Realizations of rigid C*-tensor categories as bimodules over GJS C*-algebras

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Abstract

Given an arbitrary countably generated rigid C*-tensor category, we construct a fully-faithful bi-involutive strong monoidal functor onto a subcategory of finitely generated projective bimodules over a simple, exact, separable, unital C*-algebra with unique trace. The C*-algebras involved are built from the category using the Guionnet-Jones-Shlyakhtenko construction. Out of this category of Hilbert C*-bimodules, we construct a fully-faithful bi-involutive strong monoidal functor into the category of bi-finite spherical bimodules over an interpolated free group factor. The composite of these two functors recovers the functor constructed by Brothier, Hartglass, and Penneys.

1 Introduction

Since the advent of modern subfactor theory by Jones [17], there has been considerable interest in aximotizing the standard invariant resulting from a finite-index subfactor $N \subset M$. The standard invariant has been axiomitized by Ocneanu's paragroups (for finite depth subfactors) [24], Popa's λ-lattices [29], and Jones' planar algebras [18].

A natural question is given a standard invariant, can one construct a subfactor with that standard invariant? This question was answered in the affirmative by Popa in [28] where given a λ-lattice, a finite-index II_1 subfactor $N \subset M$ was constructed whose standard invariant was the given λ-lattice.

Popa's theorem was reproven utilizing a diagrammatic argument by Guionnet, Jones, and Shlyakhtenko [10]. The first author and Penneys utilized the construction therein to state an analogous result in a C*-algebraic setting: [12] *Given a standard invariant, $\mathcal{P}_\bullet^\pm$, there is an inclusion $B_0 \subset B_1$ with finite Watatani index of simple, unital C*-algebras with unique trace satisfying: $\mathcal{P}_n^+ = B_0 \cap B_n$ and $\mathcal{P}_n^- = B_1 \cap B_{n+1}$. Here (B_n) is the Jones-Watatani tower for $B_0 \subset B_1$.*

As the $N - N$ and $M - M$ bimodules generated by $\mathrm{L}^2(M)$ as an $N - M$ bimodule form rigid C*-tensor categories, it is natural to ask to what degree an arbitrary rigid C*-tensor category can be represented by bimodules over a II_1 factor. The first author, along with Brothier and Penneys [2] proved that every rigid C*-tensor category is equivalent to a subcategory of bimodules of $L(\mathbb{F}_\infty)$ (See Example 2.10.).

Theorem ([2])**.** *Given a countably generated rigid C*-tensor category,* C*, there is a a full and faithful bi-ivolutive strong monoidal functor*

$$\mathbb{G} : \mathsf{C} \hookrightarrow \mathsf{Bim}_{\mathsf{bf}}^{\mathsf{sp}} L(\mathbb{F}_\infty).$$

It is natural to ask what sort of reconstruction theorems can be obtained in the context of projective Hilbert C*-bimodules over C*-algebras. One immediate difficulty is that due to K-theoretic obstructions, it is impossible to find a single separable unital C*-algebra B so that every countably generated rigid C*-tensor category is fully realized as a subcategory of the projective C*-bimodules over B (see the discussion before Corollary 5.14). This observation leads to the following natural question:

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Question. *Given a rigid C*-tensor category, C, is there a separable, simple, unital C*-algebra B (depending on C) so that C is realized as a full subcategory of the projective C*-bimodules of B?*

In [32], Yuan answered the above question in the affirmative under the weaker assumption of not requiring norm separability on the C*-algebra, even in the case where C is countably generated (If C has infinitely many isomorphism classes of simple objects, then Yuan's C*-algebra is the norm closure of an inductive limit of direct sums of $B(\mathcal{H})$ for $\mathcal{H}$ an infinite-dimensional Hilbert space. See [32], section 3.) In this article, we answer the above question in the affirmative:

Theorem A. *Given a countably generated rigid C*-tensor category with simple unit, C, there is a unital, simple, separable, exact C*-algebra, B_0 with unique trace, and a fully-faithful bi-involutive strong monoidal functor*

$$\mathbb{F} : \mathsf{C} \hookrightarrow \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0),$$

into the rigid C-tensor category of finitely generated projective B_0-bimodules compatible with the (unique) trace of B_0. (See Example 2.15, Equation 5.1 and [20, Definition 5.7]) Moreover, the K_0 group of B_0 is the free abelian group on the classes of simple objects in C, and the image of the simple objects of C are precisely the canonical generators of $K_0(B_0)$, when viewed as right B_0-Hilbert modules.*

Our proof of this theorem can be outlined as follows:

♣ We take off from Chapter 3 by choosing a *symmetrically self-dual* object $x \in \mathsf{C}$ (Definition 3.2) of *quantum dimension* $\delta_x > 1$, and use it to construct a graded *ground* C*-*algebra* A_∞ (Equation 3.6) from the endomorphism C*-algebras $\mathsf{C}(x^{\otimes n})$. We then build an A_∞-module X_1 and consider the *creation and annihilation operators by real vectors* inside the C*-algebra of *adjointable* endomorphisms of $\mathcal{F}(X_{1_{A_\infty}})$, the *full Fock space of* $X_{1_{A_\infty}}$. (See Definition 3.12, the discussion following Corollary 3.13 and [27].) These operators together with A_∞ generate the simple C*-algebra B_∞ (*a.k.a. Voiculescu semi-circular system*), which comes equipped with a faithful conditional expectation onto A_∞ and tracial weight Tr. We then consider certain corners $B_n \subset B_\infty$(Notation 3.21), for $n \geq 0$, which are unital, separable, simple and have a unique trace, given by $\mathrm{tr}_{B_n} := \mathrm{Tr}\,|_{B_n}$.

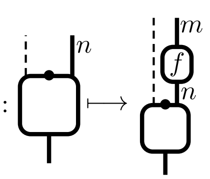
♣♣ The algebra B_0 is of central importance to us, since it acts on the *off-corners* $\{{}_lB_r\}_{l,r\in\mathbb{N}\cup 0}$ of B_∞ by means of *multiplication in the ambient* C*-*algebra* B_∞. In this framework all our actions are automatically bounded. We then construct a rigid C*-tensor category of *finitely generated projective, minimal and normalized* Hilbert C*-bimodules over B_0, denoted $\mathsf{Bim}_{\mathsf{fgp}}(B_0)$ (see Definitions 2.13 and 2.14), which includes the bimodules ${}_0B_r$, for $r \geq 0$. The simplicity of each B_n allows us to describe the *Connes' fusion* relative to B_0 of these bimodules in simple terms (Lemma 3.38) as:

$$_{B_0}({}_0B_n \underset{B_0}{\boxtimes} {}_0B_m)_{B_0} \cong_{B_0} ({}_0B_{n+m})_{B_0}.$$

This construction then induces the functor

$$\mathbb{F} : \mathsf{C}_x \hookrightarrow \mathsf{Bim}_{\mathsf{fgp}}(B_0)$$

$$x^{\otimes n} \longmapsto {}_{B_0}\,({}_0B_n)_{B_0}, \qquad \mathsf{C}(x^{\otimes n} \longrightarrow x^{\otimes m}) \ni f \longmapsto \quad \mathbb{F}(f) :$$



Here, C_x is the full subcategory of C generated by tensor powers of x . By construction, $\mathbb{F}$ will be a faithful bi-involutive strong monoidal functor, as will be shown in Chapter 4. However, showing that it is full takes some more work, and the proof is delayed to the last chapter.

♣♣♣ The next step of the proof consists of turning Hilbert C*-bimodules into *bifinite and spherical/extremal bimodules over an interpolated* II_1-*factor* M_0. This is done and explained in Chapter 5. The factor M_0 is in fact isomorphic to the *von Neumann algebra generated by* B_0, and we shall find the tools to compute it over several different representations (Lemma 3.40). In order to do so, one has to further restrict $\mathsf{Bim}_{\mathsf{fgp}}(B_0)$ to those bimodules which are *compatible with the trace* tr_{B_0}, and we denote this rigid C*-tensor subcategory by $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$. From $\mathcal{Y} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$ we obtain an *honest* Hilbert space by fusing it with $\mathrm{L}^2(B_0)$, obtaining $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$. Afterwards, we extend both left and right B_0-actions on $\mathcal{Y}$ to normal left and right actions of M_0. This process in fact defines a full and faithful bi-involutive strong monoidal functor (shown in Propositions 5.10 and 5.9)

$$_{M_0}(- \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0} : \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0) \hookrightarrow \mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0)$$

$$\mathcal{Y} \longmapsto \ _{M_0}(\mathcal{Y} \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0}, \qquad \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)(\mathcal{Y} \to \mathcal{Z}) \ni f \longmapsto f \boxtimes \mathrm{id}_{\mathrm{L}^2(B_0)}\,.$$

♣♣♣♣ Finally, we recover the functor $\mathbb{G}$ from [2] via the commuting 2-cell

$$\begin{tikzcd}
 & \mathsf{C}_x \arrow[dl, hook, "\mathbb{F}"'] \arrow[dr, hook, "\mathbb{G}"] \arrow[d, phantom, "\overset{T}{\underset{\cong}{\Rightarrow}}"] & \\
\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0) \arrow[rr, hook, "_{M_0}(-\boxtimes_{B_0}\mathrm{L}^2(B_0))_{M_0}"'] & & \mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0),
\end{tikzcd}$$

where T is a monoidal unitary natural isomorphism. It is precisely this commuting 2-cell that grants us the grounds for showing that $\mathbb{F}$ is full, and this is indeed proven by a simple finite-dimensional linear algebraic trick. (See Proposition 5.11.) We consider this step the most important part of the proof, since it provides a **conceptual and concrete connection between the realization functors** $\mathbb{F}$ and $\mathbb{G}$.

As a consequence of Theorem A (see also Remark 5.13), we found a way around the aforementioned K-theoretical obstruction to the existence of a C*-algebra B such that every rigid C*-tensor category can be realized into $\mathsf{Bim}_{\mathsf{fgp}}(B)$, by restricting our scope to unitary fusion categories: (Corollary 5.14).

Corollary B. *There exists a unital simple exact separable C*-algebra B with unique trace over which we can full and faithfully realize every unitary fusion category C into $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B)$, in the spirit of Theorem A.*

2 Background

2.1 Rigid C*-tensor categories

We shall provide a brief description and relevant examples of rigid C*-tensor categories (RC*TC). All categories in this article are assumed to be essentially small (isomorphism classes of objects form a set) and Vec enriched (not necessarily finite dimensional). We let $(\mathsf{C}, \otimes, \alpha, \mathbb{1}, \lambda, \rho)$ denote an arbitrary Vec enriched semi-simple tensor category, where $- \otimes - : \mathsf{C} \otimes \mathsf{C} \longrightarrow \mathsf{C}$ is a bilinear functor, α is the *associator*, $\mathbb{1}$ is the *monoidal/tensor unit*, and λ and ρ are the *left and right unitors*, respectively. We refer to the monoidal category simply as C. Furthermore, **we assume the monoidal unit is *simple***; i.e. its endomorphism space is one dimensional: $\mathsf{C}(\mathbb{1}) \cong$. This assumption together with semi-simplicity imply that hom spaces are finite dimensional. (For a more detailed description of (monoidal) categories see [6] and [22].) All the categories in this paper are assumed to admit direct sums and contain sub-objects; i.e. they are *Cauchy-complete.*

We further endow C with more structure:

Definition 2.1 ([15], [7])**.** *We say that C is a **dagger category** if and only if for each $a, b \in \mathsf{C}$ there is an anti-linear map*

$$\dagger : \mathsf{C}(a \to b) \to \mathsf{C}(b \to a)$$

such that

- *The map $\dagger$ is an involution; i.e. for each $a, b \in \mathsf{C}$ and every $f \in \mathsf{C}(a \to b)$ we have $(f^\dagger)^\dagger = f$.*
- *For composable morphisms f and g in C, we have $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$.*
- *Moreover, we have the identity $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.*

Furthermore, we say that $f \in \mathsf{C}(a \to b)$ is unitary if it is invertible with $f^{-1} = f^\dagger$.

Having a dagger structure on C allows us to introduce important analytical properties, as one could ask whether endomorphism $*$-algebras in C are C*-algebras. We then introduce the following definition:

Definition 2.2 ([15], [7])**.** *We say a dagger category is a* ***C*-category*** *if and only if the following conditions hold for each $a, b \in \mathsf{C}$:*

- *For every morphism $f \in \mathsf{C}(a \to b)$ there exists an endomorphism $g \in \mathsf{C}(a)$ such that $f^\dagger \circ f = g^\dagger \circ g$.*
- *The map $||\cdot|| : \mathsf{C}(a \to b) \longrightarrow [0, \infty]$ defined by*

$$||g||^2 := sup\big\{|\lambda| \geq 0 \mid (g^\dagger \circ g - \lambda \cdot id_a) \notin GL(\mathsf{C}(a))\big\}$$

  defines a (submultiplicative) C-norm. This is, the map defines a submultiplicative norm and for each morphism f as above we have $||f^\dagger \circ f|| = ||f||^2$.*

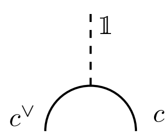
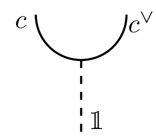
Notice that being C is a property of a dagger category and not extra structure.*

Remark 2.3. Theorem 1.2 in [14] asserts that any C*-tensor category is equivalent to a strict one. (In a strict category the associator and unitors are all identities.) We can therefore **assume that C is strict**, with no loss of generality. In lights of this result, we will often omit the associator and the unitors in our computations.

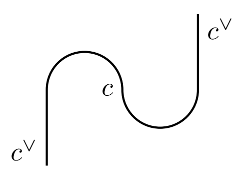
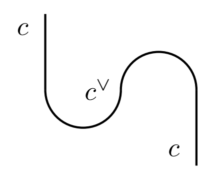
Definition 2.4 ([6])**.** *We say that C is a* ***rigid category*** *if each object has a dual and a predual. That is, for each $c \in \mathsf{C}$ there is an object $c^\vee$, the dual of c, and evaluation and coevaluation maps*

$$ev_c : c^\vee \otimes c \longrightarrow \mathbb{1} \;\text{ and }\; coev_c : \mathbb{1} \longrightarrow c \otimes c^\vee,$$

which we draw as follows:

$ev_c =$  *and* $coev_c =$  (2.1)

(From this point on, we relax the graphical calculus notation, so that we will not necessarily indicate the presence of the unit object $\mathbb{1}$ nor the dashed strand $id_{\mathbb{1}}$.) For each object $c \in \mathsf{C}$, the evaluation and coevaluation maps satisfy ***the Zig-Zag equations*** *(a.k.a. the conjugate equations in [21], Sec. 2):*

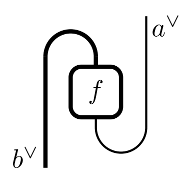
 $=$  *and*  $=$  (2.2)

Which more succinctly can be equivalently stated as

$$id_{c^\vee} = (ev_c \otimes id_{c^\vee}) \circ (id_{c^\vee} \otimes coev_c) \;\text{ and }\quad id_c = (id_c \otimes ev_c) \circ (coev_c \otimes id_c).$$

We moreover require that there exists a predual object to c, denoted by $c_\vee$, such that $(c_v)^\vee \cong c$.

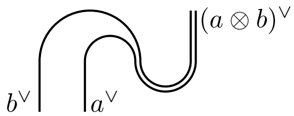
For arbitrary $c, b \in \mathsf{C}$, the dual of a map $f \in \mathsf{C}(a \to b)$ can be computed by composing with the evaluation and coevaluation maps as follows:

$f^\vee \;=$  (2.3)

where $f^\vee \in \mathsf{C}(b^\vee \to a^\vee)$ is given by $(ev_b \otimes id_{a^\vee}) \circ (id_{b^\vee} \otimes f \otimes id_{a^\vee}) \circ (id_{b^\vee} \otimes coev_a)$. As a result, these choices of duals for objects, can be arranged into a strong-monoidal ***dual functor***

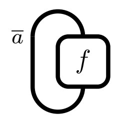
$$(\bullet)^\vee : \mathsf{C} \longrightarrow \mathsf{C}^{\text{mop}},$$

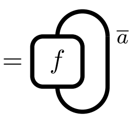
where $\mathsf{C}^{\mathrm{mop}}$ *is* C *considered with reversed arrows and tensor product. The associated tensorator is given by the natural isomorphism whose components are given by*

$\nu_{a,b} =$ 

$= (ev_b \otimes id_{(a\otimes b)^\vee}) \circ (id_{b^\vee} \otimes ev_a \otimes id_{(a\otimes b)^\vee}) \circ (id_{b^\vee} \otimes id_{a^\vee} \otimes coev_{a\otimes b}).$ (2.4)

Remark 2.5*.* Neither a dual functor nor its tensorator need to be unitary; i.e., $(f^\dagger)^\vee \neq (f^\vee)^\dagger$, nor $\nu^{-1} = \nu^\dagger$. However, for a RC*TC with simple unit, there exists a **balanced dual** for each object (see Lemma 3.9 in [31] or Prop. 3.24 in [25]); i.e., for each $c \in \mathsf{C}$, there exists a *choice* of dual $(\bar{c},\ \mathrm{ev}_c, \mathrm{coev}_c)$ for which the Zig-Zag equations hold and moreover satisfies **the balancing condition**: for an arbitrary endomorphism $f \in \mathsf{C}(a \to a)$, its **left and right traces** match; i.e

$\mathrm{ev}_a \circ (\mathrm{id}_{\overline{a}} \otimes f) \circ {\mathrm{ev}_a}^\dagger =$ 

$=$ 

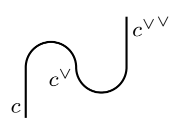
$= {\mathrm{coev}_a}^\dagger \circ (f \otimes \mathrm{id}_{\overline{a}}) \circ \mathrm{coev}_a \in .$ (2.5)

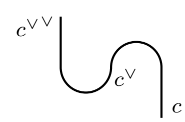
This choice of balanced dual assembles into a *unitary dual functor* for which for every morphism f we obtain $f^{\dagger^\vee} = f^{\vee^\dagger}$, and ν is unitary. We remark that this choice of dual functor is unique up to a unique natural isomorphism.

Associated to the *balanced dual functor* there is a **canonical pivotal structure**

$$\varphi : \mathrm{id}_{\mathsf{C}} \Rightarrow (\bullet)^{\vee\vee},$$

which is a **natural monoidal unitary** *spherical* (i.e. Eqn. 2.5 holds) isomorphism whose components are given by

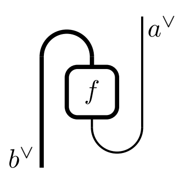
$({\mathrm{coev}_c}^\dagger \otimes \mathrm{id}_{c^{\vee\vee}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{c^\vee}) =$ 

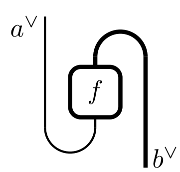
$= \varphi_c =$ 

$= (\mathrm{id}_{c^{\vee\vee}} \otimes \mathrm{ev}_c) \circ ({\mathrm{ev}_{c^\vee}}^\dagger \otimes \mathrm{id}_c).$

(2.6)

Remark 2.6*.* Using the unitary balanced dual functor in C, the naturality of φ and the canonical spherical pivotal structure, one can show that the mirrored dual of a morphism matches Eqn. 2.7: (see [2], pp. 8-10)



$= f^\vee =$ 

$= (\mathrm{id}_{a^\vee} \otimes {\mathrm{coev}_b}^\dagger) \circ (\mathrm{id}_{a^\vee} \otimes f \otimes \mathrm{id}_{b^\vee}) \circ ({\mathrm{ev}_a}^\dagger \otimes \mathrm{id}_{b^\vee}).$ (2.7)

Notice that, in light of the naturality of φ, *rotation by* 2π in C satisfies the identity $f^{\vee^\vee} \circ \varphi_a = \varphi_b \circ f$, for an arbitrary $f \in \mathsf{C}(a \to b)$.

