

A characterization of superreflexivity through embeddings of lamplighter groups

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Abstract

We prove that finite lamplighter groups $\{\mathbb{Z}_2 \wr \mathbb{Z}_n\}_{n \geq 2}$ with a standard set of generators embed with uniformly bounded distortions into any non-superreflexive Banach space, and therefore form a set of test-spaces for superreflexivity. Our proof is inspired by the well-known identification of Cayley graphs of infinite lamplighter groups with the horocyclic product of trees. We cover $\mathbb{Z}_2 \wr \mathbb{Z}_n$ by three sets with a structure similar to a horocyclic product of trees, which enables us to construct well-controlled embeddings.

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1 Introduction

One of the important directions in metric geometry is to find purely metric characterizations of interesting classes of Banach spaces. For classes of spaces determined by finite-dimensional subspaces, this direction is a part of the *Ribe program* which was described by Bourgain [5] who proved the first metric characterization of superreflexivity. See [17] for more information on the Ribe program. The goal of this paper is to prove that lamplighter groups are test spaces for superreflexivity.

Definition 1.1 ([22]). Let \mathcal{P} be a class of Banach spaces and let $T = \{T_\alpha\}_{\alpha \in A}$ be a set of metric spaces. We say that T is a set of *test spaces* for \mathcal{P} if the following two conditions are equivalent: **(1)** $X \notin \mathcal{P}$. **(2)** The spaces $\{T_\alpha\}_{\alpha \in A}$ admit bilipschitz embeddings into X with uniformly bounded distortions.

Several different sets of test-spaces for superreflexivity are known: (1) binary trees [5, 15, 3, 10], (2) binary diamond and Laakso graphs [9, 16], (3) multibranching

diamond and Laakso graphs [25], and more general families of finitely branching bundle graphs [28]. See [24] for a survey on this matter written in 2014.

In this paper we add one more item to this list: (4) the set of Cayley graphs of finite lamplighter groups. Moreover, we observe that the Cayley graph of an infinite lamplighter group also is a test space for superreflexivity.

Some of the characterizations (1)–(3) are independent in the sense that the corresponding families of test spaces do not admit bilipschitz embeddings into each other with uniformly bounded distortions. In non-obvious cases this was shown in [23, 12] for finite binary trees and diamond graphs, and in [19] for diamond and Laakso graphs.

Lamplighter groups are a very interesting class of groups which has been a rich source of important examples in geometric group theory. In 2008, Naor and Peres [18, Section 4] proved that the finite lamplighter groups $\mathbb{Z}_2 \wr \mathbb{Z}_n$, with metric defined as a word length with respect to natural sets of generators, are embeddable into L_1 with uniformly bounded distortions. The first goal of this paper is to strengthen this result and to prove their embeddability into an arbitrary nonsuperreflexive Banach space with uniformly bounded distortions. As a consequence we get a new metric characterization of superreflexivity, see Corollary 1.7.

We consider the following special case of the general wreath product construction (see, for example, [7, p. 214] for the general definition).

Definition 1.2 ([6, p. 129]). Let H and L be two groups and L^H be the set of all L -valued finitely supported functions on H . Then the *wreath product* $L \wr H$ is defined as the set $L^H \times H$ equipped with the multiplication

$$((x_h)_{h \in H}, g) \cdot ((y_h)_{h \in H}, k) := ((x_h \cdot y_{gh})_{h \in H}, gk).$$

Our main interest will be the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_n$, for $n \in \mathbb{N}$, $n \geq 2$, which we denote by \mathcal{G}_n , this group is called the *lamplighter group*, see [29] for a nice introduction including an explanation for the name.

We identify $\mathbb{Z}_2^{\mathbb{Z}_n}$ with the family of all subsets of \mathbb{Z}_n by identifying $x = (x_k)_{k \in \mathbb{Z}_n}$ with $\{j \in \mathbb{Z}_n : x_j = 1\}$, the group operation on $\mathbb{Z}_2^{\mathbb{Z}_n}$ is the symmetric difference. From now on we will abuse notation and treat an element $x \in \mathbb{Z}_2^{\mathbb{Z}_n}$ as a subset of \mathbb{Z}_n .

Considering an element $(x, k) \in \mathcal{G}_n$, we call x the *set of positions where the lamp is on*, and its complement - the *set of positions where the lamp is off*. The number $k \in \mathbb{Z}_n$ is called the *location of the lamplighter*.

It is easy to see that the elements $a = (\{0\}, 0)$ and $t = (\emptyset, 1)$ generate \mathcal{G}_n . Observe that multiplication by $a = (\{0\}, 0)$ on the right is the act of changing the lamp *at the current location of the lamplighter* and multiplication by $t = (\emptyset, 1)$ on the right is the act of the lamplighter moving one position to the next lamp in the ‘positive’ direction around the circle.

We consider the metric ρ on \mathcal{G}_n defined as the metric of the left-invariant Cayley graph with respect to the set of generators $S = \{t, ta\}$. This means that x is adjacent

to y if and only if $x = ys$ or $y = xs$ (i.e $x = ys^{-1}$) for one of the generators $s \in S$. Observe that the generator ta acts by first moving one step in the ‘positive’ direction, and then changing the state of the lamp at the final location of the lamplighter. See Section 2.2 for a more detailed description and an equivalent formula (2.5) for the metric.

