On relations between transportation cost spaces and ℓ_1

Sofiya Ostrovska and Mikhail I. Ostrovskii

June 22, 2020

Abstract

The present paper deals with some structural properties of transportation cost spaces, also known as Arens-Eells spaces, Lipschitz-free spaces and Wasserstein spaces. The main results of this work are: (1) A necessary and sufficient condition on an infinite metric space M, under which the transportation cost space on M contains an isometric copy of ℓ_1. The obtained condition is applied to answer the open questions asked by Cúth and Johanis (2017) concerning several specific metric spaces. (2) The description of the transportation cost space of a weighted finite graph G as the quotient $\ell_1(E(G))/Z(G)$, where $E(G)$ is the edge set and $Z(G)$ is the cycle space of G.

Keywords. Arens-Eells space, Banach space, earth mover distance, Kantorovich-Rubinstein distance, Lipschitz-free space, transportation cost, Wasserstein distance
2010 Mathematics Subject Classification. Primary: 46B04; Secondary: 46B20, 46B85, 91B32

1 Introduction

1.1 Definitions

Let (M,d) be a metric space. Consider a real-valued finitely supported function f on M with a zero sum, that is, $\sum_{v\in\mathrm{supp} f} f(v)=0$. A natural and important interpretation of such function, which goes back to at least Kantorovich-Gavurin [18], is to consider it as a *transportation problem*: one needs to transport certain product from locations where $f(v)>0$ to locations where $f(v)<0$. More formally, we represent f as $f=a_1(\mathbf{1}_{x_1}-\mathbf{1}_{y_1})+a_2(\mathbf{1}_{x_2}-\mathbf{1}_{y_2})+\dots+a_n(\mathbf{1}_{x_n}-\mathbf{1}_{y_n})$, where $a_i\ge 0$, $x_i,y_i\in M$, and $\mathbf{1}_u(x)$ for $u\in M$ is the *indicator function* of $\{u\}$. Any such representation of f will be called a *transportation plan* for f. The *cost* of this transportation plan (which consists in moving a_i units from x_i to y_i) is defined as $\sum_{i=1}^n a_i d(x_i,y_i)$. We denote the real vector space of all transportation problems by $\mathrm{TP}(M)$. We introduce the *transportation cost norm* $\|f\|_{\mathrm{TC}}$ of a transportation problem f as the minimal cost over all such transportation plans. It is easy to see that the minimum is attained - we consider finitely supported functions - and that $\|\cdot\|_{\mathrm{TC}}$ is a norm. The completion of this normed space is called a *transportation cost space* and is denoted by $\mathrm{TC}(M)$.

Transportation cost spaces are of interest in many areas and are studied under many different names (the most common ones are included in the keywords section). We prefer to use the term *transportation cost space* since it makes the subject of this work instantly clear to a wide circle of readers and it also reflects the historical approach leading to these notions. Interested readers can find a review of the main definitions, notions, facts, terminology and historical notes pertinent to the subject in [24].

In the theory of metric embeddings, transportation cost spaces are of interest due to the following observation by Arens and Eells [2]: The metric space M admits a canonical isometric embedding into $\mathrm{TC}(M)$, given by $v \mapsto \mathbf{1}_v - \mathbf{1}_O$, where O is a base point in M.

For background information on the transportation cost spaces we refer to [25, Chapter 10] (where such spaces are called *Lipschitz free spaces*) and [26, Chapter 3] (where such spaces are called *Arens-Eells spaces*).

1.2 Motivation and statement of results

In this work, new results pertinent to the relations between the structure of transportation cost spaces and L_1 are obtained. Previously, such relations have been studied by many researchers, see, for example, [1], [4]–[10], [13], [15], [16], [19], [20], [23], [24].

From the historical perspective, there are a few arguments in favor of studying the Banach-space-theoretical structure of transportation cost spaces. Below, some of them are provided:

(1) The linearization of the theory of cotype for metric spaces. This idea was put forth by Bill Johnson. The idea is described in [3, p. 223] and is discussed in [22].

(2) The program of using transportation cost spaces to solve some important problems of linear and nonlinear theory of Banach spaces suggested by Godefroy-Kalton [14], who used the name *Lipschitz-free spaces*. This program was substantially advanced by Kalton [17].

(3) The observation by Arens and Eells [2] (see above) shows that transportation cost spaces are natural target spaces for metric embeddings. See [25, Chapter 10].

The main results of this paper include the following outcomes:

1) A necessary and sufficient condition for containment of ℓ_1 in $\mathrm{TC}(M)$ isometrically (Theorem 2.1). This result relies on the previous studies in this direction,

namely, [7], [24, Theorem 3.1], and [19]. It is used to answer the questions in [7, Remark 10,p. 3416] which were left open in [7] and [24].

2) A generalization of the quotient over the cycle space description of $\mathrm{TC}(G)$ for an unweighted finite graph G (see [25, Proposition 10.10]) to the case of an arbitrary finite metric space. It has to be mentioned that somewhat similar descriptions for $\mathrm{TC}(\mathbb{R}^n)$ were obtained in [8] and [15].

2 Isometric copies of ℓ_1 in $\mathrm{TC}(M)$ with infinite M

In what follows, the standard terminology of matching theory [21] is used. We consider a metric space M as an infinite weighted complete graph, where the weight of each edge is defined as the distance between its ends. Let V be a subset of M of cardinality $2n$, $n \in \mathbb{N}$. If edges $\{x_iy_i\}_{i=1}^n$ with $x_i, y_i \in V$, $x_i \neq y_i$, do not have common ends, we call $\{x_iy_i\}_{i=1}^n$ a *perfect matching* of the subgraph of M spanned by V. We define the *weight* of the perfect matching $\{x_iy_i\}_{i=1}^n$ as $\sum_{i=1}^n d(x_i, y_i)$.

