On embeddings of locally finite metric spaces into ℓ_p

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Abstract

It is known that if finite subsets of a locally finite metric space M admit C-bilipschitz embeddings into ℓ_p $(1 \le p \le \infty)$, then for every $\varepsilon > 0$, the space M admits a $(C + \varepsilon)$-bilipschitz embedding into ℓ_p. The goal of this paper is to show that for $p \ne 2, \infty$ this result is sharp in the sense that ε cannot be dropped out of its statement.

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1 Introduction and Statement of Results

During the last decades, the study of bilipschitz embeddings of metric spaces into Banach spaces has become a field of intensive research with a great number of applications. The latter are not restricted to the area of Functional Analysis, but also include Graph Theory, Group Theory, and Computer Science. We refer to

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[16, 17, 18, 24, 26]. This work is focused on the study of relations between the embeddability into ℓ_p of an infinite metric space and its finite pieces. Let us recollect some needed notions.

Definition 1.1. A metric space is called *locally finite* if each ball of finite radius in it has finite cardinality.

Definition 1.2. Let (A, d_A) and (Y, d_Y) be metric spaces. Given, $1 \leq C < \infty$, a map $f : A \to Y$, is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad r d_A(u, v) \leq d_Y(f(u), f(v)) \leq rC d_A(u, v). \tag{1}$$

A map f is a *bilipschitz embedding* if it is C-bilipschitz for some $1 \leq C < \infty$. The smallest constant C for which there exists $r > 0$ such that (1) is satisfied, is called the *distortion* of f.

Unexplained terminology can be found in [15, 24].

It has been known that the bilipschitz embeddability of locally finite metric spaces into Banach spaces is finitely determined in the sense described by the following theorem.

Theorem 1.3 ([23])**.** *Let A be a locally finite metric space whose finite subsets admit bilipschitz embeddings with uniformly bounded distortions into a Banach space X. Then, A also admits a bilipschitz embedding into X.*

Theorem 1.3 has many predecessors, see [2, 3, 4, 21, 22]. Applications of this theorem to the coarse embeddings important for Geometric Group Theory and Topology are discussed in [23]. To expand on the theme, the argument of [23] yields a stronger result, namely the one stated as Theorem 1.4. In order to formulate Theorem 1.4, it is handy to employ the parameter $D(X)$ of a Banach space X introduced in [20]. Let us recollect its definition. Given a Banach space X and a real number $\alpha \geq 1$, we write:

- $D(X) \leq \alpha$ if, for any locally finite metric space A, all finite subsets of which admit bilipschitz embeddings into X with distortions $\leq C$, the space A itself admits a bilipschitz embedding into X with distortion $\leq \alpha \cdot C$;
- $D(X) = \alpha$ if α is the least number for which $D(X) \leq \alpha$;
- $D(X) = \alpha^+$ if, for every $\varepsilon > 0$, the condition $D(X) \leq \alpha + \varepsilon$ holds, while $D(X) \leq \alpha$ does not;
- $D(X) = \infty$ if $D(X) \leq \alpha$ does not hold for any $\alpha < \infty$.

In addition, we use inequalities like $D(X) < \alpha^+$ and $D(X) < \alpha$ with the natural meanings, for example $D(X) < \alpha^+$ indicates that either $D(X) = \beta$ for some $\beta \leq \alpha$ or $D(X) = \beta^+$ for some $\beta < \alpha$.