The first main result of this paper is

Theorem 1.3. *For any nonsuperreflexive Banach space X the Cayley graphs of $\mathbb{Z}_2 \wr \mathbb{Z}_n$ ($n \geq 2$) corresponding to the set $S = \{t, ta\}$ admit embeddings into X with uniformly bounded distortions.*

Remark 1.4. In [18], when proving embeddability into L_1 , Naor and Peres considered $\mathbb{Z}_2 \wr \mathbb{Z}_n$ with the set of generators equal to $\{t, a\}$ instead of $\{t, ta\}$ as we do. However, it is easy to see that the metrics induced by these two generating sets are bilipschitz equivalent to each other with a distortion 4.

In [18] Naor and Peres constructed two embeddings into L_1 , one based on irreducible representations of finite lamplighter groups, and the other motivated by what they refer to as a ‘direct geometric reasoning’. Our embedding technique is quite different from either of the embeddings in [18]. Our approach is inspired by the description of the Cayley graph of the infinite lamplighter group $\mathbb{Z}_q \wr \mathbb{Z}$, for $q \in \mathbb{N}$, $q \geq 2$, as a *horocyclic product* of two trees, introduced by Bartholdi and Woess [2, 30], and by the analysis of the metric structure of horocyclic products of trees by Stein and Taback in [27] who proved, among other results,

Theorem 1.5. [27, Corollary 10] *For any $q \in \mathbb{N}$, $q \geq 2$, the Cayley graph of $\mathbb{Z}_q \wr \mathbb{Z}$ with the set of generators $\{t, ta, \dots, ta^{q-1}\}$ admits an embedding into an ℓ_1 -sum of two q -branching trees with distortion bounded by 4.*

We refer the reader to [29] and [31] for very nice presentations of horocyclic products of trees and their applications to lamplighter groups. We give a more detailed overview of our method in Section 2.1.

Remark 1.6. It follows from Theorem 1.5, [5], and [20] that for all $q \in \mathbb{N}$, $q \geq 2$, the Cayley graph of $\mathbb{Z}_q \wr \mathbb{Z}$ admits an embedding into any non-superreflexive space with distortion independent of q , as was shown in a similar case of hyperbolic groups in [23, Section 2] (cf. also Section 2.4 below).

The bilipschitz embeddability of the infinite group $\mathbb{Z}_2 \wr \mathbb{Z}$ into any non-superreflexive space can also be derived from Theorem 1.3 and results of [5, 20].

In [14] it is proved that there exists a constant $c > 0$ so that for every $n \in \mathbb{N}$ a complete binary tree of depth cn embeds with constant distortion into \mathcal{G}_n . In Section 3 below we show an alternative simple proof of this fact using our construction.

As a consequence, and by Theorem 1.3, Remark 1.6, and [5], we obtain

Corollary 1.7. *The sequence of Cayley graphs for $\{\mathbb{Z}_2 \wr \mathbb{Z}_n\}_{n \geq 2}$ with respect to the set of generators $S = \{t, ta\}$ is a set of test-spaces for superreflexivity. The Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to any finite set of generators is a test space for superreflexivity.*

Remark 1.8. We note that Theorem 1.5 is valid for $\mathbb{Z}_q \wr \mathbb{Z}$ for all $q \in \mathbb{N}$, $q \geq 2$. As we elaborate in Section 2.1, our proof of Theorem 1.3 is inspired by the methods of Theorem 1.5, even though we do not apply its conclusion for our argument.

In our statement and proof of Theorem 1.3, for greater clarity of the presentation, we focused our attention on $\mathbb{Z}_2 \wr \mathbb{Z}_n$, but it only requires straightforward adjustments of the proof to obtain the same conclusion as in Theorem 1.3 for $\mathbb{Z}_q \wr \mathbb{Z}_n$ with the set of generators $\{t, ta, \dots, ta^{q-1}\}$, for all $q \in \mathbb{N}$, $q \geq 2$. The only difference is that one needs to use an ℓ_∞ -sum of two q -branching trees in place of two binary trees. Similarly as in [27], the value of q affects the branching of the trees, but not the number of the summands. Since for all q , a q -branching tree embeds almost isometrically into a binary tree, and since we always use an ℓ_∞ -sum of two trees, the uniform bound on distortions does not depend on $q \in \mathbb{N}$.

Since, for all $q \in \mathbb{N}$, $q \geq 2$, the Cayley graphs of $\mathbb{Z}_q \wr \mathbb{Z}_n$ and $\mathbb{Z}_q \wr \mathbb{Z}$ with the set of generators $\{t, ta, \dots, ta^{q-1}\}$ isometrically contain the Cayley graphs of $\mathbb{Z}_2 \wr \mathbb{Z}_n$ and $\mathbb{Z}_2 \wr \mathbb{Z}$ with generators $\{t, ta\}$, respectively, it follows from Corollary 1.7 and the adjusted version of Theorem 1.3, that for any $q \in \mathbb{N}$, $q \geq 2$, the sequence of Cayley graphs of $\mathbb{Z}_q \wr \mathbb{Z}_n$ with the set of generators $\{t, ta, \dots, ta^{q-1}\}$ is a set of test-spaces for superreflexivity. Similarly, for any $q \in \mathbb{N}$, $q \geq 2$, the Cayley graph of $\mathbb{Z}_q \wr \mathbb{Z}$, with respect to any finite generating set, is a test space for superreflexivity.

