty)$ such that for any separable probability measure μ, for any $C \geq 1$ and any finite dimensional subspace $Z \subset Y$, if W is any finite-dimensional subspace of $L_\infty(\mu, Z)$ such that for all $w \in W$

$$\|w\|_{L_\infty(\mu,Z)} \leq C\|w\|_{L_1(\mu,Z)}, \tag{3}$$

then W is $f(C)$-embeddable into Y.

Lemma 1.6. *In Definition 1.5 it suffices to require that the function f exists in the case where μ is the Lebesgue measure on $[0,1]$. A function f which works for some separable atomless probability measure space works for any other separable atomless probability measure space.*

Proof. The second statement follows immediately from the Carathéodory theorem [Lac74, §14] stating that all separable atomless probability measure spaces are isomorphic.

To prove the first statement assume that f is a function which works for any separable atomless probability measure space, and let (Ω, Σ, μ) be a separable probability measure space with atoms $\{a_i\}_{i=1}^M$, where M is either a positive integer or ∞. Let $\tilde{\Omega} = \Omega \backslash \{a_i\}_{i=1}^M$. For each i let $(\Omega_i, \Sigma_i, \mu_i)$ be a separable atomless measure space satisfying $\mu_i(\Omega_i) = \mu(a_i)$. We create a separable atomless probability measure space as the disjoint union

$$\tilde{\Omega} \cup \left(\bigcup_{i=1}^M \Omega_i \right)$$

with the natural σ-algebra and measure, we denote this measure by ν.

For a subspace $W \subset L_\infty(\mu, Z)$ satisfying (3) we introduce a subspace $\tilde{W} \subset L_\infty(\nu, Z)$ as the set of all vectors obtained in the following way: for each $w \in W$ we introduce $\tilde{w} \in L_\infty(\nu, Z)$ by

$$\tilde{w}(t) = \begin{cases} w(t) & \text{if } t \in \tilde{\Omega}, \\ w(a_i) & \text{if } t \in \Omega_i. \end{cases}$$

It is easy to see that $\tilde{W}$ is isometric to W, and that

$$\|\tilde{w}\|_{L_\infty(\nu,Z)} \leq C\|\tilde{w}\|_{L_1(\nu,Z)}. \tag{4}$$

Thus, the function f which can be used for the atomless measure ν, can also be used for μ. $\square$

We need the following generalization of the Bourgain discretization modulus: Given an increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$, we define $\delta^g_{X \hookrightarrow Y}(\varepsilon)$ as the supremum of δ such that for all δ-nets $\mathcal{N}_\delta$ of B_X,

$$c_Y(X) \leq g\left(\frac{1+\varepsilon}{1-\varepsilon} c_Y(\mathcal{N}_\delta)\right). \tag{5}$$

The following is a corollary of Theorem 1.3.

Corollary 1.7. *Let Y be a Banach space with the Lebesgue-Bochner discretization property. Then*

$$\delta^g_{X\hookrightarrow Y}(\varepsilon) \geq \kappa\varepsilon^2/(n^{5/2}),$$

where κ is the constant of Theorem 1.3, $g(t) := tf(t)$ and f is the function of Definition, 1.5 corresponding to a separable atomless probability measure space.

Proof. Let $\delta \leq \kappa\varepsilon^2/(n^{5/2})$ and $\mathcal{N}_\delta$ be a δ-net in an n-dimensional Banach space X. By Theorem 1.3, there exists a finite dimensional subspace $Z \subset Y$ and a finite-dimensional subspace $W \subset L_\infty(\nu, Z)$ (the image of the operator T) such that W satisfies (3) with $C = \frac{1+\varepsilon}{1-\varepsilon}c_Y(\mathcal{N}_\delta)$. Thus by the Lebesgue-Bochner discretization property of Y, $c_Y(W) \leq f(\frac{1+\varepsilon}{1-\varepsilon}c_Y(\mathcal{N}_\delta))$, and we obtain

$$c_Y(X) \leq \frac{1+\varepsilon}{1-\varepsilon}\, c_Y(\mathcal{N}_\delta) f\left(\frac{1+\varepsilon}{1-\varepsilon}c_Y(\mathcal{N}_\delta)\right). \qquad \square$$

Problem 1. *Characterize Banach spaces with the Lebesgue-Bochner discretization property.*

At the meeting of the Simons Foundation (New York City, February 20, 2015) Assaf Naor mentioned that at that time no examples of Banach spaces which do not have the Lebesgue-Bochner discretization property were known, although people who were working on this (Assaf Naor and Gideon Schechtman) believed that such examples should exist.