Definition 2.7 ([16], [25], [15])**.** *Using the dagger structure* $\dagger$ *and the unique unitary (balanced) dual functor from Remark 2.5* $(\bullet)^\vee : \mathsf{C} \longrightarrow \mathsf{C}^{\mathrm{mop}}$*, we can define an* ***involutive structure*** $((\overline{\cdot}, \nu), \varphi, r)$ *on* C *as follows:*

$$\overline{\cdot} : \mathsf{C} \longrightarrow \mathsf{C}^{\mathrm{mp}}$$
$$c \mapsto \bar{c}\ (= c^\vee), \qquad f \mapsto \overline{f} := (f^\dagger)^\vee\ (= (f^\vee)^\dagger).$$

Here, C^{mp} *is the monoidal category* C *considered with the reversed tensor product. Notice that if* $f \in \mathsf{C}(a \to b)$*, then* $\overline{f} \in \mathsf{C}(\overline{a} \to \overline{b})$*. The* ***involution*** $\overline{\cdot}$ *is a conjugate-linear strong-monoidal functor, which comes equipped with the canonical unitary pivotal structure* $\{\varphi_c : c \xrightarrow{\sim} \overline{\overline{c}}\}_{c\in\mathsf{C}}$ *satisfying* $\varphi_{\overline{c}} = \overline{\varphi_c}$*. The monoidal structure is given by natural unitaries* $\{\nu_{a,b} : \overline{a} \otimes \overline{b} \xrightarrow{\sim} \overline{a \otimes b}\}$*, and an isomorphism* $r : \mathbb{1} \xrightarrow{\sim} \overline{\mathbb{1}}$ *given by* $\lambda_{\overline{\mathbb{1}}} \circ coev_{\mathbb{1}}$*. Here* λ *is the left unitor. We limit ourselves to remark there are associativity and unitality axioms these maps must satisfy, and a more detailed view can be found in [16, Definition 2.4].*

A tensor category with an involution is called ***involutive****. And a tensor category is called* ***bi-involutive*** *([15, Definition 2.3] and [25, Definition 3.35]) if it is a dagger category with an involution and moreover, the involution $\overline{\cdot}$ is a dagger functor (i.e. for every $f \in \mathsf{C}(a \to b)$, we have $\overline{f^\dagger} = \overline{f}^\dagger$) with unitary structure isomorphisms φ, ν and r.*

Remark 2.8. Notice that the **bi-involutive structure** described above is *canonical.* (The interested reader can go to [16], Example 2.12 and the references therein for an expanded description of the structure isomorphisms.) It was shown in [25] that this bi-involutive structure is canonical and independent of the choice of unitary dual functor.

The following definition will be of importance in the last section, so we include them for the convenience of the reader.

Definition 2.9 ([25])**.** *A* ***bi-involutive functor*** *between bi-involutive RC*TCs is a dagger tensor functor $(\mathbb{F}, \mu) : \mathsf{C} \longrightarrow \mathsf{D}$ together with a natural unitary isomorphism $\chi_c : \mathbb{F}(\overline{c}) \longrightarrow \overline{\mathbb{F}(c)}$ which is monoidal with respect to μ, ν^{C} and ν^{D}, and involutive with respect to φ^{C} and φ^{D}. These properties are captured by the following commutative diagrams:*

$$\begin{tikzcd} \mathbb{F}(\overline{a}) \boxtimes \mathbb{F}(\overline{b}) \arrow[r, "\mu_{\overline{a},\overline{b}}"] \arrow[d, "\chi_a \boxtimes \chi_b"'] & \mathbb{F}(\overline{a} \otimes \overline{b}) \arrow[r, "\mathbb{F}(\nu_{a,b})"] & \mathbb{F}(\overline{b \otimes a}) \arrow[d, "\chi_{b,a}"] \\ \overline{\mathbb{F}(a)} \boxtimes \overline{\mathbb{F}(b)} \arrow[r, "\nu_{\mathbb{F}(a),\mathbb{F}(b)}"'] & \overline{\mathbb{F}(b) \boxtimes \mathbb{F}(a)} \arrow[r, "\mu_{a,b}"'] & \overline{\mathbb{F}(b \otimes a)} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}(c) \arrow[r, "\mathbb{F}(\varphi_c)"] \arrow[d, "\varphi_{\mathbb{F}(c)}"'] & \mathbb{F}(\overline{\overline{c}}) \arrow[d, "\chi_{\overline{a}}"] \\ \overline{\overline{\mathbb{F}(c)}} & \overline{\mathbb{F}(\overline{c})} \arrow[l, "\overline{\chi_c}"'] \end{tikzcd} \tag{2.8}$$

We now introduce the tensor category of *bimodules over a* II_1 *factor.* Beyond its great significance in the theory of subfactors, this example is of central importance to us as it will serve as a grounds to construct a more general RC*TC of bimodules over a C*-algebra.

Example 2.10. *For this example we closely follow [1] and [16]. For a type II_1 factor (N, τ), consider the (W*-)****tensor category of bimodules over*** *N. The tensor product is given by the Connes fusion relative tensor product [4], denoted $- \boxtimes_N -$. By restricting to the full subcategory of bifinite bimodules $\mathsf{Bim_{bf}}(N)$, we obtain a RC*TC whose structure we now describe. Let $\Omega \in L^2(N, \tau)$ be a cyclic vector in the GNS-construction. Given $H \in \mathsf{Bim_{bf}}(N)$, we say that $\xi \in H$ is (left/right-)-bounded if and only if the map $N\Omega \to H$ given by $n\Omega \mapsto \xi \lhd n$, extends to a bounded map $L_\xi : L^2(N) \to H$. The set of* ***bounded vectors*** *in H is denoted by H°, and defines a dense subset of H. Bounded vectors help us define an N-valued inner product; i.e., for $\eta, \xi \in H^\circ$, we have $\langle \eta \mid \xi \rangle_N := L_\eta^* L_\xi$. This product is indeed N-valued, as $L_\eta^* L_\xi$ commutes with the right N-action.*

The dual *of $H \in \mathsf{Bim_{bf}}(N)$ is given by $\overline{H}$, as a Hilbert space. To refer to an element in $\overline{H}$ we write $\overline{\xi} \in \overline{H}$. Right and left N-actions are given by $n \rhd \overline{\xi} \lhd m := \overline{m^* \rhd \xi \lhd n^*}$, and for $\overline{\xi}, \overline{\eta} \in \overline{H}$ their inner product is given by $\langle \overline{\xi} \mid \overline{\eta} \rangle_N := \langle \eta \mid \xi \rangle_N$. We are now able to define an evaluation map*

$$ev_H : \overline{H} \underset{N}{\boxtimes} H \to L^2(N) \text{ given on } \overline{H^\circ} \underset{N}{\otimes} H^\circ \text{ by}$$

$$\overline{\eta} \boxtimes \xi \mapsto \langle \eta \mid \xi \rangle_N,$$

which is N-bilinear and bounded.

To define coevaluation maps, we need the notion of a (finite) right N-basis, which we introduce in a more general context in Definition 2.12. By [3], such a finite set exists and we denote it by $\{\beta\} \subset H^\circ$. We then have that for every $\xi \in H^\circ$, $\xi = \sum \beta \lhd \langle \beta \mid \xi \rangle_N$. For a chosen basis $\{\beta\} \subset H^\circ$, we then have the map

$$coev_H : L^2(N) \longrightarrow H \underset{N}{\boxtimes} \overline{H} \text{ given by}$$

$$n\Omega \mapsto \sum (n \rhd \beta) \boxtimes \overline{\beta},$$

*which can be seen not to depend on the choice of basis. The maps ev_H and $coev_H$ satisfy the Zig-Zag Equations (2.2). Thus $\mathsf{Bim_{bf}}(N)$ is indeed a RC*TC, where our choice of evaluation and coevaluation maps is usually referred to as the* ***tracial evaluation and coevaluation****. One may need to renormalize these maps on irreducible bimodules so that they satisfy the balancing condition described in Equation 2.5. We further restrict to the full rigid C* tensor subcategory of spherical/extremal bimodules, denoted $\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(N)$, so that the tracial dual matches the canonical unitary dual functor.*

2.2 Hilbert C*-bimodules

We recall some useful facts about Hilbert C*-bimodules that we will use in the following sections. Throughout this section, we assume that C and D are unital C*-algebras.

Definition 2.11 ([20]). *A C-D **Hilbert C*-bimodule** $\mathcal{Y}$ is a vector space together with commuting left and right actions denoted by $- \triangleright - : C \times \mathcal{Y} \to \mathcal{Y}$ and $- \triangleleft - : \mathcal{Y} \times D \to \mathcal{Y}$. Moreover, we have the following structures:*

- *A C-valued form ${}_C\langle -, \cdot \rangle$ which is C-linear on the left and conjugate linear on the right, and*
- *a D-valued form $\langle \cdot | - \rangle_D$ which is D-linear on the right and conjugate linear on the left.*
- *Compatibility of inner products with the involution: ${}_C\langle \xi, \eta \rangle = ({}_C\langle \eta, \xi \rangle)^*$, and $\langle \xi \mid \eta \rangle_D = (\langle \eta \mid \xi \rangle_D)^*$;*

These forms provide canonical norms defined as follows:
for each $y \in \mathcal{Y}$,

$${}_C||y||^2 := ||{}_C\langle y, y \rangle||_C, \text{ and } ||y||_D^2 := ||\langle y \mid y \rangle_D||_D.$$

We also require the following identities and properties to hold for every $\xi, \eta \in \mathcal{Y}$:

- *The two norms above are complete and equivalent;*
- *The two forms are positive definite; i.e. ${}_C\langle \xi, \xi \rangle \geq 0$ on $\mathcal{Y}$ and ${}_C\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$, and similarly for the D-valued form.*
- *For each $c \in C$ and each $d \in D$, we have ${}_C\langle \xi \triangleleft d, \eta \rangle = {}_C\langle \xi, \eta \triangleleft d^* \rangle$, and $\langle c \triangleright \xi \mid \eta \rangle_D = \langle \xi \mid c^* \triangleright \eta \rangle_D$.*

For Hilbert C-bimodules ${}_C\mathcal{Y}_D$ and ${}_C\mathcal{Z}_D$, denote by $\mathcal{B}(\mathcal{Y}, \mathcal{Z})$ the Banach space of bounded C-D bilinear operators. We denote the set of **left-adjointable operators** by ${}^*\mathcal{B}(\mathcal{Y}, \mathcal{Z})$, and $\mathcal{B}^*(\mathcal{Y}, \mathcal{Z})$ denotes the set of **right-adjointable operators.** We notice that under this notation, the last bullet points amounts to saying that there is an embedding of the left C-action into the Banach space of right-adjointable operators and also that there exists an embedding of the right D-action into the space of left-adjointable operators.*

Notice that the left adjoint of a given operator need not match its right adjoint. Often times, it will be sufficient for our purposes to restrict to those operators for which these notions match, denoted **bi-adjointable operators.** In the following definitions we seek for sufficient conditions for our bimodules to form a RC*TC. We will closely follow [20] for the description of such categorical/analytic structures and properties.

Definition 2.12 ([20]). *Given a Hilbert C*-bimodule ${}_C\mathcal{Y}_D$, we say that $\{u_i\}_{i=1}^m \subset \mathcal{Y}$ is a **right D-basis** if and only if for each $\xi \in \mathcal{Y}$ we have*

$$\xi = \sum_{i=1}^m u_i \triangleleft \langle u_i \mid \xi \rangle_D.$$

*Similarly, we say $\{v_j\}_{j=1}^n \subset \mathcal{Y}$ is a **left C-basis** if and only if for each $\xi \in \mathcal{Y}$ we have*

$$\xi = \sum_{j=1}^n {}_C\langle \xi, v_j \rangle \triangleright v_j.$$

If a bimodule $\mathcal{Y}$ has both a left and a right basis, we say it is **finitely generated projective** (fgp).

Definition 2.13 ([20], Definitions 1.14 and 1.16). *Define the **right index** the and **left index** of a fgp C-D Hilbert C*-bimodule $\mathcal{Y}$ as*

$$r\text{-}Ind(\mathcal{Y}) := \sum_{i=1}^m {}_C\langle u_i, u_i \rangle, \text{ and } l\text{-}Ind(\mathcal{Y}) = \sum_{j=1}^n \langle v_j \mid v_j \rangle_D.$$

By Proposition 1.13 in [20], these sums do not depend on the choices of bases, and moreover that r-Ind $\in Z(C)^+$ and l-Ind $\in Z(D)^+$, define positive central elements in their respective C-algebras. Thus, if either C or D have trivial center, the corresponding index is then a positive real number. Moreover, we define* ***the index*** *of $\mathcal{Y}$ as*

$$\mathit{Ind}(\mathcal{Y}) = \mathit{l\text{-}Ind}(\mathcal{Y}) \cdot \mathit{r\text{-}Ind}(\mathcal{Y}).$$

We say the bimodule $\mathcal{Y}$ is ***normalized*** *if r-Ind($\mathcal{Y}$) = l-Ind($\mathcal{Y}$). By [20, Lemma 1.15] fgp bimodules over C*-algebras with trivial centers can always be normalized and therefore we will only consider these, without loss of generality.*

We shall next introduce another property for an fgp bimodule ${}_C\mathcal{Y}_D$ that will become of crucial importance when defining a tensor structure for bimodules, obtaining a multiplicative notion of (Jones') index.

Definition 2.14 ([20], Proposition 3.3 and Definition 3.5)**.** *Let ${}_C\mathcal{Y}_D$ be as above, and further assume that $Z(C) == Z(D)$. For $K := \frac{\mathit{r\text{-}Ind}(\mathcal{Y})}{\mathit{l\text{-}Ind}(\mathcal{Y})} > 0$ and each $T \in End({}_C\mathcal{Y}_D)$ we have*

$$\sum_i {}_C\langle Tu_i, u_i\rangle = K \cdot \sum_j \langle v_j | T v_j\rangle_D \in,$$

we then say that $\mathcal{Y}$ is a ***minimal bimodule****.*

Example 2.15. *We now describe a RC*TC, which is a C*-algebraic analog of the category $\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(N)$ of spherical bifinite bimodules over a II_1-factor N, outlined earlier in Example 2.10. In describing this example, we rely heavily on the results in [20]. Let B be a fixed unital C*-algebra with trivial center. Consider the category of fgp normalized and minimal Hilbert C*-bimodules over B, denoted by $\mathsf{Bim}_{\mathsf{fgp}}(B)$. Neither minimality nor being normalized are extra properties of a bimodule, since one can always remetrize to simultaneously force minimality and normality. For further details, see [20, Lemma 3.6]. Since there is no loss of generality by doing so, from this point on, we will only consider* ***fgp normalized and minimal bimodules****. The greater advantage of reducing the category to normalized and minimal bimodules is that the index is well behaved with respect to sums and products in the category, which we shall next define, and that we can structure $\mathsf{Bim}_{\mathsf{fgp}}(B)$ as a RC*TC.*

*There is a canonical way to direct sum objects in $\mathsf{Bim}_{\mathsf{fgp}}(B)$, where we emphasize that the resulting object belongs to $\mathsf{Bim}_{\mathsf{fgp}}(B)$, and the index behaves additively (Lemma 3.9 [20]). The tensor structure is given by B-**fusion** $- \boxtimes_B -$, where if $\mathcal{Y}, \mathcal{Z} \in \mathsf{Bim}_{\mathsf{fgp}}(B)$ then $\mathcal{Y} \boxtimes_B \mathcal{Z} \in \mathsf{Bim}_{fgp}(B)$ without completion (Proposition 1.23 in [20]). We remark that $\mathit{Ind}(\mathcal{Y} \boxtimes_B \mathcal{Z}) = \mathit{Ind}(\mathcal{Y}) \cdot \mathit{Ind}(\mathcal{Z})$ holds true on $\mathsf{Bim}_{\mathsf{fgp}}$ (Proposition 1.30, [20]), and that fusion is distributive with respect to sums (Lemma 1.24, [20]).*

For $\mathcal{Y}, \mathcal{Z} \in \mathsf{Bim}_{\mathsf{fgp}}(B)$, by [20, Lemma 1.10], any right (resp. left) B-module map $T : \mathcal{Y} \to \mathcal{Z}$ is automatically a finite rank operator, is bounded with respect to the right (resp. left) B-norm (and hence both norms) and is adjointable with respect to the right (resp. left) inner product, with right-adjoint T^* (resp. left-adjoint ${}^*T$). Moreover, minimality together with Proposition 3.14 in [20] ensure that if T is also adjointable with respect to the left inner product, it satisfies ${}^*T = T^*$. Therefore $\mathsf{Bim}_{\mathsf{fgp}}(B)(\mathcal{Y} \to \mathcal{Z})$ consists of ***bi-adjointable bounded*** *B-bimodule maps, with common adjoint denoted T^*. This defines a* ***dagger structure*** *on $\mathsf{Bim}_{\mathsf{fgp}}(B)$.*

We now define the ***conjugate bimodule*** *of $\mathcal{Y}$ denoted $\overline{\mathcal{Y}}$, where $\mathcal{Y} = \overline{\mathcal{Y}}$ as sets. The bimodule structure is as follows: if $\overline{y}, \overline{y'} \in \overline{\mathcal{Y}}$ and $a, b \in B$ the left and right actions are given by $a \rhd \overline{y} \lhd b := \overline{b^* \rhd y \lhd a^*}$, ${}_B\langle \overline{y}, \overline{y'}\rangle := \langle y \mid y'\rangle_B$ and $\langle \overline{y} \mid \overline{y'}\rangle_B := {}_B\langle y,\ y'\rangle$. Clearly, $\overline{\mathcal{Y}} \in \mathsf{Bim}_{\mathsf{fgp}}(B)$. We are now in position to define* ***evaluation and coevaluation****: for a given finite right B-basis $\{u_i\}_{i=1}^m$ we have maps*

$$ev_{\mathcal{Y}} : \overline{\mathcal{Y}} \underset{B}{\boxtimes} \mathcal{Y} \longrightarrow {}_B B_B, \qquad coev_{\mathcal{Y}} : {}_B B_B \longrightarrow \mathcal{Y} \underset{B}{\boxtimes} \overline{\mathcal{Y}},$$

$$\overline{\eta} \boxtimes \xi \mapsto \langle \eta \mid \xi\rangle_B \qquad b \mapsto b \rhd \sum_{i=1}^m u_i \boxtimes \overline{u_i}$$

One must check these maps belong to our category. For example, $ev_{\mathcal{Y}}$ is manifestly right B-linear so it is a bounded right-adjointable B-bimodule map. By direct computation, $ev_{\mathcal{Y}}{}^*(b) = (\sum_{j=1}^n \overline{v_j} \boxtimes v_j) \lhd b$ and *moreover* $^*\, ev_{\mathcal{Y}}\,(b) = b \triangleright \sum_{j=1}^n \overline{v_j} \boxtimes v_j$, *where* $\{v_j\}_{j=1}^n$ *is a finite left* B*-basis and* $b \in B$ *is arbitrary. Therefore,* $^*ev_{\mathcal{Y}}$ *is indeed the left-adjoint of the evaluation map. Then, necessarily,* $^*\, ev_{\mathcal{Y}}\,(b) = b \triangleright \sum_{j=1}^n \overline{v_j} \boxtimes v_j = (\sum_{j=1}^n \overline{v_j} \boxtimes v_j) \triangleleft b = ev_{\mathcal{Y}}^*$, *showing that* $ev_{\mathcal{Y}} \in \mathsf{Bim}_{\mathsf{fgp}}(B)(\overline{\mathcal{Y}} \boxtimes \mathcal{Y} \to B)$. *Similarly for* $coev_{\mathcal{Y}}$, *where* $coev_{\mathcal{Y}}^*(y \boxtimes \overline{y'}) = {}_B\langle y,\ y'\rangle = {}^*\, coev_{\mathcal{Y}}\,(y \boxtimes \overline{y'})$. *We now define* ***the dual module*** $\mathcal{Y}^\vee := \overline{\mathcal{Y}}$ *and* ***the dual of a morphism*** $T^\vee$ *is given by the canonical dual. By Proposition 4.1 in [20], under these structural maps, the category of fgp* B*-bimodules* $\mathsf{Bim}_{\mathsf{fgp}}(B)$ *is a rigid category. In fact, since we restricted to normalized bimodules,* $\mathsf{Bim}_{\mathsf{fgp}}(B)$ *has the C*-property, by Corollary 4.3 in [20].*

This choice of dual functor corresponds to the balanced unitary dual. Indeed the canonical spherical structure on $\mathsf{Bim}_{\mathit{fgp}}(B)$ *defines an honest pivotal structure so, by Proposition 3.9 in [25], the result follows. (Remarkably, that* $^*T = T^*$ *for every biadjointable map is an equivalent condition to the naturality of the pivotal structure.) Thus, we can consider the canonical* ***bi-involutive structure*** *given on objects by conjugation* $\overline{\mathcal{Y}}$ *and a morphism* $T : \mathcal{Y} \to \mathcal{Z}$, *its conjugate* $\overline{T} : \overline{\mathcal{Y}} \to \overline{\mathcal{Z}}$ *is given by* $\overline{T} := (T^*)^\vee = (T^\vee)^*$ *and satisfies* $\overline{T}(\overline{y}) = \overline{T(y)}$.