2 Proof of Theorem 1.3

2.1 Outline of the proof

To simplify notation, we assume that n is divisible by 6. It is clear that the same arguments work in general, but the formulas will be somewhat more complicated.

We cover the Cayley graph of \mathbb{Z}_n with respect to the generating set $\{\pm 1\}$ by three overlapping paths P_1, P_2, P_3 of length $\frac{2}{3}n$ each in such a way that each pair of points in \mathbb{Z}_n belongs to at least one of the paths (paths that exclude three mutually disjoint thirds of the cycle \mathbb{Z}_n work).

We consider the following three subsets of \mathcal{G}_n for $i = 1, 2, 3$,

$$\mathcal{P}_{i,n} \stackrel{\text{def}}{=} \{(x, k) \in \mathcal{G}_n \mid x \subseteq \mathbb{Z}_n, k \in P_i\}. \quad (2.1)$$

We equip $\mathcal{P}_{i,n}$ with the metric inherited from the Cayley graph of \mathcal{G}_n with respect to $S = \{t, ta\}$. We note that the union of the three sets $\mathcal{P}_{i,n}$ covers \mathcal{G}_n .

Our approach is inspired by the well-known description of infinite lamplighter groups $\mathbb{Z}_2 \wr \mathbb{Z}$ as a horocyclic product of two infinite trees [1, 2, 30] and by the

bilipschitz embedding of the word metric on $\mathbb{Z}_2 \wr \mathbb{Z}$ into an ℓ_1 -sum of two trees [27] (see Theorem 1.5 above). Our first goal is to cover $\mathbb{Z}_2 \wr \mathbb{Z}_n$ by three sets with a structure similar to a horocyclic product of trees, which will enable us to construct well-controlled embeddings.

The sets $\mathcal{P}_{i,n}$ are defined so that for all locations $k, l \in P_i$, the length of the path that is contained in P_i and connects k and l is at most twice the length of the shortest path from k to l in \mathbb{Z}_n . For this reason, the metric structure of the sets $\mathcal{P}_{i,n}$ sufficiently resembles the metric structure of a subset of $\mathbb{Z}_2 \wr \mathbb{Z}_n$ and we are able to construct sets W_n , which we think of as analogs of horocyclic products of trees, and which are bilipschitz equivalent to the sets $\mathcal{P}_{i,n}$. The sets W_n are defined as specific subsets of the ℓ_∞ -sum of two trees of depth n . To identify each element of $\mathcal{P}_{i,n}$ with an element of the Cartesian product $T_n \times T_n$ of two trees of depth n , we first mark a vertex v_0 in \mathbb{Z}_n which is the midpoint of the complement of the path P_i , that is v_0 is at the distance at least $n/6$ from any element $k \in P_i$. Modelling our description on the identification of $\mathbb{Z}_2 \wr \mathbb{Z}_n$ with the horocyclic product of trees (cf. [30, p. 419]), for every $(x, k) \in \mathcal{P}_{i,n}$ we describe the set x as the union of $x \cap I_{k,+}$ and $x \cap I_{k,-}$, where $I_{k,+}, I_{k,-}$ are two disjoint arcs in \mathbb{Z}_n both with the endpoints v_0 and k . Each set $x \cap I_{k,+}$ and $x \cap I_{k,-}$ is encoded by a sequence of 0s and 1s of length equal to the number of vertices in $I_{k,+}$ and $I_{k,-}$, respectively. This naturally encodes each element $(x, k) \in \mathcal{P}_{i,n}$ by two elements of a binary tree whose levels (in a tree) add up to n . We verify in Section 2.2 that this encoding is metrically faithful on $\mathcal{P}_{i,n}$, that is, we construct bilipschitz embeddings

$$\varphi_{i,n} : \mathcal{P}_{i,n} \rightarrow T_n \oplus_\infty T_n,$$

with uniformly bounded distortions, see Section 2.2 for details. This completes the first and main step of our proof.

The next step of our proof is a routine application of the well-known theory of Lipschitz retracts to conclude that, for each $i \in \{1, 2, 3\}$, the bilipschitz map $\varphi_{i,n} : \mathcal{P}_{i,n} \rightarrow T_n \oplus_\infty T_n$, constructed in the first step, can be extended to a Lipschitz map $\bar{\varphi}_{i,n}$ from the entire \mathcal{G}_n into an ℓ_∞ -sum of two metric trees of depth n , see Section 2.3 for details.

In the final step of our proof we define the map Φ_n from \mathcal{G}_n into an ℓ_∞ -sum of six metric trees of depth n by

$$\Phi_n(x, k) \stackrel{\text{def}}{=} (\bar{\varphi}_{1,n}(x, k), \bar{\varphi}_{2,n}(x, k), \bar{\varphi}_{3,n}(x, k)).$$

Clearly, the maps Φ_n are Lipschitz with the same Lipschitz constants as those of the maps $\bar{\varphi}_{i,n}$, for $i = 1, 2, 3$. Since the paths P_1, P_2, P_3 were chosen in such a way that for any two elements $(x, k), (y, l) \in \mathcal{G}_n$, there exists $i \in \{1, 2, 3\}$ so that $(x, k), (y, l) \in \mathcal{P}_{i,n}$, it follows that the map Φ_n is co-Lipschitz with same constant as the map $\varphi_{i,n}$ (in our construction all maps $\{\varphi_{i,n}\}_{i=1}^3$ have the same co-Lipschitz constant), see Section 2.4.

