Experimental Mathematics

ISSN: 1058-6458 (Print) 1944-950X (Online) Journal homepage: https://www.tandfonline.com/loi/uexm20

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To cite this article: Sami Assaf & Dominic Searles (2019): Kohnert Polynomials, Experimental Mathematics, DOI: 10.1080/10586458.2019.1588180

To link to this article: https://doi.org/10.1080/10586458.2019.1588180

Published online: 23 Apr 2019.

EXPERIMENTAL MATHEMATICS
https://doi.org/10.1080/10586458.2019.1588180

Kohnert Polynomials

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ABSTRACT
We associate a polynomial to any diagram of unit cells in the first quadrant of the plane using Kohnert's algorithm for moving cells down. In this way, for every weak composition one can choose a cell diagram with corresponding row-counts, with each choice giving rise to a combinatorially-defined basis of polynomials. These *Kohnert bases* provide a simultaneous generalization of Schubert polynomials and Demazure characters for the general linear group. Using the monomial and fundamental slide bases defined earlier by the authors, we show that Kohnert polynomials stabilize to quasisymmetric functions that are nonnegative on the fundamental basis for quasisymmetric functions. For initial applications, we define and study two new Kohnert bases. The elements of one basis are conjecturally Schubert-positive and stabilize to the skew-Schur functions; the elements of the other basis stabilize to a new basis of quasisymmetric functions that contains the Schur functions.

KEYWORDS
Schubert polynomials; Demazure characters; key polynomials; fundamental slide polynomials

2010 MATHEMATICS SUBJECT CLASSIFICATION
Primary 14M15; secondary 14N15; 05E05

1. Introduction

Certain homogeneous bases of the ring of polynomials are of central importance in representation theory and geometry. Foremost among these are the Schubert polynomials [Lascoux and Schützenberger 82], which are characters of Kraśkiewicz–Pragacz modules [Kraśkiewicz and Pragacz 87, Kraśkiewicz and Pragacz 04] and represent Schubert basis classes in the cohomology of the complete flag variety, and the Demazure characters [Demazure 74] (also known as key polynomials), which are the characters of Demazure modules for the general linear group. We are motivated by the question of finding other bases of polynomials that exhibit close connections to and share key properties with these important bases. Such bases may be used to understand Schubert polynomials and Demazure characters and moreover may be of independent representation-theoretic or geometric interest.

Kohnert [Kohnert 90] introduced a combinatorial model for the monomial expansion of a Demazure character. Let a be a weak composition, that is, a sequence of nonnegative integers. Kohnert's model begins with the diagram $\mathbb{D}(a)$ of a, the cell diagram in $\mathbb{N} \times \mathbb{N}$ which has a_i cells in row i, left-justified. Kohnert defined an algorithmic process on cell diagrams that moves the rightmost cell of a row down to the first available position below. The Kohnert diagrams for a are the cell diagrams that may be obtained by a (possibly empty) sequence of these Kohnert moves on $\mathbb{D}(a)$. Kohnert proved that the Demazure character for a is the generating function of the Kohnert diagrams of $\mathbb{D}(a)$.

Kohnert conjectured that the Schubert polynomials arise by applying the exact same algorithm to different initial cell diagrams, namely, the Rothe diagrams of permutations. Proofs were given by Winkel [Winkel 99, Winkel 02], though were not fully accepted due to the very technical nature of the arguments; a recent more direct proof was given by Assaf [Assaf 17] using the expansion of Schubert polynomials into Demazure characters.

In this work, we study the polynomials arising from application of Kohnert's algorithm to *any* cell diagram in $\mathbb{N} \times \mathbb{N}$; we call these polynomials *Kohnert polynomials*. By definition, Kohnert polynomials expand positively in monomials, and simultaneously generalize both Schubert polynomials and Demazure characters. Given a weak composition a, there are several different (though finitely many) Kohnert polynomials for a: in creating an initial cell diagram one

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To the memory of Axel Kohnert.

must place a_i cells in row i, but one may choose in which columns the cells are placed. If one Kohnert polynomial is chosen for every weak composition a, we call the resulting set of polynomials a *Kohnert basis* of the polynomial ring. Each Kohnert polynomial in a Kohnert basis has a unique monomial that is minimal in dominance order, hence a Kohnert basis is lower uni-triangular with respect to the basis of monomials. Thus, Kohnert bases are bases of the polynomial ring, justifying the nomenclature.

Kohnert bases thus comprise a vast collection of combinatorially-defined bases of polynomials, which includes the Schubert and Demazure character bases. To motivate and facilitate further investigation of Kohnert bases, we prove that *every* Kohnert polynomial expands positively in the monomial slide polynomials introduced in [Assaf and Searles 17]. An immediate application is that every Kohnert polynomial has a stable limit, which, in fact, is quasisymmetric and expands positively in the monomial basis of quasisymmetric functions.

We define necessary and sufficient conditions on cell diagrams for the corresponding Kohnert polynomial to expand positively in the fundamental slide basis [Assaf and Searles 17], a polynomial ring analog of Gessel's basis of fundamental quasisymmetric functions [Gessel 84]. While not every Kohnert polynomial expands positively in the fundamental slide basis, we prove that, surprisingly, the stable limit of *any* Kohnert polynomial expands positively in fundamental quasisymmetric functions. For example, the stable limits of Schubert polynomials are Stanley symmetric functions [Macdonald 91] and the stable limits of Demazure characters are Schur polynomials [Assaf and Searles 18, Lascoux and Schützenberger 90]; each of these is known to expand positively in fundamental quasisymmetric functions. Thus, by taking stable limits of Kohnert bases, one obtains new and recovers known families of fundamental-positive quasisymmetric functions. These families may or may not be bases of quasisymmetric functions; for example, the stable limits of Schubert polynomials and Demazure characters are not.

We define a simple condition on diagrams that we conjecture characterizes those diagrams for which the corresponding Kohnert polynomial expands nonnegatively as a sum of Demazure characters. Both key diagrams, indexing Demazure characters, and Rothe diagrams, indexing Schubert polynomials, satisfy the stated condition. In further support of the conjecture, the demazure condition is exactly the same as the *northwest* condition of Reiner and Shimozono [Reiner and Shimozono 95, Reiner and Shimozono 98] in their study of Specht modules associated to diagrams, suggesting a possible connection between flagged Weyl modules and Kohnert polynomials.

There are several natural choices of ways to associate a two-dimensional cell diagram to a weak composition. Demazure characters arise from left-justification, Schubert polynomials arise from choosing the Rothe diagram of the associated permutation. Using the construction of Kohnert bases, we believe that other natural choices of diagram for a weak composition will yield several new and interesting combinatorial objects, from Kohnert bases of the polynomial ring to families (or bases) of quasisymmetric functions. Initial computer experiments lead us to introduce, as a first application, two new Kohnert bases with nice combinatorial properties and surprising connections to representation theory and geometry.

The *skew polynomials* are the Kohnert polynomials associated to diagrams arising from certain rightward shifts of contiguous rows of cells. As predicted by our demazure condition, we prove that skew polynomials expand positively into Demazure characters. Based on computer evidence, we conjecture they also expand positively in Schubert polynomials, suggesting a hidden connection with geometry. Stable limits of skew polynomials are symmetric, and in fact are the skew-Schur functions.

The *lock polynomials* are the Kohnert polynomials associated to right-justified diagrams. Lock polynomials expand positively in fundamental slide polynomials (as do the Schubert polynomials and Demazure characters), and coincide with Demazure characters when the nonzero entries of a are weakly decreasing (which is also the only case when the diagrams satisfy our conjectured demazure condition). The stable limits of lock polynomials, which we call the *extended Schur functions*, are a new basis of quasisymmetric functions. By the theory of Kohnert bases, the extended Schur functions expand positively in the fundamental basis. Per the name, the extended Schur function basis contains the Schur functions as a subset, thus is a lifting of the Schur basis from symmetric to quasisymmetric functions. The description in terms of cell diagrams naturally gives rise to families of tableaux generating the lock polynomials and the extended Schur functions, similar to the definition of Kohnert tableaux for Demazure characters in [Assaf and Searles 18]. The tableau description enables us to give explicit formulas for the expansion of an extended Schur function in terms of fundamental

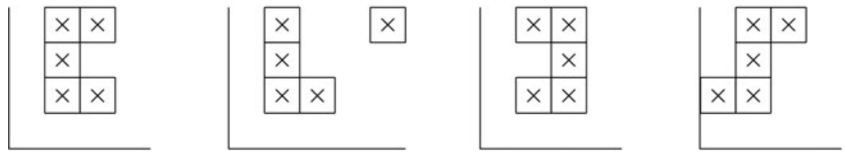

Figure 1. Four diagrams of weight $(0, 2, 1, 2)$.

quasisymmetric polynomials, and extract further interesting properties.

We expect these bases and others arising as Kohnert bases may, like the Schubert and Demazure character bases, have deep connections to representation theory and geometry.

2. Kohnert polynomials

Let $\mathbb{N}$ denote the natural numbers $\{1, 2, \ldots\}$. In Section 2.1, we review Kohnert's algorithm that generates a polynomial from a cell diagram in $\mathbb{N} \times \mathbb{N}$ and use this to define *Kohnert polynomials*. We review the motivating examples of Demazure characters in Section 2.2 and Schubert polynomials in Section 2.3, presenting both in the context of Kohnert polynomials.

2.1. Kohnert diagrams

A *diagram* is an array of finitely many cells in $\mathbb{N} \times \mathbb{N}$. The weight of a diagram D, denoted by $\mathrm{wt}(D)$, is the weak composition whose ith part is the number of cells in row i. For example, four diagrams with weight $(0, 2, 1, 2)$ are shown in Figure 1.

A diagram is called a *key diagram* if the rows are left justified. For each weak composition a, there is a unique key diagram of weight a which we call the *key diagram for a* and denote by $\mathbb{D}(a)$. For example, the leftmost diagram in Figure 1 is the key diagram for $(0, 2, 1, 2)$.

In his thesis, Kohnert [Kohnert 90] described an algorithm for generating a Demazure character, which he called a key polynomial after Lascoux and Schützenberger [Lascoux and Schützenberger 90], from a key diagram by iteratively applying certain *Kohnert moves* to the diagram.

Definition 2.1 ([Kohnert 90]). A *Kohnert move* on a diagram selects the rightmost cell of a given row and moves the cell to the first available position below in the same column (if such a position exists), jumping over other cells in its way as needed. Given a diagram D, let $\mathrm{KD}(D)$ denote the set of all diagrams that can be obtained by applying a series of Kohnert moves to D.

For example, Figure 2 shows all 16 Kohnert diagrams for the key diagram $\mathbb{D}(0, 2, 1, 2)$. For comparison, the second diagram in Figure 1 gives rise to 26 Kohnert diagrams shown in Figure 3 and the third gives rise to 9 Kohnert diagrams shown in Figure 17.

Definition 2.2. The *Kohnert polynomial* indexed by D is

$$\mathfrak{K}_D = \sum_{T \in \mathrm{KD}(D)} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}. \tag{2–1}$$

For example, from Figure 2, we see that

$$\begin{aligned}\mathfrak{K}_{\mathbb{D}(0,2,1,2)} = {} & x_1^2x_2^2x_3 + x_1^2x_2^2x_4 + x_1^2x_2x_3^2 + 2x_1^2x_2x_3x_4 \\ & + x_1^2x_2x_4^2 + x_1^2x_3^2x_4 + x_1^2x_3x_4^2 + x_1x_2^2x_3^2 \\ & + 2x_1x_2^2x_3x_4 + x_1x_2^2x_4^2 + x_1x_2x_3^2x_4 \\ & + x_1x_2x_3x_4^2 + x_2^2x_3^2x_4 + x_2^2x_3x_4^2.\end{aligned}$$

Note that the diagram of a Kohnert polynomial is not necessarily unique. For instance, if two diagrams differ by insertion or deletion of empty columns, then they necessarily give the same Kohnert polynomial. However, as demonstrated by Theorem 6.12 below, this is not sufficient. It is an interesting question to ask for necessary and sufficient conditions for two diagrams to give the same Kohnert polynomial.

Given weak compositions a and b, say that b *dominates* a, denoted by $a \leqslant b$, if $a_1 + \cdots + a_k \leqslant b_1 + \cdots + b_k$ for all k. Since Kohnert polynomials have a unique leading term that is minimal in dominance order, they provide a simple mechanism for constructing interesting bases of the polynomial ring.

Theorem 2.3. *Given any set of diagrams $\{D_a\}$, one for every weak composition, such that $\mathrm{wt}(D_a) = a$, the corresponding Kohnert polynomials $\{\mathfrak{K}_{D_a}\}$ form a basis of the polynomial ring.*

Proof. For any weak composition a and any diagram D such that $\mathrm{wt}(D_a) = a$, the corresponding Kohnert polynomial $\mathfrak{K}_D$ expands as

$$\mathfrak{K}_D = x_1^{a_1} \cdots x_n^{a_n} + \sum_{b > a} c_{a,b} x_1^{b_1} \cdots x_n^{b_n}$$

for some nonnegative integers $c_{a,b}$, where the sum is over weak compositions b that strictly dominate a. In particular, any set of Kohnert polynomials of the form

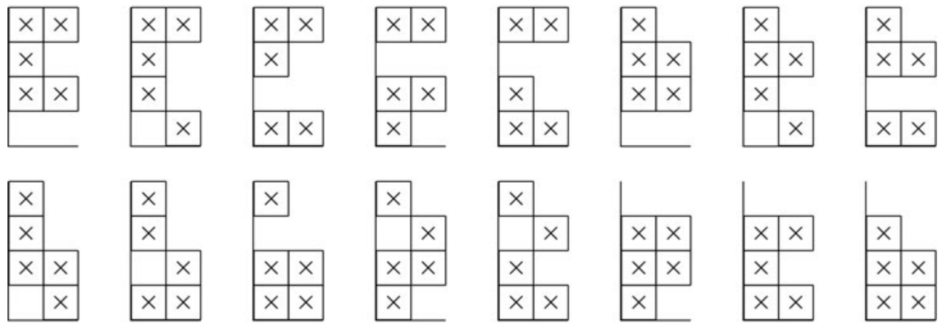

Figure 2. Kohnert diagrams for $\mathbb{D}(0,2,1,2)$.