3 Diagramatic Algebras

3.1 GJS construction

In this section we will assume that C is a fixed essentially small RC*TC. We start with an auxiliary lemma, which helps us find a special object $x \in \mathsf{C}$. From the morphisms between tensor powers of this object we will construct certain diagrammatic $*$-algebras, and an *ambient* C*-algebra, denoted B_∞ (Definition 3.15), which we will heavily rely on. The corners B_n of B_∞ will be separable simple unital C*-algebras with unique traces. Of particular importance will be the corner B_0 (Notation 3.21), as we will use it to construct a concrete rigid C*-tensor subcategory of $\mathsf{Bim}_{\mathsf{fgp}}(B_0)$, (Example 2.15) where we *fully and faithfully* represent C.

Notation 3.1. *For an object* $c \in \mathsf{C}$, *we define* δ_c *to be the* ***quantum dimension*** *of* c. *This is,* $\delta_c := coev_c{}^* \circ coev_c = ev_c \circ ev_c{}^*$, *is the (complex) value of the* c*-loop.*

Definition 3.2. *We say that an object* $x \in \mathsf{C}$ *is* ***self-dual*** *if and only if there exists a (unitary) isomorphism* $\psi : x \longrightarrow \overline{x}$. *(This isomorphism can be chosen to be unitary by polar decomposition.) Moreover, we say that* x *is* ***symmetrically self-dual*** *(ssd) ([2], Definition 2.10) if and only if is self-dual and* $ev_x \circ (\psi \otimes id_x) = coev_x{}^* \circ (id_x \otimes \psi)$. *(For more detailed treatments of self-dualities, the reader can go to [5, Theorem 3.4], [21], and [13].)*

Intuitively, in the graphical calculus, strings labeled by a ssd object need not be oriented. In the following lemma, we show there are always ssd objects. We can **fix a ssd object** $x \in \mathsf{C}$, since

Lemma 3.3. *Every RC*TC has a ssd object* x, *with* $\delta_x > 1$ *and* $x = \overline{x}$.

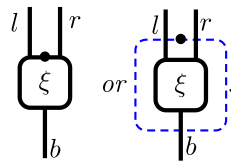
Proof. Let $\sigma \in \mathsf{C}$ be an arbitrary object (not isomorphic to the tensor unit), and define $x := \sigma \oplus \overline{\sigma}$. Clearly, $\delta_x > 1$. There is a canonical unitary map $\psi : (\sigma \oplus \overline{\sigma}) \longrightarrow (\overline{\overline{\sigma}} \oplus \overline{\sigma}) \cong \overline{x}$ given by the diagonal matrix $\mathrm{diag}(\varphi_\sigma, \mathrm{id}_{\overline{\sigma}})$, where φ is the canonical unitary pivotal structure on C. To establish symmetric self-duality, one simply observes that this condition is equivalent to $\mathrm{ev}_x = \mathrm{coev}_{\overline{x}}{}^\dagger \circ (\mathrm{id}_{\overline{x}} \otimes \varphi_x)$ and $\mathrm{ev}_{\overline{x}} \circ (\varphi_x \otimes \mathrm{id}_{\overline{x}})$, both of which hold since we are using the canonical pivotal structure. Finally, we can choose $x = \overline{x}$ by using ψ to *unitarily* redefine the balanced dual functor, if need be. □

From this point on, we reserve the symbol x to denote a fixed ssd object in C as in the statement of the previous Lemma.

Definition 3.4 ([2, 9, 12, 13])**.** *Given non-negative integers* l, r, b, *we let* $V_{b,l,r}$ *be* $\mathsf{C}(x^{\otimes b} \to x^{\otimes l} \otimes x^{\otimes r})$. *We define* Gr_∞ *whose underlying vector space is given by the external algebraic direct sum*

$$\mathrm{Gr}_\infty := \bigoplus_{l,r,b \geq 0} V_{b,l,r}.$$

An element $\xi \in V_{b,l,r}$ can be pictorially visualized as follows:

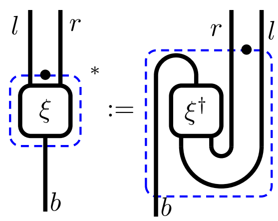


(3.1)

Here, the dot indicates how the lateral strings split on each side; i.e. there are l strings on the left and r strings on the right. We warn the reader that the same element $\xi \in V_{b,l,r}$ appears as many times as the number $(l+r)$ can be decomposed as a sum of two non-negative integers.

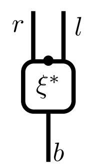
There is an **involution on** Gr_∞ denoted by $*$, which is determined by the canonical conjugate structure in C, given by its *bi-involutive structure* (See Remark 2.8):

$$* : V_{b,l,r} \longrightarrow V_{b,r,l}$$



$= (\xi^\dagger)^\vee = (\xi^\vee)^\dagger.$ (3.2)

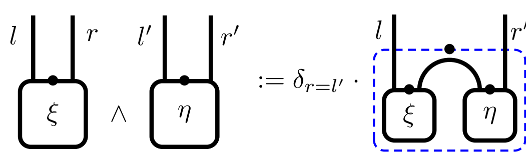
Notice how the roles of l and r switched. For notational convenience, we will denote this diagram as



Definition 3.5. *Define* $\mathrm{Gr}_{0,\infty}$ *by*

$$\mathrm{Gr}_{0,\infty} := \bigoplus_{l\geq 0, r\geq 0} V_{0,l,r}.$$

The space $\mathrm{Gr}_{0,\infty}$ *becomes an algebra when endowed with the multiplication*



(3.3)

Notice that if $l + r = l' + r'$ with $l \neq l'$, this multiplication makes $V_{b,l,r}$ orthogonal to $V_{b',l',r'}$, and this is also true for elements that are counted more than once in Gr_∞.

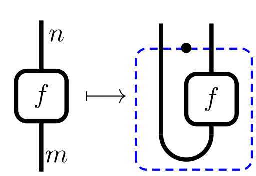
Define $\mathrm{Gr}_{0,n}$ *by*

$$\mathrm{Gr}_{0,n} := \bigoplus_{n\geq l, r\geq 0} V_{0,l,r}. \tag{3.4}$$

Note that $\mathrm{Gr}_{0,n}$ is a finite dimensional unital $*$-subalgebra of $\mathrm{Gr}_{0,\infty}$ which we describe in the following Proposition whose proof is straightforward:

Proposition 3.6. *For each $n, m \in \mathbb{N}$, we have an injective $*$-algebra homomorphism (given by* ***Frobenius Reciprocity*** *on* C*):*

$$FR : \mathsf{C}^{op}(x^{\otimes m} \to x^{\otimes n}) \xrightarrow{\sim} \mathrm{Gr}_{0,max(m,n)} \quad \textit{given by}$$



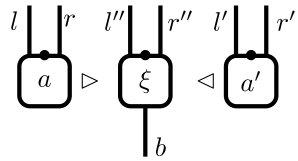
(3.5)

When $n = m$, the map FR becomes an injective $*$-algebra isomorphism.

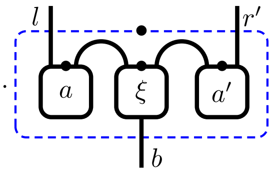
Therefore, using Frobenius Reciprocity, we endow $\mathrm{Gr}_{0,n}$ *with a norm, giving it the structure of a unital C^*-algebra. Note that the inclusion* $\mathrm{Gr}_{0,n} \hookrightarrow \mathrm{Gr}_{0,n+1}$ *is nonunital but preserves minimal projections. As such,* $\mathrm{Gr}_{0,\infty}$ *(being the inductive limit of the* $\mathrm{Gr}_{0,n}$ *) can be completed into an AF C^*-algebra which we shall denote as A_∞ with multiplication still denoted by $\wedge$. This is, as a vector space*

$$A_\infty := \overline{\varinjlim_n \mathrm{Gr}_{0,n}}. \tag{3.6}$$

We shall now complete Gr_∞ into a right Hilbert C*-module over A_∞ that will become of central importance in the sequel: For any $n, m \in \mathbb{N}$ we have left and right actions of $\mathrm{Gr}_{0,n}$ and $\mathrm{Gr}_{0,m}$ on Gr_∞ defined as follows: for $a \in \mathrm{Gr}_{0,n}$, $a' \in \mathrm{Gr}_{0,m}$ and $\xi \in \mathrm{Gr}_\infty$, we have

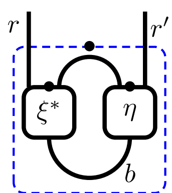


$$:= \delta_{r=l''} \cdot \delta_{r''=l'} \cdot \tag{3.7}$$



The module Gr_∞ then becomes endowed with an A_∞-valued inner product (taking values in $\mathrm{Gr}_{0,\infty}$) given by the sesquilinear extension of

$$\langle \xi \mid \eta \rangle_{A_\infty} := \delta_{l=l'} \cdot \delta_{b=b'} \cdot \tag{3.8}$$



for $\xi \in V_{b,l,r}$ and $\eta \in V_{b',l',r'}$. Note that this inner product is linear in the right variable and conjugate linear in the left.

It is a straightforward computation to prove the next Proposition, following from the pivotal spherical structure on C (see Remark 2.8):

Proposition 3.7. *The following statements hold for all diagrams $\xi, \eta \in \mathrm{Gr}_\infty$ and $a \in \mathrm{Gr}_{0,\infty}$:*

- *Right A_∞-linearity:* $\langle \xi \mid \eta \lhd a \rangle_{A_\infty} = \langle \xi \mid \eta \rangle_{A_\infty} \wedge a$;
- *The left action is right-adjointable:* $\langle \xi \mid a^* \rhd \eta \rangle_{A_\infty} = \langle a \rhd \xi \mid \eta \rangle_{A_\infty}$; *and*
- *Compatibility with adjoints:* $(\langle \xi \mid \eta \rangle_{A_\infty})^* = \langle \eta \mid \xi \rangle_{A_\infty}$.

Definition 3.8. *The vector space* Gr_∞ *can now be endowed with a C^*-norm, denoted $\| \cdot \|_{A_\infty}$, given by $\|\xi\|^2_{A_\infty} := \|\langle \xi \mid \xi \rangle_{A_\infty}\|_{A_\infty}$, where the latter norm is the norm in the C^*-algebra A_∞. (We remind the reader that this norm was introduced previously in Definition 3.28.) We therefore define $\mathcal{X}_\infty$ to be the completion of* Gr_∞ *under $\| \cdot \|_{A_\infty}$; this is*

$$\mathcal{X}_\infty := \overline{\mathrm{Gr}_\infty}^{||\cdot||_{A_\infty}}.$$

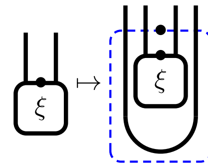
We also extend the A_∞-valued inner product to all of $\mathcal{X}_\infty \times \mathcal{X}_\infty$ in the obvious way. We will keep the same notation for this extended product.

To equip $\mathcal{X}_\infty$ with a right Hilbert A_∞-module structure (see Definition 3.28 on Hilbert bimodules and Definition 1.3.2 in [23] for Hilbert modules), we must first extend the actions to all of $\mathcal{X}_\infty$ and all of A_∞. The right $\mathrm{Gr}_{0,\infty}$-action extends to $\mathcal{X}_\infty$ since for each sequence $\xi_n \in \mathrm{Gr}_\infty$ converging to $\xi \in \mathcal{X}_\infty$, and for each $a \in \mathrm{Gr}_{0,\infty}$, a simple diagrammatic computation reveals that $||\xi_n \lhd a||^2_{A_\infty} = \delta_x \cdot ||a^* \wedge \langle \xi_n | \xi_n \rangle_{A_\infty} \wedge a||_{A_\infty} \leq \delta_x \cdot ||\xi_n||^2_{A_\infty} \cdot ||a||^2_{A_\infty}$, which is bounded above in A_∞-norm, since the ξ_n are bounded. Using the same inequality we obtain a (bounded) right A_∞-action on $\mathcal{X}_\infty$. Moreover, observe that the first and last items in Proposition 3.7 apply to the extended A_∞-action over the module $\mathcal{X}_\infty$. Thus, $(\mathcal{X}_\infty, \langle \cdot \mid \cdot \rangle_{A_\infty})$ is a right Hilbert C*-module over A_∞.

We shall now show there is a left action of A_∞ on $\mathcal{X}_\infty$ by right adjointable operators. To do so, we first bring an auxiliary lemma:

Lemma 3.9. *For $n, m \geq 0$, the collection of maps*

$$\mathrm{Gr}_{0,n} \longrightarrow \mathrm{Gr}_{0,n+m} \quad \textit{given by the linear extension of} \tag{3.9}$$

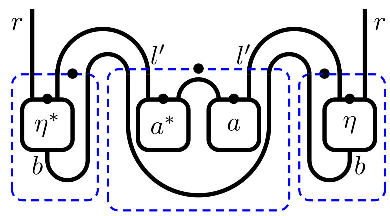


extends to an isometric C^*-algebra homomorphism $A_\infty \longrightarrow A_\infty$.

Proof. This map is clearly a densely defined injective $*$-algebra homomorphism. This map is bounded because it is a monomorphism between the finite dimensional algebras $\mathrm{Gr}_{0,n}$ and $\mathrm{Gr}_{0,n+m}$. $\square$

For arbitrary $\eta \in V_{b,l,r} \subset \mathrm{Gr}_\infty$ and $a \in \mathrm{Gr}_{0,\infty}$ we have the following diagrammatic identity:

$$\langle a \rhd \eta | a \rhd \eta \rangle_{A_\infty} = \delta_{l=r'} \cdot \tag{3.10}$$



Thus, denoting the rightmost element above as $\xi \in \mathrm{Gr}_{0,\infty}$, we then obtain that $\langle a \rhd \eta \mid a \rhd \eta \rangle_{a_\infty} = \xi^* \wedge [(a^* \wedge a)] \wedge \xi$. Here, $[a^* \wedge a]$ denotes the middle element in the diagram above. By Lemma 3.9 we obtain that $||a \rhd \eta||^2_{A_\infty} = ||\langle a \rhd \eta \mid a \rhd \eta \rangle_{A_\infty}||_{A_\infty} = ||\ \xi^* \wedge [(a^* \wedge a)] \wedge \xi||_{A_\infty} \leq ||\xi\ ||^2_{A_\infty} \cdot ||a||^2_{A_\infty} = ||\eta||^2_{A_\infty} \cdot ||a||^2_{A_\infty}$. Similarly as done for the right action, this inequality allows us to define a bounded left action of A_∞ on $\mathcal{X}_\infty$. Finally, notice that the second item in Proposition 3.7 (which extends to the left action of A_∞ on $\mathcal{X}_\infty$ by a density argument), implies that the left A_∞-action is given by right-adjointable operators. Therefore, there is a faithful embedding $([A_\infty] \rhd -) \hookrightarrow \mathcal{B}^*((\mathcal{X}_\infty)_{A_\infty})$.

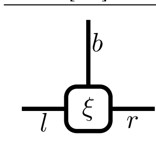
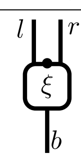
There is an $(\mathbb{N} \cup \{0\})$-graded family of right A_∞-modules of importance:

Definition 3.10. *For arbitrary $b \geq 0$, set $\mathrm{Gr}_{b,\infty}$ to be the following subspace of Gr_∞ :*

$$\mathrm{Gr}_{b,\infty} := \bigoplus_{l \geq 0, r \geq 0} V_{b,l,r},$$

and define X_b as its completion in $\mathcal{X}_\infty$. Note that each X_b is naturally a right A_∞-module with a canonical left A_∞ action (by right-adjointable operators). Also note that $X_0 = A_\infty$.

The treatment below relies heavily on the machinery in section 4 of [13] and is essentially a translation of that section into our diagrammatic language. In that article, planar algebras were used as a substitute for the graphical picture used in this article. Below is a dictionary that the reader can use when parsing the changes in notation and diagrams between [13] and this article.

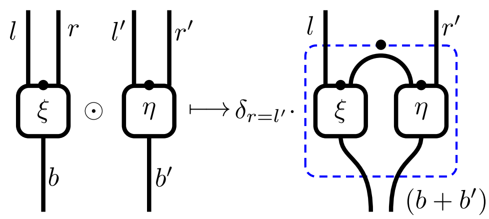
[13]	This manuscript
	
$\mathcal{P}_{l,n,r}$	$V_{b,l,r}$
$\mathcal{F}(\mathcal{P}_\bullet)$	$\mathcal{X}_\infty$
$B_n(\mathcal{P}_\bullet)$	$\mathrm{Gr}_{0,n}$
$B(\mathcal{P}_\bullet)$	A_∞
X_b	$\mathrm{Gr}_{b,\infty}$
$\mathcal{X}_b$	X_b
$L(\xi)$	$L(\xi)$(creation operators)

We have the following proposition, appearing as [13, Proposition 4.11]:

Proposition 3.11 ([13])**.** *For arbitrary $b, b' \geq 0$, the mapping*

$$U_{b,b'} : \mathrm{Gr}_{b,\infty} \odot \mathrm{Gr}_{b',\infty} \to \mathrm{Gr}_{(b+b'),\infty}$$

defined by



(3.11)

extends to a unitary isomorphism of right A_∞ Hilbert C-bimodules:*

$$U_{b,b'} : X_b \otimes_{A_\infty} X_{b'} \to X_{b+b'}.$$

Here, the relative (algebraic) tensor product $\odot$ is balanced over diagrams with no strings going up; i.e. over $\bigoplus_{b''\geq 0} \mathrm{Gr}_{b'',0,0}$.

We now introduce a construction due to Pimsner.

Definition 3.12 ([27])**.** *Given an arbitrary Hilbert C*-module $\mathcal{Y}$ over a C^*-algebra A with a left action of A as bounded, adjointable operators on $\mathcal{Y}$, one can form the* ***full Fock space*** *$\mathcal{F}(\mathcal{Y})$ given by*

$$\mathcal{F}(\mathcal{Y}) = A \oplus \bigoplus_{n=1}^{\infty} \mathcal{Y}^{\otimes_A^n}.$$

An immediate consequence from Proposition 3.11 and the previous definition gives:

Corollary 3.13. *The unitary operators in Proposition 3.11 induce a unitary*

$$U : \mathcal{F}(X_1) \to \mathcal{X}_\infty.$$

Recall that by [27], given $\mathcal{F}(\mathcal{Y})$ as above, and $\xi \in \mathcal{Y}$, one can form the **creation operator** $\ell(\xi)$ on $\mathcal{F}(\mathcal{Y})$ defined as as the linear extension of

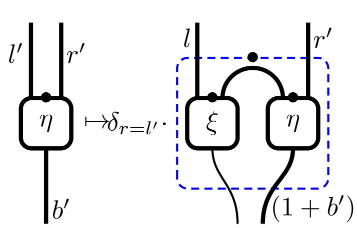
$$\begin{aligned} \ell(\xi)a &= \xi a && \text{for all } a \in A \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) &= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n && \text{for all } \xi_1, \ldots, \xi_n \in \mathcal{Y}. \end{aligned}$$

The operator $\ell(\xi)$ is bounded and right-adjointable, and its adjoint is given by

$$\begin{aligned} \ell(\xi)^*a &= 0 && \text{for all } a \in A \\ \ell(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) &= \langle \xi \mid \xi_1 \rangle_A \xi_2 \otimes \cdots \otimes \xi_n && \text{for all } \xi_1, \ldots, \xi_n \in \mathcal{Y}. \end{aligned}$$

In particular, for $\mathcal{Y} = X_1$ and for an arbitrary $\xi \in V_{1,l,r}$, consider the operator $L(\xi)$ on Gr_∞, which we define using the spaces $V_{b',l',r'}$ by:

$L(\xi)$:



(3.12)

For $\xi \in \mathrm{Gr}_{1,\infty}$, one defines $L(\xi)$ by linearity. The following is now immediate:

Proposition 3.14. *The unitary operator U in Corollary 3.13 satisfies $U^*L(\xi)U = \ell(\xi)$ for all $\xi \in \mathrm{Gr}_{1,\infty}$.*

Since one has $\|L(\xi)\| = \|\xi\|$ for all $\xi \in \mathrm{Gr}_{1,\infty}$ [27], it follows immediately that $L(\xi)$ is bounded for all $\xi \in \mathrm{Gr}_{1,\infty}$, and that one can use continuity to define $L(\xi)$ for all $\xi \in X_1$. It follows from the arguments in [13, Propositions 4.7–4.9] that the $*$-operation extends continuously on each X_n.