Theorem 2.1. *The space $\mathrm{TC}(M)$ contains ℓ_1 isometrically if and only if there exists a sequence of pairs $\{x_i, y_i\}_{i=1}^\infty$ in M, with all elements distinct, such that each set $\{x_iy_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^n$.*

Proof. **Sufficiency.** Let $\{x_i, y_i\}_{i=1}^\infty$ be such a sequence. Set $f_i = (\mathbf{1}_{x_i} - \mathbf{1}_{y_i})/d(x_i, y_i)$. Since each finite set $\{f_i\}_{i=1}^n$, $n \in \mathbb{N}$ is isometrically equivalent to the unit vector basis of ℓ_1^n by the argument of [19], we are done.

Necessity. Recall that a metric space M is called *uniformly discrete* if there exists a constant $\delta > 0$ such that

$$\forall u, v \in M \ (u \neq v) \Rightarrow (d(u, v) \geq \delta).$$

To prove the necessity, we shall consider the three cases:

(A) The space M has an accumulation point, which means that there is a sequence $\{u_i\}_{i=1}^\infty$ of distinct elements in M and $u \in M$, such that $\lim_{i\to\infty} d(u_i, u) = 0$.

(B) The space M is not uniformly discrete, but does not have an accumulation point.

(C) The space M is uniformly discrete.

The proof of the necessity will be performed according to the following steps:

- First, we derive that in Cases (A) and (B), the space M contains a sequence of pairs $\{x_i, y_i\}_{i=1}^\infty$ such that for each $n \in \mathbb{N}$ the set $\{x_iy_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^n$.

- Further, it will be shown that in Case (C) either M contains a sequence of pairs $\{x_i, y_i\}_{i=1}^{\infty}$ such that for each $n \in \mathbb{N}$ the set $\{x_i y_i\}_{i=1}^{n}$ of edges is a minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^{n}$, or $\mathrm{TC}(M)$ does not contain an isometric copy of ℓ_1.

Case (A). If M has an accumulation point u, and $\{u_i\}_{i=1}^{\infty}$ is as in (A), then either there are infinitely many pairwise disjoint pairs (i, j), $i, j \in \mathbb{N}$, $j > i$, such that all of the triangle inequalities below are strict:

$$d(u_i, u_j) < d(u, u_i) + d(u, u_j), \tag{1}$$

or, after eliminating finitely many elements of the sequence, for all of the remaining ones, the equality is reached in the triangle inequalities:

$$d(u_i, u_j) = d(u, u_i) + d(u, u_j). \tag{2}$$

To finish the proof, it suffices to select two disjoint subsequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of the sequence $\{u_i\}_{i=1}^{\infty}$ in such a way that, for each $n \in \mathbb{N}$, the set of edges $\{x_i y_i\}_{i=1}^{n}$ is the minimum weight perfect matching in the complete graph with vertices $\{x_i, y_i\}_{i=1}^{n}$.

First, this will be done in the easier case where all triangles inequalities are equalities, see (2). In this case, one may let $x_i = u_{2i-1}$ and $y_i = u_{2i}$ and check that, by virtue of equalities (2), any perfect matching in the weighted graph with the vertices $\{u_i\}_{i=1}^{2n}$ is a minimum weight matching since all of them have weight $\sum_{i=1}^{2n} d(u_i, u)$.

In the case where (1) is satisfied for infinitely many disjoint pairs $\{(i_t, j_t)\}_{t=1}^{\infty}$, we combine the fact that the differences $d(u, u_{i_t}) + d(u, u_{j_t}) - d(u_{i_t}, u_{j_t})$ are strictly positive and $\lim_{i\to\infty} d(u_i, u) = 0$, and get that one can pass to a subsequence (preserving the notation $\{(i_t, j_t)\}_{t=1}^{\infty}$ for the subsequence) in such a way that, for each t and for any k_1 and k_2 which are i_s or j_s for some $s > t$, the next inequality holds:

$$d(u_{i_t}, u_{k_1}) + d(u_{j_t}, u_{k_2}) > \sum_{m=t}^{\infty} d(u_{i_m}, u_{j_m}). \tag{3}$$

Now, let

$$x_1 = u_{i_1}, y_1 = u_{j_1}, \dots, x_n = u_{i_n}, y_n = u_{j_n}, \dots.$$

To complete the proof, it remains to check that $\{x_i y_i\}_{i=1}^{n}$ is the minimum weight perfect matching in the complete graph spanned by $\{x_i, y_i\}_{i=1}^{n}$. It will be proved inductively that $x_i y_i$, $i = 1, 2, \dots$, should be in the minimum weight perfect matching. Assume that there is a minimum weight perfect matching which does not contain $x_1 y_1$. Then we have to match $x_1 = u_{i_1}$ with some u_{k_1} and $y_1 = u_{j_1}$ with some u_{k_2}, where k_1 and k_2 are i_s or j_s for some $s \in \{2, \dots, n\}$. But then (3) implies that the sum $d(x_1, u_{k_1}) + d(y_1, u_{k_2})$ is strictly larger than the weight of the matching $\{x_i y_i\}_{i=1}^{n}$. Thus, $x_1 y_1$ should be in any minimum weight perfect matching.

It is clear that the same argument can be repeated for x_2y_2, and so on. This completes the proof in the case where (1) is satisfied for infinitely many disjoint pairs, and, thus, for the case where M has an accumulation point.

Remark 2.2. Our argument in Case (A) is close to the one in [7, Theorem 5]. For the convenience of the reader, we presented our argument in the form independent of [7].

Case (B). Since the space M is not uniformly discrete, there are sequences $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$ in M such that $u_i \neq v_i$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} d(u_i, v_i) = 0$. The following standard lemma will be applied.

Lemma 2.3. *Each sequence $\{w_i\}_{i=1}^\infty$ in a metric space either contains a Cauchy subsequence or a δ-separated subsequence, where δ is some positive number and δ-*separated *means that any two elements are at distance at least δ.*

We apply Lemma 2.3 to the sequence $\{u_i\}_{i=1}^\infty$ and keep the notation $\{u_i\}_{i=1}^\infty$ for the obtained subsequence, and the notation $\{v_i\}_{i=1}^\infty$ for the corresponding subsequence of $\{v_i\}_{i=1}^\infty$.