Theorem 1.4 ([23]). *There exists an absolute constant $D \in [1, \infty)$, such that for an arbitrary Banach space X the inequality $D(X) \leq D$ holds.*

Recently, new estimates of the parameter $D(X)$ for some classes of Banach spaces have been obtained in [20]. Recall that a family of finite-dimensional Banach spaces $\{X_n\}_{n=1}^{\infty}$ is said to be *nested* if X_n is a proper subspace of X_{n+1} for every $n \in \mathbb{N}$. For such families, an estimate for $D(X)$ from above is expressed by:

Theorem 1.5 ([20, Theorem 1.9]). *Let $1 \leq p < \infty$. If $\{X_n\}_{n=1}^{\infty}$ is a nested family of finite-dimensional Banach spaces, then $D\left((\oplus_{n=1}^{\infty} X_n)_p\right) \leq 1^+$.*

The next assertion is an immediate consequence of Theorem 1.5:

Corollary 1.6 ([20, Corollary 1.10]). *If $1 \leq p < \infty$, then $D(\ell_p) \leq 1^+$.*

It should be mentioned that the case where $p = \infty$ was discarded because the classical result of Fréchet [9] implies that $D(\ell_\infty) = 1$. Observe also that it is a well-known fact that $D(\ell_2) = 1$. Although the paper [20] contains some estimates for $D(X)$ from below, the following question was left open: whether $D(\ell_p) = 1^+$ or $D(\ell_p) = 1$ for $1 \leq p < \infty$, $p \neq 2$?

The main goal of this paper is to complete the picture by proving that $D(\ell_p) \geq 1^+$ if $p \in [1, \infty)$, $p \neq 2$. See Theorem 1.11 and Corollary 1.9. It is worth pointing out that our proofs for the cases $p = 1$ and $p > 1$ are different from each other.

Definition 1.7 ([8, Fact 7.7, p. 335]). A Banach space X is called *strictly convex* if the condition $||x+y|| = ||x|| + ||y||$ for $x, y \in X \backslash \{0\}$ implies $x = \lambda y$ for some $\lambda > 0$.

In the present work, it is shown that $D(X) > 1$ for a large class of strictly convex Banach spaces X implying that $D(X) = 1^+$ for all strictly convex Banach spaces satisfying the assumption of Theorem 1.5. To be more specific, the following statement will be proved (see Section 2):

Theorem 1.8. *Let X be a strictly convex Banach space such that all finite subsets of ℓ_2 admit isometric embeddings into X, but ℓ_2 itself does not admit an isomorphic embedding into X. Then $D(X) > 1$.*

With the help of Theorem 1.8, one derives:

Corollary 1.9. *Let $p \in (1, \infty)$, $p \neq 2$. Then every strictly convex Banach space of the form $X = (\oplus_{n=1}^{\infty} X_n)_p$, where $\{X_n\}_{n=1}^{\infty}$ is a nested sequence of finite-dimensional Banach spaces satisfies $D(X) > 1$.*

Combining Theorem 1.5 and Corollary 1.9 one obtains:

Corollary 1.10. *Let $p \in (1, \infty)$, $p \neq 2$, and let $\{X_n\}_{n=1}^{\infty}$ be a nested family of finite-dimensional strictly convex Banach spaces. Then, the space $X = (\oplus_{n=1}^{\infty} X_n)_p$ satisfies $D(X) = 1^+$. The equality $D(\ell_p) = 1^+$ for $p \in (1, \infty)$, $p \neq 2$, follows as a special case of this result.*

The case $p = 1$ is quite different because ℓ_1 is not strictly convex. This case is examined in Section 3, where we prove:

Theorem 1.11. *$D(\ell_1) > 1$.*

Juxtaposing this outcome with Theorem 1.5, we reach:

Corollary 1.12. *$D(\ell_1) = 1^+$.*

Remark 1.13. It should be mentioned that the above results are not the first known ones claiming $D(X) > 1$. Before now, results of this kind were obtained in [14, Theorem 2.9] and [20, Theorem 1.12] for some other Banach spaces and their classes.