We now introduce the C*-algebra B_∞, a corner of which (the unital simple C*-algebra B_0 from Notation 3.21) will be the main C*-algebra in this article.

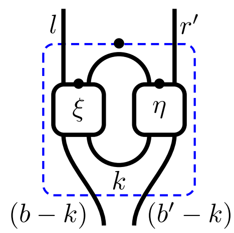
Definition 3.15. *Let $X_{\mathbb{R}}$ be the real subspace $\{\xi \in X_1 | \ \xi = \xi^*\}$. Then we define B_∞ to be the C*-algebra generated by*

$$A_\infty \cup \{L(\xi) + L(\xi)^* | \ \xi \in X_{\mathbb{R}}\} \subset \mathcal{B}^*(\mathcal{X}_\infty).$$

We will often denote the (operator) norm on B_∞ by $||\cdot||_{B_\infty}$.

There will be a convenient way to diagrammatically realize B_∞. Given $\xi \in V_{b,l,r}$, define $\pi(\xi)$ to be the operator on Gr_∞ given by (the linear extention of):

$$\pi(\xi)\eta := \delta_{r=l'} \cdot \sum_{k=0}^{\min\{b,b'\}} \qquad (3.13)$$



for $\eta \in V_{b',r',l'}$. We extend the definition of $\pi(\xi)$ linearly in ξ to all of Gr_∞. By the following Proposition ([13, Proposition 4.16]), we obtain that $\pi(\xi)$ is bounded:

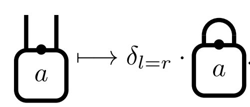
Proposition 3.16. *For each $\eta \in \mathrm{Gr}_\infty$, $\pi(\eta)$ is a polynomial in A_∞ and $\{L(\xi) + L(\xi)^* | \ \xi \in X_{\mathbb{R}}\}$. Consequently, $\pi(\eta)$ is bounded for each $\eta \in \mathrm{Gr}_\infty$.*

Remark 3.17. We notice that under this representation, π, the composition of operators satisfies $\pi(\xi') \circ \pi(\xi) = \pi(\xi' \star \xi)$, where $\xi' \in V_{b',l',r'}$, $\xi \in V_{b,l,r}$ and $- \star -$ is the **Walker Multiplication**, which diagrammatically is given by Equation 3.13 ([19]). We moreover have that Gr_∞ with the Walker product $- \star -$ is isomorphic to Gr_∞ with the **wedge/graded product** $- \wedge -$ as unital (tracial) $*$-algebras. (See [19, Lemmata 5.2 and 5.3], for further details on the traces and the isomorphism). It is straightforward to see that if $\xi \in X_{\mathbb{R}}$, then $\pi(\xi) = \xi \star - = L(\xi) + L(\xi)^*$. From this observation, we see that B_∞ is generated by A_∞ and $\{\pi(\eta) | \ \eta \in \mathrm{Gr}_\infty\}$.

3.2 A weight and a conditional expectation on B_∞

The AF $*$-algebra $\mathrm{Gr}_\infty \cap A_\infty$ can be endowed with the following positive linear functional given by the linear extension of

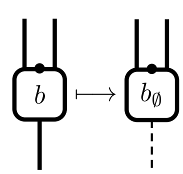
$$\Phi : \mathrm{Gr}_\infty \cap A_\infty \longrightarrow \qquad \text{given by} \qquad (3.14)$$



Proposition 3.6, allows us to see this map is simply the trace on C. (See Equation 2.5.) Therefore, the map Φ extends to a faithful, positive, semifinite, tracial weight on A_∞.

If we if we let $\Pi_0 \in \mathcal{B}(\mathcal{X}_\infty)$ be the projection from $\mathcal{X}_\infty$ onto A_∞, then from (Lemma 2.10 in [30]), Π_0 induces a faithful **conditional expectation**, $\mathbb{E}$ from B_∞ onto A_∞. Diagrammatically, this map is given by

$$\begin{aligned} \mathbb{E} : B_\infty &\longrightarrow A_\infty, \\ b &\longmapsto \Pi_0 \ b \ \Pi_0. \end{aligned} \qquad (3.15)$$



A straightforward computation shows that if $x \in \mathrm{Gr}_\infty$, then $\Pi_0 x \Pi_0 \in A_\infty$, so $\mathbb{E}$ is well defined by continuity. From [30, Lemma 2.10 and Remark 2.13] we obtain the following proposition:

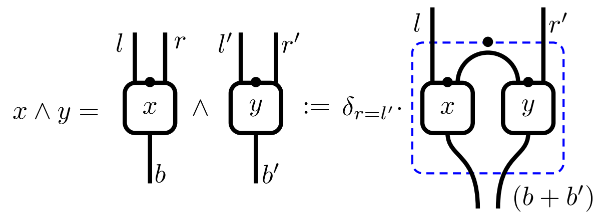
Proposition 3.18 ([13]). *The following assertions hold:*

- *The map $\mathbb{E}$ is a faithful conditional expectation from B_∞ onto A_∞.*
- *The composite map $\Phi \circ \mathbb{E}$ is a faithful, semifinite tracial weight on B_∞, which is finite on $\{\pi(\xi)|\xi \in \mathrm{Gr}_\infty\}$.*

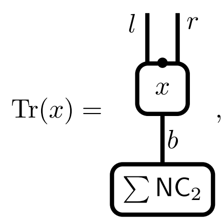
As Hilbert spaces, we have that $L^2(B_\infty, \Phi\circ\mathbb{E}) \cong \mathcal{X}_\infty \otimes_{A_\infty} L^2(A_\infty, \Phi)$. In fact, one can check diagrammatically that this equivalence is realized by the unitary W given by $W(\pi(\xi)) = \xi$ for all $\xi \in \mathrm{Gr}_\infty$. Moreover, W satisfies $W(b\eta) = bW(\eta)$ for all $b \in B_\infty$ and each $\eta \in \mathrm{Gr}_\infty$.

Remark 3.19. The above proposition allows us to canonically identify Gr_∞ as a dense subspace of B_∞, and we will suppress the π in $\pi(x)$ when the context is clear.

Remark 3.20. In order to present a more visually appealing picture of the multiplication in B_∞ we endow Gr_∞ with the following multiplication and tracial weight (sometimes refered to as the *Voiculescu trace*)



and

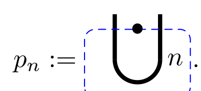


where $\sum \mathsf{NC}_2$ represents the sum over all non-crossing pairings of the b strings on the bottom of the diagram. It was shown in [19], that there is a tracial weight preserving $*$-isomoprhism Ψ from $\pi(\mathrm{Gr}_\infty)$ to Gr_∞ satisfying

$$\Phi(\pi(x)) \circ \mathbb{E} = \mathrm{Tr}(\Psi(\pi(x))) \text{ and } \Psi(\pi(x)\pi(y)) = \Psi(\pi(x)) \wedge \Psi(\pi(y)).$$

This means that the C*-algebra completion of Gr_∞ in the GNS representation of Tr is also isomorphic to B_∞. We will use this multiplication below to declutter many diagrams.

Notation 3.21. *For each $n \in \mathbb{N}$ we let $p_n \in \mathrm{Gr}_{0,n,n}$ be the element*



*We note that $(\Phi \circ \mathbb{E})$ descends to a positive, faithful linear **tracial functional** $\mathrm{Tr}_n(\bullet) = (\Phi\circ\mathbb{E})(p_n \bullet p_n)$ on the unital C*-algebra*

$$B_n := p_n \wedge B_\infty \wedge p_n \subset B_\infty,$$

which inherits a subalgebra C-norm, denoted by $||\cdot||_{B_n}$.*

By defining also $p_0 := \emptyset \in V_{0,0,0}$ to be the empty diagram, we can consider Tr_0 and the tracial unital (simple) C-algebra*

$$B_0 := p_0 \wedge B_\infty \wedge p_0 \subset B_\infty,$$

which is of special interest to us. This subalgebra also inherits a C-norm, denoted by $||\cdot||_{B_0}$. Notice that B_0 contains all diagrams with no strings on the top; i.e. $l = 0 = r$. Moreover, we observe that $1_{B_0} = \emptyset$. Finally, for $m, n \geq 0$, we define the (m,n)-th **corner of** B_∞ by ${}_nB_m := p_n \wedge B_\infty \wedge p_m$. Observe that ${}_nB_n = B_n$ for every n.*

Finally, on each B_n, we define $\mathrm{tr}_{B_n} := \mathrm{tr}_n$ to be $\mathrm{tr}_n(b) = \frac{1}{\delta_x^n}\mathrm{Tr}_n(b)$. (We remind the reader that $\delta_x > 1$ was introduced earlier on in Notation 3.1.) Notice that $\mathrm{tr}_n(p_n) = 1$.

Notation 3.22. *We denote the von Neuman closure of B_∞ by $M_\infty := B_\infty'' \subset \mathcal{B}^*(L^2(B_\infty, \mathrm{Tr}))$. Moreover, for $m, n \geq 0$, we denote the corners ${}_nM_m := p_n \wedge M_\infty \wedge p_m$ and define the Hilbert space $L^2({}_nM_m) := p_n \triangleright L^2(B_\infty) \triangleleft p_m$.*

The following lemma will be crucial to our analysis. It requires $\delta_x > 1$, which we have assumed:

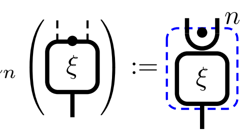
Lemma 3.23. *The following statements hold:*

- *There exists a separable and infinite-dimensional Hilbert space $\mathcal{H}$ such that $B_\infty \cong B_0 \otimes K(\mathcal{H})$ ([12], Corollary 4.15)*
- *For each $n \in \mathbb{N} \cup \{0\}$, B_∞ is simple as a C*-algebra and consequently, the corner B_n is simple too. (See Section 5 in [12].)*
- *For every $n \in \mathbb{N} \cup \{0\}$, the unital C*-algebra B_n has a unique trace given by Tr_n. (Theorem 5.5 and Corollary 5.7 in [12].)*
- *Given $n \in \mathbb{N} \cup \{0\}$, the von Neumann algebra defined by*
$$M_n := B_n'' \subseteq \mathcal{B}(L^2(B_n), \mathrm{Tr}_n)$$
is an interpolated free group factor $L(\mathbb{F}_t)$, for $t \in (1, \infty]$ whose unique trace we denote by τ_n. ([9], [2], [11])
- *For $\mathcal{S}$ any fixed set of isomorphism classes of simple objects in C_x, the full RC*TC generated by $x = \overline{x} \in \mathsf{C}$, we have $K_0(B_0) = \mathbb{Z}[\mathcal{S}]$. Moreover in this isomorphism, $\mathbb{1}_{\mathsf{C}} \mapsto B_0$. [13], [12]*

Remark 3.24. Independently of any other result of the following sections other than Lemma 3.40, one can prove that there is an isomorphism of von Neumann algebras $M_n \cong {}_nM_n$, eliminating any possible ambiguity in the definitions. We choose not to bring this result and the required notational remarks here as it would slow down the exposition. This result becomes of central importance, as it will allow us to work with all the corners ${}_nM_m$ as embedded in the same *ambient* von Neumann algebra M_∞.

3.3 The Watatani C*-tower of GJS algebras

We now consider

$\iota_n : \mathrm{Gr}_\infty \cap B_0 \to \mathrm{Gr}_\infty \cap B_n$ by ι_n  (3.16)

We now state some of the basic properties of these maps:

Proposition 3.25. *For each $n \in \mathbb{N}$ we have that*

- *the map ι_n is a $*$-algebra homomorphism, and for all $\xi \in \mathrm{Gr}_\infty \cap B_0$ we obtain $\mathrm{tr}_0(\xi) = \mathrm{tr}_n(\iota_n(\xi))$, and*
- *the map ι_n extends to an injective C*-algebra homomorphism*
$$\iota_n : B_0 \hookrightarrow B_n.$$

Proof. The first bullet point is an immediate diagrammatic calculation. As for the second bullet point, note that tr_0 and tr_n are faithful states on B_0 and B_n, respectively. Using the functional calculus, if a is any self adjoint element in a C*-algebra with faithful state ω, it follows that $\|a\| = \lim_{p\to\infty} [\omega(|a|^p)]^{1/p}$. Indeed, the map $f : t \mapsto |t|$ is continuous on the spectrum of a and moreover, $f \in \mathrm{L}^p$ for every $p \in [1, \infty]$. Thus, by the Riesz Representation Theorem there exists a positive Radon measure μ defined on the spectrum of a such that $\int f^p d\mu = \omega(|a|^p)$, and hence $\omega(|a|^p)^{1/p} = ||f||_p \to ||f||_\infty = ||a||$, as $p \to \infty$. Therefore, for all $\xi \in \mathrm{Gr}_\infty \cap B_0$,
$$\|\xi\|_{B_0}^2 = \lim_{p\to\infty} \mathrm{tr}_0((\xi^* \wedge \xi)^p)^{1/p} = \lim_{p\to\infty} \mathrm{tr}_n(\iota_n((\xi^* \wedge \xi)^p))^{1/p} = \|\iota_n(\xi)\|_{B_n}^2.$$
The second bullet point follows. $\square$

The inclusion i_n extends to a normal inclusion of von Neumann algebras $M_0 \hookrightarrow M_n$ since it is an isometry in the 2-norm, $\|\cdot\|_2$ induced by the respective traces on M_0 and M_n. Indeed, note that in a tracial von Neumann algebra, the 2-norm on bounded sets (in the operator norm) induces the strong operator topology, so it follows that i_n is strongly continuous on bounded sets.

Here are some useful facts about the inclusion of B_0 into B_n and M_0 into M_n

Proposition 3.26. *The following statements hold for every $n \in \mathbb{N}$:*

1. *The image set $i_n[M_0] \subset M_n$ is a subfactor of index δ_x^{2n}. [10]*
2. *The inclusion $i_n[B_0] \subset B_n$ is of finite type, and has Watatani index δ_x^{2n}. [12]*
3. *Relative commutants (inside the GNS-spaces $\mathcal{B}(L^2(B_n)) \cong \mathcal{B}(L^2(M_n))$) are characteized as*

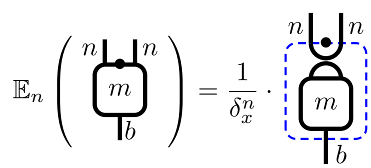
$$(i_n[M_0])' \cap M_n = \mathrm{Gr}_{0,n} = (i_n[B_0])' \cap B_n.$$

Proof. We need only prove (3) above. Indeed, from [10], $(i_n[M_0])' \cap M_n = \mathrm{Gr}_{0,n}$, and a quick diagrammatic computation shows that $\mathrm{Gr}_{0,n} \subset (i_n[B_0])' \cap B_n$. Therefore, we have:

$$\mathrm{Gr}_{0,n} \subseteq i_n(B_0)' \cap B_n = i_n(M_0)' \cap B_n \subseteq i_n(M_0)' \cap M_n = \mathrm{Gr}_{0,n}$$

so every set containment above is an equality. Note that the first equality holds since $i_n(B_0)$ is weakly dense in $i_n(M_0)$, and hence $i_n(B_0)' = i_n(M_0)'$. We should warn the reader that this equality relies on the fact that $(\mathrm{L}^2(B_0) \supset B_0'' = M_0 \cong \iota_n[B_0]'' \subset (\mathrm{L}^2(B_n)))$. This is rigorously proven in Corollary 3.41 and the discussion before it, using Lemma 3.40, independently of the results between here and there. (See also Remark 3.24.) $\square$

Definition 3.27. *For $n \in \mathbb{N}$, define $\mathbb{E}_n : M_n \to i_n[M_0]$ to be the canonical conditional expectation. Diagrammatically, we have:*



(3.17)

We immediately see that $\mathbb{E}_n$ restricts to a conditional expectation from B_n onto $i_n[B_0]$. Furthermore, Popa's entropic condition for finite index [26] implies that for all positive $b \in B_n$ we have $\mathbb{E}_n(b) \geq \frac{1}{\delta_x^n} b$. This means that for each positive $b \in B_n$ we get $\|\mathbb{E}_n(b)\|_{B_n} \geq \frac{1}{\delta_x^n}\|b\|_{B_n}$.

3.4 Concrete bimodules

Recall that as in Notation 3.21, for $n, m \geq 0$ we denote the corner $p_m \wedge B_\infty \wedge p_n$ by ${}_mB_n$. We shall now describe the B_0 bimodules we will use to construct the representation of C.

Definition 3.28. *Let $\mathcal{X} = {}_0B_1$. Then $\mathcal{X}$ is a $B_0 - B_0$ bimodule under the actions*

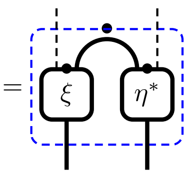
$$b \triangleright \xi \triangleleft b' = b \wedge \xi \wedge i_1(b'),$$

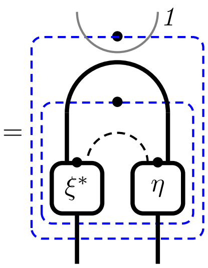
which diagrammatically respectively places b and b' to the left and right of ξ.. We place the following left and right B_0-valued inner products on $\mathcal{X}$:

$${}_{B_0}\langle \xi, \eta \rangle := \xi \wedge \eta^*, \text{ and } \langle \xi \mid \eta \rangle_{B_0} := \delta_x \cdot \mathbb{E}_1(\xi^* \wedge \eta).$$

In the right inner product, it is understood that we are canonically identifying B_0 with $i_1[B_0]$.

We will diagrammatically depict our B_0 actions using the Graded product (Remark 3.20). In terms of diagrams the above inner products are represented as follows:

$${}_{B_0}\langle \xi, \eta\rangle =$$ 

and $\langle \xi \mid \eta\rangle_{B_0} =$ 

(3.18)

Proposition 3.29. *The left and right inner products on ${}_{B_0}\mathcal{X}_{B_0}$ give uniformly equivalent norms, and ${}_{B_0}\mathcal{X}_{B_0}$ is complete under both norms.*

Proof. Let ${}_{B_0}\|\cdot\|$ be the norm under the left B_0 inner product and $\|\cdot\|_{B_0}$ be the norm under the right inner product, according to the notation established in Definition 3.28. Note that ${}_{B_0}\|\xi\|$ coincides with the norm of ξ in the C*-algebra B_∞, and also that from [26],

$${}_{B_0}\|\xi\|^2 = \|\xi^*\wedge\xi\|_{B_\infty} = \|\xi\wedge\xi^*\|_{B_\infty} \leq \delta_x^2\cdot\|\mathbb{E}_1(\xi\wedge\xi^*)\|_{B_\infty} \leq \delta_x^2\cdot {}_{B_0}\|\xi\wedge\xi^*\| = \delta_x^2\cdot\|\xi^*\wedge\xi\| = \delta_x^2\cdot {}_{B_0}\|\xi\|^2.$$

As $\|\delta_x\cdot\mathbb{E}_1(\xi\wedge\xi^*)\| = \|\xi\|^2_{B_0}$, it follows that

$$\frac{1}{\sqrt{\delta_x}}\,{}_{B_0}\|\xi\| \leq \|\xi\|_{B_0} \leq \sqrt{\delta_x}\cdot_{B_0}\|\xi\|,$$

hence the norms are equivalent. It follows that ${}_0B_1 = \mathcal{X}$ is complete in both norms. □

Notation 3.30. *More generally, given $n, l, r \geq 0$ satisfying $l + r = n$, one can define the B_0 Hilbert bimodule ${}_lB_r$ with left and right B_0 actions given by*

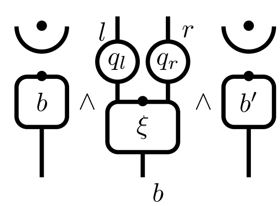
$$x \triangleright \xi \triangleleft y = i_l(x)\wedge\xi\wedge i_r(y),$$

where once more, the action is given by the multiplication inside B_∞. As usual, ${}_lB_r$ is endowed with left and right B_0 inner products given by:

- ${}_{B_0}\langle\xi,\eta\rangle = \delta_x^l\cdot\mathbb{E}_l(\xi\wedge\eta^*)$ *and*
- $\langle\xi|\eta\rangle_{B_0} = \delta_x^r\cdot\mathbb{E}_r(\xi^*\wedge\eta)$.

