ISOMORPHIC SPECTRUM AND ISOMORPHIC LENGTH OF A BANACH SPACE

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Abstract. We prove that, given any ordinal $\delta < \omega_2$, there exist a transfinite δ-sequence of separable Banach spaces $(X_\alpha)_{\alpha<\delta}$ such that X_α embeds isomorphically into X_β and contains no subspace isomorphic to X_β for all $\alpha < \beta < \delta$. All these spaces are subspaces of the Banach space $E_p = \left(\bigoplus_{n=1}^\infty \ell_p\right)_2$, where $1 \le p < 2$. Moreover, assuming Martin's axiom, we prove the same for all ordinals δ of continuum cardinality.

1. Introduction

We use the standard terminology of Banach spaces theory, see [1]. Let X and Y be Banach spaces. We write $X \hookrightarrow Y$ if X embeds isomorphically into Y, and $X \simeq Y$ if are isomorphic.

1.1. Isomorphic spectrum.

By the *isomorphic spectrum* of an infinite dimensional Banach space X we mean the set $\mathrm{sp}\,(X)$ of all isomorphic types of infinite dimensional subspaces of X.

Consider the following equivalence relation on the set $\mathcal{B}$ of separable infinite dimensional Banach spaces. We say that Banach spaces $X, Y \in \mathcal{B}$ are *equispectral* and write $X \overset{\mathrm{sp}}{\sim} Y$ provided that $X \hookrightarrow Y$ and $X \hookrightarrow Y$ (notice that Banach [2, p. 193] used a different terminology for equispectral Banach spaces X and Y, he said that X and Y have *equal linear dimension* and used the notation $\dim_l X = \dim_l Y$). It is immediate that $X \overset{\mathrm{sp}}{\sim} Y$ if and only if $\mathrm{sp}\,(X) = \mathrm{sp}\,(Y)$. It is a well known fact that $X \overset{\mathrm{sp}}{\sim} Y$ does not imply that $X \simeq Y$, however $X \simeq Y$ easily implies that $X \overset{\mathrm{sp}}{\sim} Y$. For instance, $L_1 \oplus \ell_2 \overset{\mathrm{sp}}{\sim} L_1$, however $L_1 \oplus \ell_2 \not\simeq L_1$.

Observe that if $X \in \{c_0, \ell_p : 1 \le p < \infty\}$ and Y is any infinite dimensional subspace of X then $X \overset{\mathrm{sp}}{\sim} Y$.

Denote by $\widetilde{\mathcal{B}}$ the set of all equivalence classes in $\mathcal{B}$ modulo the relation $\overset{\mathrm{sp}}{\sim}$, and for every $X \in \mathcal{B}$ by $\widetilde{X}$ we denote the equivalence class containing X.

Given Banach spaces X and Y, we write $X \prec Y$ to express that $X \hookrightarrow Y$, while $Y \not\hookrightarrow X$. It is easy to see that, for every $X_i, Y_i \in \mathcal{B}$, $i = 1, 2$ with $X_1 \overset{\mathrm{sp}}{\sim} X_2$ and $Y_1 \overset{\mathrm{sp}}{\sim} Y_2$ the relation $X_1 \prec Y_1$ is equivalent to $X_2 \prec Y_2$. So, the same relation $\prec$ is well defined on $\widetilde{\mathcal{B}}$ by setting $\mathcal{X} \prec \mathcal{Y}$ provided $X \prec Y$ for some (or, equivalently, any) representatives $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Observe that $\prec$ is a strict partial relation on $\widetilde{\mathcal{B}}$, and that $X \prec Y$ is equivalent to the strict inclusion $\mathrm{sp}\,(X) \subset \mathrm{sp}\,(Y)$.