If $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence, we consider the completion $\widetilde{M}$ of M and a point $\widetilde{u} \in \widetilde{M}$ such that $\lim_{n\to\infty} d(u_i, \widetilde{u}) = 0$, and construct, as in Case (A), sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$. Since these sequences are subsequences of $\{u_i\}_{i=1}^\infty$, the constructed subspace (see the proof of **Sufficiency**) is not only in $\mathrm{TC}(\widetilde{M})$, but also in $\mathrm{TC}(M)$.

Now, assume that $\{u_i\}_{i=1}^\infty$ is δ-separated for some $\delta > 0$. In this case, after omitting finite number of terms in the sequences $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$, we may assume that $d(u_i, v_i) < \delta/4$ for every i. Denote the obtained subsequences of $\{u_i\}_{i=1}^\infty$ and $\{v_i\}_{i=1}^\infty$ by $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$, respectively.

We have $d(x_i, y_i) < \delta/4$ and also, due to the triangle inequality and δ-separation of $\{x_i\}_{i=1}^\infty$, the inequalities $d(x_i, y_j) > 3\delta/4$ and $d(y_i, y_j) > \delta/2$ for $i \neq j$ hold. These inequalities immediately imply that, for every $n \in \mathbb{N}$ the set $\{x_iy_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^n$. This completes our proof in Case (B).

Case (C). We show that if we assume both

(i) That $\mathrm{TC}(M)$ contains a sequence $\{f_i\}_{i=1}^\infty$ which is isometrically equivalent to the unit vector basis of ℓ_1,

and

(ii) That M does not contain a sequence of pairs $\{x_i, y_i\}_{i=1}^\infty$, such that each set $\{x_iy_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^n$,

we get a contradiction.

We start with a simple case where all $\{f_i\}$ are in $\mathrm{TP}(M)$. Since our result is isometric, this will not complete the proof for the general case. However, in the easier case $\{f_i\} \subset \mathrm{TP}(M)$, the main ideas are more transparent.

For each element of the sequence f_i, we pick an optimal transportation plan (it does not have to be unique):

$$f_i = \sum_{j=1}^{m(i)} a_{j,i}(\mathbf{1}_{x_{j,i}} - \mathbf{1}_{y_{j,i}}),\ a_{j.i} > 0. \tag{4}$$

We say that $T_i := \{x_{j,i}, y_{j,i}\}_{j=1}^{m(i)}$ is the set of *transportation pairs* for f_i.

Lemma 2.4. *Each sequence $\{f_i\}_{i=1}^\infty \subset \mathrm{TP}(M)$ contains a subsequence $\{f_{i_n}\}_{n=1}^\infty$ satisfying at least one of the two conditions:*

(1) *In each T_{i_n}, there exists a transportation pair such that the obtained set of transportation pairs is pairwise disjoint.*

(2) *One can pick in each T_{i_n} a transportation pair such that, for the obtained set of transportation pairs, there is an element $x \in M$ contained in each of them.*

Proof. This lemma can be proved by considering an alternative: either there is an element $x \in M$ contained in infinitely many transportation pairs or there is no such element. □

We apply Lemma 2.4 to the sequence $\{f_i\}_{i=1}^\infty$ equivalent to the unit vector basis of ℓ_1. Assume that condition (1) is satisfied for one of its subsequences, which we still denote $\{f_i\}_{i=1}^\infty$. Without loss of generality it may be assumed that the disjoint pairs are $(x_{1,i}, y_{1,i})$, denoted by (x_i, y_i), for short. By our assumption (ii), there is $n \in \mathbb{N}$ such that $\{x_i y_i\}_{i=1}^n$ is not a minimum weight perfect matching of the graph spanned by $\{x_i, y_i\}_{i=1}^n$. Pick a minimum weight perfect matching for this graph. Interchanging the labels in some pairs (x_i, y_i) and changing the signs of the corresponding f_i if needed, one may assume that the minimum weight perfect matching is of the form $\{x_i y_{\pi(i)}\}_{i=1}^n$ for some bijection $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$.

Let $a_i > 0$ be the quantity transported from x_i to y_i in the plans (4) (written in a different notation). Set $a = \min_{1 \le i \le n} a_i > 0$.

Now, consider the vector $\sum_{i=1}^n f_i$ and construct the following transportation plan for it: the plan is close to being the sum of plans (4), but

$$\sum_{i=1}^n a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i}) \tag{5}$$

in it is replaced by

$$\sum_{i=1}^{n}(a_i - a)(\mathbf{1}_{x_i} - \mathbf{1}_{y_i}) + \sum_{i=1}^{n} a(\mathbf{1}_{x_i} - \mathbf{1}_{y_{\pi(i)}}). \tag{6}$$

Since $\{x_i y_{\pi(i)}\}_{i=1}^n$ is a minimum weight perfect matching while $\{x_i y_i\}_{i=1}^n$ is not, the obtained transportation plan has a strictly smaller cost than the sum of the plans (4). This leads to

$$\left\|\sum_{i=1}^{n} f_i\right\|_{\mathrm{TC}} < \sum_{i=1}^{n} \|f_i\|_{\mathrm{TC}},$$

which is a contradiction with the hypothesis (i) that $\{f_i\}$ is isometrically equivalent to the unit vector basis of ℓ_1.