Hence to finish the proof of Theorem 1.3, it is enough to verify that for all $n \in \mathbb{N}$, the ℓ_∞ -sum of six metric trees of depth n embeds into any non-superreflexive Banach space X with uniformly bounded distortions. This follows readily by known techniques and results on bilipschitz embeddability of trees into any non-superreflexive Banach space, and on extension of bilipschitz embeddings into Banach spaces from vertex sets to graphs to the corresponding 1-dimensional complexes. In fact it even suffices to prove the existence of Lipschitz maps that satisfy slightly weaker requirements, see Section 2.4 for details.

2.2 Step 1

We define the paths P_1 , P_2 and P_3 , to be arcs of lengths $\frac{2}{3}n$ in \mathbb{Z}_n with endpoints $[\frac{1}{6}n, \frac{5}{6}n]$, $[\frac{1}{2}n, \frac{1}{6}n]$ and $[\frac{5}{6}n, \frac{1}{2}n]$, respectively (recall that we assumed that n is divisible by 6).

Let $\theta_n : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be the ‘rotation’ of \mathbb{Z}_n by an arc of length $n/3$, i.e. for each $k \in \mathbb{Z}_n$, $\theta_n(k) = k + \frac{n}{3}$, where the addition is in the sense of \mathbb{Z}_n . The rotation θ_n induces the isometry $\overline{\theta}_n$ of \mathcal{G}_n onto itself, defined by

$$\overline{\theta}_n(x, k) = (\theta_n(x), \theta_n(k)),$$

where, as usual, $\theta_n(x)$ denotes the image of the set x under the action of θ_n .

Note, that the paths P_1 , P_2 , P_3 , satisfy $P_2 = \theta_n(P_1)$ and $P_3 = \theta_n^2(P_1)$, and the metric spaces $\{\mathcal{P}_{i,n}\}_{i=1}^3$, defined by (2.1), are isometric to each other, specifically

$$\mathcal{P}_{2,n} = \overline{\theta}_n(\mathcal{P}_{1,n}), \quad \mathcal{P}_{3,n} = \overline{\theta}_n^2(\mathcal{P}_{1,n}). \quad (2.2)$$

Let T_n be a binary tree of depth n , that is, T_n is the graph whose vertices are labelled by sequences of 0s and 1s of lengths $\leq n$ with the usual graph distance d_T . We consider the ℓ_∞ -sum $T_n \oplus_\infty T_n$ defined as the Cartesian product $T_n \times T_n$ endowed with the metric

$$d_\infty((A_1, A_2), (B_1, B_2)) \stackrel{\text{def}}{=} \max\{d_T(A_1, B_1), d_T(A_2, B_2)\}.$$

We define

$$W_n \stackrel{\text{def}}{=} \left\{ (A_1, A_2) \in T_n \oplus_\infty T_n : |A_1| + |A_2| = n, \quad |A_1|, |A_2| \in \left[\frac{1}{6}n, \frac{5}{6}n \right] \right\},$$

where $|A|$ denotes the length of the sequence $A \in T_n$.

It will be convenient to also use the distance d_1 on W_n which is 2-equivalent with d_∞ .

$$d_1((A_1, A_2), (B_1, B_2)) \stackrel{\text{def}}{=} d_T(A_1, B_1) + d_T(A_2, B_2).$$

We will show that there exist bijections $\varphi_{1,n}$ from $\{(\mathcal{P}_{1,n}, d_\infty)\}_n$ onto $\{(W_n, d_\infty)\}_n$ which have uniformly bounded distortions.

We define a map $\varphi_{1,n} : \mathcal{P}_{1,n} \rightarrow W_n$ as follows: for any $(x, k) \in \mathcal{P}_{1,n}$, the element $\varphi_{1,n}(x, k) \stackrel{\text{def}}{=} (A_1, A_2)$, where $A_1 = (a_{1,0}, \dots, a_{1,k-1})$ is a sequence of length k and $A_2 = (a_{2,1}, \dots, a_{2,n-k})$ is a sequence of length $n - k$, defined by

$$a_{1,j} = \begin{cases} 1 & \text{if } j \in x, \\ 0 & \text{if } j \notin x, \end{cases} \quad a_{2,i} = \begin{cases} 1 & \text{if } n - i \in x, \\ 0 & \text{if } n - i \notin x. \end{cases}$$

It is clear that the map $\varphi_{1,n}$ is one-to-one and onto. We will show that $\varphi_{1,n}$ is a bilipschitz isomorphism of $(\mathcal{P}_{1,n}, \rho)$ and (W_n, d_∞) .