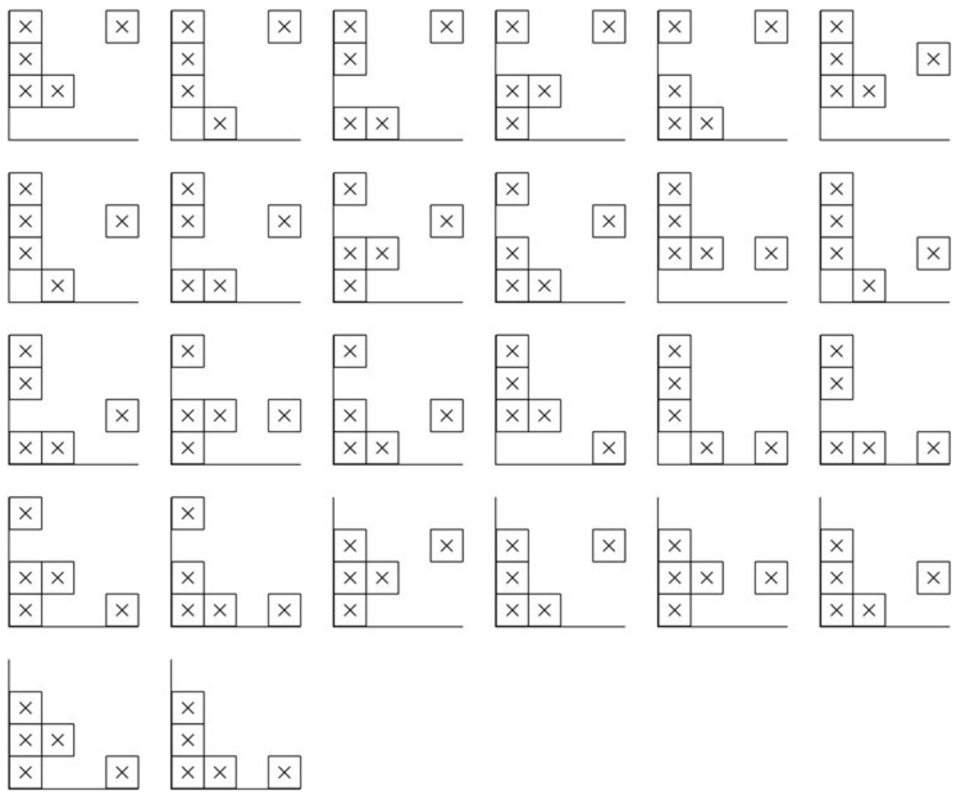

Figure 3. Kohnert diagrams for $\mathbb{D}(143625)$.

$\{\mathfrak{K}_{D_a}\}$ where $\mathrm{wt}(D_a) = a$ is lower uni-triangular with respect to monomials, and thus is also a basis. □

As this concept is central to the current study, we introduce the following terminology.

Definition 2.4. A basis $\{\mathfrak{B}_a\}$ for polynomials is a *Kohnert basis* if each element $\mathfrak{B}_a$ can be realized as a Kohnert polynomial for some diagram D with $\mathrm{wt}(D) = a$.

Two important examples of Kohnert bases are Demazure characters and Schubert polynomials, discussed below. In addition to proving general positivity results for Kohnert polynomials, we demonstrate the power of this paradigm by giving a new example of a Kohnert basis in Section 6.

2.2. Demazure characters

Kohnert's original motivation for studying key diagrams arose from characters of Demazure modules for the general linear group [Demazure 74], which may be regarded as truncations of irreducible characters [Demazure 74]. These polynomials were studied combinatorially by Lascoux and Schützenberger [Lascoux and Schützenberger 90], who call them *key polynomials*. For a nice survey of the combinatorial aspects, see [Reiner and Shimozono 95]; for a recent treatment from Kohnert's perspective, see [Assaf and Searles 18].

The original definition for Demazure characters is in terms of *divided difference operators*, denoted by ∂_i, defined on a polynomial f by

$$\partial_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}, \tag{2–2}$$

where s_i is the simple transposition interchanging i and $i+1$ and it acts on polynomials by interchanging x_i and x_{i+1}. Extending this, we may define a linear operator π_i on polynomials by

$$\pi_i f = \partial_i (x_i f). \tag{2–3}$$

Given a permutation w, we may define

$$\partial_w = \partial_{s_1} \ldots \partial_{s_k}$$
$$\pi_w = \pi_{s_1} \ldots \pi_{s_k}$$

for any expression $s_1 \ldots s_k = w$ with k minimal. It can be shown that both ∂_w and π_w are independent of the choice of reduced expression.

Definition 2.5. Given a weak composition a, the *Demazure character* κ_{a} is

$$\kappa_a = \pi_{w(a)} x^{\mathrm{sort}(a)}, \tag{2–4}$$

where $\mathrm{sort}(a)$ is the weakly decreasing rearrangement of a, $w(a)$ is the shortest permutation that sorts a and x^b denotes the monomial $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$.

For example, for $a = (0,2,1,2)$, we have $\mathrm{sort}(a) = (2,2,1,0)$ and $w(a) = 2431$, and so

$$\begin{aligned}
\kappa_{(0,2,1,2)} &= \pi_1 \pi_2 \pi_3 \pi_2 \left(x_1^2 x_2^2 x_3\right) \\
&= \pi_1 \pi_2 \pi_3 \left(x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2\right) \\
&= \pi_1 \pi_2 (x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + x_1^2 x_2 x_3^2 \\
&\quad + x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_4^2) \\
&= \pi_1 (x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + x_1^2 x_2 x_3^2 \\
&\quad + 2 x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_4^2 + x_1^2 x_3^2 x_4 + x_1^2 x_3 x_4^2) \\
&= x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_4 + x_1^2 x_2 x_3^2 \\
&\quad + 2 x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_4^2 + x_1^2 x_3^2 x_4 + x_1^2 x_3 x_4^2 \\
&\quad + x_1 x_2^2 x_3^2 + 2 x_1 x_2^2 x_3 x_4 + x_1 x_2^2 x_4^2 \\
&\quad + x_1 x_2 x_3^2 x_4 + x_1 x_2 x_3 x_4^2 + x_2^2 x_3^2 x_4 + x_2^2 x_3 x_4^2.
\end{aligned}$$

Notice that the final computation agrees with $\mathfrak{K}_{\mathbb{D}(a)}$ computed earlier.

Theorem 2.6 ([Kohnert 90]). *The Demazure character κ_a is equal to the Kohnert polynomial $\mathfrak{K}_{\mathbb{I}(a)}$, i.e.*

$$\kappa_a = \mathfrak{K}_{\mathbb{I}(a)}, \tag{2–5}$$

where $\mathbb{I}(a)$ is the key diagram for the indexing composition a.

Kohnert's algorithm for key diagrams precisely gives the monomial expansion of a Demazure character. Therefore, Kohnert polynomials are a generalization of Demazure characters.

Macdonald [Macdonald 91] noted that when a is weakly increasing of length n, we have $\kappa_a = s_{\mathrm{rev}(a)}(x_1, \ldots, x_n)$, where s_λ is the *Schur polynomial* that gives the irreducible characters for the general linear group. This follows as well from Demazure's perspective [Demazure 74] since the Demazure modules interpolate between the highest weight space and the full irreducible highest weight module. The Demazure characters are obtained from the irreducible characters by truncating, and so they are, in general, only partially symmetric. However, they are well-defined under stabilization and in the limit converge to the *Schur functions*. This result is implicit in [Lascoux and Schützenberger 90] and explicit in [Assaf and Searles 18].

Proposition 2.7. *For a weak composition a, we have*

$$\lim_{m \to \infty} \kappa_{0^m \times a}(x_1, \ldots, x_m, 0, \ldots, 0) = s_{\mathrm{sort}(a)}(x_1, x_2, \ldots), \tag{2–6}$$

where $0^m \times a$ denotes the weak composition obtained by prepending m zeros to a.

We will see below that Kohnert polynomials also stabilize, though not, in general, to symmetric functions.

2.3. Schubert polynomials

Schubert polynomials were introduced by Lascoux and Schützenberger [Lascoux and Schützenberger 82] as polynomial representatives for Schubert classes in the cohomology ring of the flag manifold for the general linear group. That is, they are polynomials indexed by permutations whose structure constants precisely correspond to those for the distinguished linear basis of the cohomology ring. They are defined by the divided difference operators, which Fulton [Fulton 92] showed have deep connections to modern intersection theory.

Definition 2.8 ([Lascoux and Schützenberger 82]). Given a permutation w, the *Schubert polynomial* $\mathfrak{S}_w$ is given by

$$\mathfrak{S}_w = \partial_{w^{-1} w_0} \left(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\right), \tag{2–7}$$

where $w_0 = n \cdots 21$ is the longest permutation of length $\binom{n}{2}$.

For example, for $w = 143625$, we have $w^{-1} w_0 = 462351$, and so

$$\begin{aligned}\mathfrak{S}_{143625} &= \partial_1\partial_2\partial_3\partial_4\partial_5\partial_4\partial_2\partial_3\partial_1\partial_2\left(x_1^5x_2^4x_3^3x_2^2x_1\right)\\ &= x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^3x_3x_4 + 2x_1^2x_2^2x_3 + 2x_1^2x_2^2x_4\\ &\quad + x_1^2x_2x_3^2 + 3x_1^2x_2x_3x_4\\ &\quad + x_1^2x_2x_4^2 + x_1^2x_3^2x_4 + x_1^2x_3x_4^2 + x_1x_2^3x_3\\ &\quad + x_1x_2^3x_4 + x_1x_2^2x_3^2 + 3x_1x_2^2x_3x_4\\ &\quad + x_1x_2^2x_4^2 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2 + x_2^3x_3x_4\\ &\quad + x_2^2x_3^2x_4 + x_2^2x_3x_4^2.\end{aligned}$$

For a permutation w with a unique descent at position k, we have $\mathfrak{S}_w = s_\lambda(x_1, \ldots, x_k)$, where λ is the partition given by $\lambda_{k-i+1} = w_i - k$. In particular, Schubert polynomials contain the Schur polynomials as a special case. In certain cases, including this so-called *grassmannian* case, a Schubert polynomial is equal to a Demazure character.

The *Rothe diagram* of a permutation w, denoted by $\mathbb{D}(w)$, is given by

$$\mathbb{D}(w) = \left\{(i, w_j) | i<j \text{ and } w_i>w_j\right\}. \tag{2–8}$$

For example, the middle diagram in Figure 1 is the Rothe diagram for 143625. Macdonald [Macdonald 91] used the Rothe diagram of a permutation to characterize precisely when a Schubert polynomial is equal to a Demazure character. Lascoux and Schützenberger [Lascoux and Schützenberger 85] first gave such a characterization in terms of pattern avoidance, and they termed permutations w for which $\mathfrak{S}_w = \kappa_a$ *vexillary* permutations.

Proposition 2.9 ([Macdonald 91]). *Given a permutation w, the following are equivalent*

i. *the row support of any two columns of $\mathbb{D}(w)$ are nested sets;*
ii. *the column support of any two rows of $\mathbb{D}(w)$ are nested sets;*
iii. *the Schubert polynomial $\mathfrak{S}_w$ is equal to a key polynomial.*

When $\mathfrak{S}_w = \kappa_a$, we have $a = \mathrm{wt}(\mathbb{D}(w))$.

Kohnert observed that his algorithm can be used on the Rothe diagram of a vexillary permutation to compute the Schubert polynomial, and he asserted that his rule worked for Schubert polynomials in general. For example, Figure 3 gives the Kohnert diagrams for $\mathbb{D}(143625)$, where we have deleted the empty column on the left since doing so does not affect the Kohnert polynomial. Note that the corresponding Kohnert polynomial is precisely the Schubert polynomial for 143625.

Two proofs of Kohnert's rule for Schubert polynomials appear in the literature by Winkel [Winkel 99, Winkel 02], though given the obscure and intricate nature of the arguments, they are not widely accepted. A direct, bijective proof by Assaf [Assaf 17] utilizes the expansion of Schubert polynomials into Demazure characters.

Theorem 2.10 ([Assaf 17, Winkel 99, Winkel 02]). *The Schubert polynomial $\mathfrak{S}_w$ is given by the Kohnert polynomial*

$$\mathfrak{S}_w = \mathfrak{K}_{\mathbb{D}(w)}, \tag{2–9}$$

where $\mathbb{D}(w)$ is the Rothe diagram for the indexing permutation w.

While the Schubert polynomials contain the Schur polynomials, and so are also a polynomial generalization of Schur polynomials, we argue that this fact has more to do with the result that Schubert polynomials expand as nonnegative sums of Demazure characters, and the latter naturally contains Schur polynomials.