We note that the argument which uses the Fubini and Carathéodory theorems to show that $L_p(L_p)$ is isometric to L_p (for suitable measure spaces) fails for other function spaces, even in a certain 'isomorphic' form (see [BBS02, Appendix]). For some spaces a very strong opposite of the situation in the L_p-case happens: Raynaud [Ray89] proved that when $L_\varphi([0,1],\mu)$ is an Orlicz space that is not isomorphic to some L_p and does not contain c_0 or ℓ_1, then for any $r \in [1,\infty)$, the space $\ell_r(L_\varphi)$ (and thus also $L_\varphi([0,1],\mu,L_\varphi)$) not only does not embed in $L_\varphi([0,1],\mu)$, but is not even crudely finitely representable in it.

In general, if E is a Banach function space on a measure space (Ω,μ), the structure of the E-valued Bochner space $E(\Omega,\mu,E)$ can be very different from the structure of the space E, see [Rea90], [BBS02], [FPP08]. We refer the reader to [BBS02] for a detailed discussion and history of related results.

In this paper we show (Proposition 2.2) that there is a class of Banach spaces which do not have the Lebesgue-Bochner discretization property and observe that this class contains the space constructed by Figiel [Fig72].

We also find some examples, besides L_p, of Banach spaces that have the Lebesgue-Bochner discretization property. An easy observation is that the Lebesgue-Bochner spaces $L_p(E)$, where E is any Banach space, have the Lebesgue-Bochner discretization property. It is interesting that even the finite direct sums of such spaces have the Lebesgue-Bochner discretization property, see Proposition 3.1. We would like to mention that many well-known and important spaces are of the form $L_p(E)$. In particular, the mixed norm Lebesgue spaces L^P introduced in [BP61] are such and thus have the Lebesgue-Bochner discretization property. For $P=(p_1,\dots,p_m)\in[1,\infty)^m$, the space L^P consists of measurable functions f on $\Omega=\prod_{i=1}^m(\Omega_j,\mu_j)$, the norm defined by

$$\|f\|_P := \left(\int \dots \left(\int \left(\int |f(t_1,\dots,t_n)|^{p_1}d\mu_1\right)^{p_2/p_1} d\mu_2\right)^{p_3/p_2} \dots d\mu_m\right)^{1/p_m}.$$

Mixed norm spaces of this type arise naturally in harmonic and functional analysis. Such norms (and their generalizations that use other function space norms in place of the L_{p_j}-norms) are used, for example, to study Fourier and Sobolev inequalities and embeddings of Sobolev spaces. The properties and applications of mixed norm spaces are extensively studied in the literature, see e.g. [GS16, CS16, DPS10] and their references.

2 Finitely squarable Banach spaces

Definition 2.1. An infinite-dimensional Banach space Y is called *finitely squarable* if there exists a constant C such that for every finite-dimensional subspace $Z\subset Y$, the direct sum $Z\oplus_\infty Z$ admits a linear embedding into Y with distortion bounded by C.

The first examples of Banach spaces which are not finitely squarable were constructed by Figiel [Fig72]. An easy observation is that a Banach space Y which is isomorphic to $Y\oplus Y$ is finitely squarable. The converse it false. In fact, both of the earliest examples of Banach spaces which are not isomorphic to their squares, the James [Jam50] quasireflexive space J [BP60] and $c(\omega_1)$ [Sem60], are finitely squarable, and for a very simple reason: they have trivial cotype. For the James space this was proved in [GJ73], for $c(\omega_1)$ this is obvious. Modern Banach space theory provides much more sophisticated examples of finitely squarable spaces which are not isomorphic to their squares, for example, the Argyros-Haydon space [AH11].

Proposition 2.2. *Any space which is not finitely squarable does not have the Lebesgue-Bochner discretization property.*

Proof. Let Z be a subspace of Y for which $Z\oplus_\infty Z$ is "very far" from a subspace of Y.

We introduce the following subspace $W\subset L_\infty([0,1],Z)$: it consists of all Z-valued functions which are constant on the first half and constant on the second half, but these constants can be different vectors of Z. It is clear that this space is isometric to $Z\oplus_\infty Z$. It is also clear that the $L_1([0,1],Z)$ norm on this subspace is 2-equivalent to the L_∞-norm. The conclusion follows. □

This proposition makes the following problem important:

Problem 2. *Does there exist a finitely squarable space which does not have the Lebesgue-Bochner discretization property?*

We conjecture that the answer to Problem 2 is positive.