Next, suppose that condition (2) of Lemma 2.4 is satisfied for a subsequence of $\{f_i\}_{i=1}^{\infty}$, which we still denote $\{f_i\}_{i=1}^{\infty}$. Relabelling and changing the signs of f_i if needed, it may be assumed that $\{x_{1,i}, y_{1,i}\}_{i=1}^{\infty}$ are such that all $x_{1,i}$ are the same, let us denote all of them by x. We claim that if $\{f_i\}_{i=1}^{\infty}$ are isometrically equivalent to the unit vector basis of ℓ_1, then

$$\forall i, j \in \mathbb{N} \quad d(x, y_{1,i}) + d(x, y_{1,j}) = d(y_{1,i}, y_{1,j}). \tag{7}$$

Assume the contrary, that is,

$$\exists i, j \in \mathbb{N} \quad d(x, y_{1,i}) + d(x, y_{1,j}) > d(y_{1,i}, y_{1,j}). \tag{8}$$

Let $a = \min\{a_{1,i}, a_{1,j}\}$. Consider the function $f_i - f_j$, subtract the corresponding plans (4), and make the following modification in the resulting transportation plan. We replace the difference

$$a_{1,i}(\mathbf{1}_x - \mathbf{1}_{y_{1,i}}) - a_{1,j}(\mathbf{1}_x - \mathbf{1}_{y_{1,j}})$$

by

$$(a_{1,i} - a)(\mathbf{1}_x - \mathbf{1}_{y_{1,i}}) - (a_{1,j} - a)(\mathbf{1}_x - \mathbf{1}_{y_{1,j}}) + a(\mathbf{1}_{y_{1,j}} - \mathbf{1}_{y_{1,i}}).$$

The strict inequality (8) implies that it is a strictly better plan. Thus, $\|f_i - f_j\|_{\mathrm{TC}} < \|f_i\|_{\mathrm{TC}} + \|f_j\|_{\mathrm{TC}}$, and this contradiction proves (7).

Now, we introduce a new sequence $\{\widetilde{x}_i, \widetilde{y}_i\}_{i=1}^{\infty}$ by setting $\widetilde{x}_i = y_{1,2i-1}$ and $\widetilde{y}_i = y_{1,2i}$. As in the case (A)(2), it is easy to see that (7) implies that the sequence of pairs $\{\widetilde{x}_i, \widetilde{y}_i\}_{i=1}^{\infty}$ satisfies the condition in (ii), and we get a contradiction. This completes the proof in the case $\{f_i\} \subset \mathrm{TP}(M)$.

In the rest of this proof, our aim is to generalize the argument we just presented to the case where elements $\{f_i\}_{i=1}^{\infty}$ are not in $\mathrm{TP}(M)$ but in its completion $\mathrm{TC}(M)$.

By the standard description of the completion (see [11, Section 3.11.4]), each element $f \in \mathrm{TC}(M)$ can be presented as a series of the form

$$f = \sum_{k=1}^{\infty} \left(\sum_{i=s_k+1}^{s_{k+1}} a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i}) \right) \tag{9}$$

for some $0 = s_1 < s_2 < \cdots < s_k < \ldots$, $\{x_i\} \subset M$, $\{y_i\} \subset M$, and $\{a_i\} \subset \mathbb{R}^+$ with

$$\sum_{k=1}^{\infty} \left\| \sum_{i=s_k+1}^{s_{k+1}} a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i}) \right\|_{\mathrm{TC}} < \infty. \tag{10}$$

Furthermore, the norm $\|f\|_{\mathrm{TC}}$ is equal to the infimum of sums of the form (10) over all representations (9), in which we assume that the sums in brackets are optimal transportation plans for the corresponding elements of $\mathrm{TP}(M)$.

We are now considering Case (C) where the space M is uniformly discrete. Let

$$\delta = \inf_{x \neq y \in M} d(x, y). \tag{11}$$

For $f \in \mathrm{TP}(M)$, we introduce $\|f\|_1 = \sum_{v \in \mathrm{supp} f} |f(v)|$ and extend this norm to functions on M with countable supports and absolutely summable collections of values.

For $f \in \mathrm{TP}(M)$ the amount of the product which is to be delivered is $\|f\|_1/2$ and the vector f is in the kernel of the linear functional on finitely supported vectors defined as the sum of the values. By (11), we get that $\delta\|f\|_1/2 \leq \|f\|_{\mathrm{TC}}$ for any $f \in \mathrm{TP}(M)$. For this reason, the space $\mathrm{TC}(M)$ is continuously embedded in $\ell_1(M)$, and is in the kernel of the functional defined as the sum of all coordinates (this functional is naturally defined for $f \in \ell_1(M)$).

Lemma 2.5. *Let $h \in \mathrm{TC}(M)$ and $g \in \mathrm{TP}(M)$ be such that the diameter of the support of g be $\leq D$. Then $\|g + h\|_1 \geq \frac{2}{D}(\|g\|_{\mathrm{TC}} - \|h\|_{\mathrm{TC}})$.*

Note that the conclusion of Lemma 2.5 is nontrivial only if $\|h\|_{\mathrm{TC}} < \|g\|_{\mathrm{TC}}$.

Proof of Lemma 2.5. If $h \notin \mathrm{TP}(M)$, we can approximate h arbitrarily well - both in $\|\cdot\|_{\mathrm{TC}}$ and $\|\cdot\|_1$ - by vectors belonging to $\mathrm{TP}(M)$. For this reason, we may assume that $h \in \mathrm{TP}(M)$.

Let us write an optimal transportation plan for h:

$$h = a_1(\mathbf{1}_{u_1} - \mathbf{1}_{v_1}) + a_2(\mathbf{1}_{u_2} - \mathbf{1}_{v_2}) + \cdots + a_n(\mathbf{1}_{u_n} - \mathbf{1}_{v_n}), \quad a_i > 0. \tag{12}$$

By combining the corresponding terms, we may and shall assume that none of the v_i is equal to any of the u_j. We split the transportation plan in (12) into two sums: (a) The sum h_1 which contains the terms $a_i(\mathbf{1}_{u_i} - \mathbf{1}_{v_i})$ with at most one of the elements u_i, v_i being in the support of g; (b) The sum h_2 which contains those terms $a_i(\mathbf{1}_{u_i} - \mathbf{1}_{v_i})$ for which both of the elements u_i, v_i are in the support of g.

The important and easy-to-see observation is that $\|g + h_1 + h_2\|_1 \geq \|g + h_2\|_1$. This is because each term of h_1 can decrease the value of $g + h_2$ at some point, but it adds the same amount elsewhere.