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By the solution of the homogeneous Banach space problem obtained by a combination of results of Gowers [5, 6] and Komorowski–Tomczak-Jaegermann [9, 10], ℓ_2 is the unique element X of $\mathcal{B}$ with $\mathrm{sp}\,(X) = \{X\}$. Although the spaces c_0 and ℓ_p with $1 \le p < \infty$, $p \ne 2$ have more than one-element isomorphic spectrum, all of them are equispectral, as mentioned above. So, $\widetilde{c_0}$ and $\widetilde{\ell_p}$ with $1 \le p < \infty$ are minimal elements of $\widetilde{\mathcal{B}}$. On the other hand, it is easy to see that $\widetilde{C[0,1]}$ is the unique maximal element of $\widetilde{\mathcal{B}}$, which is, moreover, the greatest element of $\widetilde{\mathcal{B}}$.

1.2. **Set-theoretical preliminaries.** We use the standard set-theoretical terminology and notation of [7], where the reader can also find necessary background. By $\mathfrak{c}$ we denote the cardinality of continuum. We say that A *meets* B provided that $A \cap B \ne \emptyset$.

Let $(M, <)$ be a partially ordered set. Following [11], the *length* of M is defined to be the supremum of ordinals α which are isomorphic to a subset of M, and is denoted by $L(M)$. For instance, $L(\alpha) = \alpha$ for every ordinal α and $L(\mathbb{R}) = \omega_1$.

Let ω_α be any infinite cardinal. We endow the power-set $\mathcal{P}(\omega_\alpha)$ with the partial order $A < B$ if and only if $|A \setminus B| < \aleph_\alpha = |B \setminus A|$.

Let us recall the statement of Martin's axiom (MA). A subset D of a partially ordered set P is said to be *dense* if for every $p \in P$ there is $d \in D$ such that $d \le p$. A subset $Q \subseteq P$ is said to be *consistent* provided for every finite subset $F \subseteq Q$ there exists $p \in P$ such that $p \le f$ for every $f \in F$. Elements p, q of P are said to be *consistent* if the two-element subset $\{p, q\}$ is consistent. A subset $Q \subseteq P$ consisting of more than two elements is said to be *pairwise inconsistent* if every two distinct elements of Q are not consistent. P is said to have the *countable chain condition* (CCC in short) if every pairwise inconsistent subset of P is at most countable.

Martin's axiom. *Let P be a partially ordered set possessing the CCC. Let $\mathfrak{M}$ be a collection of dense subsets of P of cardinality $< \mathfrak{c}$. Then there exists a consistent subset $Q \subseteq P$ which meets every element of $\mathfrak{M}$.*

We remark that MA is independent of the usual axioms ZFC. It follows from the Continuum Hypothesis (CH) and sometimes allows to extend results, previously established under the assumption of CH.

We need the following combinatorial lemma proved in [11].

Lemma 1.1. (1) *For every regular cardinal ω_δ one has $L(\mathcal{P}(\omega_\delta)) \ge \omega_{\delta+2}$.*
(2) *Let ω_c be the cardinal of cardinality $\mathfrak{c}$. Then* (MA) $L(\mathcal{P}(\omega_0)) = \omega_{c+1}$.

Here (MA) in item (2) means that the proof of (2) uses Martin's axiom.

1.3. **Isomorphic length of a Banach space.** Let X be a separable infinite dimensional Banach space. By the *isomorphic length* of X we mean the length of the subset $\widetilde{\mathcal{B}}_X$ of the partially ordered set $\widetilde{\mathcal{B}}$ consisting of all equivalence classes containing all infinite dimensional subspaces of X: $IL(X) = L(\widetilde{\mathcal{B}}_X)$. Since by the above $\widetilde{\mathcal{B}}_{\ell_p}$ and $\widetilde{\mathcal{B}}_{c_0}$ are singletons, we have that $IL(\ell_p) = IL(c_0) = 1$ for every $p \in [1, +\infty)$. In the next section, we show that for $E_p = \left(\bigoplus_{n=1}^\infty \ell_p\right)_2$ with $1 \le p < 2$ one has $IL(E_p) \ge \omega_2$, and Martin's axiom implies that $IL(E_p) = \omega_{c+1}$. Of course, the same could be said about the universal Banach space $C[0,1]$, which has the maximal possible length.