3 Examples of spaces with the Lebesgue-Bochner discretization property

In this section we provide some examples of spaces having the Lebesgue-Bochner discretization property. In all proofs below we use the notation of Definition 1.5. That is: Y is a Banach space, $C > 0$, $Z \subset Y$ is a finite dimensional subspace of Y. Since we consider separable probability measures, by the Carathéodory theorem [Lac74], we may assume that W is a finite-dimensional subspace of $L_\infty([0,1],\mu,Z)$ such that for all $w \in W$

$$\frac{1}{C}\|w\|_{L_\infty([0,1],\mu,Z)} \leq \|w\|_{L_1([0,1],\mu,Z)} \leq \|w\|_{L_\infty([0,1],\mu,Z)}. \tag{6}$$

Since W is finite dimensional, for any $\varepsilon > 0$, there exists a subspace $\tilde{W} \subseteq L_\infty([0,1],\mu,Z)$ with Banach-Mazur distance from W less than $1+\varepsilon$, such that $\tilde{W}$ is spanned by simple functions and all $w \in \tilde{W}$ satisfy (6) with C replaced by $(1+\varepsilon)C$. Thus, without loss of generality, we may assume that W is spanned by simple functions which are constant on elements $\{\Delta_i\}_{i=1}^n$ of some partition of $[0,1]$ into sets of measure $\frac{1}{n}$. Thus we can denote elements $w \in W$ as

$$w = (w_1, \ldots, w_n),$$

meaning that $w = \sum_{i=1}^n \mathbf{1}_{\Delta_i} \otimes w_i$. For all $w \in W$ we have $\|w\|_{L_\infty([0,1],\mu,Z)} = \max_{1\leq i\leq n}\|w_i\|_Z$, and for all p, $1 \leq p < \infty$, we have

$$\|w\|_{L_p([0,1],\mu,Z)} = \Big(\frac{1}{n}\sum_{i=1}^n \|w_i\|_Z^p\Big)^{\frac{1}{p}}.$$

Given any $p \in [1,\infty]$, $k \in \mathbb{N}$, and any Banach spaces $E_1, \ldots, E_k$, by $L_p^k(E_1,\ldots,E_k)$ we denote the Banach space of all k-tuples $(a_1,\ldots,a_k)$ such that $a_j \in E_j$ for all $j \in [k] \stackrel{\text{def}}{=} \{1,\ldots,k\}$, endowed with the norm

$$\|(a_1,\ldots,a_k)\|_{L_p^k(E_1,\ldots,E_k)} := \Big(\frac{1}{k}\sum_{i=1}^k \|a_i\|_{E_i}^p\Big)^{\frac{1}{p}}, \text{ if } p < \infty,$$
$$\|(a_1,\ldots,a_k)\|_{L_\infty^k(E_1,\ldots,E_k)} := \max_{1\leq i\leq k}\|a_i\|_{E_i}.$$

If the spaces $E_1,\ldots,E_k$ are equal to the same space E, we denote $L_p^k(E_1,\ldots,E_k)$ by $L_p^k(E)$.

Proposition 3.1. *Let $k \in \mathbb{N}$, $p, q_1, \ldots, q_k \geq 1$, $X_1,\ldots,X_k$ be any Banach spaces, and for each $j \in [k]$, let (Ω_j,μ_j) be any atomless separable measure space, with finite or infinite measure, or $\Omega_j = \mathbb{N}$, and μ_j be the counting measure. Then the space*

$$Y = L_p^k(L_{q_1}(\Omega_1,\mu_1,X_1), L_{q_2}(\Omega_2,\mu_2,X_2), \ldots, L_{q_k}(\Omega_k,\mu_k,X_k))$$

has the Lebesgue-Bochner discretization property with $f(C) \leq k^{4-3/p}C$.

Note that since the constant $f(C)$ in Definition 1.5 can depend on k, the fact that Y is an L_p^k-sum is not essential. Essential is the fact that Y is a finite direct sum.

Proof of Proposition 3.1. To simplify notation we will omit the measure spaces when writing the symbol for a Lebesgue-Bochner space, i.e. we will write $L_{q_j}(X_j)$ instead of $L_{q_j}(\Omega_j, \mu_j, X_j)$ with the understanding that for all $j \in [k]$, the measure spaces are those fixed in the statement of the proposition.