Macdonald [Macdonald 91] showed that Schubert polynomials also stabilize and that their stable limits are the Stanley symmetric functions [Stanley 84] introduced by Stanley to study reduced expressions for a permutation. Stanley [Stanley 84] proved that these functions are symmetric, and Edelman and Greene [Edelman and Greene 87] showed that they are Schur positive. The Schur positivity also follows from Demazure positivity of Schubert polynomials in light of Proposition 2.7.

3. Monomial slide expansions

We begin our study of Kohnert polynomials by investigating their expansion in the *monomial slide basis* introduced in [Assaf and Searles 17]. In Section 3.1, we review quasisymmetric polynomials and monomial slide polynomials. In Section 3.2, we show that every Kohnert polynomial expands nonnegatively into the monomial slide basis allowing us to determine, in particular, when a Kohnert polynomial is quasisymmetric. In Section 3.3, we use the stable limit of monomial slide polynomials to define *Kohnert quasisymmetric functions*, which are the well-defined stable limits of Kohnert polynomials.

3.1. Monomial slide polynomials

A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is *quasisymmetric* if for every (strong) composition (i.e., sequence of positive integers) $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $\ell \leqslant n$, we have

$$\left[x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell} | f\right] = \left[x_{j_1}^{\alpha_1} \cdots x_{j_\ell}^{\alpha_\ell} | f\right] \tag{3–1}$$

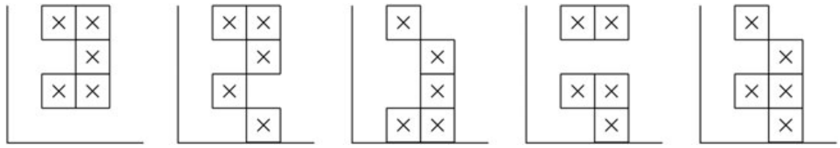

Figure 4. The set MKD(D) for D the leftmost diagram above.

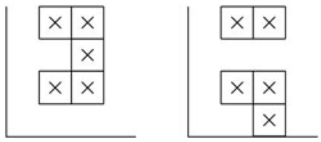

Figure 5. The set QYKD(D) of quasi-Yamanouchi Kohnert diagrams for D the leftmost diagram above.

for any two sequences $1 \leqslant i_1 < \cdots < i_\ell \leqslant n$ and $1 \leqslant j_1 < \cdots < j_\ell \leqslant n$, where $[x_{i_1}^{\alpha_1} \ldots x_{i_\ell}^{\alpha_\ell} | f]$ denotes the multiplicity of the monomial $x_{i_1}^{\alpha_1} \ldots x_{i_\ell}$ in the monomial expansion of f.

The ring of quasisymmetric functions plays a central role in algebraic combinatorics. Gessel [Gessel 84] initiated the study of quasisymmetric polynomials by introducing the monomial quasisymmetric functions that give an integral basis.

Given a weak composition a, let flat(a) denote the strong composition obtained by removing all zero parts from a. For example, $\text{flat}(0,2,1,0,2) = (2,1,2)$.

Definition 3.1 ([Gessel 84]). The monomial quasisymmetric polynomial indexed by α is

$$M_\alpha(x_1, \ldots, x_n) = \sum_{\text{flat}(b) = \alpha} x_1^{b_1} \cdots x_n^{b_n}, \tag{3–2}$$

where the sum is over all weak compositions of length n whose flattening gives α.

For example, take $\alpha = (2,1,2)$ and restricting to 4 variables, we have

$$M_{(2,1,2)}(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3^2 + x_1^2 x_2 x_4^2 + x_1^2 x_3 x_4^2 + x_2^2 x_3 x_4^2.$$

Assaf and Searles [Assaf and Searles 17] introduced a new basis for the polynomial ring, called *monomial slide polynomials*, that gives a natural polynomial generalization of monomial quasisymmetric functions.

Definition 3.2 ([Assaf and Searles 17]). The *monomial slide polynomial* indexed by a is

$$\mathfrak{M}_a = \sum_{\substack{b \geqslant a \\ \text{flat}(b) = \text{flat}(a)}} x_1^{b_1} \cdots x_n^{b_n}, \tag{3–3}$$

where $b \geqslant a$ means $b_1 + \cdots + b_k \geqslant a_1 + \cdots + a_k$ for all $k = 1, \ldots, n$.

For example, taking $a = (2,0,1,2)$, which implies 4 variables, we compute

$$\mathfrak{M}_{(2,0,1,2)}(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3^2 + x_1^2 x_2 x_4^2 + x_1^2 x_3 x_4^2.$$

As this example illustrates, monomial slide polynomials are not, in general, quasisymmetric. The following result characterizes when a monomial slide polynomial is quasisymmetric.

Proposition 3.3 ([Assaf and Searles 17]). *For a weak composition a of length n, $\mathfrak{M}_a$ is quasisymmetric in $x_1, \ldots, x_n$ if and only if $a_j \neq 0$ whenever $a_i \neq 0$ for some $i < j$. Moreover, in this case we have $\mathfrak{M}_a = M_{\text{flat}(a)}(x_1, \ldots, x_n)$.*

The monomial slide polynomials are a lifting of monomial quasisymmetric polynomials to the full polynomial ring. Remarkably, their structure constants are non-negative and generalize the quasi-shuffle product of Hoffman [Hoffman 00].

Theorem 3.4 ([Assaf and Searles 17]). *The monomial slide polynomials $\{\mathfrak{M}_a\}$ are a basis of the polynomial ring with structure constants*

$$\mathfrak{M}_a \mathfrak{M}_b = \sum_c [c \mid a \uplus b] \mathfrak{M}_c, \tag{3–4}$$

where $[c \mid a \uplus b]$ is the coefficient of c in the quasi-slide product $a \uplus b$. In particular, $[c \mid a \uplus b]$ is a non-negative integer.

Unlike Demazure characters and Schubert polynomials, they are not a Kohnert basis.

Proposition 3.5. *Monomial slide polynomials are not a Kohnert basis.*

Proof. For any diagram D of weight (0, 2), we claim $\mathfrak{K}_D \neq \mathfrak{M}_{(0,2)}$. If $\text{wt}(D) = (0,2)$, then KD(D) must have a diagram of weight (1, 1) by pushing the rightmost box in row 2. Therefore $\mathfrak{K}_D$ will contain the monomial $x_1 x_2$, which does not appear in $\mathfrak{M}_{(0,2)} = x_2^2 + x_1^2$. Therefore $\mathfrak{M}_{(0,2)}$ is not a Kohnert polynomial. □

3.2. Kohnert polynomials are monomial slide positive

For every row index $r \geqslant 1$, define an operator Up_r on diagrams that raises all cells in row r up to row $r+1$. While this is not, in general, well-defined, we will

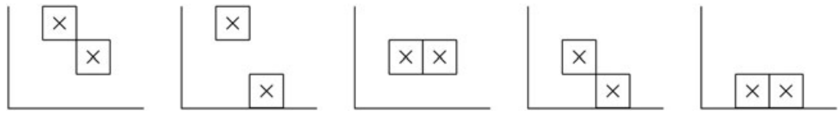

Figure 6. The Kohnert diagrams for the leftmost diagram above, for which the Kohnert polynomial is not fundamental slide positive.

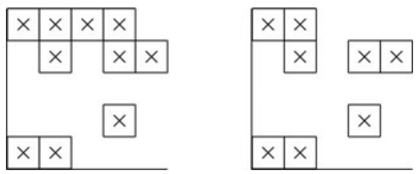

Figure 7. An example (left) and non-example (right) of the fundamental property.

only apply Up_r when no cell in row r sits immediately below a cell in row $r+1$. The following definition allows us to relate Kohnert polynomials with monomial slide polynomials.

Definition 3.6. For a diagram D, define the subset $\mathrm{MKD}(D)$ of Kohnert diagrams for D by

$$\mathrm{MKD}(D) = \{T \in \mathrm{KD}(D) | \mathrm{Up}_r(T) \notin \mathrm{KD}(D) \forall r \text{ such that } (r+1, c) \notin T \ \forall c\}. \tag{3–5}$$

For example, for D the third diagram in Figure 1, Figure 4 shows the set $\mathrm{MKD}(D)$. Notice that

$$\mathfrak{K}_D = \mathfrak{M}_{(0,2,1,2)} + \mathfrak{M}_{(1,1,1,2)} + \mathfrak{M}_{(2,1,1,1)} + \mathfrak{M}_{(1,2,0,2)} + \mathfrak{M}_{(1,2,1,1)},$$

which corresponds precisely to the weights of the diagrams in $\mathrm{MKD}(D)$.

Theorem 3.7. *Given any diagram D, we have*

$$\mathfrak{K}_D = \sum_{T \in \mathrm{MKD}(D)} \mathfrak{M}_{\mathrm{wt}(T)}. \tag{3–6}$$

In particular, Kohnert polynomials expand nonnegatively into monomial slide polynomials.

Proof. For $U \in \mathrm{KD}(D)$, define $\mathrm{Up}(U)$ to be the diagram resulting from applying Up_i operators sequentially to U, under the restriction that at each step the resulting diagram is in $\mathrm{KD}(D)$, until no Up_i operator can be applied without leaving $\mathrm{KD}(D)$. The diagram $\mathrm{Up}(U)$ is well-defined: Up_i only moves the cells in row i to the empty row $i+1$, so rows of cells retain their relative order, and if both $\mathrm{Up}_i(U), \mathrm{Up}_j(U) \in KD_D$ then $\mathrm{Up}_i\mathrm{Up}_j(U) \in KD_D$ and $\mathrm{Up}_j\mathrm{Up}_i(U) = \mathrm{Up}_i\mathrm{Up}_j(U)$ so it does not matter in what order the operators are applied. By Definition 3.6, $\mathrm{Up}(U) \in \mathrm{MKD}(D)$ and if $U \in \mathrm{MKD}(D)$, then $\mathrm{Up}(U) = U$. Hence Up is a retraction of $\mathrm{KD}(D)$ onto $\mathrm{MKD}(D)$, and thus partitions $\mathrm{KD}(D)$ into equivalence classes, each of which contains exactly one element of $\mathrm{MKD}(D)$.

To complete the proof, we need to show that

$$\sum_{U:\mathrm{Up}(U)=T} x^{\mathrm{wt}(U)} = \mathfrak{M}_{\mathrm{wt}(T)}.$$

Suppose $\mathrm{Up}(U) = T$. Since Up_i moves an entire row without consolidating rows, we have $\mathrm{flat}(U) = \mathrm{flat}(T)$ for any such U. Moreover, since rows move up, we also have $\mathrm{wt}(U) \geqslant \mathrm{wt}(T)$. Finally, for any weak composition b such that $\mathrm{flat}(b) = \mathrm{flat}(\mathrm{wt}(T))$ and $b \geqslant \mathrm{wt}(T)$, we may construct $U \in \mathrm{KD}(D)$ such that $\mathrm{wt}(b) = U$ and $\mathrm{Up}(U) = T$ uniquely as follows. First suppose T has only one row. Then this row can be moved down, cell by cell from right to left until all cells sit in the row indexed by the nonzero part of b; this is a sequence of Kohnert moves. Now suppose T has $k > 1$ rows, and assume inductively that U can be constructed whenever T has fewer than k rows. Begin by moving the lowest row of T down to the row indexed by the leftmost nonzero part of b. Let b' be the weak composition obtained by setting the leftmost nonzero entry of b equal to zero, and T' the diagram obtained by deleting the lowest row of T. By induction, we may construct a diagram U' such that $\mathrm{wt}(U') = b'$ and $\mathrm{Up}(U') = T'$. Since, we begin by moving the lowest row of T down to the row of the leftmost nonzero entry of b, we may follow by constructing U' from the remaining T'; the result is a diagram U of weight b' such that $\mathrm{Up}(U) = T$, as required. □

Combining Theorem 3.7 and Proposition 3.3, we have the following characterization of when a Kohnert polynomial is quasisymmetric.

Proposition 3.8. *Let D be a diagram with highest occupied row r. The polynomial* $\mathfrak{K}_{0^m \times D}$ *is quasisymmetric for all* $m \geqslant 0$ *if and only if the set of columns containing cells of D in row i is a subset of the set of columns containing cells of D in row* $i+1$, *for all* $i < m + r$.

Proof. Suppose there is a cell in row $i < r$ of D which has no cell immediately above it. If row $i+1$ is empty, then the monomial M obtained from $x^{\mathrm{wt}(D)}$ by replacing each x_i with x_{i+1} does not belong to $\mathcal{K}_D$, hence

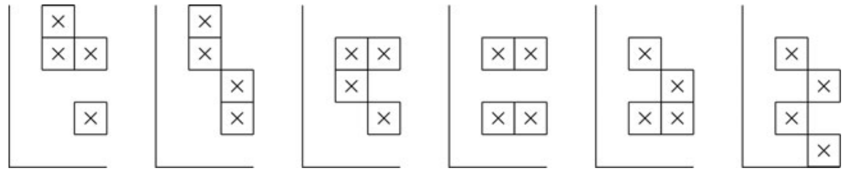

Figure 8. The elements of MKD(D) for D the leftmost diagram, which is not fundamental.

$\mathcal{K}_D$ is not quasisymmetric. If row $i+1$ of D is nonempty, consider $0 \times D$. Then from right to left, perform a single Kohnert move on all cells in row $i+2$ of $0 \times D$ (the reason for passing to $0 \times D$ is to ensure that these Kohnert moves can be performed). Let N be the associated monomial of the resulting Kohnert diagram. Then the monomial obtained from N by replacing x_j with x_{j+1} for all $j \leqslant i$ does not belong to $\mathcal{K}_{0\times D}$, hence $\mathcal{K}_{0\times D}$ is not quasisymmetric.