It remains to estimate $\|g + h_2\|_1$. Notice that

$$\|g + h_2\|_{\mathrm{TC}} \geq \|g\|_{\mathrm{TC}} - \|h_2\|_{\mathrm{TC}} \geq \|g\|_{\mathrm{TC}} - \|h\|_{\mathrm{TC}},$$

and that - due to the choice of h_2 - the diameter of the support of $g + h_2$ also does not exceed D. It remains to observe that $\|g + h_2\|_{\mathrm{TC}}$ does not exceed the amount of product which is to be moved - that is, $\|g + h_2\|_1/2$ - times the maximal distance which this product has to travel - that is, D. Therefore,

$$\|g + h_2\|_1 \geq \frac{2}{D} \|g + h_2\|_{\mathrm{TC}},$$

whence the conclusion follows. □

It has to be pointed out that Lemma 2.5 together with equation (9) along with the fact that $\|f\|_{\mathrm{TC}}$ is the infimum of sums (10) implies that $m_i := \|f_i\|_1 > 0$. Furthermore, it is easy to see that equation (9) and the fact that $\|f\|_{\mathrm{TC}}$ is the infimum of sums (10) imply that, for each $i, m \in \mathbb{N}$, one can write $f_i = S_i^m + R_i^m$ where $S_i^m \in \mathrm{TP}(M)$, $R_i^m \in \mathrm{TC}(M)$, $1 - 2^{-m} \leq \|S_i^m\|_{\mathrm{TC}} \leq 1 + 2^{-m}$, $\|R_i^m\|_{\mathrm{TC}} \leq 2^{-m+1}$, and $\|R_i^m\|_1 \leq m_i/8$. Using the last inequality, one arrives at $\|S_i^m\|_1 \geq 7m_i/8$.

Writing f_i^+ and f_i^- for the non-negative and non-positive parts of f_i, we have $\|f_i^+\|_1 = \|f_i^-\|_1 = m_i/2$, and therefore we can select in the support of f_i^+ a finite subset V_i^+ such that $\sum_{v \in V_i^+} f_i(v) \geq \frac{7m_i}{16}$ and in the support of f_i^- a finite subset V_i^- such that $\sum_{v \in V_i^-} f_i(v) \leq -\frac{7m_i}{16}$.

Let i be fixed for a moment. As is well known (see [26, Proposition 3.16]), for each m we can pick an optimal transportation plan

$$S_i^m = \sum_{j=1}^{N_m} a_j^m (\mathbf{1}_{x_j^m} - \mathbf{1}_{y_j^m}),\ a_j^m > 0,$$

such that $S_i^m(x_j^m) > 0$ and $S_i^m(y_j^m) < 0$ for every $j = 1, \ldots, N_m$. For each of $\{S_i^m\}_{m=1}^\infty$, create a matrix whose columns are labelled by elements of V_i^+ and whose rows are labelled by elements of V_i^-. In the intersection of the column corresponding to x and the row corresponding to y, we record the amount of product which is moved in such an optimal plan from x to y, while we put 0 if nothing is moved. We claim that the sum of all entries of the obtained matrix is at least $\frac{4m_i}{16} = \frac{m_i}{4}$, i.e. $\sum_{j \in J_m} a_j^m \geq \frac{m_i}{4}$ where $J_m = \{j \leq N_m :\ (x_j^m, y_j^m) \in V_i^+ \times V_i^-\}$.

To prove this, let us introduce the following functions on V_i^+ and V_i^-, respectively:

$$P_i^m(v) = \begin{cases} S_i^m(v) & \text{if } S_i^m(v) > 0 \text{ and } v \in V_i^+ \\ 0 & \text{for all other } v \in V_i^+. \end{cases}$$

$$N_i^m(v) = \begin{cases} S_i^m(v) & \text{if } S_i^m(v) < 0 \text{ and } v \in V_i^- \\ 0 & \text{for all other } v \in V_i^-. \end{cases}$$

It has to be shown that in any optimal transportation plan for S_i^m, a nontrivial part of product which is available at points of V_i^+, that is P_i^m, should be transported to satisfy the need at points of V_i^-, that is N_i^m. Evidently, each unit of product available in the transportation problem S_i^m at points where $P_i^m > 0$ should be moved to the points where $S_i^m(v) < 0$. These points can be the ones where $N_i^m(v) < 0$, but can be also some other points where $S_i^m(v) < 0$. To estimate the amount which should be moved to the points where $N_i^m(v) < 0$ we need some inequalities. Since $a^+ + b^+ \geq (a+b)^+$, we have

$$\|P_i^m\|_1 \geq \|f_i^+\|_1 - \frac{m_i}{16} - \|(R_i^m)^+\|_1 \geq \frac{6m_i}{16}.$$

To see that some of these $\|P_i^m\|_1$ units have to go to the points where $N_i^m < 0$, we need to prove that outside V_i^- the total amount of negative values of S_i^m is relatively small. In fact, since $S_i^m = f_i - R_i^m$ such values can occur either at the points v where $f_i(v) < 0$, but $v \notin V_i^-$, or because of subtraction of $(R_i^m)^+$. Therefore, the total amount of this need is $\leq \frac{2m_i}{16}$. Thus at least $\frac{4m_i}{16} = \frac{m_i}{4}$ of the product available at the points where $P_i^m(v) > 0$ should be transported to the points where $N_i^m(v) < 0$, as claimed.

Since the matrix described in the paragraph where we introduced V_i^+ and V_i^-, is finite and a sum of its entries is at least $\frac{m_i}{4}$, there exists a subsequence of $\{S_i^m\}_{m=1}^\infty$ such that, for some choice of $x_i \in V_i^+$ and $y_i \in V_i^-$, the amount of transported product (with respect to the picked above optimal transportation plan for S_i^m) from x_i to y_i will be at least $\varepsilon_i > 0$, which is a positive number depending only on i. Since subsequences $\{S_i^m\}_{m=1}^\infty$ and $\{R_i^m\}_{m=1}^\infty$ also satisfy the defining inequalities for $\{S_i^m\}_{m=1}^\infty$ and $\{R_i^m\}_{m=1}^\infty$, we may assume without loss of generality, that the subsequences are $\{S_i^m\}_{m=1}^\infty$ and $\{R_i^m\}_{m=1}^\infty$ themselves.