Note that if at least one of q_j is equal to ∞, the space Y has trivial cotype and thus has the Lebesgue-Bochner discretization property. In the following we assume that $q_j < \infty$ for all $j \in [k]$.

First, we prove that in the case where $p = 1$, we have $f(C) \leq kC$.

We denote $Y_1 = L_1^k(L_{q_1}(X_1), L_{q_2}(X_2), \ldots, L_{q_k}(X_k))$. Using the discussion and notation preceding Proposition 3.1, we see that it suffices to prove that any subspace $W \subseteq L_\infty^n(Y_1)$ satisfying

$$\forall w \in W \qquad \|w\|_{L_\infty^n(Y_1)} \leq C\|w\|_{L_1^n(Y_1)} \tag{7}$$

admits a kC-isomorphic embedding into Y_1.

Let $n \in \mathbb{N}$, $w = (w_1, \ldots, w_n) \in W \subseteq L_\infty^n(Y_1)$ and, for $i \in [n]$, $w_i = (w_{ij})_{j=1}^k \in L_1^k(L_{q_1}(X_1), L_{q_2}(X_2), \ldots, L_{q_k}(X_k))$, where, for all $i \in [n]$, $j \in [k]$, $w_{ij} \in L_{q_j}(X_j)$. We will define a map φ from $L_\infty^n(Y_1)$ to Y_1 such that for all $w \in L_\infty^n(Y_1)$, we have

$$\|\varphi(w)\|_{Y_1} = \frac{1}{k}\sum_{j=1}^k \left(\frac{1}{n}\sum_{i=1}^n \|w_{ij}\|_{q_j}^{q_j}\right)^{\frac{1}{q_j}}. \tag{8}$$

For each $j \in [k]$, we select n mutually disjoint subsets $\{\Omega_{j\nu}\}_{\nu=1}^n$ of Ω_j such that for each $\nu \in [n]$ there exists a constant $a_{j\nu} > 0$ and a surjective isometry $T_{j\nu} : L_{q_j}(\Omega_j, \mu_j) \to L_{q_j}(\Omega_{j\nu}, a_{j\nu}\mu_j)$. It is well-known that when (Ω_j, μ_j) is atomless or equal to $\mathbb{N}$ with the counting measure, then such choices are possible, and that the isometry $T_{j\nu}$ can be naturally extended to the isometry $\bar{T}_{j\nu}$ from the Lebesgue-Bochner space $L_{q_j}(\Omega_j, \mu_j, X_j)$ onto $L_{q_j}(\Omega_{j\nu}, a_{j\nu}\mu_j, X_j)$, cf. e.g. [DU77].

We define the map $\varphi_j : L_\infty^n(L_{q_j}(X_j), \ldots, L_{q_j}(X_j)) \to L_{q_j}(X_j)$ by setting for all $(x_1, \ldots, x_n) \in L_\infty^n(L_{q_j}(X_j), \ldots, L_{q_j}(X_j))$

$$\varphi_j(x_1, \ldots, x_n) := \sum_{\nu=1}^n \left(\frac{a_{j\nu}}{n}\right)^{\frac{1}{q_j}} \bar{T}_{j\nu}x_\nu.$$

Since the sets $\{\Omega_{j\nu}\}_{\nu=1}^n$ are mutually disjoint, we get

$$\|\varphi_j(x_1, \ldots, x_n)\|_{q_j} = \left(\frac{1}{n}\sum_{\nu=1}^n a_{j\nu}\|\bar{T}_{j\nu}x_\nu\|_{q_j}^{q_j}\right)^{\frac{1}{q_j}} = \left(\frac{1}{n}\sum_{\nu=1}^n \|x_\nu\|_{q_j}^{q_j}\right)^{\frac{1}{q_j}}. \tag{9}$$

Next, given $w = (w_1, \ldots, w_n) \in L_\infty^n(Y_1)$ where, for $i \in [n]$, $w_i = (w_{ij})_{j=1}^k \in Y_1$, we define $\varphi(w) \in Y_1$ by setting

$$\varphi((w_i)_{i=1}^n) = \Big(\varphi_j\big((w_{ij})_{i=1}^n\big)\Big)_{j=1}^k.$$

By (9), equality (8) is satisfied.