Again, the two inner products give equivalent complete norms on ${}_lB_r$. (See Proposition 3.29.) Up to B_0-valued inner product preserving B_0 Hilbert C-bimodule isomorphisms, ${}_lB_r$* ***is independent of the decomposition of*** *n* ***as*** *$(l + r)$.*

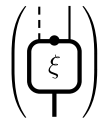
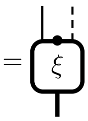
We can also describe sub-objects of the ${}_lB_r$: if $q_l \leq p_l = 1_{B_l}$ and $q_r \leq p_r = 1_{B_r}$ are orthogonal projections in B_l and B_r, respectively, the span of diagrams of the form



defines a pre-Hilbert C B_0-bimodule $(q_l\wedge {}_lB_r\wedge q_r)$. The left and right B_0-valued inner products and B_0 actions are the obvious ones.*

Notation 3.31. *We shall now describe the* ***the conjugate bimodule***, $\overline{\mathcal{X}}$*. (See Definition 2.7, Example 2.15, and also [20, p 3443].) To do so, we will first show that* $\mathcal{X} = {}_0B_1$ *is isomorphic to* ${}_1B_0$ *by defining the map* ϕ*, which* ***pushes the dot to the right***:

$$\phi : \bigoplus_{n=0}^{\infty} V_{n,0,1} \longrightarrow \bigoplus_{n=0}^{\infty} V_{n,1,0} \quad by \tag{3.19}$$

ϕ () = 

It then follows that for all $\xi, \eta \in_0 B_1$,

- *The inner products are related as* ${}_{B_0}\langle \phi(\xi),\ \phi(\eta)\rangle = i_1\left[{}_{B_0}\langle \xi,\ \eta\rangle\right]$ *and* $\langle \xi \mid \eta\rangle_{B_0} = i_1\left[\langle \phi(\xi) \mid \phi(\eta)\rangle_{B_0}\right]$,
- *the map* ϕ *extends to a unitary* B_0*-bimodule isomorphism of* ${}_0B_1$ *onto* ${}_1B_0$*, and*
- *for any* $\xi \in \mathcal{X}$*, we have that* $\phi(\xi)^* = \phi^{-1}(\xi^*)$.

More generally, observe that an obvious definitions of ϕ*, analogous to that in Equation 3.19 will exhibit* B_0*-bilinear unitaries with similar properties realizing* ${}_0B_n \cong \ldots \cong\ {}_lB_r\ \cong \ldots \cong\ {}_nB_0$*, where* $l + r = n$*, which we still denote by* ϕ*. Thus, for every* $n \in \mathbb{N}$*, we have*

$$\phi^n : {}_0B_n \longrightarrow {}_nB_0$$

is a unitary B_0*-bimodule isomorphism. Additionally, we introduce the* ***congujation map***

$$\begin{aligned} (\,\bar{\cdot}\,) : {}_0B_n &\longrightarrow \overline{{}_0B_n} \\ \xi &\mapsto \phi^{-n}(\xi^*) := \overline{\xi}. \end{aligned}$$

Notice that this defines an B_0*-conjugate linear unitary isomorphism. Using an analogous property to the third bullet point, it follows that* $\overline{\overline{\xi}} = \xi$ *on* ${}_0B_n$*, so* ${}_0B_n$ *is identical to its double dual as* B_0*-bimodules. Finally, the isomorphism*

$$\begin{aligned} \phi^n : \overline{{}_0B_n} &\longrightarrow {}_nB_0 \\ \overline{\xi} &\mapsto \xi^*, \end{aligned}$$

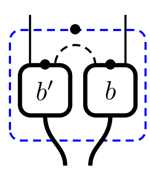
is also a B_0*-bimodule unitary isomorphism.*

The following propositions will establish the most important algebraic properties of the bimodule $\mathcal{X}$:

Proposition 3.32. *The bimodule* ${}_{B_0}\mathcal{X}_{B_0}$ *has finite left and right* B_0*-bases; i.e.* $\mathcal{X}$ *is fgp. (See Definition 2.12.)*

Proof. We first show that $\mathcal{X}$ has a finite left B_0-basis. Note that for $\xi, v \in \mathcal{X}$, an expression of the form ${}_{B_0}\langle \xi, v\rangle \triangleright v$ is the same as the product $\xi \wedge v^* \wedge v$. Observe that $\xi \wedge v^* \in B_0$ and $v^* \wedge v \in B_1$.

Consider $I := {}_1B_0 \cdot {}_0B_1 := \text{span}\{b'b|\ b' \in {}_1B_0 \text{ and } b \in {}_0B_1\}$. Note that I is a two-sided ideal in B_1 containing all diagrams of the form:



(3.20)

If I were proper, then since B_1 is unital, it follows that the closure of I, $\overline{I}$ would also be a proper ideal in B_1. However, since B_1 is simple (Lemma 3.23), it follows that $\overline{I} = B_1$, hence $I = B_1$. Therefore, there exists finite sets $\{w_j\}_{j=1}^n, \{w'_j\}_{j=1}^n \subset \mathcal{X}$ such that

$$p_1 = 1_{B_1} = \sum_{j=1}^{n} w_j^* \wedge w'_j. \tag{3.21}$$

From this equation, it follows that for any $\xi \in \mathcal{X}$,

$$\xi = \sum_{j=1}^{n} \xi \wedge w_j^* \wedge w_j' = \sum_{j=1}^{n} {}_{B_0}\langle \xi, w_j \rangle \triangleright w_j'.$$

By Lemma 1.7 in [20], these two sets can be made the same. This is, there exists a finite set $\{v_j\}_{j=1}^n \subset \mathcal{X}$ such that for any $\xi \in \mathcal{X}$,

$$\xi = \sum_{j=1}^{n} \xi \wedge v_j^* \wedge v_j = \sum_{j=1}^{n} {}_{B_0}\langle \xi, v_j \rangle \triangleright v_j.$$

showing $\{v_j\}_{j=1}^n$ defines a left B_0-basis for $\mathcal{X}$. Moreover, one can easily show that this set also satisfies

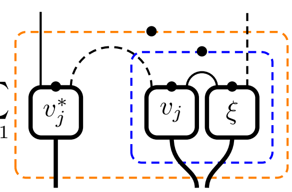
$$\sum_{j=1}^{n} v_j^* \wedge v_j = p_1 = 1_{B_1}. \tag{3.22}$$

Now it suffices to exhibit a right B_0-basis for $\mathcal{X}$. To do so, we will first show that $\{\zeta_j := \phi[\phi(v_j)^*]\}_{j=1}^n$ is a right basis for ${}_1B_0$. By the last property of the map ϕ, that *pushes the dot to the right* introduced in Remark 3.31, we obtain that $\zeta_j = v_j^*$. We then have that

$$1_{B_1} = \sum_{j=1}^{n} \zeta_j \wedge \zeta_j^*,$$

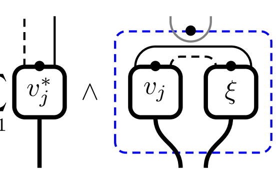
so for an arbitrary $\xi \in \mathcal{X} = {}_0B_1$ we obtain that

$$\phi(\xi) = 1_{B_1} \wedge \phi(\xi) = \sum_{j=1}^{n} \zeta_j \triangleleft \langle \zeta_j \mid \phi(\xi) \rangle_{B_0} = \sum_{j=1}^{n}$$



therefore, by the B_0-linearity of ϕ^{-1}, we get

$$\xi = \sum_{j=1}^{n}$$



$$= \sum_{j=1}^{n} [\phi(v_j)^*] \triangleleft \langle [\phi(v_j)^*] \mid \xi \rangle_{B_0}.$$

Which proves that $\{\phi(v_j)^* := u_j\}_{j=1}^n$ is a right B_0-basis for $\mathcal{X}$.

□

Proposition 3.33. *The left and right indices of $\mathcal{X}$ as a $B_0 - B_0$ Hilbert bimodule match. More precisely, we have that r-Ind$(\mathcal{X}) = \delta_x =$ l-Ind$(\mathcal{X})$. Therefore, $\mathcal{X}$ is normalized. (See Definition 2.13 and the following paragraph.)*

Proof. Let $\{v_j\}_{j=1}^n$ be a left B_0-basis of $\mathcal{X} = {}_0B_1$. Using the decomposition $B_1 = {}_1B_0 \wedge {}_0B_1$ described in the proof of Proposition 3.29, one can easily see that in B_1, the v_j can be chosen to satisfy $\sum_{j=1}^n v_j^* v_j = 1_{B_1}$. This means that

$$\sum_{j=1}^{n} \langle v_j \mid v_j \rangle_{B_0} = \delta_x \cdot \sum_{j=1}^{n} \mathbb{E}_1(v_j^* \wedge v_j) = \delta_x \cdot \mathbb{E}_1\left(\sum_{j=1}^{n} v_j^* \wedge v_j\right) = \delta_x \cdot \mathbb{E}_1(1_{B_1}) = \delta_x.$$

As for the right index, use the basis constructed in the proof of Proposition 3.32 and proceed as above. □

To show that the bimodule $\mathcal{X}$ is minimal (Definition 2.14) we will need the next proposition:

Proposition 3.34. *Considering the right Hilbert B_0-bimodule $\mathcal{X}_{B_0} = {}_0B_{1B_0}$, we have $\mathcal{B}^*(\mathcal{X}_{B_0}) = \mathcal{K}(\mathcal{X}_{B_0}) = B_1$ where B_1 acts with its canonical left action. Picturing $\mathcal{X}$ as a left Hilbert B_0 module, ${}_{B_0}\mathcal{X}$, we have $\mathcal{B}^*({}_{B0}\mathcal{X}) = \mathcal{K}({}_{B0}\mathcal{X}) = B_1^{op}$, where B_1 acts with its canonical right action.*

Proof. Consider ${}_1B_0 = \phi[\mathcal{X}]$ and arbitary elements $\zeta, \xi, \eta \in \mathcal{X}$. Recall that the right inner product is given by $\langle \phi(\eta) \mid \phi(\zeta)\rangle_{B_0} = \phi(\eta)^* \wedge \phi(\zeta)$. Since we know by Proposition 3.32 that $\mathcal{X}$ is fgp, by Lemma 1.7 in [20], it follows that $\mathcal{B}^*(\mathcal{X}_{B_0}) = \mathcal{K}^*(\mathcal{X}_{B_0})$, which in turn is generated by the rank 1 operators $|\phi(\xi)\rangle\langle\phi(\eta)|(\phi(\zeta)) = \phi(\xi) \lhd \langle \phi(\eta) \mid \phi(\zeta)\rangle_{B_0} = \phi(\xi) \wedge \phi(\eta)^* \wedge \phi(\zeta)$. This means that $|\phi(\xi)\rangle\langle\phi(\eta)|$ is the multiplication operator $\phi(\xi) \wedge \phi(\eta)^*$. As discussed in the proof of Proposition 3.32, B_1 is the linear space spanned by these products. The first statement follows, and the second statement follows by a similar argument. $\square$

Remark 3.35. Similar reasoning will show that $\mathcal{B}^*({}_lB_r) \cong \mathcal{B}^*({}_nB_0) = B_n$, whenever $n = (l + r)$. If $\mathcal{Y}$ is a $B_0 - B_0$ submodule of ${}_nB_0$, then

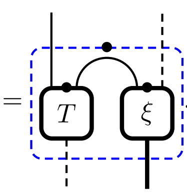
$$\mathcal{Y} = p[{}_nB_0]$$

for some projection $p \in B_n \cap i_n[B_0]' = \mathrm{Gr}_{0,n}$. It follows that if $x \in \mathcal{B}^*(\mathcal{Y})$, then x can be extended to be a bounded operator $\tilde{x}$ on ${}_nB_0$ satisfying $\tilde{x} = 0$ on $(1-p)[{}_nB_0]$. It is easily seen that $\tilde{x}$ is adjointable, $\tilde{x}^* = 0$ on $(1-p)[{}_nB_0]$ and $\tilde{x}^* = x^*$ on $\mathcal{Y}$. Consequently, it follows that $\tilde{x} \in pB_np$. Conversely, any operator $z \in pB_np$ is in $\mathcal{B}^*(\mathcal{Y})$; therefore $\mathcal{B}^*(\mathcal{Y}) = pB_np$.

Corollary 3.36. *The endomorphism algebra of B_0-bimodule maps on $\mathcal{X} = {}_0B_1$ is characterized as*

$$\mathrm{End}_{B_0-B_0}(\mathcal{X}) \cong B_1 \cap \mathrm{Gr}_{0,1}\,.$$

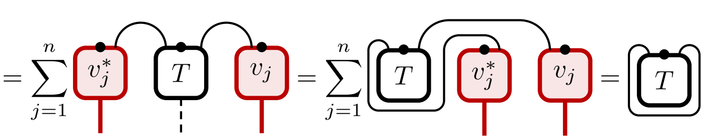
Proof. We replace $\mathcal{X}$ by the isomorphic bimodule $\phi[\mathcal{X}] = {}_1B_0$. Let $\xi \in {}_0B_1$ be arbitrary, and note that if $T \in \mathrm{Gr}_{0,1} \cap B_1$, then clearly $T \in \mathrm{End}_{B_0-B_0}({}_1B_0)$ via the usual action:

$T(\phi(\xi)) =$ .

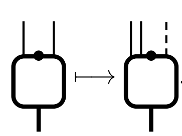
Conversely, we claim that if $T \in \mathrm{End}_{B_0-B_0}({}_1B_0)$, then $T \in \mathrm{Gr}_{0,1} \cap B_1$. On the one hand, in particular $T \in \mathcal{B}^*({}_1B_{0B_0})$, so $T \in B_1$, by Proposition 3.34. On the other, $T \in \mathrm{End}_{B_0-B_0}({}_1B_0)$ implies $T \in \mathcal{B}^*({}_1B_{0B_0}) \cap (i_1[B_0])'$. As a consequence, $T \in B_1 \cap (\mathcal{B}^*({}_1B_{0B_0}) \cap (i_i[B_0])') = B_1 \cap (i_i[B_0])' = B_1 \cap \mathrm{Gr}_{0,1}$, in light of (the proof of) Proposition 3.26. $\square$

Proposition 3.37. *The Hilbert B_0-bimodule $\mathcal{X}$ is minimal (See Definition 2.14).*

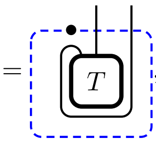
Proof. Let $T \in \mathrm{End}_{B_0-B_0}(\phi[\mathcal{X}]) = \mathrm{Gr}_{0,1} \cap B_1$, where again $\phi[\mathcal{X}] = \phi[{}_0B_1] = {}_1B_0$. Let $\{v_j\}_{j=1}^n$ be the left B_0-basis for ${}_0B_1$ constructed in the proof of Proposition 3.32, and pretending for for the moment that each v_i is actually in Gr_∞, we have (formally):

$$\sum_{j=1}^n \langle \phi(v_j) \mid T(\phi(v_j))\rangle_{B_0} = $$  $\in \mathbb{C}$.

Notice that we are making use of the maps ϕ, that *push the dot to the right* $\phi : {}_lB_r \longrightarrow {}_{(l+1)}B_{(r-1)}$, defined in Notation 3.31. In our particular case and to make this computation rigorous, we will make use of

$\phi : B_1 \longrightarrow {}_2B_0$ given by .

Approximate each v_j in norm by a sequence $\{v_j^k\}_{k=1}^{\infty} \subset \mathrm{Gr}_\infty \cap {}_0B_1$. If we let T' be the element

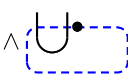
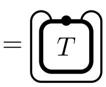
$T' :=$  ,

then the above diagrams imply that for each $k \in \mathbb{N}$,

$$\sum_{j=1}^{n} \langle \phi(v_j^k) \mid T[\phi(v_j^k)] \rangle_{B_0} = T' \wedge \phi\left[(v_j^k)^* v_j^k\right].$$

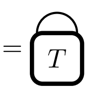
Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} \sum_{j=1}^{n} \langle \phi(v_j^k) \mid T[\phi(v_j^k)] \rangle_{B_0} = T' \wedge \tag{3.23}$$

 $=$ 

Performing a similar computation for the the right B_0-basis for ${}_1B_0$ given by $\{\phi[\phi(v_j)^* = v_j^*]\}_{j=1}^n$ we obtain that

$$\sum_{j=1^n} {}_{B_0}\langle\, T(\phi[\phi(v_j)^*]), (\phi[\phi(v_j)^*])\,\rangle = \tag{3.24}$$

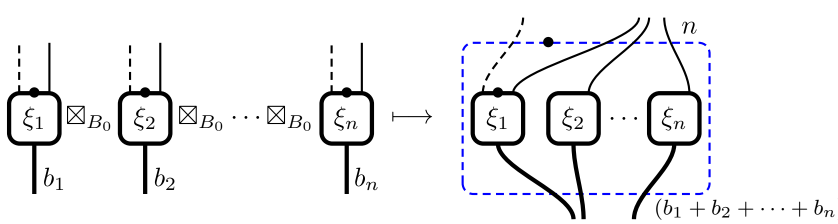


To conclude, we must show that the diagrams obtained in Equation (3.23) and Equation (3.24) are equal. Since T directly comes from sums of diagrams in the category C, using Frobenius reciprocity (Proposition 3.6) together with the canonical spherical structure of C (Remark 2.8), we immediatly see that these two diagrams correspond to the left and right traces of T, which are identical. This observation completes the proof. $\square$

We now study tensor powers of our bimodule ${}_{B_0}\mathcal{X}_{B_0}$. In the following lemma we describe **the bimodule structure of the n-fold fusion $\mathcal{X}^{\boxtimes_{B_0} n}$**, which will turn out extremely useful in the sequel. For a more detailed view on this fusion with respect to B_0, we refer the reader to Example 2.15 above.

Lemma 3.38 ([13], Proposition 4.11)**.** *For each $n \in \mathbb{N}$, there is a B_0-bilinear unitary, which on diagrams is given by (See also Proposition 4.11 in [13].)*

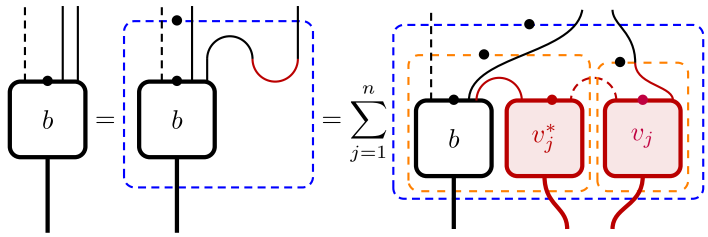
$$\Psi_n : \ {}_{B_0}(\mathcal{X}^{\boxtimes_{B_0} n})_{B_0} \longrightarrow \ {}_{B_0}({}_0B_n)_{B_0}$$



. (3.25)

Proof. It is clear that Ψ_n is well-defined and B_0-bilinear on $[\mathrm{Gr}_\infty \cap {}_0B_1]^{\odot n}$. Here, we are balancing our tensor product $\odot$ by $\mathrm{Gr}_\infty \cap B_0$. From the definitions of the left and right B_0-valued inner products on $\mathcal{X}^{\boxtimes_{B_0} n}$ and Notation 3.30, it is easy to see that Ψ_n is isometric on diagrams. To extend Ψ_n to a unitary to all of $\mathcal{X}^{\boxtimes_{B_0} n}$, we observe that $[\mathrm{Gr}_\infty \cap {}_0B_1]^{\odot n} \subset \mathcal{X}^{\odot n}$ is dense, and that the latter space is clearly dense in $\mathcal{X}^{\boxtimes_{B_0} n}$. Middle B_0-linearity now automatically follows from $\mathrm{Gr}_\infty \cap B_0$-linearity.