From here on, we follow the same line of argument as in the first part of the proof (for Case (C)). Using the same reasoning as in Lemma 2.4, one obtains, after taking subsequences in i the following alternatives: (1) the pairs $\{x_i, y_i\}$ are disjoint; (2) they all have a common element (changing signs of f_i we can assume that all of x_i are the same).

If alternative (1) holds, using the assumption (ii), we conclude that there exists a finite subcollection $\{x_i, y_i\}_{i=1}^n$ such that in the subgraph spanned by it, there is a perfect matching with a smaller weight. After that one can apply the same "improvement of a transportation plan" as we used when replacing (5) by (6). This improvement shows that there is $\tau_n > 0$ such that

$$\|S_1^m + \cdots + S_n^m\|_{\mathrm{TC}} \leq \|S_1^m\|_{\mathrm{TC}} + \cdots + \|S_n^m\|_{\mathrm{TC}} - \tau_n. \tag{13}$$

A crucial issue is that this holds for every $m \in \mathbb{N}$. More precisely, formula (6) shows that τ_n can be chosen to be the product of $\min_{1\leq i\leq n} \varepsilon_i$ and the difference between the weights of the matching $\{x_i y_i\}_{i=1}^n$ and the minimum weight perfect matching in the subgraph spanned by $\{x_i, y_i\}_{i=1}^n$.

Therefore, one obtains

$$\begin{aligned}\|f_1 + \cdots + f_n\|_{\mathrm{TC}} &\leq \|S_1^m + \cdots + S_n^m\|_{\mathrm{TC}} + \|R_1^m + \cdots + R_n^m\|_{\mathrm{TC}} \\ &\leq n \cdot (1 + 2^{-m}) - \tau_n + n2^{-m+1}.\end{aligned}$$

Since this inequality holds for every m and τ_n does not depend on m, we can pick m in such a way that the number in the rightmost side of the last inequality is $< n$. This gives a contradiction with the assumption that $\{f_i\}$ is isometrically equivalent to the unit vector basis of ℓ_1.

Finally, consider alternative (2): the pairs $\{x_i, y_i\}$ have a common point. As above, one may assume that this common point coincides with all of x_i and denote it by x. Similarly to the argument above, the goal is to prove the equalities in some of the triangle inequalities. Now the desired equalities are:

$$d(x, y_i) + d(x, y_j) = d(y_i, y_j) \quad \text{for } i \neq j. \tag{14}$$

The proof goes according to the same steps as above. If one of the triangle inequalities is strict, that is $d(x, y_i) + d(x, y_j) > d(x_i, x_j)$, we can find $\tau_{i,j} > 0$ such that $\|S_i^m - S_j^m\|_{\mathrm{TC}} \leq \|S_i^m\|_{\mathrm{TC}} + \|S_j^m\|_{\mathrm{TC}} - \tau_{i,j}$ for every m. From here, we derive $\|f_i - f_j\|_{\mathrm{TC}} < \|f_i\|_{\mathrm{TC}} + \|f_j\|_{\mathrm{TC}}$, contrary to the assumption that $\{f_i\}$ is isometrically equivalent to the unit vector basis of ℓ_1.

After establishing (14), we complete the proof as in the previous case. □

Example 2.6. *As an application of Theorem 2.1, we use it to answer the questions on isometric presence of ℓ_1 in $\mathrm{TC}(M)$ for metric spaces M listed in [7, Remark 10,p. 3416], for which the answer has not been known. In all of the examples $M = \{v_n\}_{n=1}^\infty$. The metrics on M are defined for $n > k$ as follows:*

(a) $\rho(v_k, v_n) = k + n - \frac{1}{k}$

(b) $\rho(v_k, v_n) = 2 - \frac{1}{k} + \frac{1}{n}$

(c) $\rho(v_k, v_n) = 2 - \frac{1}{k} - \frac{1}{2n}$

(d) $\rho(v_k, v_n) = 1 + \frac{1}{n}$

(e) $\rho(v_k, v_n) = 1 + \frac{1}{2k} + \frac{1}{n}$

Using Theorem 2.1 we can prove that in all of the examples the answer is negative - the corresponding transportation cost spaces do not contain isometric copies of ℓ_1. In each of the cases, we prove that, for any selected sequence $\{x_i, y_i\}_{i=1}^{\infty}$ of pairs of distinct elements in the metric space, one can find m such that the set $\{x_i y_i\}_{i=1}^{m}$ of edges is not a minimum weight perfect matching in the complete graph spanned by $\{x_i, y_i\}_{i=1}^{m}$ (with weight of each edge equal to the distance between its ends).

The main observation here is that no matter how the sequence $\{x_i, y_i\}_{i=1}^{\infty}$ is selected, it is possible to pick two pairs (x_j, y_j) and (x_m, y_m), $j < m$, such that the indices of vertices x_m and y_m in the sequence $\{v_n\}_{n=1}^{\infty}$ are larger than the indices of x_j and y_j. Without loss of generality we may assume that indices of x_j, y_j, x_m, y_m are $q_1 < q_2 < q_3 < q_4$, respectively.

This will immediately imply the desired conclusion of the previous paragraph as soon as it will be derived that $d(x_j, x_m) + d(y_j, y_m) < d(x_j, y_j) + d(x_m, y_m)$. Hence, it remains to verify this inequality in cases (a)-(e). Indeed, direct calculations lead to the following inequalities, which are obviously true:

(a) $q_1 + q_3 - \dfrac{1}{q_1} + q_2 + q_4 - \dfrac{1}{q_2} < q_1 + q_2 - \dfrac{1}{q_1} + q_3 + q_4 - \dfrac{1}{q_3}$ as $\dfrac{1}{q_3} < \dfrac{1}{q_2}$.