Conversely, if the rows of D are nested with smaller rows below larger ones, then each $T \in \mathrm{MKD}(0^m \times D)$ will satisfy the condition that if there is a cell in row $i<m+r$ of T, then there is a cell in row $i+1$ of T. Therefore, by Proposition 3.3, each term in the monomial slide polynomial expansion will be quasisymmetric. □

Notice that in Proposition 3.8 we characterize only when $\mathfrak{K}_{0^m\times D}$ is quasisymmetric *for all* $m \geqslant 0$. The reason for this caveat is that Kohnert polynomials for diagrams close to the x-axis can be quasisymmetric due to lack of room to move rows down. For example, taking any right-justified diagram with at least one cell in rows $1, 2, \ldots, r$ and no cells above row r, the corresponding Kohnert polynomial will be quasisymmetric in $x_1, x_2, \ldots, x_r$ trivially, since it is equal to a single monomial with all variables appearing with positive exponent. In order to avoid these somewhat artificial cases and to connect with quasisymmetric *functions* below, we are primarily interested in quasisymmetry that exists even when empty rows are inserted below the diagram.

3.3. Kohnert quasisymmetric functions

The ring of quasisymmetric functions is the inverse limit of quasisymmetric polynomials. The monomial and fundamental quasisymmetric polynomials stabilize to the monomial and fundamental quasisymmetric functions when the number of variables tends to infinity.

Assaf and Searles [Assaf and Searles 17] showed that the monomial slide polynomials stabilize and that their stable limits are precisely the monomial quasisymmetric functions.

Theorem 3.9 ([Assaf and Searles 17]). *For a weak composition a, we have*

$$\lim_{m\to\infty} \mathfrak{M}_{0^m\times a}(x_1, \ldots, x_m, 0, \ldots, 0) = M_{\mathrm{flat}(a)}(x_1, x_2, \ldots), \tag{3–7}$$

where $0^m \times a$ denotes the weak composition obtained by prepending m 0's to a.

Let $0^m \times D$ denote the diagram of D shifted up vertically by m rows. For example, once again taking D to be the third diagram in Figure 1, we may compute

$$\begin{aligned}\mathfrak{K}_{0\times D} = {} & \mathfrak{M}_{(0,0,2,1,2)} + \mathfrak{M}_{(0,1,1,1,2)} + \mathfrak{M}_{(1,1,1,0,2)} + \mathfrak{M}_{(0,2,1,1,1)} \\ & + \mathfrak{M}_{(0,1,2,0,2)} + \mathfrak{M}_{(0,1,2,1,1)} + \mathfrak{M}_{(1,1,2,0,1)} + 2\mathfrak{M}_{(1,1,1,1,1)}.\end{aligned}$$

Moreover, for any $m \geqslant 0$, we have the following expansion,

$$\begin{aligned}\mathfrak{K}_{0^{m+2}\times D} = {} & \mathfrak{M}_{0^m\times(0,0,0,2,1,2)} + \mathfrak{M}_{0^m\times(0,0,1,1,1,2)} + \mathfrak{M}_{0^m\times(0,1,1,1,0,2)} \\ & + \mathfrak{M}_{0^m\times(0,0,2,1,1,1)} + \mathfrak{M}_{0^m\times(0,0,1,2,0,2)} + \mathfrak{M}_{0^m\times(0,0,1,2,1,1)} \\ & + \mathfrak{M}_{0^m\times(0,1,1,2,0,1)} + 2\mathfrak{M}_{0^m\times(0,1,1,1,1,1)} + \mathfrak{M}_{0^m\times(1,1,1,1,0,1)}.\end{aligned}$$

In particular, the monomial slide expansion of the Kohnert polynomial eventually stabilizes. Inspired by this, we may consider the stable limit of Kohnert polynomials in the following sense.

Definition 3.10. The *Kohnert quasisymmetric* function indexed by D is

$$\mathcal{K}_D(X) = \lim_{m\to\infty} \mathfrak{K}_{0^m\times D}(x_1, \ldots, x_m, 0, \ldots, 0), \tag{3–8}$$

where $0^m \times D$ denotes the diagram of D shifted up vertically by m rows.

For example, continuing with D the third diagram in Figure 1, we have

$$\begin{aligned}\mathcal{K}_D = {} & M_{(2,1,2)} + 2M_{(1,1,1,2)} + M_{(2,1,1,1)} + M_{(1,2,2)} \\ & + M_{(1,2,1,1)} + M_{(1,1,2,1)} + 3M_{(1,1,1,1,1)}.\end{aligned}$$

Theorem 3.11. *For any diagram D, $\mathcal{K}_D(X)$ is a well-defined quasisymmetric function that expands nonnegatively into the monomial quasisymmetric functions.*

Proof. For $T \in \mathrm{MKD}(D)$, the map $T \mapsto 0 \times T$ induces an injection $\mathrm{MKD}(D) \to \mathrm{MKD}(0 \times D)$ since

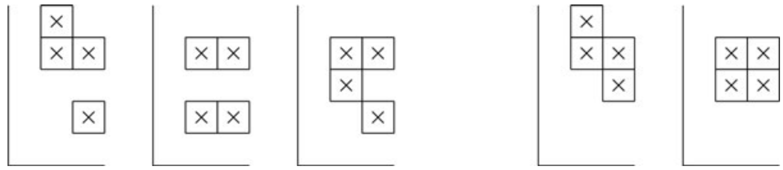

Figure 9. The three quasi-Yamanouchi Kohnert diagrams for the leftmost diagram (left), which is not itself fundamental, and the two quasi-Yamanouchi Kohnert diagrams for the fourth diagram (right), which is fundamental.

Definition 3.6 is a local condition about the cells in relative positions.

For any diagram D, if $T \in \mathrm{KD}(0^m \times D)$ has no cells in row $j < m$ and row $j-1$ is nonempty, then $\mathrm{Up}_{j-1}(T) \in \mathrm{KD}(0^m \times D)$, so by Definition 3.6, $T \notin \mathrm{MKD}(0^m \times D)$. In particular, if the bottom row of a diagram $T \in \mathrm{MKD}(0^m \times D)$ is occupied, then so are all rows $j \leqslant m$. The only possible terms not in the image of the injection $\mathrm{MKD}(0^m \times D) \to \mathrm{MKD}(0^{m+1} \times D)$ are those $T \in \mathrm{MKD}(0^{m+1} \times D)$ with at least one cell in the bottom row. Therefore, if D is a diagram with m cells, then the injection $\mathrm{MKD}(0^m \times D) \to \mathrm{MKD}(0^{m+1} \times D)$ must be a bijection.

Therefore, the result follows from Theorem 3.7 and Theorem 3.9. □

We review two motivating examples of this stability. For D a Rothe diagram for w, the Kohnert polynomial is a Schubert polynomial and the stable limit, as shown by Macdonald [Macdonald 91], is the Stanley symmetric function [Stanley 84] introduced by Stanley to enumerate reduced expressions for a permutation. Implicit in the work of Lascoux and Schützenberger [Lascoux and Schützenberger 90] and explicit in [Assaf and Searles 18], the Kohnert polynomial of a key diagram stabilizes to the Schur function indexed by the partition to which the weak composition sorts. Both of these examples have stable limits that are symmetric functions, but Kohnert quasisymmetric functions are not always symmetric. In Section 6, we consider a new Kohnert basis that gives rise to an interesting new basis for quasisymmetric functions.

As remarked earlier, two diagrams that differ by insertion or deletion of empty columns give rise to the same Kohnert polynomial. In the stable limit, we can strengthen this with the following.

Proposition 3.12. *Given two diagrams D and D' that differ by insertion or deletion of empty rows, we have $\mathcal{K}_D = \mathcal{K}_{D'}$.*

Proof. Fix a positive integer m. Then $\mathcal{K}_D(x_1, \ldots, x_m, 0, 0, \ldots)$ is the weighted sum of all Kohnert diagrams of D whose highest cell is weakly below row m, and similarly for $\mathcal{K}_{D'}(x_1, \ldots, x_m, 0, 0, \ldots)$. Slightly abusing notation, let $\mathrm{flat}(D)$ be the diagram obtained by deleting all empty rows from D. Then any Kohnert diagram of $0^m \times D$ whose highest cell is weakly below row m is also a Kohnert diagram of $0^m \times \mathrm{flat}(D)$. That is, we have an injective map from the subset of $\mathrm{KD}(0^m \times D)$ with no cell above row m to $\mathrm{KD}(0^m \times \mathrm{flat}(D))$. By definition $\mathrm{flat}(D) = \mathrm{flat}(D')$, therefore $\mathcal{K}_D(x_1, \ldots, x_m, 0, 0, \ldots) = \mathcal{K}_{D'}(x_1, \ldots, x_m, 0, 0, \ldots)$. The statement then follows by letting $m \to \infty$. □

4. Fundamental slide expansions

Strengthening the results of the previous section, we next investigate the fundamental slide expansion of a Kohnert polynomial, which is not always nonnegative. In Section 4.1, we review the basis of fundamental slide polynomials introduced in [Assaf and Searles 17]. In Section 4.2, we characterize those diagrams for which the corresponding Kohnert polynomial expands nonnegatively into the fundamental slide basis. Further, we conjecture a simple condition on diagrams that ensures the corresponding Kohnert polynomial expands nonnegatively into the Demazure character basis. In Section 4.3, we prove the surprising fact that, while some Kohnert polynomials are not positive on the fundamental slide basis, every Kohnert quasisymmetric function is positive on the fundamental quasisymmetric function basis.

4.1. Fundamental slide polynomials

Gessel [Gessel 84] introduced another basis for quasisymmetric functions that is closely related to Schur functions.

Given two compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_m)$ such that $\alpha_1 + \cdots + \alpha_\ell = \beta_1 + \cdots + \beta_m$, say that β *refines* α if there exist indices $1 = i_0 < i_1 < \cdots < i_\ell$ such that for all $0 \leqslant j < \ell$ we have $\beta_{i_j+1} + \cdots + \beta_{i_{j+1}} = \alpha_{j+1}$. For example, (1, 2, 2) refines (3, 2) but does not refine (2, 3).

Definition 4.1 ([Gessel 84]). The *fundamental quasisymmetric polynomial* indexed by α is

$$F_\alpha(x_1,\dots,x_n) = \sum_{\beta \text{ refines } \alpha} M_\beta(x_1,\dots,x_n) = \sum_{\text{flat}(b) \text{ refines } \alpha} x_1^{b_1}\cdots x_n^{b_n}, \quad (4\text{–}1)$$

where the latter sum is over all weak compositions of length n whose flattening refines α.

For example, taking $\alpha = (2,1,2)$ and restricting to 4 variables, we have

$$\begin{aligned} F_{(2,1,2)}(x_1,x_2,x_3,x_4) &= M_{(2,1,2)}(x_1,x_2,x_3,x_4) \\ &\quad + M_{(1,1,1,2)}(x_1,x_2,x_3,x_4) \\ &\quad + M_{(2,1,1,1)}(x_1,x_2,x_3,x_4), \\ &= x_1^2x_2x_3^2 + x_1^2x_2x_4^2 + x_1^2x_3x_4^2 \\ &\quad + x_2^2x_3x_4^2 + x_1x_2x_3x_4^2 + x_1^2x_2x_3x_4. \end{aligned}$$

The fundamental quasisymmetric functions inspired the *fundamental slide polynomials* of Assaf and Searles [Assaf and Searles 17], analogous to the relationship between the monomial slide polynomials and the monomial quasisymmetric polynomials.

Definition 4.2 ([Assaf and Searles 17]). The *fundamental slide polynomial* indexed by a is

$$\mathfrak{F}_a = \sum_{\substack{b \geqslant a \\ \text{flat}(b) \text{ refines flat}(a)}} x_1^{b_1}\cdots x_n^{b_n}, \quad (4\text{–}2)$$

where $b \geqslant a$ means $b_1 + \cdots + b_k \geqslant a_1 + \cdots + a_k$ for all $k = 1,\dots,n$.

For example, taking $a = (2,0,1,2)$, which implies 4 variables, we compute

$$\begin{aligned} \mathfrak{F}_{(2,0,1,2)} &= \mathfrak{M}_{(2,0,1,2)} + \mathfrak{M}_{(2,1,1,1)} = x_1^2x_2x_3^2 + x_1^2x_2x_4^2 \\ &\quad + x_1^2x_3x_4^2 + x_1^2x_2x_3x_4. \end{aligned}$$

The fundamental slide polynomials generalize the fundamental quasisymmetric polynomials.

Proposition 4.3 ([Assaf and Searles 17]). *For a weak composition a of length n, $\mathfrak{F}_a$ is quasisymmetric in $x_1,\dots,x_n$ if and only if $a_j \neq 0$ whenever $a_i \neq 0$ for some $i < j$. Moreover, in this case we have $\mathfrak{F}_a = F_{\text{flat}(a)}(x_1,\dots,x_n)$.*

The fundamental slide polynomials give a basis for the polynomial ring [Assaf and Searles 17]. Remarkably, their structure constants are nonnegative and generalize the shuffle product of Eilenberg and Mac Lane [Eilenberg and Lane 53].

Theorem 4.4 ([Assaf and Searles 17]). *The fundamental slide polynomials $\{\mathfrak{F}_a\}$ are a basis of the polynomial ring with structure constants*

$$\mathfrak{F}_a\mathfrak{F}_b = \sum_c [c|a ⧢ b]\mathfrak{F}_c, \quad (4\text{–}3)$$

where $[c|a ⧢ b]$ is the coefficient of c in the slide product $a ⧢ b$. In particular, $[c|a ⧢ b]$ is a nonnegative integer.