For any $A, B \in T_n$, we denote by $\text{lgca}(A, B)$ the length of the greatest common ancestor of A and B in T_n . In this notation,

$$d_T(A, B) = (|A| - \text{lgca}(A, B)) + (|B| - \text{lgca}(A, B)),$$

and for $(A_1, A_2), (B_1, B_2) \in W_n$,

$$\begin{aligned} d_1((A_1, A_2), (B_1, B_2)) &= d_T(A_1, B_1) + d_T(A_2, B_2) \\ &= 2(n - (\text{lgca}(A_1, B_1) + \text{lgca}(A_2, B_2))). \end{aligned} \tag{2.3}$$

In particular, for all $(A_1, A_2), (B_1, B_2) \in W_n$ we have

$$d_1((A_1, A_2), (B_1, B_2)) \leq 2n. \tag{2.4}$$

To continue we need to estimate the distance $\rho((x, k), (y, l))$, where $(x, k), (y, l) \in \mathcal{G}_n$ and ρ is the distance in the Cayley graph of \mathcal{G}_n with the generating set $\{t, ta\}$. We use the following observation: to get from (x, k) to (y, l) we need

- to traverse at least one of the two paths from k to l on the n -cycle (the graph of \mathbb{Z}_n with respect to the generating set $\{\pm 1\}$);
- to visit all positions $j \in \mathbb{Z}_n$ which are not on the selected path, but which belong to $x \triangle y$, and to change the state of all lamps at these positions.

Denote by p_1 and p_2 the lengths of the two distinct paths from k to l on \mathbb{Z}_n , and by g_1 and g_2 – the sizes of the largest “gaps” in these paths, that is, the largest distances between distinct vertices for which there is no element of $x \triangle y$ in between (observe that g_1 and g_2 are at least 1 each). With this notation it is easy to see the validity of the leftmost inequality in

$$\begin{aligned} \min\{p_1 + 2(p_2 - g_2), p_2 + 2(p_1 - g_1)\} &\leq \rho((x, k), (y, l)) \\ &\leq \min\{p_1 + 2(p_2 - g_2), p_2 + 2(p_1 - g_1)\} + 2. \end{aligned} \tag{2.5}$$

The rightmost inequality in (2.5) holds because discrepancies with the equality in (2.5) can occur only at one of the endpoints of the ‘interval’ on \mathbb{Z}_n consisting

of all vertices that are visited by an optimal tour from k to l that establishes the distance between (x, k) and (y, l) . One of the cases when the distance exceeds the minimum by 2 is the following: the distance from k to l in the positive direction is significantly smaller than in the negative direction and $x\Delta y = \{k, l\}$. In this case, if we start at k we need to do one step in the negative direction in order to change the status of the lamp at k , and then head back in the positive direction. Note that the position l is reached by the step ta in order to change the status of the lamp there, so no additional steps are needed at this endpoint.

For the rest of the proof we fix $(x, k), (y, l) \in \mathcal{P}_{1,n}$, and $(A_1, A_2) = \varphi_{1,n}(x, k)$, $(B_1, B_2) = \varphi_{1,n}(y, l)$ in W_n .

Observation 2.1. *The sum $\text{lgca}(A_1, B_1) + \text{lgca}(A_2, B_2)$ is equal to the number of vertices of the set E constructed as a union of two, possibly empty, ‘intervals’ in \mathbb{Z}_n . One of the intervals starts at $0 \in \mathbb{Z}_n$, goes in the ‘positive’ direction and ends at the first vertex which belongs to $x\Delta y \cup \{k, l\}$, excluding this vertex, in particular if $0 \in x\Delta y \cup \{k, l\}$, then the interval is empty. The other interval starts at $(n-1) \in \mathbb{Z}_n$, goes in the ‘negative’ direction and ends at the first vertex which is in $x\Delta y \cup \{k, l\}$, this interval includes its end if and only if it does not belong to $x\Delta y$. Since the number of vertices in the interval E is equal to either its length $|E|$, if E is empty, or to $|E| + 1$, otherwise, we obtain*

$$|E| \leq \text{lgca}(A_1, B_1) + \text{lgca}(A_2, B_2) \leq |E| + 1. \quad (2.6)$$

Note that (2.3) and (2.6) immediately imply that

$$d_1(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l)) \geq 2(n - g),$$

where g is the number of vertices in the largest ‘interval’ I in \mathbb{Z}_n which is disjoint with $\{k, l\} \cup x\Delta y$. Since it is clear that there exists a ‘tour’ of the lamplighter that travels from k to l , passes through every vertex of the difference $x\Delta y$, visits each vertex at most twice, and stays outside I except possibly one vertex (see the paragraph after (2.5)), for all pairs $(x, k), (y, l) \in \mathcal{P}_{1,n}$ we get the inequality

$$\begin{aligned} \rho((x, k), (y, l)) &\leq 2 + 2(n - g) \leq 2 + d_1(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l)) \\ &\leq 2 + 2d_\infty(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l)). \end{aligned}$$

To get the inequality in the other direction we observe that $\rho((x, k), (y, l))$ is at least $\tau - 1$, where τ is the number of vertices in the smallest ‘interval’ J on \mathbb{Z}_n that contains all elements of $\{k, l\} \cup x\Delta y$. Note that J is nonempty, and thus $\tau = |J| + 1$.

If the interval J does not contain 0, then J and the interval E constructed in Observation 2.1 have at most one point in common, and the union $J \cup E$ covers \mathbb{Z}_n .