We will show that for all $w \in L_\infty^n(Y_1)$, we have

$$\|w\|_{L_1^n(Y_1)} \le \|\varphi(w)\|_{Y_1} \le k\|w\|_{L_\infty^n(Y_1)}. \tag{10}$$

To prove the leftmost inequality, we write

$$\begin{aligned}
\|w\|_{L_1^n(Y_1)} &= \frac{1}{n}\sum_{i=1}^n \|w_i\|_{Y_1} = \frac{1}{n}\sum_{i=1}^n \Big(\frac{1}{k}\sum_{j=1}^k \|w_{ij}\|_{q_j}\Big) \\
&= \frac{1}{k}\sum_{j=1}^k \Big(\frac{1}{n}\sum_{i=1}^n \|w_{ij}\|_{q_j}\Big) \le \frac{1}{k}\sum_{j=1}^k \Big(\frac{1}{n}\sum_{i=1}^n \|w_{ij}\|_{q_j}^{q_j}\Big)^{\frac{1}{q_j}} \\
&= \|\varphi(w)\|_{Y_1},
\end{aligned}$$

where the inequality follows from the classical theorem on averages ([HLP52, Theorem 16]) applied to each of the summands with the corresponding exponent q_j, for $1 \le j \le k$, respectively.

To prove the rightmost inequality, for each $j \in [k]$, let $i_j \in [n]$ be such that $\|w_{i_j j}\|_{q_j} = \max_{1\le i\le n} \|w_{ij}\|_{q_j}$. Then

$$\begin{aligned}
\|\varphi(w)\|_{Y_1} &= \frac{1}{k}\sum_{j=1}^k \Big(\frac{1}{n}\sum_{i=1}^n \|w_{ij}\|_{q_j}^{q_j}\Big)^{\frac{1}{q_j}} \le \frac{1}{k}\sum_{j=1}^k \Big(\max_{1\le i\le n} \|w_{ij}\|_{q_j}\Big) \\
&= \frac{1}{k}\sum_{j=1}^k \|w_{i_j j}\|_{q_j} \le \frac{1}{k}\sum_{j=1}^k \Big(\sum_{l=1}^k \|w_{i_j l}\|_{q_j}\Big) \\
&= \sum_{j=1}^k \Big(\frac{1}{k}\sum_{l=1}^k \|w_{i_j l}\|_{q_j}\Big) = \sum_{j=1}^k \|w_{i_j}\|_{Y_1} \\
&\le k \max_{1\le i\le n} \|w_i\|_{Y_1} = k\|w\|_{L_\infty^n(Y_1)}.
\end{aligned}$$

Thus (10) holds, and therefore, by (7),

$$\frac{1}{C}\|w\|_{L_\infty^n(Y_1)} \le \|\varphi(w)\|_{Y_1} \le k\|w\|_{L_\infty^n(Y_1)}.$$

Thus φ is a kC-isomorphic embedding of W into Y_1, which ends the proof in the case where $p = 1$.

For the general case, suppose that $p > 1$ and $Y = L_p^k(L_{q_1}(X_1), L_{q_2}(X_2), \dots, L_{q_k}(X_k))$. Then spaces Y and Y_1 are equal as sets, and for all $y \in Y$ we have

$$\|y\|_{Y_1} \le \|y\|_Y \le b\|y\|_{Y_1}, \tag{11}$$

where $b = k^{1-1/p}$. Let W be a subspace of $L_\infty^n(Y)$ satisfying (7). Then for every $w \in W$ we have

$$\|w\|_{L_\infty^n(Y_1)} \le \|w\|_{L_\infty^n(Y)} \le C\|w\|_{L_1^n(Y)} \le Cb\|w\|_{L_1^n(Y_1)}. \tag{12}$$

In the first part of the proof we defined the map $\varphi : L_\infty^n(Y_1) \to Y_1$ such that for all $w \in L_\infty^n(Y_1)$ we have (10). Therefore, by combining (10) and (11), we obtain for all $w \in L_\infty^n(Y)$,

$$\begin{aligned}\frac{1}{b}\|w\|_{L_1^n(Y)} &\le \|w\|_{L_1^n(Y_1)} \le \|\varphi(w)\|_{Y_1}\\ &\le \|\varphi(w)\|_Y\\ &\le b\|\varphi(w)\|_{Y_1} \le bk\|w\|_{L_\infty^n(Y_1)} \le bk\|w\|_{L_\infty^n(Y)}.\end{aligned}$$

Thus, by (12), for all $w \in W$ we have

$$\frac{1}{Cb^2}\|w\|_{L_\infty^n(Y)} \le \|\varphi(w)\|_Y \le bk\|w\|_{L_\infty^n(Y)}.$$

Since $b = k^{1-1/p}$, this means that φ is a $(Ck^{4-3/p})$-isomorphic embedding of W into Y. $\square$

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