(b) $2 - \dfrac{1}{q_1} + \dfrac{1}{q_3} + 2 - \dfrac{1}{q_2} + \dfrac{1}{q_4} < 2 - \dfrac{1}{q_1} + \dfrac{1}{q_2} + 2 - \dfrac{1}{q_3} + \dfrac{1}{q_4}$ as $\dfrac{2}{q_3} < \dfrac{2}{q_2}$.

(c) $2 - \dfrac{1}{q_1} - \dfrac{1}{2q_3} + 2 - \dfrac{1}{q_2} - \dfrac{1}{2q_4} < 2 - \dfrac{1}{q_1} - \dfrac{1}{2q_2} + 2 - \dfrac{1}{q_3} - \dfrac{1}{2q_4}$ as $\dfrac{1}{2q_3} < \dfrac{1}{2q_2}$.

(d) $1 + \dfrac{1}{q_3} + 1 + \dfrac{1}{q_4} < 1 + \dfrac{1}{q_2} + 1 + \dfrac{1}{q_4}$ as $\dfrac{1}{q_3} < \dfrac{1}{q_2}$.

(e) $1 + \dfrac{1}{2q_1} + \dfrac{1}{q_3} + 1 + \dfrac{1}{2q_2} + \dfrac{1}{q_4} < 1 + \dfrac{1}{2q_1} + \dfrac{1}{q_2} + 1 + \dfrac{1}{2q_3} + \dfrac{1}{q_4}$ as $\dfrac{1}{2q_3} < \dfrac{1}{2q_2}$.

3 Canonical description of $\mathrm{TC}(M)$ as a quotient of ℓ_1 for finite metric space M

The goal of this section is a generalization for an arbitrary finite metric space of the known description [25, Proposition 10.10] of transportation cost spaces for finite unweighted graphs as quotients of finite-dimensional ℓ_1.

Let M be a finite metric space with n elements. It can be viewed as a weighted complete graph K_n, where the weight of the edge joining u and v is the distance $d(u, v)$. The weighted ℓ_1-space on the edge set $E(K_n)$ will be introduced as follows. Given $f : E(K_n) \to \mathbb{R}$, denote by f_{uv} the value of this function on the edge uv. The norm of f is defined as:

$$\|f\|_{1,d} := \sum_{uv \in E(K_n)} |f_{uv}| d(u, v).$$

The normed space obtained in this way will be denoted by $\ell_{1,d} = \ell_{1,d}(E(K_n))$. It can be readily seen that it is an $\frac{n(n-1)}{2}$-dimensional space isometric to $\ell_1^{n(n-1)/2}$.

Further, let us fix an orientation on the edges of K_n. Notice that only intermediate objects and results rather than final outcomes will depend on it. For this reason, it is customary to say that we select a *reference orientation*. Consider a cycle C in K_n and pick one of the two possible orientations of C satisfying the following condition: each vertex of C is a head of exactly one edge and a tail of exactly one edge. Having done so, we introduce the *signed indicator function* $\chi_C \in \ell_{1,d}$ of the cycle C by

$$\chi_C(e) = \begin{cases} 1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ are the same} \\ -1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ are different} \\ 0 & \text{if } e \notin C, \end{cases} \tag{15}$$

where e is used to denote edges in K_n.

The span of this set of functions in $\ell_{1,d}$ is denoted by Z and called the *cycle space* (or the *flow space* in some sources). The following assertion holds.

Theorem 3.1. $\mathrm{TC}(M) = \ell_{1,d}/Z$.

Proof. Since the spaces $\ell_{1,d}/Z$ and $\mathrm{TC}(M)$ are finite-dimensional, it suffices to show that the dual space $(\ell_{1,d}/Z)^*$ can be in a natural way identified with the space $\mathrm{Lip}_0(M)$, which is known to coincide with $(\mathrm{TC}(M))^*$, see [25, Theorem 10.2].

To begin with, let us introduce the spaces $\ell_{\infty,d}$ and $\ell_{2,d}$ as spaces of real-valued functions on $E(K_n)$ with the norms

$$\|f\|_{\infty,d} = \max_{uv\in E(K_n)} \frac{|f_{uv}|}{d(u,v)}$$

and

$$\|f\|_{2,d} = \left(\sum_{uv\in E(K_n)} |f_{uv}|^2 \right)^{\frac{1}{2}}, \tag{16}$$

respectively.

It is clear that $\ell_{2,d}$ is an inner product space, in which the notion of orthogonality is naturally defined. We denote the inner product inducing the norm (16) by $\langle\cdot,\cdot\rangle$.

The subspace of $\ell_{2,d}$ orthogonal to the cycle space Z is denoted by B and is called the *cut space* or *cut subspace*. Observe that by virtue of (16), B does not really depend on the distance d, but only on the size of M. One has a direct, orthogonal in $\ell_{2,d}$, decomposition

$$\ell_{2,d} = Z \oplus B. \tag{17}$$

Next, we apply the standard duality result, which, generally speaking, states that that the dual of the quotient space $\mathcal{X}/\mathcal{Y}$ is isometric to the subspace $\mathcal{Y}^\perp := \{f \in \mathcal{X}^* : \forall y \in \mathcal{Y},\ f(y) = 0\}$. Observing that our choice of norms on $\ell_{1,d}$ and $\ell_{\infty,d}$ is such that $\ell_{\infty,d} = (\ell_{1,d})^*$ with the pairing given by $g(f) = \langle g, f\rangle$, one concludes that the dual space of the quotient space $\ell_{1,d}/Z$ is naturally isometric to the space B_∞, where B_∞ stands for the space B endowed with its $\ell_{\infty,d}$-norm.

To complete our argument it is convenient to use another description of B. Denote by $\ell_2(M)$ the space $\mathbb{R}^M$ with its Euclidean norm. Let D be defined as a matrix whose rows are labelled using elements of M, whose columns are labelled using (oriented) edges of K_n and the ve-entry is given by

$$d_{ve} = \begin{cases} 1, & \text{if } v \text{ is the head of } e, \\ -1, & \text{if } v \text{ is the tail of } e, \\ 0, & \text{if } v \text{ is not incident to } e. \end{cases}$$

The description of B which we are going to use is that that B is the image of $\ell_2(M)$ under the action of D^T with D^T being the transpose of the matrix D. See [25, p. 315] noting that the result described there holds, in particular, for the cut space of the complete graph.