To conclude the proof, it suffices to show that Ψ_n is surjective. To see this, recall Lemma 3.23 asserting that B_1 is simple and unital, and moreover, the unit element in B_1 is a finite sum $1_{B_1} = p_1 = \sum_{j=1}^n v_j^* \wedge v_j$, where each $v_j \in \mathcal{X}$. (See Equation 3.22.) We shall now proceed by means of induction. If $n = 2$, for an arbitrary element $b \in {}_0B_2$, we have (formally):



(3.26)

Notice that in the last term of the above equation, the elements inside the orange boxes are elements in $\mathcal{X}$. Thus, the element $\sum_{j=1}^{n}(b \wedge v_j^*) \boxtimes v_j \in \mathcal{X} \boxtimes_{B_0} \mathcal{X}$ maps to the arbitrarily chosen b under Ψ_2. This proves the claim for $n = 2$. The inductive step of the proof is the same as the base case, with the single difference that now b has n strings to the right. These details are simple and thus not explained here. $\square$

Notation 3.39. *From the fgp B_0-bimodules ${}_lB_r$ we can perform a type of* ***GNS-construction*** *to obtain a Hilbert space. For arbitrary $i, j \geq 0$, we define $L^2({}_lB_r) := L^2({}_0B_{l+r}, \mathrm{Tr}_{l+r})$ as the completion of the pre-Hilbert space whose underlying vector space is given by ${}_0B_{l+r}$ and its complex-valued inner product is given by*

$$\langle \eta \mid \xi \rangle_{L^2({}_0B_{l+r})} := \mathrm{Tr}_{l+r}\left(\langle \eta \mid \xi \rangle_{B_0}\right).$$

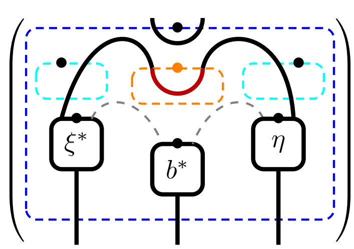
For $\eta \in {}_lB_r$, we write $\eta \cdot \Omega$, to denote its corresponding image in the complete space $L^2({}_lB_r)$. Notice that using the maps ϕ from Notation 3.31 we can turn this space into a B_n-B_m bimodule. In fact, it is easily seen that ${}_{B_l}(L^2({}_lB_r))_{B_r} \cong_{B_l} (p_l \triangleright L^2(B_\infty) \triangleleft p_r)_{B_r}$. Furthermore, for the case B_n, where the module is an algebra, ϕ induces a B_n-unitary isomorphism between the standard GNS-construction $L^2(B_n, \mathrm{Tr}_n)$ and our $L^2({}_0B_{(n+n)})$.

The purpose for introducing these Hilbert spaces is to **compare the relative commutants of the C*-algebras B_n over different representations**; i.e. the relation between $M_0 = B_0'' \subset \mathcal{B}(\mathrm{L}^2(B_0))$ and $i_n[B_0]'' \subset \mathcal{B}(\mathrm{L}^2(B_n))$. (These von Neumann algebras were introduced earlier in Lemma 3.23.) In doing so, we invoke a lemma communicated by André Henriques which we simply state here:

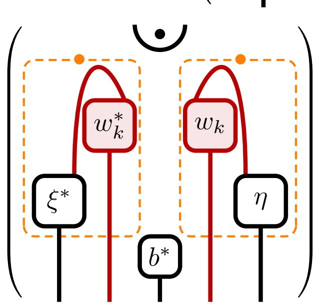
Lemma 3.40. *Let B be a unital $*$-algebra acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and $M = B'' \subset \mathcal{B}(K)$. Assume there exists a dense subspace $\mathcal{H}_\circ \subset \mathcal{H}$ such that for every $\eta, \xi \in \mathcal{H}_\circ$, there exists $x, y \in \mathcal{K}$ such that $\langle b \triangleright \xi \mid \eta \rangle_{\mathcal{H}} = \langle b \triangleright x \mid y \rangle_{\mathcal{K}}$ holds for all $b \in B$. Then the representation of B on $\mathcal{H}$ extends to a normal representation of M on $\mathcal{H}$.*

We now check that the hypotheses of the previous lemma apply to B_0 acting on $\mathrm{L}^2(B_0)$ and B_0 acting on $\mathrm{L}^2({}_0B_n)$. For an arbitrary $n \geq 0$, any $\xi, \eta \in {}_0B_n \cdot \Omega$, and any arbitrary $b \in B_0$ we have that

$$\langle b \triangleright \xi \mid \eta \rangle_{\mathrm{L}^2({}_0B_n)} = \mathrm{Tr}\left(\langle b \triangleright \xi \mid \eta \rangle_{B_0}\right) = \mathrm{Tr}_n$$



$$= \sum_{k=1}^{K} \mathrm{Tr}_n$$



$$= \sum_{k=1}^{K} \mathrm{Tr}_n\left(\langle b \triangleright [w_k \wedge \xi] \mid [v_k \wedge \eta] \rangle_{B_0}\right)$$

Here, $\{w_k\}_{k=1}^K$ is a finite left B_0-basis for the fgp bimodule ${}_nB_0 \cong \mathcal{X}^{\boxtimes n}$, giving us that $1_{B_n} = p_n = \sum_k w_k^* \wedge w_k$, similarly as in Equation 3.21.

We immediately obtain the following:

Corollary 3.41. *For $n \geq 0$, we have an isomorphism of von Neumann algebras*

$$\mathcal{B}(L^2(B_0)) \supset M_0 \cong B_0'' \subset \mathcal{B}\left(L^2({}_0B_n)\right).$$

We close this section with a proposition that will be very useful in Chapter 4:

Proposition 3.42. *Let $n = l + r$. On ${}_lB_r$, one has*

$$\mathrm{tr}(\langle \xi \mid \eta \rangle_{B_0}) = \mathrm{tr}({}_{B_0}\langle \eta, \xi \rangle)$$

Proof. We will work on ${}_nB_0$. Note that in this case, if $\xi, \eta \in {}_nB_0$, then $\eta^*\xi \in B_0$ and $\xi \wedge \eta^* \in B_n$. If we utilize the maps Φ and $\mathbb{E}$ from the discussion before Proposition 3.18, we have:

$$\mathrm{tr}(\langle \xi \mid \eta \rangle_{B_0}) = \mathrm{tr}(\eta^* \wedge \xi) = (\Phi \circ \mathbb{E})(\eta^* \wedge \xi) = (\Phi \circ \mathbb{E})(\xi \wedge \eta^*) = \delta_x^n \cdot \mathrm{tr}_n(\xi \wedge \eta^*)) = \delta_x^n \cdot \mathrm{tr}(\mathbb{E}_n(\xi \wedge \eta^*)) = \mathrm{tr}_n({}_{B_0}\langle \eta, \xi \rangle)$$

□

4 Realizing a RC*TC as Hilbert C*-Bimodules

In this chapter we will show that the constructions of Section 3.4 induce a full bi-involutive (strong-)monoidal functor of the form

$$\mathbb{F} : \mathsf{C}_x \hookrightarrow \mathsf{Bim}_{\mathsf{fgp}}(B_0).$$

As a reminder, B_0 is a **separable simple tracial unital exact C*-algebra**, (see Proposition 3.23) as in Notation 3.21, and every $\mathcal{X} \in \mathsf{Bim}_{\mathsf{fgp}}(B_0)$ is a **minimal** (Definition 2.14 and Proposition 3.37) **normalized** (Definition 2.13 and Proposition 3.33) **finitely generated projective (fgp)** (Definition 3.29 and Proposition 3.32) **Hilbert C*-bimodule** (Definition 3.28). We direct the reader to Section 2.2 with special attention to Example 2.15 for a more detailed viewpoint on the RC*TC of bimodules $\mathsf{Bim}_{\mathsf{fgp}}(B_0)$. We moreover assume that C is an **essentially small strict RC*TC with simple unit and its canonical bi-involutive and spherical structures**. We moreover **fix a symmetrically self-dual object** $x = \overline{x} \in \mathsf{C}$, (Lemma 3.3) of dimension $\delta_x > 1$ and we denote by C_x the full RC*TC of C, whose objects are all tensor powers of x.

Let us now start constructing the functor $\mathbb{F}$. For a given $n \in \mathbb{N}$ we define (see Lemma 3.38)

$$\mathbb{F}(x^{\otimes n}) := {}_{B_0}\left({}_0B_n\right)_{B_0} (\cong {}_{B_0}(\mathcal{X}^{\boxtimes_{B_0} n})_{B_0}). \tag{4.1}$$

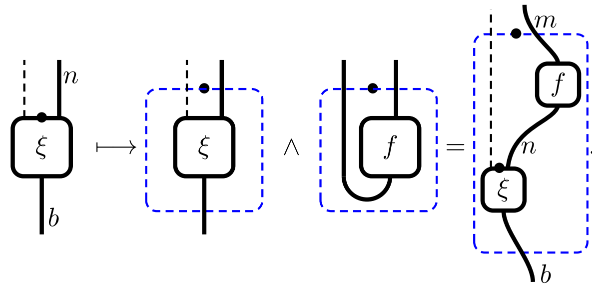
We shall now define $\mathbb{F}$ on the morphism spaces. To do so, we make use of the embedding of diagrams coming from C into the C*-algebra A_∞, as described in Proposition 3.6. (We remind the reader that the construction of A_∞ was done in the paragraphs preceeding Proposition 3.7.) For $n, m \in \mathbb{N}$ and $f \in \mathsf{C}(x^{\otimes n} \to x^{\otimes m})$, we define the action of $\mathbb{F}(f)$ on diagrams by only using the multiplication internal to the C*-algebra $A_\infty \subseteq B_\infty \subseteq \mathcal{B}(\mathcal{X}_\infty)$. In the sequel, we will often simply write f instead of $FR(f)$, directly identifying diagrams in the category with those in A_∞.

For a diagram $\xi \in {}_0B_n \cap \mathrm{Gr}_\infty$, we define

$$\mathbb{F}(f)(\xi) := \xi \wedge f,$$

corresponding to the (linear extension of the) following diagrammatic computation:

$$\mathbb{F}(f) : (\mathrm{Gr}_\infty \cap {}_0B_n) \longrightarrow (\mathrm{Gr}_\infty \cap {}_0B_m)$$



(4.2)

The next result will allow us to extend the definition of $\mathbb{F}(f)$ to all of ${}_0B_n$.

Lemma 4.1. *For every $n, m \in \mathbb{N}$, we have that*

1. *the subset $\mathrm{Gr}_\infty \cap {}_0B_n \subset {}_0B_n$ is dense.*
2. *Given $f \in \mathsf{C}(x^{\otimes n} \to x^{\otimes m})$, the map $\mathbb{F}(f) : \mathrm{Gr}_\infty \cap {}_0B_n \to \mathrm{Gr}_\infty \cap {}_0B_m$ is bounded and $||f||_{A_\infty} = ||f||_{\mathsf{C}}$ (see Definition 2.2), and therefore extends to a bounded (right-adjointable) map $\mathbb{F}(f) \in \mathsf{Bim}_{\mathsf{fgp}}(B_0)({}_0B_n \to {}_0B_m)$, with $\mathbb{F}(f)^* = \mathbb{F}(f^\dagger)$, so $\mathbb{F}$ is a dagger functor.*

Proof. Statement (1) was proven above. (See Remark 3.19 and the Proposition preceeding it.) We shall now prove (2). This proof can also be found in Proposition 4.16 in [13], but we give it here for the convenience of the reader. By definition, on diagrams we have that $\mathbb{F}(f)(-) = (-) \wedge (f)$. The bound is now obvious since this map is given by multiplication in the Banach algebra A_∞, and it moreover corresponds to the operator of right creation by f in the full Fock space picture. (Definition 3.12 and Proposition 3.14.) A diagrammatic computation will reveal that $\langle \mathbb{F}(f)\xi \mid \eta\rangle_{B_0} = \langle \xi \mid \mathbb{F}(f^\dagger)\eta\rangle_{B_0}$. Thus, $\mathbb{F}(f)$ is right-adjointable with $\mathbb{F}(f)^* = \mathbb{F}(f^\dagger)$. □

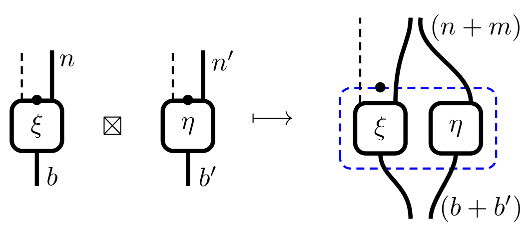
We now describe the **tensorator and unit** data for $\mathbb{F}$:

$$\iota^{\mathbb{F}} : \mathbb{1}_{\mathsf{Bim}_{\mathsf{fgp}}(B_0)} \longrightarrow \mathbb{F}(\mathbb{1}_{\mathsf{C}}),$$

given by the identity on B_0. For arbitrary $n,\ m \geq 0$ we define

$$\mu^{\mathbb{F}}_{n,m} : {}_0B_n \underset{B_0}{\boxtimes} {}_0B_m \longrightarrow {}_0B_{(n+m)},$$

by extending the following diagramatic composition (and omitting the unitors):



(4.3)

Notice that if f and g are morphisms in C, then $\mu^{\mathbb{F}}(f,g) = \mathrm{id}_{\mathbb{1}} \otimes f \otimes g$ on diagrams. So far, in defining $\mu^{\mathbb{F}}_{n,m}$ we have only used the (unitary) associators and the unitors in $\mathsf{Bim}_{\mathsf{fgp}}(B_0)$, so naturality in n and m automatically holds, alongside with unitarity. Therefore, by Lemma 4.1, $\mu^{\mathbb{F}}_{n,m}$ extends to a unitary isomorphism ${}_0B_n \boxtimes_{B_0} {}_0B_m \xrightarrow{\sim} {}_0B_{(n+m)}$, which is again natural in both n and m. Similarly, we can also extend $\mu^{\mathbb{F}}(f,g)$ from diagrams to all of ${}_0B_n \boxtimes_{B_0} {}_0B_m$, since this map is just a series of vertical and horizontal compositions in C. It is clear that bi-adjointability for $\mu^{\mathbb{F}}(f,g)$ holds on diagrams, so the extension of $\mu^{\mathbb{F}}$ is also bi-adjointable. This therefore establishes the following Proposition.

Proposition 4.2. *The functor $(\mathbb{F}, \mu^{\mathbb{F}}, \iota^{\mathbb{F}})$ is strong monoidal.*

Remark 4.3. A direct diagrammatic argument demonstrates the functor $\mathbb{F}$ is faithful. This, however, is a general fact about strong-monoidal dagger functors between rigid C*-categories with simple unit.

We can further unveil more structure on the functor $\mathbb{F}$.

Proposition 4.4. *The dagger tensor functor* $(\mathbb{F}, \mu^{\mathbb{F}}, \iota^{\mathbb{F}})$ *is bi-involutive with structure maps*

$$\begin{aligned} \chi_n : \mathbb{F}(\overline{x^{\otimes n}}) = {}_0B_n &\longrightarrow \overline{\mathbb{F}(x^{\otimes n})} = \overline{{}_0B_n} \\ \eta &\longmapsto \phi^{-n}(\eta^*), \end{aligned}$$

for every $n \in \mathbb{N}$.

Proof. Clearly each χ_n is a bounded bi-adjointable bimodule isomorphism, since it is the composition of such maps. That $\{\chi_n\}$ is a monoidal natural transformation for which Diagrams (2.8) commute is a straightforward diagrammatic verification, and showing each χ_n is unitary follows from the properties of the map ϕ introduced in Notation 3.31 and simple diagrammatic considerations. $\square$

We shall postpone the proof that this functor is full until the end of the next chapter, where we develop the tools necessary to prove it.

5 Hilbertifying C*-Bimodules

For the reminder of this article, we will further restrict our scope to the category $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0) \subset \mathsf{Bim}_{\mathsf{fgp}}(B_0)$ **consisting of those bimodules which are compatible with the trace** [20, Definition 5.7]; i.e., $\mathcal{Y} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$ if and only if for each $\xi, \eta \in \mathcal{Y}$ the following identity holds:

$$\operatorname{tr}_{B_0}\left(\langle \eta \mid \xi \rangle_{B_0}\right) = \operatorname{tr}_{B_0}\left({}_{B_0}\langle \xi, \eta \rangle\right). \tag{5.1}$$

Notice this subcategory is still a RC*TC, since the fusion of bimodules compatible with the trace is again compatible.

The goal of this section is to construct a fully-faithful bi-involutive strong monoidal functor of the form:

$${}_{M_0}(- \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0} : \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0) \longrightarrow \mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0)$$

$$\mathcal{Y} \longmapsto {}_{M_0}(\mathcal{Y} \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0} \tag{5.2}$$

$$\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)(\mathcal{Y} \to \mathcal{Z}) \ni f \mapsto f \boxtimes \mathrm{id}_{\mathrm{L}^2(B_0)} \tag{5.3}$$

(See Example 2.10 for a description of $\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0.)$)

We shall now explain how to obtain the M_0-bimodules. For a fixed but arbitrary $\mathcal{Y} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$, consider the *algebraic* B_0-balanced *tensor product* $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ endowed with the (densely-defined) **sesquilinear form**:

$$\langle \xi \boxtimes a\Omega,\ \eta \boxtimes b\Omega\rangle := \langle \langle \eta \mid \xi \rangle_{B_0} \triangleright a\Omega,\ b\Omega\rangle_{\mathrm{L}^2(B_0)} := \operatorname{tr}_{B_0}\left(\ b^*\langle \eta \mid \xi \rangle_{B_0} a\ \right), \quad \text{for all } a, b \in B_0, \text{and } \xi, \eta \in \mathcal{Y}. \tag{5.4}$$

Notice that $\langle \xi \boxtimes a\Omega,\ \eta \boxtimes \Omega\rangle = \langle [\xi \triangleleft a] \boxtimes \Omega, [\eta \triangleleft b] \boxtimes \Omega\rangle$. We can moreover endow this space with a right B_0-action given by

$$(\xi \boxtimes a\Omega) \triangleleft b := (\xi \boxtimes ab\Omega).$$

This action is bounded. Indeed, for arbitrary $\xi \in \mathcal{Y}$ and $b \in B_0$, we have

$$||\xi \boxtimes b\Omega||_2^2 = \operatorname{tr}_{B_0}(\langle \xi \triangleleft b \mid \xi \triangleleft \rangle_{B_0}) = \operatorname{tr}_{B_0}(\ {}_{B_0}\langle \xi \triangleleft b, \xi \triangleleft b\rangle) \leq ||b||^2 \cdot \operatorname{tr}_{B_0}(\langle \xi \mid \xi \rangle_{B_0}) = ||b||^2 \cdot ||\xi \boxtimes \Omega||_2^2.$$

Here, we used the repeatedly used the compatibility with the trace condition, Equation (5.1), together with Lemma 1.26 in [20].

Remark 5.1. The algebraic B_0-balanced tensor product space $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$, is a Hilbert space so it needs no further completion. Indeed, since $\mathcal{Y}$ is fgp, it then has a finite right B_0-basis $\{u_i\}_{i=1}^n$. By observing that $p = \sum_i |u_i\rangle\langle u_i|$ is a self-adjoint idempotent, we can then realize $\mathcal{Y} \cong p[B_0] \subset {B_0}^{\oplus n}$ as right B_0-modules via

$$y = \sum_i u_i \lhd \langle u_i \mid y\rangle_{B_0} \longmapsto \left(\sum_{j=1}^n \langle u_i \mid u_j\rangle_{B_0} \cdot \langle u_j \mid y\rangle_{B_0} \right)_{i=1}^n .$$

We now observe that the orthogonal projection $p \boxtimes \mathrm{id}_{\mathrm{L}^2(B_0)} \in \mathcal{B}^*\left({B_0}^{\oplus n} \boxtimes_{B_0} \mathrm{L}^2(B_0)\right)$ realizes $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ as a (necessarily closed) direct summand of the Hilbert space ${B_0}^{\oplus n} \boxtimes_{B_0} \mathrm{L}^2(B_0) \cong \mathrm{L}^2(B_0)^{\oplus n}$.

We can therefore extend this action from $\mathcal{Y} \boxtimes \Omega$ to the whole Hilbert space $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$. To see this **right B_0-action extends to a (normal) right action of** M_0, we use Lemma 3.40, observing that for the dense subspace $\mathcal{Y} \boxtimes \mathrm{L}^2(B_0)$ spanned by the vectors $\xi \otimes b\Omega$, the following identity $\langle \xi \boxtimes a\Omega \lhd b,\ \eta \boxtimes c\Omega\rangle = \mathrm{tr}(c^* \langle \eta \mid \xi\rangle_{B_0} ab) = \langle (\langle \eta \mid \xi\rangle_{B_0} \rhd a\Omega) \lhd b,\ c\Omega\rangle$ holds.

There is a densely defined **left B_0-action** given by

$$b \rhd (\xi \boxtimes c\Omega) := (b \rhd \xi) \boxtimes (c\Omega).$$

In the following we will show that this action is bounded on $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ and how to extend this to a normal left M_0-action. The distinctive elements that make this extension work are **the existence of a a positive definite trace on the *right-adjointable operators*** and a **faithful, trace-preserving conditional expectation**, which we shall respectively denote by:

$$\mathrm{Tr} : \mathcal{B}^*(\mathcal{Y}) \longrightarrow \qquad \mathcal{E} : \mathcal{B}^*(\mathcal{Y}) \longrightarrow B_0,$$

Remark 5.2. Recall that since $\mathcal{Y}$ is fgp, it then follows that $\mathcal{B}^*(\mathcal{Y}) = \mathcal{K}^*(\mathcal{Y}) = \mathrm{FinRan}(\mathcal{Y})$. We can therefore characterize right-adjointable operators as finite combinations of *ket-bras*, $|\xi\rangle\langle\eta|$, for $\xi, \eta \in \mathcal{Y}$.