2 Proof of Theorem 1.8

Prior to presenting the proof of Theorem 1.8, let us provide some auxiliary information. By developing the notion of a linear triple [6, p. 56], we introduce the following:

Definition 2.1. A collection $r = \{r_i\}_{i=1}^n$, $n \geq 3$, of points in a metric space (A, d_A) is called a *linear tuple* if the sequence $\{d_A(r_i, r_1)\}_{i=1}^n$ is strictly increasing and if, for $1 \leq i < j < k \leq n$, the equality below holds:

$$d_A(r_i, r_k) = d_A(r_i, r_j) + d_A(r_j, r_k). \tag{2}$$

A *linear triple* is a linear tuple with $n = 3$.

Lemma 2.2. *An isometric image of a linear tuple $r = \{r_i\}_{i=1}^n$ in a strictly convex Banach space is contained in the line segment joining the images of r_1 and r_n.*

Proof. It suffices to prove the lemma for linear triples, and then to use this result for all triples of the form $\{r_1, r_i, r_n\}$, $i = 2, \ldots, n-1$.

It may be assumed, without loss of generality, that the image of r_1 is 0. Let $0, x$, and z be the images of the linear triple under an isometric embedding. Equality (2) and the assumption that the embedding is isometric imply that $||x||+||z-x|| = ||z||$. By Definition 1.7, this yields $x = \lambda(z - x)$ for some $\lambda > 0$. The conclusion that x belongs to the line segment joining 0 and z follows. □

For the sequel, the next fact is needed (by B_Z we denote the unit ball of a Banach space Z):

Lemma 2.3. *Let Z be a finite-dimensional Banach space and F be a Banach space. Then, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, Z, F) > 0$ such that if a δ-net in B_Z admits an isometric embedding into F, then F contains a subspace whose Banach-Mazur distance to Z does not exceed $(1 + \varepsilon)$.*

Lemma 2.3 is an immediate consequence of Bourgain's discretization theorem [7]. It should be emphasized that this theorem provides a much stronger claim because Bourgain found an explicit estimate for δ as a function of ε and the dimension of

Z; besides in Bourgain's theorem, the distortion of embedding of Z is estimated in terms of distortion of embedding of a δ-net of B_Z. See [5, 10] for simplifications of Bourgain's proof, see also its presentation in [24, Section 9.2]. Meanwhile, the existence of $\delta(\varepsilon, Z, F)$ can be derived from earlier results of Ribe [25] and Heinrich and Mankiewicz [11], see [10, p. 818].

Proof of Theorem 1.8. Denote the unit vector basis of ℓ_2 by $\{e_i\}_{i=1}^{\infty}$. Our intention is to find a locally finite subset M of ℓ_2 in such a way that:

(A) M contains a $\delta(\frac{1}{n}, \ell_2^n, X)$-net M_n of a shifted unit ball $y_n + B_{\ell_2^n}$, $n \in \mathbb{N}$; and

(B) There exists a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of positive numbers such that, if $T : M \to X$ is an isometry and $T(0) = 0$, then the image of $T(M_n)$ is contained in the linear span of $\{T(\alpha_1 e_1), \dots, T(\alpha_n e_n)\}$.

The following result is to be applied.

Theorem 2.4 ([13]). *Let B be a normed linear space. Then, a necessary and sufficient condition that B be isomorphic to an inner product space is that there exists a constant $k \geq 1$ such that, for each finite dimensional subspace J of B, there exists a linear mapping H_J of J into a Hilbert space satisfying $(1/k)||x|| \leq ||H_J x|| \leq k||x||$ for each x in J.*

Theorem 2.4 will be used to show that the existence of a set M satisfying both conditions **(A)** and **(B)** will prove Theorem 1.8, by whose assumption finite subsets of M admit isometric embeddings into X. What is left is to establish that M itself does not admit an isometric embedding into X.

In fact, such an embedding T could be assumed to satisfy $T(0) = 0$. Combining condition **(A)** with Lemma 2.3, one concludes that the subspace spanned by $T(M_n)$ contains another one, which is $\left(1 + \frac{1}{n}\right)$-isomorphic to ℓ_2^n. By condition **(B)**, the latter subspace has to coincide with the linear span of $\{T(\alpha_1 e_1), \dots, T(\alpha_n e_n)\}$.