As was the case for the monomial slide polynomials, the fundamental slide polynomials are also not a Kohnert basis.

Proposition 4.5. *The fundamental slide polynomials are not a Kohnert basis.*

Proof. For any diagram D of weight (0, 2, 1), we claim $\mathfrak{K}_D \neq \mathfrak{F}_{(0,2,1)}$. If this is a Kohnert polynomial $\mathfrak{K}_D$, then D must have two cells in row 2 and one in row 3. Let D be any such diagram. Since there is only one cell, say c in row 3 of D, this cell is the rightmost in its row. If c is in the same column as some cell in row 2 then applying a Kohnert move to c yields a diagram of weight (1, 2, 0), otherwise applying a Kohnert move to c yields a diagram of weight (0, 3, 0). Thus $\mathfrak{K}_D$ must have one of $x_1x_2^2$ or x_2^3 appear as a term. However, neither of these terms appears in $\mathfrak{F}_{(0,2,1)} = x_2^2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1^2x_2$, and so $\mathfrak{F}_{(0,2,1)}$ cannot be a Kohnert polynomial. □

4.2. Fundamental diagrams

One motivation for defining and studying the fundamental slide polynomials is a refined expansion of Schubert polynomials [Assaf and Searles 17]. This formula utilizes the *pipe dream* model for the monomial expansion of Schubert polynomials given by Bergeron and Billey [Bergeron and Billey 93] based on the *compatible sequences* model due to Billey, Jockusch, and Stanley [Billey et al. 93].

Theorem 4.6 ([Assaf and Searles 17]). *For w a permutation, we have*

$$\mathfrak{S}_w = \sum_{P \in \text{QPD}(w)} \mathfrak{F}_{\text{wt}(P)}, \quad (4\text{–}4)$$

where the sum is over quasi-Yamanouchi pipe dreams for w.

For example, the Schubert polynomial for the permutation 143625 is

$$\begin{aligned} \mathfrak{S}_{143625} &= \mathfrak{F}_{(0,2,1,2)} + \mathfrak{F}_{(1,2,0,2)} + \mathfrak{F}_{(0,2,2,1)} + \mathfrak{F}_{(0,3,1,1)} \\ &\quad + \mathfrak{F}_{(1,2,1,1)} + \mathfrak{F}_{(1,3,0,1)} + \mathfrak{F}_{(2,2,0,1)}. \end{aligned}$$

Assaf and Searles [Assaf and Searles 18] also show that the Demazure characters have a natural decomposition into fundamental slide polynomials. This

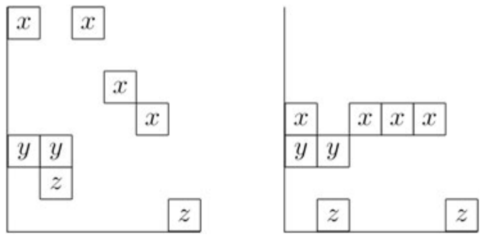

Figure 10. A split diagram U (left) and the diagram drop(U) (right). In each diagram the threading is given by labeling of cells with x, y, and z.

formula utilized Kohnert's model for Demazure characters [Kohnert 90].

Theorem 4.7 ([Assaf and Searles 18]). *For a weak composition a, we have*

$$\kappa_a = \sum_{T \in \mathrm{QKT}(a)} \mathfrak{F}_{\mathrm{wt}(T)}, \tag{4–5}$$

where the sum is over all quasi-Yamanouchi Kohnert tableaux for a.

For example, the Demazure character for the weak composition $(0,2,1,2)$ decomposes as

$$\kappa_{(0,2,1,2)} = \mathfrak{F}_{(0,2,1,2)} + \mathfrak{F}_{(1,2,0,2)} + \mathfrak{F}_{(0,2,2,1)} + \mathfrak{F}_{(1,2,1,1)}.$$

Generalizing these two examples, along with the common notion of *quasi-Yamanouchi* used in both expansions, we have the following.

Definition 4.8. For a diagram D, define the subset of quasi-Yamanouchi Kohnert diagrams for D, denoted by $\mathrm{QYKD}(D)$, by

$$\mathrm{QYKD}(D) = \left\{ T \in \mathrm{KD}(D) \;\middle|\; \begin{array}{c} \mathrm{Up}_r(T) \notin \mathrm{KD}(D) \forall r \\ \text{such that all cells in} \\ \text{row } r+1 \text{ lie strictly left} \\ \text{of all cells in row } r \end{array} \right\}. \tag{4–6}$$

Note that $\mathrm{QYKD}(D) \subseteq \mathrm{MKD}(D)$. For example, for D the third diagram in Figure 1, Figure 5 shows the set $\mathrm{QYKD}(D)$. Notice that

$$\mathfrak{K}_D = \mathfrak{F}_{(0,2,1,2)} + \mathfrak{F}_{(1,2,0,2)},$$

which corresponds precisely to the weights of the diagrams in $\mathrm{QYKD}(D)$.

Similarly, seven of the Kohnert diagrams in Figure 3 are in $\mathrm{QYKD}(\mathbb{D}(143625))$ and four of the Kohnert diagrams in Figure 2 are in $\mathrm{QYKD}(\mathbb{D}(0,2,1,2))$. The fundamental slide generating polynomials of these two sets are $\mathfrak{S}_{143625}$ and $\kappa_{(0,2,1,2)}$, respectively.

Similar to the proof of Theorem 3.7, we wish to consolidate Kohnert diagrams into equivalence classes, each of which contains a unique quasi-

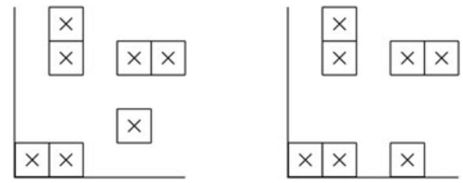

Figure 11. The left diagram is not southwest; the right is southwest.

Yamanouchi Kohnert diagram, so that the fundamental slide expansion of the corresponding Kohnert polynomials is precisely given by the quasi-Yamanouchi Kohnert diagrams. However, unlike the case with monomial slide polynomials, Kohnert polynomials are not, in general, fundamental slide positive. For example, taking D to be the left diagram in Figure 6, the corresponding Kohnert polynomial expands as

$$\mathfrak{K}_D = \mathfrak{M}_{(0,1,1)} + \mathfrak{M}_{(0,2,0)} = \mathfrak{F}_{(0,1,1)} + \mathfrak{F}_{(0,2,0)} - \mathfrak{F}_{(1,1,0)}.$$

The impediment to fundamental slide positivity is captured by the following notion.

Definition 4.9. A diagram is split if there exist rows $r_1 < r_2$ and columns $c_1 < c_2$ such that there are cells in positions (r_2, c_1) and (r_1, c_2) but no cells in rows r for $r_1 < r < r_2$ and no cells in positions (r_1, c) for $c < c_2$ or (r_2, c) for $c > c_1$. In this case, we call the cells (r_2, c_1) and (r_1, c_2) a split pair.

That is, a diagram is split if it contains two cells with one strictly northwest of the other such that no other cells lie between them in the reading order that reads left to right along rows, starting with the highest row. For example, the first, second and fourth diagrams in Figure 6 are split, and neither of the diagrams in Figure 5 is split. Indeed, none of the diagrams in $\mathrm{QYKD}(\mathbb{D}(143625))$ nor in $\mathrm{QYKD}(\mathbb{D}(0,2,1,2))$ is split.

Lemma 4.10. *Let $U \in \mathrm{KD}(D)$ be such that both $S = \mathrm{Up}_{i_k} \cdots \mathrm{Up}_{i_1}(U)$ and $T = \mathrm{Up}_{j_l} \cdots \mathrm{Up}_{j_1}(U)$ raise rows only when all cells of the row above lie strictly to the left. If $S, T \in \mathrm{QYKD}(D)$ and $S \neq T$, then at least one of S, T is split.*

Proof. Suppose T is non-split. Since each raising of a row either moves a row up or consolidates two rows, for $U \in \mathrm{KD}(D)$, if U has a path to $T \in \mathrm{QYKD}(D)$, then the rows of T are unions of the rows of U, taken in order and moved weakly up. Suppose that U has another raising path to S. Consider the highest row, say r_2, in which S and T differ, say with T having t cells and S having s cells in row r_2. Without loss of generality, we may assume $s<t$. Then T must have consolidated more rows of U into its row r_2. Therefore, row r_2 of S must consist of the s leftmost cells of row r_2 of T. Set c_1 to be the column of the sth cell from the left in row r_2 of T (equivalently, S), and let c_2 be the column of the next cell to its right. Let $r_1<r_2$ be the highest nonempty of S below r_2. Then the leftmost cell of r_1 in S must lie in column c_2. In particular, S has cells in positions (r_2, c_1) and (r_1, c_2) with no cells in between, and so S is split. □

Note that if split quasi-Yamanouchi diagrams for D exist, then a Kohnert diagram for D can have different raising paths resulting in different quasi-Yamanouchi diagrams, although at most one of these resulting diagrams can be non-split. For example, taking D to be the leftmost diagram in Figure 6, consider U to be the fourth diagram from the left. Then T has two raising paths, namely $\mathrm{Up}_1\mathrm{Up}_2(T)$ which terminates in the leftmost diagram and $\mathrm{Up}_1(T)$ which terminates in the third diagram, both of which are quasi-Yamanouchi.

While Lemma 4.10 is sufficient to guarantee that raising paths converge to the same quasi-Yamanouchi diagram in the case that no quasi-Yamanouchi diagram is split, it is possible this may hold even if split quasi-Yamanouchi diagrams exist. For instance, if we take D to be the fourth diagram from the left in Figure 6, then $\mathrm{KD}(D) = \mathrm{QYKD}(D)$ consists of the fourth and fifth diagrams, neither of which can raise. Even though the fourth diagram is split, all raising paths to any QYKD are trivially unique. However, this is somewhat accidental since shifting D up one row gives the leftmost diagram in Figure 6 which we have just seen fails to have well-defined quasi-Yamanouchi raisings.

Definition 4.11. A diagram D is fundamental if for each cell (r, c) of D that is leftmost in its row, either there is a cell in position $(r+1, c)$, or for each column $c'<c$ and for all $k\geqslant 1$ we have

$$\#\{(s,c') \in D | r<s\leqslant r+k\} \leqslant \#\{(s,c) \in D | r<s\leqslant r+k\}. \tag{4–7}$$

For example, see Figure 7. The diagram on the left is fundamental. The diagram on the right is not fundamental: the box in row 2, column 4 fails the condition with respect to the second column when $k=3$.

In both key diagrams and Rothe diagrams, a cell that is leftmost in its row cannot have any cell strictly above and strictly left, whence both are examples of fundamental diagrams. The significance of Definition 4.11 is that fundamental diagrams are precisely those whose quasi-Yamanouchi diagrams are all non-split, with the exception of certain diagrams with nonempty rows close to the x-axis for which split diagrams cannot arise given insufficient room to move rows down. To prove this, we begin with the following.

Lemma 4.12. *If D has a split quasi-Yamanouchi Kohnert diagram, then D has a split quasi-Yamanouchi Kohnert diagram in which the bottom-right cell of the split pair is in its original row in D.*

Proof. Take any split $T \in \mathrm{QYKD}(D)$. By definition, *some* cell in the row of the bottom-right cell of the split pair is in its original row in D, otherwise this row could be raised in T, contradicting that T is a quasi-Yamanouchi Kohnert diagram. Let c be the leftmost such cell. If c is not leftmost in its row, then from left to right, perform reverse Kohnert moves (no jumps) on the cells to the left of c in the row of c until they either reach their original row or land to the right of an existing cell, whichever happens first. This process creates another quasi-Yamanouchi Kohnert diagram for D, and, in particular, it is split on cell c and the cell that was closest (on the left) to c in T. □

By Lemma 4.10, for D a fundamental diagram and for any $U \in \mathrm{KD}(D)$, we may define the de-standardization of U, denoted by $\mathrm{dst}_D(U)$, to be the result of any maximal length raising path. Note that $\mathrm{dst}_D(U)$ is necessarily a quasi-Yamanouchi Kohnert diagram for D.

Theorem 4.13. *If a diagram D is fundamental, then no quasi-Yamanouchi Kohnert diagram for D is split. Conversely, if no quasi-Yamanouchi Kohnert diagram for $0^{|D|} \times D$ is split, then D is fundamental.*

Proof. First suppose that D is not fundamental. We will construct a split quasi-Yamanouchi Kohnert diagram of $0^{|D|} \times D$. We may assume there are no cells strictly above and strictly right of the cell (r, c) in $0^{|D|} \times D$: if there are, then one can push all such cells down to be weakly below row r, so that the top-left of all these cells now lies in row r. These cells are now all "anchored" on the cell in position (r, c), so the result is a quasi-Yamanouchi Kohnert diagram of $0^{|D|} \times D$.

Select the rightmost column $c'<c$ and then smallest k such that the condition fails. In what follows, we work entirely in the rectangle between rows r and $r+k$ (inclusive) and between columns c' and c (inclusive). Working from right to left, push cells in all columns $c', c'+1, \ldots c-1$ downwards (jumping over other cells if necessary) so that each of these cells ends up

in the same row as some cell in column c. This is possible since the condition is satisfied on these columns. Now, all rows $r+1, \ldots, r+k$ that have no cell in column c' also have no cell in any column to the right of c'. Since we never moved any cell in column c, these rows cannot be de-standardized, so the result is a quasi-Yamanouchi Kohnert diagram of $0^{|D|} \times D$.