Thus the length of J is at least $n - |E| - 1$. In this case, by (2.3) and (2.6), we get

$$\begin{aligned}\rho((x, k), (y, l)) &\geq \tau - 1 \geq n - |E| - 1 \\ &\geq \frac{1}{2}(d_1(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l))) - 1 \\ &\geq \frac{1}{2}(d_\infty(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l))) - 1.\end{aligned}$$

If the interval J contains 0, then it contains at least $\frac{1}{3}n - 1$ vertices, because by the definition of $\mathcal{P}_{1,n}$ we have $k, l \in P_1 = [\frac{1}{6}n, \frac{5}{6}n]$. So in this case, by (2.4), we get

$$\begin{aligned}\rho((x, k), (y, l)) &\geq \frac{1}{3}n - 2 \\ &\geq \frac{1}{6} \max\{d_1((A_1, A_2), (B_1, B_2)) : (A_1, A_2), (B_1, B_2) \in W_n\} - 2 \\ &\geq \frac{1}{6}(d_\infty(\varphi_{1,n}(x, k), \varphi_{1,n}(y, l))) - 2.\end{aligned}$$

This ends the proof that the bijections $\varphi_{1,n} : (\mathcal{P}_{1,n}, d_\infty) \rightarrow (W_n, d_\infty)$ have uniformly bounded distortions.

Since, by (2.2), the spaces $\mathcal{P}_{2,n}$ and $\mathcal{P}_{3,n}$ are isometric to $\mathcal{P}_{1,n}$, the maps $\varphi_{2,n} \stackrel{\text{def}}{=} \varphi_{1,n} \circ \bar{\theta}_n$, and $\varphi_{3,n} \stackrel{\text{def}}{=} \varphi_{1,n} \circ \bar{\theta}_n^2$ are bijections from $\mathcal{P}_{2,n}$ and $\mathcal{P}_{3,n}$, respectively, onto (W_n, d_∞) , and they have the same Lipschitz and co-Lipschitz constants as the maps $\varphi_{1,n}$.

2.3 Step 2

We will apply the well-known theory of Lipschitz retracts.

For basic theory of Lipschitz retracts in metric spaces see e.g. [4, Chapter 1] and [11, Propositions 2.1, 2.2, and the comment at the top of page 303]. For the convenience of the reader, we briefly recall the definitions and results that we use.

A metric space M is called *injective* if for every metric space B , every $A \subseteq B$, and every Lipschitz function $f : A \rightarrow M$, there exists a Lipschitz extension of f , that is, a function $\bar{f} : B \rightarrow M$, so that $\bar{f}|_A = f$ and $\text{Lip}(\bar{f}) = \text{Lip}(f)$.

A metric space X is called a λ -*absolute Lipschitz retract* (where $1 \leq \lambda < \infty$) provided that whenever X is isometrically contained in a metric space Y , there exists a retraction, r , from Y onto X , with $\text{Lip}(r) \leq \lambda$.

Theorem 2.2 ([11, Proposition 2.2]). *A metric space M is injective if and only if it is an 1-absolute Lipschitz retract.*

A metric space M is said to have the *binary intersection property* if every collection of mutually intersecting closed balls in M has a common point.

A metric space M is said to be *metrically convex* if for every $x_0, x_1 \in M$ and for every $0 < t < 1$ there is a point $x_t \in M$ such that $d(x_0, x_t) = td(x_0, x_1)$ and $d(x_1, x_t) = (1 - t)d(x_0, x_1)$.

Proposition 2.3 ([4, Proposition 1.4]). *A metric space M is an 1-absolute Lipschitz retract if and only if it is metrically convex and has the binary intersection property.*

A metric space M is called a *metric tree* if it is complete, metrically convex, and for any pair of vertices there is a unique continuous curve joining them in M . Given a tree T in Graph Theory sense, one can construct the corresponding metric tree by attaching between any two adjacent vertices mutually disjoint curves isometric to the interval $(0, 1)$.

Proposition 2.4 (Corollary of [8, Lemma 2.1]). *A metric binary tree of any height is a 1-absolute Lipschitz retract.*

Observation 2.5. *The binary intersection property and metric convexity are preserved under ℓ_∞ -sums.*

Corollary 2.6. *Let Π_n be the ℓ_∞ -sum of two metric binary trees of depth n each. Then Π_n is a 1-absolute Lipschitz retract.*

As an immediate consequence we obtain the objective of Step 2.

Corollary 2.7. *For $i = 1, 2, 3$, let $\varphi_{i,n} : \mathcal{P}_{i,n} \rightarrow W_n$ be the maps defined in Step 1. Since $W_n \subseteq T_n \oplus_\infty T_n \subseteq \Pi_n$, there exist Lipschitz extensions of $\varphi_{i,n}$ to maps $\bar{\varphi}_{i,n} : \mathcal{G}_n \rightarrow \Pi_n$ which have uniformly bounded Lipschitz constants.*

2.4 The Final Step of the proof of Theorem 1.3

By [5] binary trees $\{T_n\}_{n=1}^\infty$ admit bilipschitz embeddings into any nonsuperreflexive space X with uniformly bounded distortions (see also [26] and [24]), and, without loss of generality, we may assume that these embeddings do not decrease any distances.