Let B be the linear span of $\{T(\alpha_i e_i)\}_{i=1}^{\infty}$. By the conclusion of the previous paragraph, the normed linear space B satisfies the condition of Theorem 2.4 with $k = 2$. Thus, the closure of B in X is isomorphic to ℓ_2, which is a contradiction.

Set

$$M = \left(\bigcup_{n=1}^{\infty} M_n \right) \bigcup \{0\},$$

where M_n are finite sets constructed in the way described hereinafter.

Denote by $R_i, i \in \mathbb{N}$, the positive rays generated by e_i, that is, $R_i = \{\alpha e_i : \alpha \geq 0\}$. Let M_1 be the $\delta(1, \ell_2^1, X)$-net in the line segment $[0, 2e_1]$, where we assume that M_1 includes e_1. It is clear that M_1 satisfies **(A)**.

For $n > 1$ sets $\{M_n\}_{n=1}^{\infty}$ will be constructed inductively. Suppose that we have already created $M_1, \dots, M_{n-1}$. To construct M_n, we pick points $s_i^n \in R_i$, $1 \leq i \leq n$, and one more point, $s_{n+1}^n \in R_n$ - so that R_n contains both s_n^n and s_{n+1}^n - in such a way that $\mathrm{conv}(\{s_i^n\}_{i=1}^{n+1})$ is at distance at least n from the origin, and $\mathrm{conv}(\{s_i^n\}_{i=1}^{n+1})$ contains a shift $y_n + B_{\ell_2^n}$ of the unit ball (for some y_n). This is clearly possible.

Next, we select a $\delta(\frac{1}{n}, \ell_2^n, X)$-net $\mathcal{N}_n$ in this shifted unit ball $y_n + B_{\ell_2^n}$ and include it in M_n together with $\{s_i^n\}_{i=1}^{n+1}$. At this point, it is evident that condition **(A)** is satisfied.

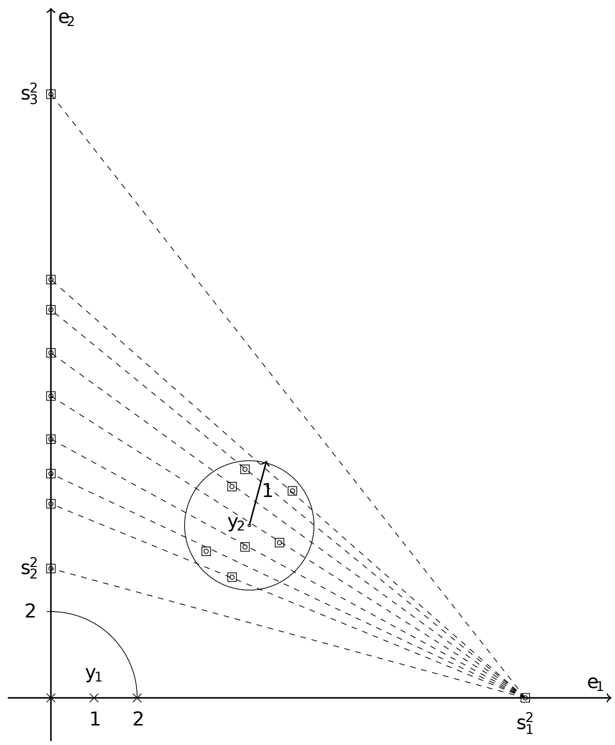

Figure 1: M_1 and M_2.