2. Transfinite $\prec$-increasing sequences of spaces

Theorem 2.1. *Let $1 \le p < 2$ and $E_p = \left(\bigoplus_{n=1}^{\infty} \ell_p\right)_2$. Then*

(1) *for every ordinal γ of cardinality $\aleph_1$ there is a transfinite sequence $(X_\alpha)_{\alpha<\gamma}$ of subspaces of E_p such that $X_\alpha \prec X_\beta$ for all $\alpha < \beta < \gamma$.*
(2) (MA) *for every ordinal γ of cardinality $\mathfrak{c}$ there is a transfinite sequence $(X_\alpha)_{\alpha<\gamma}$ of subspaces of E_p such that $X_\alpha \prec X_\beta$ for all $\alpha < \beta < \gamma$.*

Proof. Let $(p_n)_{n=1}^{\infty}$ be any sequence on numbers with $p < p_1 < p_2 < \dots$ and $\lim_{n\to\infty} p_n = 2$.

Lemma 2.2. *For every finite dimensional Banach space X and every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every into isomorphism $T : \ell_{p_n}^m \to X \oplus_2 \left(\bigoplus_{j>n} \ell_{p_j}\right)_2$ one has $\|T\|\|T^{-1}\| \ge n$.*

Proof of Lemma 2.2. Recall the standard definition (see, for example, [12, p. 54]): A Banach space Z is said to have *Rademacher type p*, $1 \le p \le 2$ (or just *type p*) if there exists a constant $T_p(Z) < \infty$ such that for every $k \in \mathbb{N}$ and for every $x_1, \dots, x_k \in Z$,

$$\left(\int_0^1 \left\|\sum_{i=1}^k r_i(t)x_i\right\|_Z^p dt\right)^{1/p} \le T_p(Z)\left(\sum_{i=1}^k \|x_i\|_Z^p\right)^{1/p}, \tag{2.1}$$

where $\{r_i\}$ are Rademacher functions.

The Khinchin-Kahane inequality (see e.g. [12, p. 57]) implies that we can replace $\left(\int_0^1 \left\|\sum_{i=1}^k r_i(t)x_i\right\|_Z^p dt\right)^{1/p}$ by $\left(\int_0^1 \left\|\sum_{i=1}^k r_i(t)x_i\right\|_Z^2 dt\right)^{1/2}$ in the left-hand side of inequality (2.1), it will not change the class of spaces of type p, but may change the constant $T_p(Z)$, let us denote this new constant $T_{p,2}(Z)$.

Now we shall check (recall that $p \le 2$) that the fact that spaces $\{Z_n\}_{n=1}^{\infty}$ have type p with uniformly bounded constants $\{T_{p,2}(Z_n)\}_{n=1}^{\infty}$, then $Z := \left(\bigoplus_{n=1}^{\infty} Z_n\right)_2$ also has type p with constant $T_{p,2}(Z)$ bounded from above by $\mathbf{T} := \sup_n T_{p,2}(Z_n)$.

So let $z_i = \{z_{i,n}\}_{n=1}^{\infty} \in \left(\bigoplus_{n=1}^{\infty} Z_n\right)_2$, so $z_{i,n} \in Z_n$. We have

$$\begin{aligned}
\left(\int_0^1 \left\|\sum_{i=1}^k r_i(t)z_i\right\|_Z^2 dt\right)^{1/2} &= \left(\int_0^1 \sum_{n=1}^{\infty} \left\|\sum_{i=1}^k r_i(t)z_{i,n}\right\|_{Z_n}^2 dt\right)^{1/2} \\
&\le \mathbf{T}\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^k \|z_{i,n}\|_{Z_n}^p\right)^{2/p}\right)^{1/2} \\
&\le \mathbf{T}\left(\sum_{i=1}^{k}\left(\sum_{n=1}^\infty \|z_{i,n}\|_{Z_n}^2\right)^{p/2}\right)^{1/p} \\
&= \mathbf{T}\left(\sum_{i=1}^k \|z_i\|_Z^p\right)^{1/p},
\end{aligned}$$

where in the first line we use the definition of Z as a direct sum; in the second line we use the fact that Z_n have type p with constant $\mathbf{T}$; in the third line we use the triangle inequality for the space $\ell_{2/p}$ (recall that $2/p \geq 1$), and in the last line we use the definition of Z again.