In lights of the above Remark, we can explicitly define these maps as the -linear extensions of

$$\mathrm{Tr}\left(\ |\xi\rangle\langle\eta|\ \right) := \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot \mathrm{tr}_{B_0}\left[\ \langle \eta \mid \xi\rangle_{B_0}\right] \quad \text{and} \quad \mathcal{E}\left(\ |\xi\rangle\langle\eta|\ \right) := \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot {}_{B_0}\langle \xi, \eta\rangle. \tag{5.5}$$

Recall that since $\mathcal{Y} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$, it is already normalized (see Definition 2.13 and Example 2.15), so that $\text{l-Ind}(\mathcal{Y}) = \sqrt{\mathrm{Ind}(\mathcal{Y})} = \text{r-Ind}(\mathcal{Y}) \in Z(B_0)^+ = \mathbb{R}_{\geq 0}$.

We provide a summary of the properties of Tr and $\mathcal{E}$ in the form of a proposition:

Proposition 5.3. *The following statements hold on* $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$*:*

1. *The map* Tr *is tracial;*
2. *The trace* Tr *is positive definite;*
3. *The mapping $\mathcal{E}$ is B_0-bilinear and unital;*
4. *In fact, $\mathcal{E}$ is a fully faithful map; and*
5. *The conditional expectation $\mathcal{E}$ preserves the trace, and therefore, for each $\xi, \eta \in \mathcal{Y}$, and each $a, b \in B_0$*

$$\langle \xi \boxtimes a\Omega,\ \eta \boxtimes b\Omega\rangle = \mathrm{tr}\left(\ {}_{B_0}\langle \xi \lhd a,\ \eta \lhd b\rangle\ \right) = \langle \Omega,\ {}_{B_0}\langle \eta \lhd b,\ \xi \lhd a\rangle\Omega\rangle. \tag{5.6}$$

Notice that by Lemma 3.23, B_0 has a unique trace, so for some $\lambda > 0$, we have that $\mathrm{Tr}|_{B_0} = \lambda \cdot \mathrm{tr}_{B_0}$. *Since $id_{\mathcal{Y}} = \sum_j |v_j\rangle\langle v_j|$ for a left B_0-basis $\{v_j\}_{j=1}^n$ for $\mathcal{Y}$, $\lambda = \sqrt{Ind(\mathcal{Y})}$.*

Proof. 1. This is a direct computation.

2. An arbitrary element $x \in \mathcal{B}^*(\mathcal{Y}) = \mathrm{FinRan}(\mathcal{Y})$ can be written as $x = \sum_{r=1}^{t} |\xi_r\rangle\langle\eta_r|$. Suppose $\{u_i\}_{i=1}^m$ is a right basis for $\mathcal{Y}_{B_0}$. Assuming $x \cdot x^*$ has zero trace and by first expanding the η's with respect to this basis, and then expanding the ξ's, we obtain that

$$\begin{aligned}
0 &= \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot \mathrm{Tr}(x \cdot x^*) = \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot \sum_{r,s=1}^{t} \mathrm{Tr}\left(|\xi_r\rangle\langle\eta_r| \circ |\eta_s\rangle\langle\xi_s|\right) \\
&= \sum_{r,s=1}^{t} \mathrm{tr}_{B_0}\left(\langle\xi_s \mid \xi_r\rangle_{B_0} \cdot \sum_{i,j=1}^{m} \langle u_i \mid \eta_r\rangle_{B_0}^* \cdot \langle u_i \mid u_j\rangle_{B_0} \cdot \langle u_j \mid \eta_s\rangle_{B_0}\right) \\
&= \sum_{r,s=1}^{t} \mathrm{tr}_{B_0}\left(\sum_{k=1}^{m}\left[\sum_{j=1}^{m}\langle\eta_s \mid u_j\rangle_{B_0} \cdot \langle u_j \mid u_k\rangle_{B_0}\right]^* \cdot \langle\xi_s \mid \xi_r\rangle_{B_0} \cdot \left[\sum_{i=1}^{m}\langle\eta_r \mid u_i\rangle_{B_0} \cdot \langle u_i \mid u_k\rangle_{B_0}\right]\right) \\
&= \sum_{r,s=1}^{t}\sum_{k=1}^{m} \mathrm{tr}_{B_0}\left(b_{ks}^* \cdot \langle\xi_s \mid \xi_r\rangle_{B_0} \cdot b_{rk}\right) = \sum_{k=1}^{m} \mathrm{tr}_{B_0}\left(\sum_{l,l'}^{m} c_{lk}^* \cdot \langle u_i \mid u_j\rangle_{B_0} \cdot c_{l'k}\right).
\end{aligned}$$

Here, $\left(\sum_{i=1}^{m}\langle\eta_r \mid u_i\rangle_{B_0} \cdot \langle u_i \mid u_k\rangle_{B_0}\right) =: (b_{rk}) \in \mathrm{Mat}_{T\times m}(B_0)$, and $\sum_{r=1}^{t}\langle u_l \mid \xi_r\rangle_{B_0} \cdot b_{rk} =: c_{lk} \in \mathrm{Mat}_{m\times m}(B_0)$. Notice that in the third equality we used the fact that the matrix $(\langle u_i \mid u_j\rangle_{B_0})_{ij}$ is an idempotent. The last equality is then a positive definite quadratic form in B^m, and therefore conclude that each of the k terms in the sum is zero, by the faithfulness of tr_{B_0}. We then obtain that for each $k, i \in \{1, \ldots, m\}$

$$0 = \sum_{j=1}^{m}\langle u_i \mid u_j\rangle_{B_0} \cdot c_{jk} = \sum_{r=1}^{t}\langle u_i \mid \xi_r\rangle_{B_0} \cdot \langle\eta_r \mid u_k\rangle_{B_0}.$$

This implies that $\sum_{r=1}^{t}|\xi_r\rangle\langle\eta_r| = 0$, thus establishing that Tr is positive definite.

3. The left B_0-linearity is obvious. The right bilinearity follows from

$$\begin{aligned}
\mathcal{E}(|\xi\rangle\langle\eta| \cdot b) = \mathcal{E}(|\xi\rangle\langle b^* \rhd \eta|) = {}_{B_0}\langle\xi, b^* \rhd \eta\rangle &= \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}}\,{}_{B_0}\langle b^* \rhd \eta, \xi\rangle^* \\
&= \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}}(b^* \cdot_{B_0} \langle\eta, \xi\rangle)^* = \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}}\,{}_{B_0}\langle\xi, \eta\rangle \cdot b = \mathcal{E}(|\xi\rangle\langle\eta|) \cdot b,
\end{aligned}$$

for all $\xi, \eta \in \mathcal{Y}$ and $b \in B_0$. Unitality follows from the fact that $\mathrm{id}_{\mathcal{Y}} = \sum_{i=1}^{m}|u_i\rangle\langle u_i|$, where $\{u_i\}_{i=1,\ldots m}$ is a right B_0-basis for $\mathcal{Y}_B$.

4. Applying Lemma 1.26 in [20] to the full bimodule ${}_{B_0}\mathcal{Y}_{B_0}$ implies $\mathcal{E}$ is a conditional expectation (of index finite type). (The bimodule ${}_{B_0}\mathcal{Y}_{B_0}$ is full since B_0 is simple.) This map is clearly fully faithful, from the definition of the inner products.

5. By compatibility with the trace, Equation (5.1), we have that

$$\mathrm{Tr}\left(\ |\xi\rangle\langle\eta|\ \right) = \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot \mathrm{tr}_{B_0}\left(\ \langle\eta \mid \xi\rangle_{B_0}\ \right) = \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot \mathrm{tr}_{B_0}\left(\ {}_{B_0}\langle\xi,\ \eta\rangle\ \right) = \mathrm{tr}_{B_0}\left(\ \mathcal{E}(|\xi\rangle\langle\eta|)\ \right).$$

□

Remark 5.4. Notice that all the bimodules ${}_lB_r$ introduced in Notation 3.30 define objects in $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$, in light of Proposition 3.42.

Proposition 5.5. *The left B_0-action on $\mathcal{Y} \boxtimes \Omega$ extends to a bounded left B_0-action on $\mathcal{Y} \boxtimes_{B_0} L^2(B_0)$.*

Proof. This follows from the fact that $\mathcal{Y}$ is fgp. Indeed, for arbitrary $b \in B_0$ and $y \in \mathcal{Y}$, we have that $||b \triangleright (y \boxtimes \Omega)||_2^2 = \mathrm{tr}(\langle b \triangleright y \mid b \triangleright y \rangle_{B_0}) \leq \mathrm{tr}(||b||^2 \cdot \langle y \mid y \rangle_{B_0}) = ||b||^2 \cdot ||y \boxtimes \Omega||_2^2$. The inequality is a direct application of (3) in [20, Proposition 1.12], stating that $\langle b \triangleright y \mid b \triangleright y \rangle_{B_0} \leq ||b||^2 \cdot \langle y \mid y \rangle_{B_0}$. $\square$

We now extend the left B_0-action on $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ to a normal left-M_0 action by using Lemma 3.40. For the dense subspace $\mathcal{Y} \boxtimes \Omega$ spanned by the vectors $\xi \boxtimes a\Omega$, since the conditional expectation preserves the trace, we obtain

$$\langle b \triangleright (\xi \boxtimes a\Omega),\ \eta \boxtimes c\Omega \rangle = \mathrm{tr}_{B_0}\left({}_{B_0}\langle b \triangleright \xi \triangleleft a,\ \eta \triangleleft c \rangle\right) = \mathrm{tr}_{B_0}\left(b \cdot {}_{B_0}\langle \xi \triangleleft a,\ \eta \triangleleft c \rangle\right) = \langle b\Omega,\ {}_{B_0}\langle \eta \triangleleft c,\ \xi \triangleleft a \rangle \cdot \Omega \rangle$$

Hence **the left B_0-action on $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ extends to a normal left** M_0-action.

Remark 5.6. For the remainder of this chapter will need to further exploit the structure of M_0-bimodules of the form $\mathcal{Z} \boxtimes_{B_0} \mathrm{L}^2(B_0)$, and their tensor products. We say that **a vector $\xi \in \mathcal{Z} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ is (left) bounded** if and only if the mappping

$$L_\xi : M_0\Omega \to \mathcal{Z} \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0), \text{ given by } m\Omega \mapsto m \blacktriangleright \xi$$

extends to a bounded map defined on $\mathrm{L}^2(B_0) \cong \mathrm{L}^2(M_0)$. If ξ, η are bounded vectors, then the composite map $L_\eta^* \circ L_\xi$ makes sense and takes values in $M_0\Omega$. This defines a **left M_0-valued inner product** on the set bounded vectors, denoted $\langle \eta \mid \xi \rangle_{M_0}$. (For a proof see [1] or [16].) One can easily see that for an arbitrary $z \in \mathcal{Z}$, the vector $z \boxtimes \Omega \in \mathcal{Z} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ is bounded, and hence $\langle z \mid z \rangle_{M_0} \in M_0$. A computation will reveal that $L_{z\boxtimes\Omega}^* : \mathcal{Z} \boxtimes_{B_0} \mathrm{L}^2(B_0) \to \mathrm{L}^2(M_0)$ is given by $y \boxtimes \Omega \mapsto {}_{B_0}\langle y, z \rangle\Omega$. It then follows that $\langle z \boxtimes \Omega \mid z \boxtimes \Omega \rangle_{M_0} = {}_{B_0}\langle z, z \rangle$.

We now show that ${}_{M_0}(- \boxtimes_{B_0} \mathrm{L}^2(B_0))_{M_0}$ **defines a bi-involutive fully-faithful strong monoidal functor** into the claimed target category. Again, we need to be certain that the von Neumann closure of B_0 is independent of the space over which we represent it.

Lemma 5.7. *The σ-weak topologies on the left (resp. right) action of B_0'' coming from the left (resp. right) action on $L^2(B_0)$ and the action on $\mathcal{Y} \boxtimes_{B_0} L^2(B_0)$ agree.*

Proof. Let's first show the statement corresponding to the left actions. Let $(B_0 \triangleright)$ denote the image of B_0 as a subset of $\mathcal{B}(\mathrm{L}^2(B_0))$, and let $(B_0 \blacktriangleright)$ that of B_0 considered as a subset of $\mathcal{B}(\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0))$. By Lemma 3.40, we can consider the von Neumann algebra $(M_0 \blacktriangleright) \subset \mathcal{B}(\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0))$. Moreover, this lemma tells us that for every $b \in B_0$ we have that $(b \blacktriangleright) = ((b \triangleright) \blacktriangleright)$. So we automatically obtain that $(B \blacktriangleright)'' \subseteq (M \blacktriangleright)$, since both are von Neumann algebras containing the same copy of B_0. Now, since $\blacktriangleright$ defines a σ-weak continuous map $\blacktriangleright : M \to \mathcal{B}(\mathcal{Y} \underset{B_0}{\otimes} \mathrm{L}^2(B_0))$, it then follows that $\blacktriangleright$ commutes with taking σ-weak-limits. Thus, for an arbitrary $m \in M$ and a net b_λ such that $b_\lambda \triangleright \longrightarrow m$ σ-weak, we have the following chain of equalities:

$$\mathrm{WOT} - \lim_\lambda b_\lambda \blacktriangleright = \mathrm{WOT} - \lim_\lambda ((b_\lambda \triangleright) \blacktriangleright) = (\mathrm{WOT} - \lim_\lambda b_\lambda \triangleright) \blacktriangleright = m \blacktriangleright,$$

thus establishing that $(B_0 \blacktriangleright)'' = (M \blacktriangleright) \subset \mathcal{B}(\mathcal{Y} \underset{B_0}{\otimes} \mathrm{L}^2(B_0))$. The proof of the corresponding statement for the right actions is straightforward and thus we omit it. $\square$

Proposition 5.8. *The assignment ${}_{M_0}(- \boxtimes_{B_0} L^2(B_0))_{M_0}$ defines a faithful dagger functor out of $\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$.*

Proof. Let $\mathcal{Y}, \mathcal{Z} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$ and $f : \mathcal{Y} \to \mathcal{Z}$ be a fixed but arbitrary B_0-bimodule map. We denote $\tilde{f} := f \boxtimes \mathrm{id}_{\mathrm{L}^2(B_0)}$. It is clear that $\tilde{f}$ is B_0-M_0 bilinear and bounded. It is also evident that ${}_{M_0}(- \boxtimes_{B_0} \mathrm{L}^2(B_0))_{M_0}$ respects composition of maps and the dagger structures, so it only remains to check that $\tilde{f}$ is in fact M_0-bilinear. Let m be an arbitrary element in M_0, and $(b_n) \subset B$ a sequence such that $\mathrm{SOT} - \lim_n (b_n \blacktriangleright) = m \blacktriangleright$. we then have that

$$\begin{aligned}
m \blacktriangleright \tilde{f}(y \boxtimes \Omega) &= \| \cdot \|_2 - \lim_n b_n \blacktriangleright \tilde{f}(y \boxtimes \Omega) = \| \cdot \|_2 - \lim_n \tilde{f}((b_n \triangleright y) \boxtimes \Omega) \\
&= \tilde{f}\left(\| \cdot \|_2 - \lim_n ((b_n \triangleright y) \otimes \Omega)\right) = \tilde{f}\left(|| \cdot ||_2 - \lim_n b_n \blacktriangleright (y \otimes \Omega)\right) \\
&= \tilde{f}(m \blacktriangleright (y \boxtimes \Omega)).
\end{aligned}$$

Faithfulness follows since the trace tr_{B_0} is faithful. Notice that we are arguing using sequences converging in the SOT, as opposed to nets. Indeed, on $||\cdot||_\infty$-bounded sets, the $||\cdot||_2$-norm topology on $B_0\Omega$ is equivalent to the SOT on B_0. Thus, by Kaplansky Density Theorem, we can replace SOT converging nets by $||\cdot||_2$-converging sequences which are automatically bounded in $||\cdot||_\infty$. $\square$

Proposition 5.9. *The functor* ${}_{M_0}(-\boxtimes_{B_0} L^2(B_0))_{M_0}$ *is strong monoidal with tensorator given by the linear extension of:*

$$\mu_{\mathcal{Y},\mathcal{Z}} : (\mathcal{Y} \underset{B_0}{\boxtimes} L^2(B_0)) \underset{M_0}{\boxtimes} (\mathcal{Z} \underset{B_0}{\boxtimes} L^2(B_0)) \longrightarrow ((\mathcal{Y} \underset{B_0}{\boxtimes} \mathcal{Z}) \underset{B_0}{\boxtimes} L^2(B_0))$$

mapping $(\xi \boxtimes \Omega) \boxtimes (\eta \boxtimes \Omega) \mapsto (\xi \boxtimes \eta) \boxtimes \Omega$.

Proof. Let $y \in \mathcal{Y}$ and $x \in \mathcal{Z}$ be fixed but arbitrary. We shall first show that this map is well-defined. For an arbitrary operator $m \blacktriangleright\in M_0$, consider a bounded net $(b_i) \subset B_0$ such that $b_i \blacktriangleright\to m \blacktriangleright$ SOT, as $i \to \infty$. We then obtain that

$$||((y \lhd m) \boxtimes z)) \boxtimes \Omega - y \boxtimes (b_i \rhd z) \boxtimes \Omega||_2^2 = \lim_i ||(y \boxtimes (b_i - m) \rhd z) \boxtimes \Omega||_2^2 \leq ||y||_B^2 \cdot ||(b_i - m) \rhd z \boxtimes \Omega||_2^2 = 0.$$

We shall next show that μ is an isometry. By Remark 5.6 and Equation (1.1) in [1] defining the product on the M_0-relative fusion of two bimodules in $\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0)$, we have the following chain of equalities:

$$\begin{aligned}
\langle (y \boxtimes \Omega) \boxtimes (z \boxtimes \Omega),\ (y \boxtimes \Omega) \boxtimes (z \boxtimes \Omega)\rangle &= \langle (y \boxtimes \Omega) \blacktriangleleft \langle z \boxtimes \Omega \mid z \boxtimes \Omega\rangle_{M_0},\ y \boxtimes \Omega\rangle \\
&= \mathrm{tr}_{B_0}(\langle y \mid y \lhd {}_{B_0}\langle z, z\rangle\rangle_{B_0}) = \mathrm{tr}_{B_0}(\langle y \mid y\rangle_{B_0} \cdot {}_{B_0}\langle z, z\rangle) \\
&= \mathrm{tr}_{B_0}({}_{B_0}\langle \langle y \mid y\rangle_{B_0} \rhd z \mid z\rangle) = \mathrm{tr}_{B_0}(\langle z \mid \langle y \mid y\rangle_{B_0} \rhd z\rangle_{B_0}) \\
&= \mathrm{tr}_{B_0}(\langle y \boxtimes z \mid y \boxtimes z\rangle_{B_0}) = \langle (y \boxtimes z) \boxtimes \Omega, (y \boxtimes z) \boxtimes \Omega\rangle.
\end{aligned}$$

Thus, this map defines an isometry. Since $\mu_{\mathcal{Y},\mathcal{Z}}$ obviously has dense range, then it extends to a unitary M_0-bimodule map. As a word of warning, to use the inner product on the M_0-fusion bimodule, one needs to know *a priori* that $\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ and $\mathcal{Z} \boxtimes_{B_0} \mathrm{L}^2(B_0)$ define spherical/extremal bifinite M_0-bimodules; i.e. they are objects in $\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0)$. This is independently demonstrated as part of the proof of Proposition 5.11, but we postpone it not to slow down the exposition.

It remains to verify that μ defines a natural transformation. A simple computation will show that naturality holds when restricted to the dense subspace $(\mathcal{Y} \boxtimes_{B_0} \Omega) \boxtimes_{B_0} (\mathcal{Z} \boxtimes_{B_0} \Omega)$, and this will suffice, since all the transformations involved are bounded. This completes the proof. $\square$

Proposition 5.10. *The functor* $- \boxtimes_{B_0} L^2(B_0)$ *is bi-involutive with structure maps*

$$\chi_{\mathcal{Y}} : \overline{\mathcal{Y}} \underset{B_0}{\boxtimes} L^2(B_0) \longrightarrow \overline{\mathcal{Y} \underset{B_0}{\boxtimes} L^2(B_0)} \text{ given on } \overline{\mathcal{Y}} \boxtimes \Omega \text{ by}$$
$$\overline{y} \boxtimes \Omega \longmapsto \overline{y \boxtimes \Omega},$$

for $\mathcal{Y} \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$.