To ensure that condition **(B)** is also satisfied - as it will be seen later - we add, for each element $z \in \mathcal{N}_n$, finitely many additional elements of $\operatorname{conv}(\{s_i^n\}_{i=1}^{n+1})$ to M_n according to the procedure suggested below:

- If $z \in \{s_i^n\}_{i=1}^{n+1}$, there is nothing to include. If $z \notin \{s_i^n\}_{i=1}^{n+1}$, we find and include in M_n an element $w_1(z)$ in a convex hull of an n-element subset $W_1(z)$ of $\{s_i^n\}_{i=1}^{n+1}$ with z being on the line segment joining $w_1(z)$ and $s_i^n \in \left(\{s_i^n\}_{i=1}^{n+1} \backslash W_1(z)\right)$. One of the ways of doing this for M_2 is shown in Figure 1.
- If $w_1(z) \in \{s_i^n\}_{i=1}^{n+1}$, there is nothing else to include. If $w_1(z) \notin \{s_i^n\}_{i=1}^{n+1}$, we find and include in M_n an element $w_2(z)$ in a convex hull of an $(n-1)$-element subset $W_2(z)$ of $\{s_i^n\}_{i=1}^{n+1}$ such that $w_1(z)$ is on the line segment joining $w_2(z)$ and $s_i^n \in \left(\{s_i^n\}_{i=1}^{n+1} \backslash W_2(z)\right)$.
- We continue in an obvious way.

- If we do not terminate the process in one of the previous steps, we arrive at the situation when $w_n(z)$ is in a convex hull of a 2-element subset of $\{s_i^n\}_{i=1}^{n+1}$, and hence it is on some line segment of the form $[s_i^n, s_j^n]$. At this point we stop.

It has already been stated that condition **(A)** is satisfied for M. Now, let us verify condition **(B)**. To do this, it suffices to prove that, for each isometry $T : (M_n \cup \{0\}) \to X$ satisfying $T(0) = 0$, the image $T(M_n)$ is contained in the linear span of $\{T(s_1^n), \dots, T(s_n^n)\}$. This condition looks slightly different from the one in **(B)**. However, defining $\{\alpha_i\}_{i=1}^\infty$ by $\alpha_1 = 1$ and $\alpha_i e_i = s_i^i$ one can see that in essence the conditions are equivalent because, by Lemma 2.2, the images $\{T(s_i^n)\}_{n=i}^\infty$ are multiples of each other.

To show that $T(M_n)$ is contained in the linear span L of $\{T(s_1^n), \dots, T(s_n^n)\}$, the procedure outlined underneath is applied, where in each step Lemma 2.2 is used.

- Since $0, s_n^n$, and s_{n+1}^n form a linear triple, and $T(0) = 0$, we have $T(s_{n+1}^n) \in L$.
- Whenever $w_n(z)$ is defined, one has $T(w_n(z)) \in L$ because $w_n(z) \in [s_i^n, s_j^n]$.
- Likewise, for each z such that $w_{n-1}(z)$ is defined, one obtains $T(w_{n-1}(z)) \in L$ since $w_{n-1}(z)$ is in the line segment joining $w_n(z)$ and one of s_i^n.
- We proceed in a straightforward way until we get $T(z) \in L$.

In addition, it is easy to see that the assumption that $\mathrm{conv}(\{s_i^n\}_{i=1}^{n+1})$ is at distance at least n from the origin together with the fact that each set M_n is finite and is contained in $\mathrm{conv}(\{s_i^n\}_{i=1}^{n+1})$, implies that the set $\cup_{n=1}^\infty M_n$ is locally finite. $\square$

Proof of Corollary 1.9. To check that this X satisfies the conditions of Theorem 1.8 two well-known facts come in handy:

(1) Each finite subset of $L_p[0,1]$ admits an isometric embedding into ℓ_p, see [1].
(2) The space $L_p[0,1]$ contains a subspace isometric to ℓ_2, see [12, p. 16].