In [23, pp.157-158], the Mazur's method of constructing basic sequences (see [13, pp. 4-5]) was used to prove the existence of the embeddings j_n , which do not decrease distances and have uniformly bounded distortions, of an ℓ_∞ -sum of six copies of T_n into any non-superreflexive space X

$$j_n : T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n \rightarrow X.$$

Using [22, Lemma 3.3] or a direct argument, one can extend the embeddings j_n to bilipschitz embeddings of $\Pi_n \oplus_\infty \Pi_n \oplus_\infty \Pi_n$ into X with uniformly bounded distortions (recall that Π_n denotes the ℓ_∞ -sum of two metric binary trees of depth n each).

However, for our purpose, it suffices to extend the embeddings j_n to Lipschitz maps from $\Pi_n \oplus_\infty \Pi_n \oplus_\infty \Pi_n$ into X with uniformly bounded Lipschitz constants. This can be done directly: we extend the maps from a binary tree to the metric edges using 'linear interpolation': a point u_t ($0 < t < 1$) on the edge joining vertices u_0

and u_1 in the metric tree corresponding to T_n with $d(u_0, u_t) = t$ and $d(u_t, u_1) = 1 - t$ is mapped onto the corresponding convex combination of the images of u_0 and u_1 .

We denote the obtained Lipschitz maps by

$$E_n : \Pi_n \oplus_\infty \Pi_n \oplus_\infty \Pi_n \rightarrow X.$$

By construction, the maps $\{E_n\}_n$ are non-contractive on $T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n$, and their Lipschitz constants are bounded by a universal constant C .

We are now ready to define the embeddings $F_n : \mathcal{G}_n \rightarrow X$. We put

$$F_n(x, k) \stackrel{\text{def}}{=} E_n(\bar{\varphi}_{1,n}(x, k), \bar{\varphi}_{2,n}(x, k), \bar{\varphi}_{3,n}(x, k)).$$

It follows from Step 1 and our construction that the Lipschitz constants of the maps F_n are uniformly bounded. To prove that maps F_n are uniformly co-Lipschitz, we observe that for any $(x, k), (y, l) \in \mathcal{G}_n$, there exists $i \in \{1, 2, 3\}$ such that both (x, k) and (y, l) are in $\mathcal{P}_{i,n}$. By Step 1 and since the restriction of E_n to $T_n \oplus_\infty T_n \oplus_\infty T_n \oplus_\infty T_n$ is noncontractive, we get

$$\|F_n(x, k) - F_n(y, l)\|_X \geq d_\infty(\varphi_{i,n}(x, k), \varphi_{i,n}(y, l)) \geq \frac{1}{4}\rho((x, k), (y, l)),$$

which completes the proof of Theorem 1.3.

3 Proof of Corollary 1.7

For finite groups. As we mentioned in the Introduction, it is enough to prove that for all n , the Cayley graph of \mathcal{G}_n contains a subset that is bilipschitz equivalent with an absolute constant to the binary tree of depth $n/2$. As mentioned earlier, this fact was proved in [14] for trees of depth n/c , for some $c > 0$, but below we obtain it as a direct consequence of our construction in Section 2.2.

Indeed, let W_n be the set defined in Section 2.2, and let U be the subset of W_n consisting of all pairs $(A, B) \in W_n$ so that $|A| \leq \frac{n}{2}$, and B is a sequence consisting of $n - |A|$ zeroes. The distance in (W_n, d_∞) between any two elements of the set U is bounded below by the tree distance of the corresponding A s, and is bounded above by twice the tree distance of the corresponding A s. Thus U is 2-equivalent to the binary tree of depth $n/2$. Since, by Step 1, $\mathcal{P}_{1,n} \subseteq \mathcal{G}_n$ is bilipschitz equivalent with W_n , and by [5], the proof for finite groups is complete.

For the infinite group, by Remark 1.6, we only need to show that the bilipschitz embeddability of $\mathbb{Z}_2 \wr \mathbb{Z}$ into a Banach space X implies that X is nonsuperreflexive. This follows similarly as in the case of finite groups. By [30], the Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$ with the generating set $S = \{t, ta\}$ coincides with the horocyclic product of two infinite binary trees, i.e. infinite trees whose every vertex has degree 3, and the identification is obtained by mapping every element $(x, k) \in \mathbb{Z}_2 \wr \mathbb{Z}$ to an element

(A_1, A_2, k) of two infinite sequences of 0s and 1s with an (arbitrary) finite number of nonzero terms, cf. also [29]. By Theorem 1.5, the metric on $\mathbb{Z}_2 \wr \mathbb{Z}$ is bilipschitz equivalent with the metric inherited from the ℓ_1 -sum of two tree metrics. Thus, as in the finite case, we see that $\mathbb{Z}_2 \wr \mathbb{Z}$ contains a bilipschitz copy of an infinite rooted binary tree by taking the set of all elements of the form (A_1, A_2, k) , where $k = 0, 1, 2, \dots$, A_1 is any sequence of 0s and 1s so that all terms with indices larger than k are equal to 0, and all terms in the sequence A_2 are equal to 0.

The fact that bilipschitz embeddability of a binary tree into X implies nonsuper-reflexivity of X follows from [5], see also [3].