Proof. We need to show this is a family of unitary isomorphisms satisfying the conditions in Definition (2.9). By compatibility with the trace condition (Equation 5.1) we have the following chain of equalities:

$$\begin{aligned}
\langle \chi_{\mathcal{Y}}(\overline{y} \boxtimes \Omega) \mid \overline{x \boxtimes \Omega}\rangle &= \langle \overline{y \boxtimes \Omega} \mid \overline{x \boxtimes \Omega}\rangle = \langle x \boxtimes \Omega \mid y \boxtimes \Omega\rangle \\
&= \mathrm{tr}_{B_0}(\langle x \mid y\rangle_{B_0}) = \mathrm{tr}_{B_0}({}_{B_0}\langle y,\ x\rangle) = \mathrm{tr}_{B_0}(\langle \overline{y} \mid \overline{x}\rangle_{B_0}) \\
&= \langle \overline{y} \boxtimes \Omega \mid \overline{x} \boxtimes \Omega\rangle = \langle \overline{y} \boxtimes \Omega \mid \chi_{\mathcal{Y}}^{-1}(\overline{x \boxtimes \Omega})\rangle,
\end{aligned}$$

proving that $\chi_{\mathcal{Y}}$ is right-adjointable and that it extends to a unitary. To show that that this family of unitaries $\{\chi_{\mathcal{Y}}\}$ is natural, monoidal and that Diagrams (2.8) commute is a matter of simple diagrammatic computation. This completes the proof. $\square$

Proposition 5.11. *The following 2-cell commutes:*

$$\begin{tikzcd} & \mathsf{C}_x \arrow[dl, "\mathbb{F}"'] \arrow[dr, "\mathbb{G}"] \arrow[d, phantom, "\overset{T}{\underset{\cong}{\Rightarrow}}"] & \\ \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0) \arrow[rr, hook, "-\boxtimes_{B_0} L^2(B_0)"'] & & \mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0). \end{tikzcd} \tag{5.7}$$

Here, $T = \{T_n\}_{n\geq 0}$ is a monoidal unitary natural isomorphism whose components are

$$T_n : ({}_0B_n) \underset{B_0}{\boxtimes} L^2(B_0) \longrightarrow \mathbb{G}(x^{\otimes n}) = p_0 \rhd L^2({}_0M_n) \lhd p_n \text{ given on } {}_0B_n \boxtimes \Omega \text{ by}$$

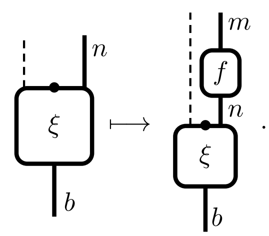
$$(p_0 \wedge b \wedge p_n) \boxtimes \Omega \longmapsto p_0 \rhd b\Omega \lhd p_n = (p_0 \wedge b \wedge p_n)\Omega,$$

and $\mathbb{G}$ is the fully faithful bi-involutive strong-monoidal functor constructed on [2]. In in our language, $\mathbb{G}$ is given on objects by

$$x^{\otimes n} \longmapsto {}_{M_0}\left(p_0 \rhd L^2({}_0M_n) \lhd p_n\right)_{M_0}$$

and on morphisms

$$(f : x^{\otimes n} \longrightarrow x^{\otimes m}) \longmapsto \quad \mathbb{G}(f) : \tag{5.8}$$



It then follows that $\mathbb{F}$ is full; i.e.,

$$\mathbb{F}[\mathsf{C}(x^{\otimes n} \to x^{\otimes m})] = \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)(\mathcal{X}^{\boxtimes_{B_0} n} \to \mathcal{X}^{\boxtimes_{B_0} m}).$$

Proof. Each of the maps T_n is densely defined and is clearly B_0-bilinear and isometric with dense range. Thus, each T_n extends to a B_0-unitary. For an arbitrary map $f \in \mathsf{C}(x^{\otimes n} \to x^{\otimes m})$, checking that $(T_m \circ (-\boxtimes_{B_0} \mathrm{L}^2(B_0)) \circ \mathbb{F})(f) = (\mathbb{G} \circ T_n)(f)$ is an easy computation. The tensorator μ^T is given by extending juxtaposition of diagrams, similarly as was previously done for $\mu^{\mathbb{F}}$. Therefore, establishing Diagram 5.7 commutes, as well as proving that $\mathbb{F}$ does take values in the bifinite spherical bimodules over M_0 up to a monoidal unitary natural isomorphism.

To show that $\mathbb{F}$ is full we implement a finite-dimensional linear algebraic trick: Since $\mathbb{G}$ is full and faithful between rigid categories, we have that

$$\dim[\mathsf{C}(x^{\otimes n} \to x^{\otimes m})] = \dim[\mathsf{Bim}^{\mathsf{sp}}_{\mathsf{bf}}(M_0)(\mathbb{G}(x^{\otimes n}) \to \mathbb{G}(x^{\otimes m}))] =: D < \infty.$$

Since $\mathbb{F}$ and $(- \boxtimes_{B_0} \mathrm{L}^2(B_0))$ are faithful, we obtain

$$D = \dim[\mathbb{F}[\mathsf{C}(x^{\otimes n} \to x^{\otimes m})]] \leq \dim[\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}({}_0B_n \to {}_0B_m)] = \dim[\mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}({}_0B_n \to {}_0B_m) \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0)] \leq D.$$

Thus all inequalities must be simultaneously satified, proving $\mathbb{F}$ is full. This completes the proof. $\square$

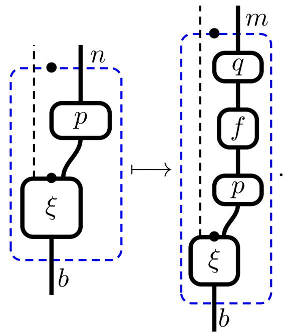
Now we **unitarily Cauchy-complete** $\mathbb{F}$ on C_x: **on subobjects** of tensor powers of x: for a given $n \in \mathbb{N}$, a projection $p \in \mathsf{C}(x^{\otimes n} \to x^{\otimes n})$ determines a sub-object of $x^{\otimes n}$, denoted $p[x^{\otimes n}] \in \mathsf{C}_x$, we then define

$$\mathbb{F}(p[x^{\otimes n}]) := \mathbb{F}(p)[{}_0B_n] \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0),$$

and **linearly extend** the definition of $\mathbb{F}$ **over direct sums**. Here, $\mathbb{F}(p)[\ {}_0B_n] \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$ denotes the range of $\mathbb{F}(p)$. We shall now define $\mathbb{F}(f)$, where $f : p[x^{\otimes n}] \to q[x^{\otimes m}]$ is a morphism between projections $p \in \mathsf{C}(x^{\otimes n})$ and $q \in \mathsf{C}(x^{\otimes m})$. We define

$$\mathbb{F}(f) : \mathbb{F}(p)[{}_0B_n] \longrightarrow \mathbb{F}(q)[{}_0B_m]$$

as the extension of the mapping acting on a diagram $\xi \in \mathrm{Gr}_\infty \cap {}_0B_n$ by:

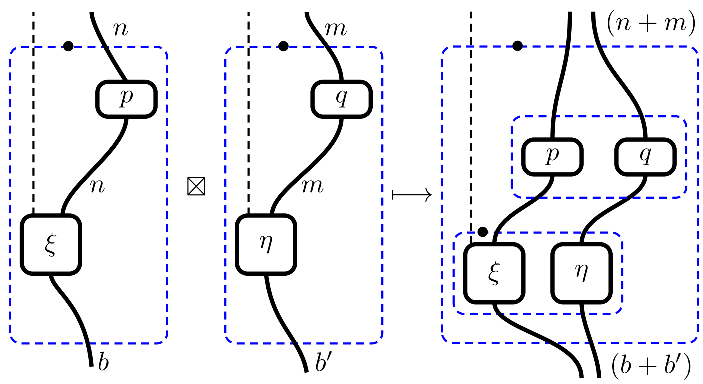


(5.9)

We now extend the **tensorator data** for $\mathbb{F}$. For arbitrary $n,\ m \geq 0$, and projections $p \in \mathsf{C}(x^{\otimes n} \to x^{\otimes n})$ and $q \in \mathsf{C}(x^{\otimes m} \to x^{\otimes m})$ we define

$$\mu^{\mathbb{F}}_{p,q} : \mathbb{F}(p)[\,{}_0B_n] \boxtimes_{B_0} \mathbb{F}(q)[\,{}_0B_m] \longrightarrow \mathbb{F}(p \otimes q)[\,{}_0B_{n+m}],$$

by extending the following diagramatic composition (and omitting the unitors):



(5.10)

Notice that if p' and q' are projections in the Cauchy completion of C_x, and if $f : p \to p'$ and $g : q \to q'$ are morphisms in C_x, then $\mu^{\mathbb{F}}(f, g) = \mathrm{id}_{\mathbb{1}} \otimes f \otimes g$ on diagrams. By Lemma 4.1, $\mu^{\mathbb{F}}_{p,q}$ extends to a unitary isomorphism $\mathbb{F}(p)[{}_0B_n] \boxtimes_{B_0} \mathbb{F}(q)[{}_0B_m] \xrightarrow{\sim} \mathbb{F}(p \otimes q)[{}_0B_{(n+m)}]$, and is clearly natural in p and q. We can also extend $\mu^{\mathbb{F}}(f, g)$ from diagrams to all of $\mathbb{F}(p)[{}_0B_n] \boxtimes_{B_0} \mathbb{F}(q)[{}_0B_m]$, since this map is just a series of vertical and horizontal compositions in the category. It is clear that bi-adjointability for $\mu^{\mathbb{F}}(f, g)$ holds on diagrams, so the extension of $\mu^{\mathbb{F}}$ is also bi-adjointable.

We are now ready to state the final form of the most important result of this manuscript, which summarizes this whole article:

Theorem 5.12. *Given a singly-generated small RC*TC, C_x with simple unit with generator $x \in \mathsf{C}_x$, there exists a unital, simple, separable, exact C*-algebra B_0 with unique trace, and a fully-faithful bi-involutive strong monoidal functor*

$$\mathbb{F} : \mathsf{C}_x \hookrightarrow \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0),$$

into a full subcategory of fgp minimal and normalized B_0-bimodules. Moreover, the K_0 group of B_0 is the free abelian group on the classes of simple objects in C_x, and the image of the simple objects of C_x are precisely the canonical generators of $K_0(B_0)$.

Remark 5.13. One can extend the result of Theorem 5.12 by allowing C to be countably generated. Indeed if C is countably generated, then it is generated by a countable set of symmetric self-dual objects $x_1, x_2, \cdots$. Given a word w a word in $x_1, x_2, \cdots$ we define x_w by

$$x_w = x_{i_1} \otimes \cdots \otimes x_{i_k} \text{ provided } w = x_{i_1} \cdots x_{i_j}.$$

Given words u, v, and w, one now defines $V_{u,v,w}$ by $\mathsf{C}(x_u \to x_w \otimes x_w)$. If we define W to be the set of all words in x_1, x_2, $\cdots$, one can form

$$\mathrm{Gr}_\infty = \bigoplus_{u,v,w \in W} V_{u,v,w},$$

and endow it with the $*$-structure and multiplication as described in Sections 3.1 and 3.2, where we now color the strings according to the labels $x_1, x_2, \cdots$. Let B_∞ be the C*-algebra generated by Gr_∞, and let $B_0 = p_0 \wedge B_\infty \wedge p_0$, where p_0 is the empty diagram. It is also true that B_0 is a unital, separable, simple, exact C*-algebra.

If $w \in W$, we define

$$X_w = \overline{\bigoplus_{v \in W} V_{v,w,\emptyset}}^{\|\cdot\|}$$

then $X_w \in \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$ via the actions and inner products as described in Section 3.4. Moreover, the mapping $x_w \longmapsto X_w$ induces a fully-faithful bi-involutive strong monoidal functor

$$\mathbb{F} : \mathsf{C} \hookrightarrow \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0).$$

We note that there is no separable C*-algebra, B with unique trace, over which we can fully realize every RC*TC as a full subcategory of Hilbert C*-bimodules. This stems from the fact that the K_0 group of any such B must be countable, and thus there are only countably many values for the traces of projections in $B \otimes K(\mathcal{H})$. On the other hand, amongst all of the countably generated fusion categories, any $\delta \geq 2$ can be the dimension of a generating object. However, our main theorem does allow us to establish this result, if we restrict our scope to **unitary fusion categories**:

Corollary 5.14. *There exists a unital simple exact separable C*-algebra B with unique trace over which we can represent every unitary fusion category C in the spirit of Theorem 5.12.*

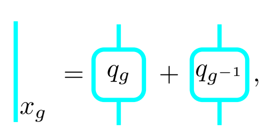
Proof. Up to equivalence, there are countably many fusion categories. Let $\mathsf{C}_1, \mathsf{C}_2, \cdots$ be a set of representatives for these fusion categories, and let C be their free product (see [8] for a description). Then C is a small RC*TC with simple unit, and each C_k embeds into C in the obvious way. We can therefore apply the construction in Remark 5.13 to C to obtain B. $\square$

6 A concrete example: $\mathsf{Vec}_{\mathsf{fd}}(G)$

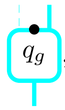
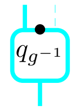
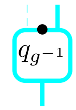
We now develop an explicit illustration of Theorem 5.12. We will describe the C*-algebra B_0 and the bimodules ${}_0B_1$ (Notation 3.21) utilized in the statement of the theorem, which in turn where introduced in Section 4.

Let G be a countable discrete group and choose $\mathsf{C} = \mathsf{Vec}_{\mathsf{fd}}(G)$, the tensor category of finite dimensional G-graded vector spaces. A complete set of representatives for the simple objects of C is given by $\{_g\}_{g \in G}$, where $\mathsf{C}(_g \to _h) \cong \delta_{g=h} \cdot$. It is easy to see that the balanced unitary dual functor on C is determined by its action on simples, and is given by $\overline{_g} =_{g^{-1}}$, (see Definition 2.4 and Remark 2.5). By letting $x_g :=_g \oplus_{g^{-1}}$ for each $g \in G$, we obtain a set of symmetrically self-dual objects (Definition 3.2) of quantum dimension (Notation 3.1) $\delta_{x_g} = 2 > 1$ that generate C. We will identify x_g with $x_{g^{-1}}$.

Following Section 3, we can construct similar graded algebras as those in Definition 3.4, where the strings on the bottom and on the sides are labeled by finite words in the symbols $\{x_g\}_{g \in G}$. We can use either the structure of Definition 3.15 or that of Remark 3.17, in conjunction with Remark 5.13, to construct a simple C*-algebra B_∞, by completing the tracial graded *-algebras arising from the diagrams in C. The only difference with the algebras from Section 3 is that B_∞ now contains strings colored by the elements of G as in 5.13. Note that the following identity holds:



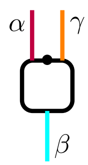
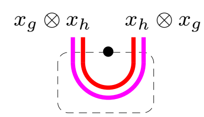
where q_g is the orthogonal projection onto $_g$ in $\mathsf{C}(x_g \to x_g)$. Our C*-algebra, B_∞ is generated by A_∞ together with the elements Y_g given by:

$Y_g =$ , and notice that $Y_g^* =$  $\neq Y_{g^{-1}} =$ .

The elements (Y_g) are free with amalgamation over A_∞ with respect to $\mathbb{E}$ and form an A_∞-valued circular family [30]. This describes B_∞ as an amalgamated free product. The unital C*-algebra $B_0 = {}_0B_0$ is the corner $p_0 \wedge B_\infty \wedge p_0$, where p_0 is the orthogonal projection corresponding to the empty diagram, and notice that B_0 contains all diagrams of the form:

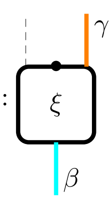
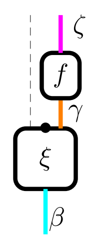
 $\in V_{\emptyset,\beta,\emptyset} \subset B_0$,

where β denotes a word in $\{x_g\}_{g\in G}$. We note that $m \in B_0$ will equal zero unless β contains a copy of $_e$, where $e \in G$ is the identity. This means that the only nonzero components after decomposing m as a direct sum over $\{_g\}_{g\in G}$ are given by words $q_{g_1} \cdots q_{g_m}$ where $g_1 \cdots g_m = e$. For a word ζ in $\{x_g\}_{g\in G}$, let p_ζ be the orthogonal projection corresponding to colored composition of cups in the symbols of ζ. We describe an example of such a projection below. More generally, for arbitrary words α and γ in $\{x_g\}_{g\in G}$, we consider the corners ${}_\alpha B_\gamma = p_\alpha \wedge B_\infty \wedge p_\gamma$. We now describe the diagrammatic elemtents in ${}_\alpha B_\gamma$, and depict p_ζ in the case where $\zeta = x_g \otimes x_h$:

 $\in V_{\alpha,\beta,\gamma} \subset B_0$, and $p_\zeta =$ .

Notice that these diagrams can be endowed with the structure of a tracial graded *-algebra, similarly as in Definition 3.4, 3.2, and Remark 3.20.

We now construct the bimodules used in the representation $\mathsf{F} : \mathsf{C} \to \mathsf{Bim}^{\mathsf{tr}}_{\mathsf{fgp}}(B_0)$. For each $n \in \mathbb{N}$ and each $g \in G$, let ${}^g_0B_n = p_0 \wedge B_\infty \wedge p_{g^n}$, and consider the assignment:

$x_g \longmapsto {}^g_0B_1$ $\qquad$ $\mathsf{C}(\gamma \to \zeta) \ni f \longmapsto \mathbb{F}(f) :$  $\longmapsto$ .

There are obvious (bounded) left and right actions of B_0 on each ${}^g_0B_n$, with left and right inner products similar to those introduced in Notation 3.30, giving ${}^g_0B_n$ the structure of a Hilbert C*-bimodule over B_0, that is compatible with the trace. (Compare with Proposition 3.42.)

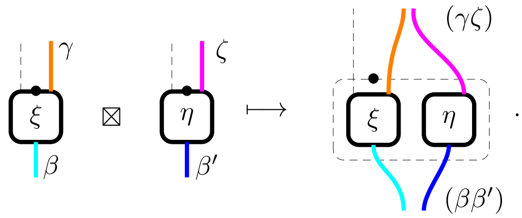
The **unit map** for $\mathbb{F}$ is

$$\iota^{\mathbb{F}} : \mathbb{1}_{\mathsf{Bim}_{\mathsf{fgp}}(B_0)} \longrightarrow \mathbb{F}(\mathbb{1}_{\mathsf{C}}),$$

given by the identity on B_0. We now describe the components $\mu^{\mathbb{F}}_{\gamma,\zeta}$ of **the tensorator**, where γ and ζ are words in $\{x_g\}_{g\in G}$. We notice this is an extended definition from that of Equation 4.3, where we were assuming every string has the same color. We define

$$\mu^{\mathbb{F}}_{\gamma,\zeta} : {}_0B_\gamma \underset{B_0}{\boxtimes} {}_0B_\zeta \longrightarrow {}_0B_{(\gamma\zeta)},$$

by extending the following diagramatic composition:



(6.1)

That $\mu^{\mathbb{F}}_{\gamma,\zeta}$ is a B_0 bilinear and middle B_0-linear isometry, follows from the equation

$$\mu^{\mathbb{F}}_{\gamma,\zeta}(\xi \boxtimes \eta) = \phi_\gamma^{-1}[\phi_\gamma(\xi) \wedge \eta],$$

where ϕ_γ is *the map that moves the dot to the right of* γ, extending the maps defined in Equation 3.19, inside Notation 3.31. Proving surjectivity of the tensorator follows an identical argument to the proof of Lemma 3.38, with the difference that now the strings are multi-colored. Thus, $\mu^{\mathbb{F}}$ is a B_0-bilinear unitary and is therefore a unitary natural transformation.

It now only remains to unitarily Cauchy complete the functor $\mathbb{F}$ induced by the assignment above. The B_0-B_0 sub-bimodules of ${}^{g}_{0}B_n$, for $g \in G$ and $n \in \mathbb{N}$ can be described along the same lines as in the discussion preceeding Equation 5.9, in Section 5, The extension of the tensorator to multi-colored bimodules is done similarly as in Equation 5.10, where once again the strings are now labeled by words in the $\{x_g\}_{\{}g \in G\}$.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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