Combining **(1)** and **(2)** we conclude that all finite subsets of ℓ_2 are isometric to subsets of ℓ_p, and, thence, to subsets of X. On the other hand, it is known that each infinite-dimensional subspace of X contains a subspace isomorphic to ℓ_p (this can be done using a slight variation of the argument used to prove [15, Proposition 2.a.2]), and as such it is not isomorphic to ℓ_2. $\square$

Remark 2.5. The first part of the proof of Theorem 1.8 demonstrates that its statement can be strengthened by replacing the condition "ℓ_2 does not admit an isomorphic embedding into X" by "there is $\alpha > 1$ such that X does not contain a subspace whose Banach-Mazur distance to ℓ_2 does not exceed α". It is known [19] that the latter condition is strictly weaker. In addition, it is not difficult to see that although Joichi did not formally state the pertinent modification of the main result of [13], it arises from the proof.

3 The case of ℓ_1

Proof of Theorem 1.11. Recall [1] that each finite subset of $L_1(-\infty,\infty)$ admits an isometric embedding into ℓ_1. To prove Theorem 1.11 we construct in $L_1(-\infty,\infty)$ a locally finite metric space M such that its isometric embeddability into ℓ_1 would imply that ℓ_1 contains a unit vector x which, for every $n\in\mathbb{N}$, can be represented as a sum of 2^n vectors with pairwise disjoint supports and of norm 2^{-n} each. This leads to a contradiction: consider the maximal in absolute value coordinate of the vector x, let it be α. If, for some $n\in\mathbb{N}$, $|\alpha|>2^{-n}$, it is clearly impossible to partition the vector into 2^n vectors of norm 2^{-n} each with pairwise disjoint supports.

The starting point of the construction is the fact that the indicator function $\mathbf{1}_{(0,1]}$ has, for each $n\in\mathbb{N}$, a representation as a sum of 2^n pairwise disjoint vectors of norm 2^{-n}. To be specific, we adopt the writing:

- $\mathbf{1}_{(0,1]}=d_0+d_1$, where $d_0=\mathbf{1}_{(0,\frac{1}{2}]}, d_1=\mathbf{1}_{(\frac{1}{2},1]}$
- $\mathbf{1}_{(0,1]}=d_{00}+d_{01}+d_{10}+d_{11}$, where $d_{00}=\mathbf{1}_{(0,\frac{1}{4}]}, d_{01}=\mathbf{1}_{(\frac{1}{4},\frac{1}{2}]}, d_{10}=\mathbf{1}_{(\frac{1}{2},\frac{3}{4}]}, d_{11}=\mathbf{1}_{(\frac{3}{4},1]}$
- We carry on in an obvious way.

In the sequel, the following notation will be employed: let $d=\mathbf{1}_{(0,1]}$ and denote the functions introduced above by d_σ, where σ is a finite string of 0's and 1's. Denote by $\ell(\sigma)$ the length of the string σ. For each $\sigma=\{\sigma_i\}_{i=1}^{\ell(\sigma)}$, the subinterval $I(\sigma)$ of $(0,1]$ is defined by:

$$I(\sigma)=\left(\sum_{i=1}^{\ell(\sigma)}\sigma_i 2^{-i},\ 2^{-\ell(\sigma)}+\sum_{i=1}^{\ell(\sigma)}\sigma_i 2^{-i}\right].$$

With this notation $d_\sigma=\mathbf{1}_{I(\sigma)}$ and the mentioned above representation of $\mathbf{1}_{(0,1]}$ as a sum of 2^n terms can be written as:

$$d=\sum_{\sigma,\ \ell(\sigma)=n} d_\sigma,$$

where the summands are disjointly supported. Now, denote by $\mathcal{T}$ the set of all finite strings of 0's and 1's. It is obvious that $\{d_\sigma\}_{\sigma\in\mathcal{T}}$ is not a locally finite set. Nonetheless, we can add to $\{d_\sigma\}$ pairwise disjoint functions in such a way that a locally finite subset of $L_1(-\infty,\infty)$ will be obtained, and the existence of an isometric embedding of this set into ℓ_1 would imply the existence in ℓ_1 of a vector x with the properties described at the beginning of the proof.

First, opt for an injective map Ψ from the collection of all finite strings of 0's and 1's into $\mathbb{Z}\backslash\{0\}$.