Now we return to the proof of Lemma 2.2. Since X is finite-dimensional, it has type p_{n+1} with sufficiently large constant. We need the well-known fact that ℓ_p has type p if $p \in [1,2]$ (see e.g. [12, p. 63]) and an easy-to-see fact (consider the unit vectors) that ℓ_p does not have a larger type.

We conclude that $X \oplus_2 \big(\bigoplus_{j>n} \ell_{p_j}\big)_2$ has type p_{n+1} with some constant C, but ℓ_{p_n} does not have type p_{n+1}. Therefore the type constant of $\ell_{p_n}^m$ for type p_{n+1} and sufficiently large m is $> Cn$. It is easy to see that this implies that for every into isomorphism $T : \ell_{p_n}^m \to X \oplus_2 \big(\bigoplus_{j>n} \ell_{p_j}\big)_2$ one has $\|T\|\|T^{-1}\| \geq n$. □

We continue the proof of Theorem 2.1. Using Lemma 2.2, construct recurrently a sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers so that

for every $n \in \mathbb{N}$ and every into isomorphism

$$U : \ell_{p_n}^{m_n} \to \Big(\bigoplus_{i=1}^{n-1} \ell_{p_i}^{m_i}\Big)_2 \oplus_2 \Big(\bigoplus_{j>n} \ell_{p_j}\Big)_2 \tag{2.2}$$

one has $\|U\|\|U^{-1}\| \geq n$.

It is known that for every $\varepsilon > 0$, every $m \in \mathbb{N}$ and every $q \in (p, 2]$ there exists a subspace F of ℓ_p which is $(1+\varepsilon)$-isomorphic to ℓ_q^m (see [8] for tight estimates of the parameters involved, the result itself follows from [4]). Using this fact for $\varepsilon = 1$, $m = m_n$ and $q = p_n$, for every $n \in \mathbb{N}$ we choose a subspace F_n of n-th summand of E_p (which is isometric to ℓ_p) which is 2-isomorphic to $\ell_{p_n}^{m_n}$, say, by means of an isomorphism $J_n : F_n \to \ell_{p_n}^{m_n}$ with $\|J_n\|\|J_n^{-1}\| \leq 2$.

Fix any ordinal γ of cardinality $\aleph_1$ (or $\mathfrak{c}$, respectively). Using items (1) and (2) of Lemma 1.1, respectively, choose a transfinite sequence $(N_\alpha)_{\alpha<\gamma}$ of subsets of $\mathbb{N}$ so that $|N_\alpha \setminus N_\beta| < \aleph_0 = |N_\beta \setminus N_\alpha|$ for all $\alpha < \beta < \gamma$. For each $\alpha < \gamma$ set

$$X_\alpha = \Big(\bigoplus_{n\in N_\alpha} F_n\Big)_2.$$

We consider each X_α as a subspace of E_p. Let us show that $(X_\alpha)_{\alpha<\gamma}$ has the desired properties. Fix any $\alpha < \beta < \gamma$. Set $N' = N_\alpha \setminus N_\beta$, $N'' = N_\alpha \cap N_\beta$, $N''' = N_\beta \setminus N_\alpha$. Then $N_\alpha = N' \sqcup N''$, $N_\beta = N'' \sqcup N'''$, $|N'| < \aleph_0 = |N'''|$. Hence,

$$X_\alpha = \Big(\bigoplus_{n\in N'} F_n\Big)_2 \oplus_2 \Big(\bigoplus_{n\in N''} F_n\Big)_2, \ X_\beta = \Big(\bigoplus_{n\in N''} F_n\Big)_2 \oplus_2 \Big(\bigoplus_{n\in N'''} F_n\Big)_2. \tag{2.3}$$

Since $|N'| < \aleph_0 = |N'''|$, we have that

$$\dim\Big(\bigoplus_{n\in N'} F_n\Big)_2 < \infty = \dim\Big(\bigoplus_{n\in N'''} F_n\Big)_2$$

and hence, X_α embeds isomorphically into X_β.