Therefore, each $b \in B$ can be represented as $b = D^T f$, implying that $b(uv) = h(u) - h(v)$ for some $h : M \to \mathbb{R}$ and all oriented edges uv, where u is the head and v is the tail. It is clear that addition of a constant to the function h does not change $D^T h$, so one may assume $h(O) = 0$, that is, $h \in \mathrm{Lip}_0(M)$. Clearly, the Lipschitz constant of h is equal to

$$\mathrm{Lip}(h) = \max_{uv \in E(K_n)} \frac{|h(u) - h(v)|}{d(u,v)} = \|b\|_{\infty,d}.$$

Thus, we have established a natural isometry between B_∞ and $\mathrm{Lip}_0(M)$. □

Acknowledgement

The second author gratefully acknowledges the support by the National Science Foundation grant NSF DMS-1700176 and by St. John's University. We would like to thank the referee for the careful reading of the paper and numerous corrections.

References

[1] R. J. Aliaga, C. Petitjean, A. Procházka, Embeddings of Lipschitz-free spaces into ℓ_1, `arXiv:1909.05285`.

[2] R. F. Arens, J. Eells, Jr., On embedding uniform and topological spaces, *Pacific J. Math.*, **6** (1956), 397–403.

[3] J. Bourgain, The metrical interpretation of superreflexivity in Banach spaces, *Israel J. Math.*, **56** (1986), no. 2, 222–230.

[4] M. S. Charikar, Similarity estimation techniques from rounding algorithms. *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, 380–388, ACM, New York, 2002.

[5] M. Cúth, M. Doucha, Lipschitz-free spaces over ultrametric spaces. *Mediterr. J. Math.* **13** (2016), no. 4, 1893–1906.

[6] M. Cúth, M. Doucha, P. Wojtaszczyk, On the structure of Lipschitz-free spaces. *Proc. Amer. Math. Soc.* **144** (2016), no. 9, 3833–3846.

[7] M. Cúth, M. Johanis, Isometric embedding of ℓ_1 into Lipschitz-free spaces and ℓ_∞ into their duals. *Proc. Amer. Math. Soc.* **145** (2017), no. 8, 3409–3421.

[8] M. Cúth, O. F. K. Kalenda, P. Kaplický, Isometric representation of Lipschitz-free spaces over convex domains in finite-dimensional spaces. *Mathematika* **63** (2017), no. 2, 538–552.

[9] A. Dalet, P.L. Kaufmann, A. Procházka, Characterization of metric spaces whose free space is isometric to ℓ_1. *Bull. Belg. Math. Soc. Simon Stevin* **23** (2016), no. 3, 391–400.

[10] S. J. Dilworth, D. Kutzarova, M. I. Ostrovskii, Lipschitz-free spaces on finite metric spaces, *Canad. J. Math.*, 72 (2020), 774–804, `DOI:10.4153/S0008414X19000087`.

[11] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[12] M. Fréchet, Les dimensions d'un ensemble abstrait, *Math. Ann.*, **68** (1910), no. 2, 145–168.

[13] A. Godard, Tree metrics and their Lipschitz-free spaces, *Proc. Amer. Math. Soc.*, **138** (2010), no. 12, 4311–4320.

[14] G. Godefroy, N. J. Kalton, Lipschitz-free Banach spaces, *Studia Math.*, **159** (2003), no. 1, 121–141.

[15] G. Godefroy, N. Lerner, Some natural subspaces and quotient spaces of L_1. *Adv. Oper. Theory* **3** (2018), no. 1, 61–74.

[16] P. Indyk, N. Thaper, Fast image retrieval via embeddings, in: *ICCV 03: Proceedings of the 3rd International Workshop on Statistical and Computational Theories of Vision*, 2003.

[17] N. J. Kalton, Spaces of Lipschitz and Hölder functions and their applications, *Collect. Math.*, **55** (2004), 171–217.

[18] L. V. Kantorovich, M. K. Gavurin, Application of mathematical methods in the analysis of cargo flows (Russian), in: *Problems of improving of transport efficiency*, USSR Academy of Sciences Publishers, Moscow, 1949, pp. 110–138.

[19] S. S. Khan, M. Mim, M. I. Ostrovskii, Isometric copies of ℓ_∞^n and ℓ_1^n in transportation cost spaces on finite metric spaces, in: *The Mathematical Legacy of Victor Lomonosov. Operator Theory*, pp. 189–203, De Gruyter, 2020; `DOI:10.1515/9783110656756-014`.

[20] S. Khot, A. Naor, Nonembeddability theorems via Fourier analysis, *Math. Ann.*, **334** (2006), 821–852.

[21] L. Lovász, M. D. Plummer, *Matching theory.* Corrected reprint of the 1986 original. AMS Chelsea Publishing, Providence, RI, 2009.

[22] A. Naor, Metric dimension reduction: a snapshot of the Ribe program, *Proc. Int. Cong. of Math. - 2018*, Rio de Janeiro, Vol. **1**, 759–838.

[23] A. Naor, G. Schechtman, Planar Earthmover is not in L_1, *SIAM J. Computing*, **37** (2007), 804–826.

[24] S. Ostrovska, M. I. Ostrovskii, Generalized transportation cost spaces, *Mediterr. J. Math.*, **16** (2019), no. 6, Paper No. 157.

[25] M. I. Ostrovskii, *Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces*, de Gruyter Studies in Mathematics, **49**. Walter de Gruyter & Co., Berlin, 2013.

[26] N. Weaver, *Lipschitz algebras*, Second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

Department of Mathematics, Atilim University, 06830 Incek, Ankara, TURKEY

E-mail address: `sofia.ostrovska@atilim.edu.tr`

Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, NY 11439, USA

E-mail address: `ostrovsm@stjohns.edu`