Now consider the top cell C of column c' (in rows weakly lower than $r+k$). This cell C has no cell to its right in its row, in particular, there is no cell in the same row in column c, since this would contradict the minimality of k. Also, by minimality of k and the argument of the previous paragraph, every row strictly between row r and the row of C either has cells in both columns c' and c, or neither. To complete the construction, take the cell C and move it downwards, jumping over all cells in column c', rows $r+1, \ldots, r+k$. The cell C ends in the row immediately below the lowest cell of column c that is above (r, c), which, by assumption, is in row $r+2$ or higher. By construction this is a quasi-Yamanouchi Kohnert diagram for $0^{|D|} \times D$, and it is split over the cell C and the cell in position (r, c).

For the other direction, suppose the D satisfies the fundamental condition. We claim the cell (r, c) can never be the lower-right cell in the split pair of a split quasi-Yamanouchi Kohnert diagram of D. In particular D has no split quasi-Yamanouchi Kohnert diagram where the bottom-right cell is in its original position, so by Lemma 4.12 D has no split quasi-Yamanouchi Kohnert diagrams at all. To prove this claim, we need to show that no cell strictly above and left of (r, c) in D can ever get strictly below the lowest cell of D in the positions $(r+1, c), (r+2, c), \ldots (r+k, c)$ via a series of Kohnert moves, while remaining strictly above row r. This is true if D has a cell in position $(r+1, c)$. Otherwise, consider any column c' strictly left of column c. The condition states that for any cell in column c (above row r), there are at least as many cells of D weakly below this cell in column c as there are in column c'. Performing Kohnert moves on the cells of column c does not alter this, so we may suppose we do not perform any Kohnert moves on column c. Moreover, for simplicity, we may assume there are no cells of D in columns between c' and c and above row r, since any such cells are to the right of column c' and so only impede cells of column c' from moving downwards. Suppose we can perform a Kohnert move on a cell C of column c'. This means there is no cell in column c in the row of C, therefore the condition implies there are strictly more cells in column c strictly below C than there are in column c' strictly below C.

Now perform the Kohnert move on C. The condition is preserved for cells of c' above the original position of C and strictly below the position where C lands. For cell C itself, the condition is preserved since if C jumps over, say, ℓ cells for some $\ell \geqslant 0$, then C necessarily moves from being strictly above to strictly below weakly fewer than ℓ cells of column c (with equality only if the all of the first ℓ positions in column c strictly below the row of C are occupied by cells). Finally, consider any cell that C jumps over. By definition, all such cells exist in a column interval immediately below C. Therefore, if any one of these cells, say X, met the condition with equality, C would necessarily fail the condition since there are strictly more cells in column c' than there are in column c, in the rows between the row of X and the row of C (inclusive). Therefore, for any such cell X, there are strictly more cells of D weakly below X in column c than there are in column c' (above row r). Thus, when C moves from above to below X, the condition is maintained on X. Therefore, the condition is preserved under Kohnert moves on column c', which in particular implies that no cell strictly above and left of (r, c) in D can ever get strictly below the lowest cell of D in the positions $(r+1, c), (r+2, c), \ldots (r+k, c)$. □

Combining Lemma 4.10 and Theorem 4.13, we may give the fundamental slide expansion of a Kohnert polynomial indexed by a fundamental diagram.

Theorem 4.14. *Given a fundamental diagram D, we have*

$$\mathfrak{K}_D = \sum_{T \in \mathrm{QYKD}(D)} \mathfrak{F}_{\mathrm{wt}(T)}. \tag{4–8}$$

In particular, these Kohnert polynomials expand non-negatively into fundamental slide polynomials.

Proof. Recall for $U \in \mathrm{KD}(D), \mathrm{dst}_D(U) \in \mathrm{QYKD}(D)$ is the result of any maximal length raising path. If $\mathrm{dst}_D(U) = T$, then $\mathrm{wt}(U) \geqslant \mathrm{wt}(T)$ and $\mathrm{flat}(\mathrm{wt}(U))$ refines $\mathrm{flat}(\mathrm{wt}(T))$ since T is obtained recursively by moving *all* cells in row $i-1$ of U to row i in U. Conversely, we claim that given $T \in \mathrm{QYKD}(D)$, for every weak composition b of length n such that $b \geqslant \mathrm{wt}(T)$ and $\mathrm{flat}(b)$ refines $\mathrm{flat}(\mathrm{wt}(T))$, there is a unique $U \in \mathrm{KD}(D)$ with $\mathrm{wt}(U) = b$ such that $\mathrm{dst}_D(U) = T$. From the claim, we have

$$\sum_{U \in \mathrm{dst}_D^{-1}(T)} x^{\mathrm{wt}(U)} = \mathfrak{F}_{\mathrm{wt}(T)},$$

from which theorem follows. To construct U from b and T, for $j = 1, \ldots, n$, if $\mathrm{wt}(T)_j = b_{i_{j-1}+1} + \cdots + b_{i_j}$, then, from right to left, move the first $b_{i_{j-1}+1}$ cells down to row $i_{j-1}+1$, the next $b_{i_{j-1}+2}$ cells down to row $i_{j-1}+2$, and so on. Each of these moves is a valid Kohnert move with no cells jumping over any others. Existence is proved, and uniqueness follows from the lack of choice at every step. □

Both Schubert polynomials and Demazure characters expand non-negatively into fundamental slide polynomials, with the former indexed by quasi-Yamanouchi pipe dreams and the latter by quasi-Yamanouchi Kohnert tableaux. Since both Rothe diagrams and key diagrams are fundamental, the expansion in (4–8) gives a common generalization of these results.

4.3. Kohnert quasisymmetric functions are fundamental positive

Assaf and Searles [Assaf and Searles 17] showed that the fundamental slide polynomials stabilize and that their stable limits are precisely the fundamental quasisymmetric functions.

Theorem 4.15 ([Assaf and Searles 17]). *For a weak composition a, we have*

$$\lim_{m\to\infty} \mathfrak{F}_{0^m\times a}(x_1,\dots,x_m,0,\dots,0) = F_{\mathrm{flat}(a)}(x_1,x_2,\dots), \tag{4–9}$$

where $0^m\times a$ denotes the weak composition obtained by prepending m 0's to a.

In order to remove extraneous redundancy from the stable limits, we say that a diagram is *flat* if there is no empty row below a nonempty row. For each diagram D, we define $\mathrm{flat}(D)$ to be the diagram obtained by removing empty rows. For any diagrams C, D such that $\mathrm{flat}(C)=\mathrm{flat}(D)$, we have $\mathcal{K}_D(X)=\mathcal{K}_C(X)$. In particular, in the stable limit, it is enough to consider flat diagrams. It is an interesting, though difficult, question to characterize when two Kohnert quasisymmetric functions are equal. We offer the following partial solution that allows us to consider only flattened, fundamental diagrams.

Lemma 4.16. *Given a diagram D and a row index i such that all cells in row $i+1$ lie strictly left of all cells in row i, we have $\mathcal{K}_D(X)=\mathcal{K}_{\mathrm{Up}_i(D)}(X)$.*

Proof. It is enough to show $\mathfrak{K}_{0^{m+|D|+1}\times D}(x_1,\dots x_m) = \mathfrak{K}_{0^{m+|D|+1}\times \mathrm{Up}_i(D)}(x_1,\dots x_m)$ for all m. Specifically, let $\mathrm{KD}_i(D)$ denote the subset of $\mathrm{KD}(D)$ consisting of diagrams having no cells in row $i+1$ or higher. We will show $\mathrm{KD}_m(0^{m+|D|+1}\times D)=\mathrm{KD}_m(0^{m+|D|+1}\times \mathrm{Up}_i(D))$.

Since $0^{m+|D|+1}\times D$ is a Kohnert diagram of $0^{m+|D|+1}\times \mathrm{Up}_i(D)$, we have $\mathrm{KD}_m(0^{m+|D|+1}\times D)\subset \mathrm{KD}_m(0^{m+|D|+1}\times \mathrm{Up}_i(D))$. To see the other containment, observe that $0^{m+|D|}\times \mathrm{Up}_i(D)$ is a Kohnert diagram of $0^{m+|D|+1}\times D$, formed by dropping all cells in row $i+1$ of D down to row i and dropping all cells not in rows i or $i+1$ down one row. □

For example, letting D be the leftmost diagram in Figure 8, we may compute the fundamental quasisymmetric function expansion of the Kohnert quasisymmetric function by

$$\begin{aligned}\mathcal{K}_D &= F_{\mathrm{flat}(0,1,0,2,1)}+F_{\mathrm{flat}(0,2,0,2,0)}+F_{\mathrm{flat}(0,1,1,2,0)}-F_{\mathrm{flat}(1,1,0,2,0)}\\ &= F_{(1,2,1)}+F_{(2,2)}.\end{aligned}$$

Notice that each of the split quasi-Yamanouchi Kohnert diagrams is canceled in the limit.

Compare this with the fourth diagram, say D', in Figure 9, which is fundamental. For this diagram there are two quasi-Yamanouchi Kohnert diagrams, and so we have

$$\mathcal{K}_{D'} = F_{\mathrm{flat}(0,0,1,2,1)}+F_{\mathrm{flat}(0,0,2,2,0)} = F_{(1,2,1)}+F_{(2,2)},$$

which coincides with $\mathcal{K}_D$ computed above, illustrating Lemma 4.16 since $D'=\mathrm{Up}_2(D)$.

Despite the restriction of Theorem 4.14 to fundamental diagrams, positivity in the stable limit holds in general (Figure 10).

Definition 4.17. For any diagram U, the threading of U is the partitioning of the cells of U into equivalence classes called threads, defined as follows. Let $c_1,c_2,\dots,c_m$ be the cells of U read left to right, starting at the highest row. For $i>1$, c_i is in the same thread as c_{i-1} if and only if c_i lies strictly to the right of c_{i-1} in U.

We refer to the *length* of a thread is the number of cells in that thread.

By construction, if c_{i-1},c_i are consecutive (in reading order) cells of U that lie in different threads, then c_i must lie in a strictly lower row than c_{i-1}. In particular, the rightmost cells of all threads lie in distinct rows, making the following well-defined.

Definition 4.18. For any diagram U, define $\mathrm{drop}(U)$ to be the diagram obtained by placing all cells of each thread in the same row as the rightmost cell in that thread. For example, see Figure 10.

Lemma 4.19. *Let D be a diagram and $U\in\mathrm{QYKD}(D)$. Then*

1. $\mathrm{drop}(U)\in\mathrm{QYKD}(D)$;
2. *the sequence of thread lengths (from highest thread to lowest thread) is the same in U and* $\mathrm{drop}(U)$;

3. *the threads in* drop(U) *are exactly the rows of* drop(U);
4. *U is non-split if and only if* drop$(U) = U$.

Proof. Notice that if c_{i-1}, c_i are consecutive (in reading order) cells of U that lie in the same thread, then c_i must lie strictly to the right of and weakly below c_{i-1}. Suppose that c_{i-1} lies strictly above c_i. Then from the reading order, there is no cell right of c_{i-1} in its row nor left of c_i in its row nor in a row strictly between those of c_{i-1} and c_i. Therefore, we may apply a Kohnert move to c_{i-1} resulting in it lying exactly one row lower with no cell to its left. In particular, this move preserves the reading order, and as such preserves the threading. Iterating this process one cell at a time until all cells of each thread lie in the same row results in drop(U). The threads are maintained at each step, proving (2), and, in the terminal case, proving (3). Since each move is a Kohnert move, we have drop$(U) \in \mathrm{KD}(D)$ and shows (2). Moreover, by construction the rows that have a cell of drop(U) are a subset of the rows that have a cell of U. Since $U \in \mathrm{QYKD}(D)$, no row of U can be raised, and since the rows of drop(U) are a subset of those of U, no row of drop(U) can be raised, proving (1).

Finally, to see (4), we have drop$(U) \neq U$ if and only if a thread uses two consecutive cells in different rows, which is true if and only if the lower cell is strictly southeast of the higher cell and there are no cells in between them in reading order, i.e., if and only if U is split. □

In order to obtain a nonnegative formula for the fundamental quasisymmetric expansion of the Kohnert quasisymmetric function for a diagram D, we will sum over only the non-split quasi-Yamanouchi diagrams. To justify this, we will show that all contributions from a split quasi-Yamanouchi diagram $U \in \mathrm{QYKD}(D)$ are subsumed in the limit by drop(U), which, by Lemma 4.19, is non-split.

Lemma 4.20. *Let D be a diagram. Suppose there is a raising path for a diagram $E \in KD(D)$ to $U \in QYKD(D)$. If all cells of E are weakly below the lowest cell of drop(U), then there is a raising path from E to drop(U).*

Proof. The key point is that if there is a raising path from E to U, then the sequence of thread lengths is the same in E and U. This follows because when cells are moved in a raising path, the position of cells in reading order is unchanged, and no cell is moved from weakly left to strictly right of any other cell or vice versa. Therefore, by Lemma 4.19, the sequence of thread lengths of E is the same as that of drop(U).