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References

- [1] L. Bartholdi, M. Neuhauser, W. Woess, Horocyclic products of trees. *J. Eur. Math. Soc. (JEMS)* **10** (2008), no. 3, 771–816.
- [2] L. Bartholdi, W. Woess, Spectral computations on lamplighter groups and Diestel-Leader graphs. *J. Fourier Anal. Appl.* **11** (2005), no. 2, 175–202.
- [3] F. Baudier, Metrical characterization of super-reflexivity and linear type of Banach spaces, *Archiv Math.*, **89** (2007), no. 5, 419–429.
- [4] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis*. Vol. **1**. American Mathematical Society Colloquium Publications, **48**. American Mathematical Society, Providence, RI, 2000.
- [5] J. Bourgain, The metrical interpretation of superreflexivity in Banach spaces, *Israel J. Math.*, **56** (1986), no. 2, 222–230.
- [6] C. Drutu, M. Kapovich, *Geometric group theory*. With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, **63**. American Mathematical Society, Providence, RI, 2018.
- [7] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [8] W.B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman, Lipschitz quotients from metric trees and from Banach spaces containing ℓ_1 . *J. Funct. Anal.* **194** (2002), no. 2, 332–346.
- [9] W.B. Johnson, G. Schechtman, Diamond graphs and super-reflexivity, *J. Topol. Anal.*, **1** (2009), no. 2, 177–189.

- [10] B. Kloeckner, Yet another short proof of the Bourgain’s distortion estimate for embedding of trees into uniformly convex Banach spaces, *Israel J. Math.*, **200** (2014), no. 1, 419–422.
- [11] U. Lang, Injective hulls of certain discrete metric spaces and groups. *J. Topol. Anal.* **5** (2013), no. 3, 297–331.
- [12] S.L. Leung, S. Nelson, S. Ostrovska, M.I. Ostrovskii, Distortion of embeddings of binary trees into diamond graphs. *Proc. Amer. Math. Soc.* **146** (2018), no. 2, 695–704.
- [13] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. **92**. Springer-Verlag, Berlin-New York, 1977.
- [14] R. Lyons, R. Pemantle, Y. Peres, Random walks on the lamplighter group. *Ann. Probab.* **24** (1996), no. 4, 1993–2006.
- [15] J. Matoušek, On embedding trees into uniformly convex Banach spaces, *Israel J. Math.*, **114** (1999), 221–237.
- [16] M. Mendel, A. Naor, Markov convexity and local rigidity of distorted metrics, *J. Eur. Math. Soc. (JEMS)*, **15** (2013), no. 1, 287–337; Conference version: *Computational geometry* (SCG’08), 49–58, ACM, New York, 2008.
- [17] A. Naor, An introduction to the Ribe program, *Jpn. J. Math.*, **7** (2012), no. 2, 167–233.
- [18] A. Naor, Y. Peres, Embeddings of discrete groups and the speed of random walks. *Int. Math. Res. Not. IMRN* 2008, Art. ID rnn 076, 34 pp.
- [19] S. Ostrovska, M.I. Ostrovskii, Nonexistence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs, *Discrete Math.*, **340** (2017), no. 2, 9–17.
- [20] M.I. Ostrovskii, Embeddability of locally finite metric spaces into Banach spaces is finitely determined, *Proc. Amer. Math. Soc.*, **140** (2012), 2721–2730.
- [21] M.I. Ostrovskii, *Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces*, de Gruyter Studies in Mathematics, **49**. Walter de Gruyter & Co., Berlin, 2013.
- [22] M.I. Ostrovskii, Different forms of metric characterizations of classes of Banach spaces, *Houston. J. Math.*, **39** (2013), no. 3, 889–906.
- [23] M.I. Ostrovskii, Metric characterizations of superreflexivity in terms of word hyperbolic groups and finite graphs. *Anal. Geom. Metr. Spaces* **2** (2014), 154–168.

- [24] M. I. Ostrovskii, Metric characterizations of some classes of Banach spaces, in: *Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory*, Celebrating Cora Sadosky's life, M. C. Pereyra, S. Marcantognini, A. M. Stokolos, W. U. Romero (Eds.), Association for Women in Mathematics Series, Vol. 4, pp. 307–347, Springer-Verlag, Berlin, 2016.
- [25] M. I. Ostrovskii, B. Randrianantoanina, A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces. *J. Funct. Anal.* **273** (2017), no. 2, 598–651.
- [26] G. Pisier, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics **155**. Cambridge, Press Cambridge University Press, 2016.
- [27] M. Stein, J. Taback, *Metric properties of Diestel-Leader groups*. Michigan Math. J. **62** (2013), no. 2, 365–386.
- [28] A. Swift, *A coding of bundle graphs and their embeddings into Banach spaces*. *Mathematika* **64** (2018), no. 3, 847–874.
- [29] J. Taback *Lamplighter groups*. in: M. Clay, D. Margalit (Editors), *Office hours with a geometric group theorist*, 310–330, Princeton Univ. Press, Princeton, NJ, 2017.
- [30] W. Woess, Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions. *Combin. Probab. Comput.* **14** (2005), no. 3, 415–433.
- [31] W. Woess, What is a horocyclic product, and how is it related to lamplighters? *Internat. Math. Nachrichten of the Austrian Math. Soc.* **224** (2013), 1–27; see also a corrected version on the author's website.

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