Now, we consider the set M satisfying the conditions: It contains both functions d and 0, and, in addition, it includes all sums $f_\sigma:=d_\sigma+\ell(\sigma)\cdot\mathbf{1}_{(\Psi(\sigma),\Psi(\sigma)+1]}$, where $\sigma\in\mathcal{T}$.

Since $\ell(\sigma)$ is less than any fixed constant only for finitely many strings σ, this set is a locally finite subset of $L_1(-\infty,\infty)$. It has to be shown that isometric embeddability of this set into ℓ_1 implies the existence in ℓ_1 of a vector x with the properties stated in the first paragraph of the proof, thus resulting in a contradiction.

Indeed, if there is an isometric embedding of M into ℓ_1, then there is an isometric embedding F which maps 0 to 0 and - as it will be proved - in such a case $x = F(d)$ is the desired vector. More elaborately put, the existence of such an isometric embedding implies that there exist vectors $\{x_\sigma\}_{\sigma\in\mathcal{T}}$ so that, for each $n \in \mathbb{N}$, the vectors $\{x_\sigma\}_{\ell(\sigma)=n}$ are disjointly supported, have norm 2^{-n}, and

$$x = F(d) = \sum_{\sigma,\ \ell(\sigma)=n} x_\sigma.$$

Each element $a = \sum_{i=1}^{\infty} a_i e_i$ of ℓ_1 can be considered as a possibly infinite union of intervals in the coordinate plane which join $(i,0)$ and (i,a_i). The total length of all intervals is equal to $||a||$.

The proposed construction guarantees that if $\ell(\sigma)=n$, then $||f_\sigma - d|| = ||f_\sigma|| + ||d|| - 2\cdot 2^{-n}$. Since F is an isometry, $F(0) = 0$, and $F(d) = x$, this implies $||F(f_\sigma) - x|| = ||F(f_\sigma)|| + ||x|| - 2\cdot 2^{-n}$. Consequently, the total length of intersections of the intervals corresponding to x and to $F(f_\sigma)$ is 2^{-n} for $\sigma \in \{0,1\}^n$.

On the other hand, if $\sigma \neq \tau$ and $\ell(\sigma) = \ell(\tau) = n$, the functions f_σ and f_τ are disjointly supported and, therefore, $||f_\sigma - f_\tau|| = ||f_\tau - 0|| + ||f_\sigma - 0||$. As a result, $||F(f_\sigma) - F(f_\tau)|| = ||F(f_\sigma)|| + ||F(f_\tau)||$. This means that the intersections of the intervals corresponding to $F(f_\sigma)$ and $F(f_\tau)$ have total length 0. It does not immediately imply that vectors $F(f_\sigma)$ and $F(f_\tau)$ are disjointly supported: one can imagine, for example, that $F(f_\sigma)$ contains the interval joining $(i,0)$ and $(i,\frac{1}{4})$ and $F(f_\tau)$ contains the interval joining $(i,0)$ and $(i,-\frac{1}{4})$.

Let us define the vector x_σ for σ satisfying $\ell(\sigma) = n$ as a vector for which the corresponding intervals are intersections of the intervals corresponding to x and to $F(f_\sigma)$. The previous paragraphs imply that x_σ and x_τ satisfy $||x_\sigma|| = ||x_\tau|| = 2^{-n}$ and have disjoint supports when $\ell(\sigma) = \ell(\tau) = n$ and $\sigma \neq \tau$ (for the latter we use the fact that the interval corresponding to x at i can have 'positive' or 'negative' part, but not both).

Finally, let $s = \sum_{\sigma,\ \ell(\sigma)=n} x_\sigma$. With the preceding arguments, we conclude that $||s|| = 1$ and $|s_i| \le |x_i|$ for each $i \in \mathbb{N}$. Thus, $s = x$, and the desired decomposition of x is completed. $\square$

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