Now suppose E has every cell weakly below the lowest row of drop(U). We construct a raising path from E to drop(U) as follows. Select any thread of E that uses cells in more than one row. Raise all cells in the row of the rightmost cell of the thread. Continue this process until all cells in the thread have been raised to the row of the leftmost cell in the thread. Now perform this process with all remaining threads of E that use cells in more than one row. In the resulting diagram E', the threads are exactly the rows, and the sequence of thread lengths is the same as in drop(U). Since all rows of E' lie below all rows of drop(U), we may now raise rows one by one to obtain drop(U). Every step of this procedure is legitimate: all such moves are (a sequence of) reverse Kohnert moves, hence any intermediate diagram at any stage in this process may be obtained from drop$(U) \in \mathrm{QYKD}(D)$ via Kohnert moves; in particular, we never leave $\mathrm{KD}(D)$ during this procedure. □

Theorem 4.21. *For any diagram D and any m at least as great as the number of cells of D, we have*

$$\mathcal{K}_D = \sum_{\substack{T \in \mathrm{QYKD}(0^m \times D) \\ T \text{ non-split}}} F_{\mathrm{flat}(\mathrm{wt}(T))}. \qquad (4\text{–}10)$$

In particular, Kohnert quasisymmetric functions expand non-negatively into fundamental quasisymmetric functions.

Proof. From Definition 4.8, for m at least the number of cells of D, no $T \in \mathrm{QYKD}(0^{m+1} \times D)$ has a cell in the bottom row. So $\mathrm{QYKD}(0^m \times D)$ is stable in the sense that $\mathrm{QYKD}(0^{m+1} \times D)$ consists of exactly the elements of $\mathrm{QYKD}(0^m \times D)$ with every cell raised by one row.

Partition the Kohnert diagrams of $0^m \times D$ into $\mathrm{KD}(0^m \times D) = \mathrm{KD}_{NSp}(0^m \times D) \sqcup \mathrm{KD}_{Sp}(0^m \times D)$, where $\mathrm{KD}_{NSp}(0^m \times D)$ are those diagrams that have a raising path to a non-split element of $\mathrm{QYKD}(0^m \times D)$, and $\mathrm{KD}_{Sp}(0^m \times D)$ are those that can only raise to split elements of $\mathrm{QYKD}(0^m \times D)$. Thus, we have

$$\mathfrak{K}_D = \sum_{E \in \mathrm{KD}_{NSp}(0^m \times D)} x^{\mathrm{wt}(E)} + \sum_{V \in \mathrm{KD}_{Sp}(0^m \times D)} x^{\mathrm{wt}(V)}. \qquad (4\text{–}11)$$

First consider $\mathrm{KD}_{NSp}(0^m \times D)$. By Lemma 4.10, if a Kohnert diagram of $0^m \times D$ raises to a non-split element $T \in \mathrm{QYKD}(0^m \times D)$ then it cannot raise to any other non-split element of $\mathrm{QYKD}(0^m \times D)$. Hence $\mathrm{KD}_{NSp}(0^m \times D)$ is partitioned further into the sets of

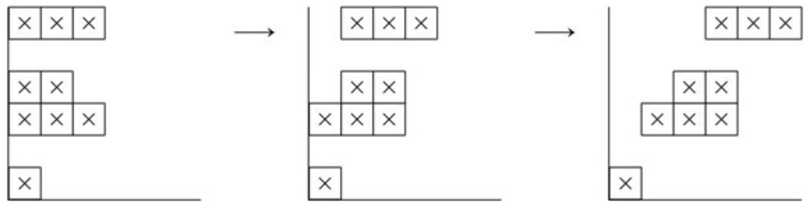

Figure 12. Illustration of construction of the skew diagram $\mathbb{S}(1,0,3,2,0,3)$.

Kohnert diagrams that can raise to each non-split QYKD. Moreover, as in the proof of Theorem 4.14, the monomials corresponding to the set of all Kohnert diagrams of $0^m \times D$ that can raise to a given non-split element $T \in \mathrm{QYKD}(0^m \times D)$ (equivalently, those Kohnert diagrams obtained from T by Kohnert moves without jumping or moving a cell into a row containing another cell) comprise the fundamental slide polynomial $\mathfrak{F}_{\mathrm{wt}(T)}$. Therefore, we can rewrite the left sum in (4–11) as

$$\sum_{E \in \mathrm{KD}_{NSp}(0^m \times D)} x^{\mathrm{wt}(E)} = \sum_{\substack{T \in \mathrm{QYKD}(0^m \times D) \\ T \text{ non-split}}} \mathfrak{F}_{\mathrm{wt}(T)}. \qquad (4\text{–}12)$$

By Theorem 4.15, in the limit this becomes a sum of fundamental quasisymmetric functions indexed by the flattened weights of the non-split QYKDs.

Now consider $\mathrm{KD}_{Sp}(0^m \times D)$. By Lemma 4.19, for any $U \in \mathrm{QYKD}(0^m \times D)$ we have $\mathrm{drop}(U) \in \mathrm{QYKD}(0^m \times D)$ and $\mathrm{drop}(U)$ is non-split, hence $\mathrm{KD}_{Sp}(0^m \times D)$ consists of Kohnert diagrams of $0^m \times D$ that have a raising path to a split U but no raising path to $\mathrm{drop}(U)$. By Lemma 4.20, any diagram that raises to $U \in \mathrm{QYKD}(0^m \times D)$ but not to $\mathrm{drop}(U)$ must have a cell above the lowest row of $\mathrm{drop}(U)$. In general, therefore, all diagrams in $\mathrm{KD}_{Sp}(0^m \times D)$ have a cell above the lowest row that is occupied by some element of $\mathrm{QYKD}(0^m \times D)$. By Definition 4.8, this index of this row is greater than $m-|D|$. Therefore, if $V \in \mathrm{KD}_{Sp}(0^m \times D)$ then $x^{\mathrm{wt}(V)}$ is divisible by some variable x_r where $r \geqslant m-|D|$. Since $|D|$ is constant, r grows without bound as $m \to \infty$. Therefore, in the limit, every term in the second sum in (4–11) vanishes, leaving only (4–12). The theorem follows. □

Since the fundamental quasisymmetric expansion is governed by the non-split quasi-Yamanouchi diagrams, we have the following converse to Theorem 4.14 using Theorems 4.13 and 4.21.

Corollary 4.22. *Given a diagram D that is not fundamental, the Kohnert polynomial $\mathfrak{K}_{0^{|D|} \times D}$ is not nonnegative on the fundamental slide basis.*

Theorem 4.21 is the strongest result one can expect in that we know of no other bases for quasisymmetric functions on which all Kohnert quasisymmetric functions are nonnegative.

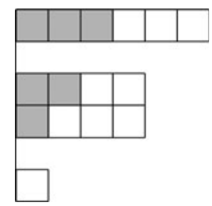

Figure 13. Illustration of the partitions $\lambda = (6,4,4,1)$ and $\mu = (3,2,1)$ associated to the skew diagram $\mathbb{S}(1,0,3,2,0,3)$.

5. Demazure character expansions

Our two motivating examples for general Kohnert polynomials are Schubert polynomials and Demazure characters. Generalizing Proposition 2.9 that characterizes when a Schubert polynomial is equal to a Demazure character, Lascoux and Schützenberger [Lascoux and Schützenberger 90] proved that Schubert polynomials always expand as a nonnegative integral sum of Demazure characters. Thus, it is natural to explore the question of when a general Kohnert polynomial expands as a nonnegative integral sum of Demazure characters.

5.1. Demazure diagrams

Inspired by these two important bases, Schubert polynomials and Demazure characters, we have the following simple condition on diagrams.

Definition 5.1. A diagram D is southwest if whenever a pair of cells (r_2, c_1) and (r_1, c_2) are in D, where $r_1 < r_2$ and $c_1 < c_2$, then the cell (r_1, c_1) is also in D.

For example, the diagram on the left of Figure 11 is not southwest since it contains cells in positions (4, 2) and (2, 4) but not the cell in position (2, 2). In contrast, the diagram on the right of Figure 11 is southwest, precisely because this impediment has been removed.

In particular, a southwest diagram is necessarily fundamental. Conversely, a fundamental diagram need not be southwest. For example, all four diagrams in

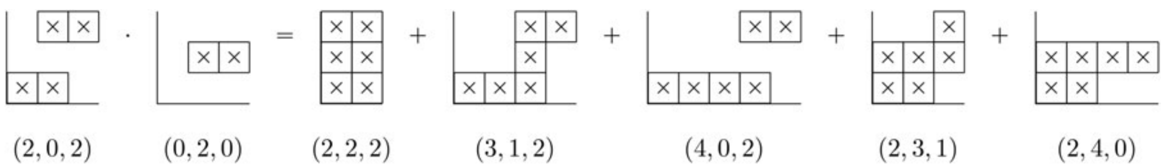

Figure 14. Skew diagrams illustrating the skew polynomial expansion of $\mathfrak{K}_{\mathbb{S}(2,0,2)} \cdot \mathfrak{K}_{\mathbb{S}(0,2,0)}$.

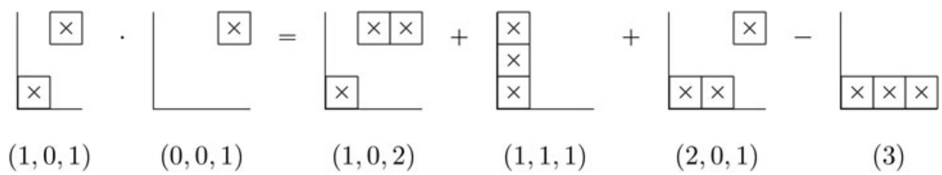

Figure 15. Skew diagrams illustrating the skew polynomial expansion of $\mathfrak{K}_{\mathbb{S}(1,0,1)} \cdot \mathfrak{K}_{\mathbb{S}(0,0,1)}$.

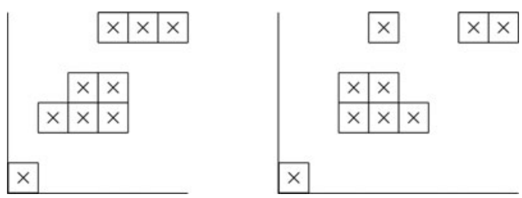

Figure 16. The skew diagram $\mathbb{S}(1,0,3,2,0,3)$ and Rothe diagram $\mathbb{D}(216539478)$.

Figure 1 are fundamental, but the third is not southwest.

Our motivation for the southwest condition comes from Schubert polynomials and Demazure characters, since both composition diagrams and Rothe diagrams are southwest. Moreover, extensive computer experimentation supports the following conjecture that we believe provides an enormous class of Kohnert polynomials that have representation-theoretic and geometric significance.

Conjecture 5.2. *Given a southwest diagram D, the Kohnert polynomial $\mathfrak{K}_D$ expands non-negatively into Demazure characters.*

In further support of Conjecture 5.2, the southwest condition is exactly the same as the *northwest* condition of Reiner and Shimozono [Reiner and Shimozono 95, Reiner and Shimozono 98] in their study of Specht modules associated to diagrams. This suggests a connection between flagged Weyl modules and Kohnert polynomials, conjecturally that Kohnert polynomials give the characters of flagged Weyl modules.

In light of Proposition 2.7, which states that Demazure characters stabilize to Schur functions, Conjecture 5.2 gives an enormous class of Kohnert quasisymmetric functions that are *Schur positive*. This would be striking given that not all Kohnert quasisymmetric functions are symmetric let alone Schur positive and shows the potential power of Kohnert polynomials.

5.2. Skew polynomials

We now give a new and nontrivial example of a southwest Kohnert basis that further supports Conjecture 5.2.

Definition 5.3. For a weak composition a, the skew diagram $\mathbb{S}(a)$ is constructed as follows:

- left justify a_i cells in row i,
- for j from 1 to n such that $a_j>0$, take $i<j$ maximal such that $a_i>0$, and if $a_i>a_j$, then shift rows kj rightward by a_i-a_j columns,
- shift each row j rightward by $\#\{i<j|a_i=0\}$ columns.

The *skew polynomial* is the Kohnert polynomial $\mathfrak{K}_{\mathbb{S}(a)}$.

For example, we construct the skew diagram $\mathbb{S}(1,0,3,2,0,3)$ from the composition diagram $\mathbb{C}(1,0,3,2,0,3)$ by shifting rows $k\geqslant 4$ rightward by 1 column since $a_3-a_4=1$, then shifting rows 3, 4 rightward by one since $a_2=0$ and row 6 rightward by two since $a_2=a_5=0$. These steps are illustrated in Figure 12.

Proposition 5.4. *Skew diagrams are southwest.*

Proof. Failure of the southwest condition ensures the existence of a pair of cells (r_1, c_2) and (r_2, c_1) in D, where $r_1<r_2$ and $c_1<c_2$, such that the position (r_1, c_1) is not a cell of D. Such a configuration is impossible in a skew diagram D since rows of cells have no internal gaps and the leftmost cell in a lower row is weakly left of the leftmost cell in a higher row. Thus $(r_2,c_1)\in D$ implies $(r_1,c_1)\in D$ whenever row r_1 is nonempty. □

Conjecture 5.2 implies that skew polynomials should expand nonnegatively in Demazure characters. Indeed, this fact follows using the machinery of weak dual equivalence developed in [Assaf 17].

Theorem 5.5. *Skew polynomials* $\{\mathfrak{K}_{\mathbb{S}(a)}\}$ *form a basis of* $\mathbb{Z}[x_1, x_2, ..., x_n]$*, expand nonnegatively in demazure characters, and stabilize to Schur positive symmetric functions.*

Proof. By Theorem 2.3, since skew polynomials are a Kohnert basis, they are lower uni-triangular with respect to monomials, and as such form a basis for the polynomial ring.