Prove that X_β does not embed isomorphically into X_α. Assume, on the contrary, that there is an into isomorphism $T : X_\beta \to X_\alpha$. Take any $n_0 \in N'''$ and consider the restriction $T_{n_0} = T|_{F_{n_0}}$ of T to F_{n_0}.

Observe that

$$X_\alpha \subseteq \Big(\bigoplus_{i=1}^{n_0-1} F_i\Big)_2 \oplus_2 \Big(\bigoplus_{j>n_0} \ell_{p_j}\Big)_2. \tag{2.4}$$

Let

$$S:\Big(\bigoplus_{i=1}^{n-1}F_i\Big)_2\oplus_2\Big(\bigoplus_{j>n}\ell_{p_j}\Big)_2\to\Big(\bigoplus_{i=1}^{n-1}\ell_{p_i}^{m_i}\Big)_2\oplus_2\Big(\bigoplus_{j>n}\ell_{p_j}\Big)_2$$

be an operator which sends $\big((f_i)_{i=1}^{n_0-1},g\big)$ to $\big((J_if_i)_{i=1}^{n_0-1},g\big)$. Since J_i are isomorphisms with $\|J_i\|\|J_i^{-1}\|\le 2$, so is S with $\|S\|\|S^{-1}\|\le 2$. Hence,

$$\|T\|\|T^{-1}\|\ge\|T_0\|\|T_0^{-1}\|\ge\frac{1}{2}\|S\circ T_0\|\|(S\circ T_0)^{-1}\|\overset{\text{by}(2.2)}{\ge}\frac{1}{2}n_0.$$

This is impossible for large enough $n_0\in N'''$. □

The next corollary follows from Theorem 2.1 and the observation that a separable infinite dimensional Banach space X has only continuum many closed subspaces, and hence, $IL(X)\le\omega_{c+1}$.

Corollary 2.3. (MA) $IL(E_p)=IL\big(C[0,1]\big)=\omega_{c+1}$.

It would be interesting to find the isomorphic length of the classical spaces $L_p=L_p[0,1]$.

Problem 1. *Evaluate $IL(L_p)$ for $1\le p<\infty$, $p\ne 2$.*

The embeddability of L_r into L_p for $1\le p<r\le 2$ [4] together with impossibility of the embedding L_p into L_r for the same values of p,r [2, p. 206] imply the inequality $IL(L_p)\ge\omega_1$ for $1\le p<2$, because every countable ordinal $\alpha<\omega_1$ is isomorphic to a subset of any interval (a,b) in the reverse order. The same inequality $IL(L_p)\ge\omega_1$ for all values $1\le p<\infty$, $p\ne 2$ is a corollary of the following result.

Theorem 2.4 (Bourgain, Rosenthal, Schechtman [3])**.** *Let $1<p<\infty$, $p\ne 2$. There exists a family $(X_\alpha^p)_{\alpha<\omega_1}$ of complemented subspaces of L_p so that for all $\alpha<\beta<\omega_1$ one has $X_\alpha^p\prec X_\beta^p$. Moreover, if B is a separable Banach space such that $X_\alpha^p\hookrightarrow B$ for all $\alpha<\omega_1$ then $L_p\hookrightarrow B$.*

Observe that Theorem 2.4 gives a strictly $\prec$-increasing ω_1-sequence of subspaces of L_p for $1<p<2$ directly. The same holds also for $p=1$ due to the fact ([4]) that L_r $(1<r<2)$ embeds isometrically into L_1. On the other hand, the argument based on embeddability/non-embeddability of L_r into L_p does not provide an uncountable sequence. However, both arguments provide the same estimate for $IL(L_p)$ if $1<p<2$.

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