In [Assaf 17] (Theorem 4.10), Assaf proves a generalized Littlewood–Richardson rule that expands a *skew key polynomial*, indexed by a composition with a partition shape removed from the northwest corner, as a nonnegative integral sum of Demazure characters (therein called key polynomials). The definition for these skew key polynomials, [Assaf 17] (Definition 4.7), uses standard key tableaux. The bijection between standard key tableaux and quasi-Yamanouchi Kohnert tableaux stated in [Assaf 17] (Definition 3.14) and proved in [Assaf 17] (Theorem 3.15) together with the bijection from the latter to quasi-Yamanouchi Kohnert diagrams stated in [Assaf and Searles 18] (Definition 2.5) and proved in [Assaf and Searles 18] (Theorem 2.8) establishes the equivalence of skew key polynomials with the Kohnert polynomials defined by the indexing shape. Thus, each skew polynomial is a skew key polynomial, and so expands nonnegatively into Demazure characters.

Finally, by Proposition 2.7 Demazure characters stabilize to Schur functions, hence the stable limit of a skew polynomial is a Schur-positive symmetric function. □

For example, the skew diagram $\mathbb{S}(1,0,3,2,0,3)$ can be realized as the composition diagram $\mathbb{D}(1,0,4,4,3,6)$ skewed by the partition $(0,0,1,2,3,3)$, and so the polynomial equivalence states

$$\mathfrak{K}_{\mathbb{S}(1,0,3,2,0,3)} = \kappa_{(1,0,4,4,3,6)\backslash(0,0,1,2,3,3)}.$$

Thanks to the stability results for Kohnert polynomials, we have the following.

Corollary 5.6. *Let a be a weak composition. Let* λ *be the partition given by the flattening of the weight of the skew diagram for a, i.e.*

$$\lambda = \mathrm{rev}(\mathrm{flat}(\mathrm{wt}(\mathbb{S}(a)))),$$

and let μ *be the partition given by*

$$\mu_i = \lambda_i - \mathrm{flat}(a)_i.$$

Then the stable limit of the skew polynomial indexed by a is

$$\mathcal{K}_{\mathbb{S}(a)} = \lim_{m\to\infty} \mathfrak{K}_{\mathbb{S}(a)} = s_{\lambda/\mu}$$

In particular, skew polynomials are a polynomial generalization of skew Schur functions.

For example, Figure 13 illustrates the computation of λ and $\mu \subset \lambda$ in establishing the following limit,

$$\mathcal{K}_{\mathbb{S}(1,0,3,2,0,3)} = s_{(6,4,4,1)\backslash(3,2,1)}.$$

5.3. Applications of skew polynomials

Unlike Schubert polynomials, whose structure constants enumerate points in a suitable intersection of Schubert varieties and as such as known to be nonnegative, Demazure characters often have negative structure constants. For example,

$$\kappa_{(2,0,2)}\kappa_{(0,2,0)} = \kappa_{(2,2,2)} + \kappa_{(3,1,2)} + \kappa_{(4,0,2)} + \kappa_{(2,3,1)} - \kappa_{(3,2,1)} + \kappa_{(2,4,0)} - \kappa_{(4,2,0)}.$$

Interestingly, the structure constants for skew polynomials are often nonnegative. For example, Figure 14 illustrates the following expansion,

$$\mathfrak{K}_{\mathbb{S}(2,0,2)} \cdot \mathfrak{K}_{\mathbb{S}(0,2,0)} = \mathfrak{K}_{\mathbb{S}(2,2,2)} + \mathfrak{K}_{\mathbb{S}(3,1,2)} + \mathfrak{K}_{\mathbb{S}(4,0,2)} + \mathfrak{K}_{\mathbb{S}(2,3,1)} + \mathfrak{K}_{\mathbb{S}(2,4,0)}.$$

In light of Corollary 5.6, this gives the following nonnegative expansion of a product of skew Schur function into skew Schur functions,

$$s_{(3,2)/(1)} \cdot s_{(2)} = s_{(2,2,2)} + s_{(4,3,3)/(2,2)} + s_{(5,4)/(3)} + s_{(3,3,2)/(2)} + s_{(4,2)}.$$

Since skew Schur functions over determine a basis for symmetric functions, this expansion is surprising.

However, such a nice expansion does not always hold. For example, Figure 15 illustrates the following signed expansion,

$$\mathfrak{K}_{\mathbb{S}(1,0,1)} \cdot \mathfrak{K}_{\mathbb{S}(0,0,1)} = \mathfrak{K}_{\mathbb{S}(1,0,2)} + \mathfrak{K}_{\mathbb{S}(1,1,1)} + \mathfrak{K}_{\mathbb{S}(2,0,1)} - \mathfrak{K}_{\mathbb{S}(3,0,0)}.$$

In this case, Corollary 5.6 still applies and gives the following signed expansion,

$$s_{(2,1)/(1)} \cdot s_{(1)} = s_{(3,1)/(1)} + s_{(1,1,1)} + s_{(3,2)/(2)} - s_{(3)}.$$

Despite the signs, the canonical expansion of a product of skew Schur functions into skew Schur functions is still interesting, and signs appearing can be natural (e.g. see [Assaf and McNamara 11]). Therefore, exploring the structure constants for skew polynomials is a worthwhile endeavor.

Shifting to a more positive direction, skew polynomials correspond with Demazure characters in the

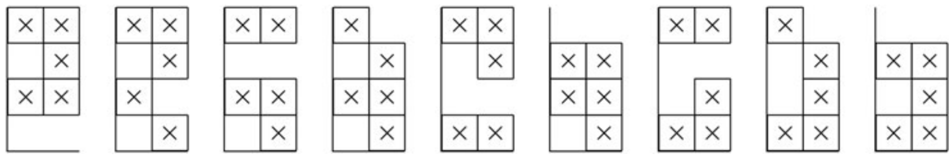

Figure 17. Kohnert diagrams for $\mathbb{D}(0,2,1,2)$.

case when a is weakly increasing, in which case both are Schur polynomials. Skew polynomials also correspond with Schubert polynomials in certain cases, even outside of the above coincidence with Schur polynomials. For instance, we have the following nonobvious coincidence,

$$\mathfrak{K}_{\mathbb{S}(1,0,3,2,0,3)} = \mathfrak{S}_{216539478}.$$

Comparing the skew diagram $\mathbb{S}(1,0,3,2,0,3)$ and Rothe diagram $\mathbb{D}(216539478)$ as in Figure 16, the diagrams themselves are somewhat different yet the resulting Kohnert polynomials coincide.

While this coincidence of skew polynomials and Schubert polynomials does not hold in general, we conjecture that Schubert polynomials are, in fact, nested between Demazure characters and skew polynomials in the following sense.

Conjecture 5.7. *Skew polynomials expand as nonnegative sums of Schubert polynomials.*

Conjecture 5.7 has been verified for degree up to 10 in up to 6 variables. If true, this conjecture is highly suggestive that skew polynomials are a combinatorial shadow of representation-theoretic and geometric objects yet to be discovered.

6. Extending Schur functions to the ring of quasisymmetric functions

We now demonstrate the construction of another Kohnert basis, this one not southwest, with interesting properties. In Section 6.1 we define the new Kohnert basis of *lock polynomials* and apply our previous results to give explicit formulas for the monomial and fundamental slide expansions. In Section 6.2, we demonstrate how to generate a tableaux model from the corresponding Kohnert diagrams and use this to prove a special case when lock polynomials and Demazure characters coincide. In Section 6.3, we consider the stable limits of lock polynomials, which we term *extended Schur functions*, since they contain Schur functions and give a basis for quasisymmetric functions.

6.1. Lock polynomials

We now define a new Kohnert basis. Using the machinery of Kohnert polynomials, this requires only that for each weak composition a, we make a choice for the columns in which we place the a_i boxes in row i. For each weak composition a, there is a unique right-justified diagram of weight a which we call the *lock diagram for* a and denoted by $\mathbb{L}(a)$. For example, the third diagram in Figure 1 is the lock diagram for $(0,2,1,2)$. The Kohnert diagrams for this diagram are shown in Figure 17.

Definition 6.1. The *lock polynomial indexed* by a is

$$\mathfrak{L}_a = \mathfrak{K}_{\mathbb{L}(a)}, \qquad (6\text{–}1)$$

where $\mathbb{L}(a)$ is the right justified diagram of weight a.

For example, from Figure 17, we see that

$$\mathfrak{L}_{(0,2,1,2)} = x_1^2x_2x_3^2 + x_1^2x_2x_3x_4 + x_1^2x_2x_4^2 + x_1^2x_3x_4^2 + x_1x_2^2x_3^2 + x_1x_2^2x_3x_4 + x_1x_2^2x_4^2 + x_1x_2x_3x_4^2 + x_2^2x_3x_4^2.$$

By Theorem 2.3, since lock polynomials are a Kohnert basis, they are, in particular, a basis of polynomials.

Corollary 6.2. *The lock polynomials form a basis for the polynomial ring that is lower uni-triangular with respect to monomials.*

By Theorem 3.7, we may express lock polynomials more compactly in the monomial slide basis as follows.

Corollary 6.3. *Lock polynomials expand non-negatively into monomial slide polynomials by*

$$\mathfrak{L}_a = \sum\nolimits_{T\in \mathrm{MKD}(\mathbb{L}(a))} \mathfrak{M}_{\mathrm{wt}(T)}. \qquad (6\text{–}2)$$

For the previous example, refined to MKD in Figure 4, we have

$$\mathfrak{L}_{(0,2,1,2)} = \mathfrak{M}_{(0,2,1,2)} + \mathfrak{M}_{(1,1,1,2)} + \mathfrak{M}_{(2,1,1,1)} + \mathfrak{M}_{(1,2,0,2)} + \mathfrak{M}_{(1,2,1,1)}.$$

Even more powerful, by Theorem 4.14 we have the following.

Corollary 6.4. *Lock polynomials expand non-negatively into fundamental slide polynomials by*

$$\mathfrak{L}_a = \sum\nolimits_{T\in \mathrm{QYKD}(\mathbb{L}(a))} \mathfrak{F}_{\mathrm{wt}(T)}. \qquad (6\text{–}3)$$

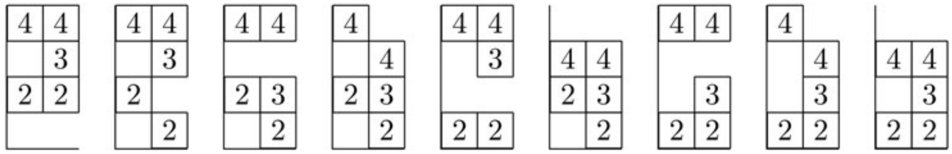

Figure 18. The nine lock tableaux for $\mathbb{D}(0, 2, 1, 2)$.

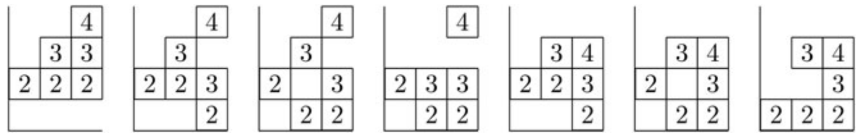

Figure 19. The set QLT$(0, 3, 2, 1)$ of quasi-Yamanouchi lock tableaux of content $(0, 3, 2, 1)$.

Proof. Given a lock diagram $\mathbb{L}(a)$, for every column $c \leqslant \max(a)$ and for any nonempty collection of rows, there are at least as many cells in those rows in column c as there are in column $c-1$. Therefore (4–7) is always satisfied, and so lock diagrams are fundamental by Definition 4.11. Thus Theorem 4.14 applies. □

Returning again to our example, with the fundamental diagrams shown in Figure 5, we have

$$\mathfrak{L}_{(0,2,1,2)} = \mathfrak{F}_{(0,2,1,2)} + \mathfrak{F}_{(1,2,0,2)}.$$

Unlike Schubert polynomials and, trivially, Demazure characters, the lock polynomials do not expand non-negatively into Demazure characters. For example,

$$\mathfrak{L}_{(0,2,1,2)} = \kappa_{(0,2,1,2)} - \kappa_{(0,2,2,1)} + \kappa_{(1,2,2,0)} - \kappa_{(2,2,1,0)}.$$

In light of Conjecture 5.2, this comes as no surprise since lock diagrams are not, in general, southwest.

We remark that lock polynomials do not have non-negative expansions into other familiar bases of the polynomial ring, including quasi-key polynomials [Assaf and Searles 18] and Demazure atoms [Lascoux and Schützenberger 90].

6.2. Lock tableaux

Kohnert's rule allows for easy computations, but the potential redundancy of two different sequences of Kohnert moves arriving at the same diagram can be problematic. In [Assaf and Searles 18], the authors gave a static description of *Kohnert tableaux* for key diagrams by tracking from where each cell in a Kohnert diagram came in the key diagram, and when Kohnert's algorithm gives multiple possibilities for this, fixing a canonical choice. This was done via a canonical labeling of a Kohnert diagram coming from a key diagram, resulting in a simple rule to determine readily if a given diagram can arise as a Kohnert diagram for a key diagram. We extend this procedure to lock diagrams below.

Definition 6.5. Given a weak composition a of length n, a lock tableau of content a is a diagram filled with entries $1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$, one per cell, satisfying the following conditions:

i. there is exactly one i in each column from $\max(a) - a_i + 1$ through $\max(a)$;
ii. each entry in row i is at least i;
iii. the cells with entry i weakly descend from left to right;
iv. the labeling strictly decreases down columns.

Denote the set of lock tableaux of content a by $\mathrm{LT}(a)$.

For example, the lock tableaux of content $(0, 2, 1, 2)$ are shown in Figure 18. Compare this with the Kohnert diagrams for $\mathbb{L}(0, 2, 1, 2)$ shown in Figure 17.

The definition of *Kohnert tableaux* in [Assaf and Searles 18], the analogous model for key diagrams, differs from Definition 6.5 only in condition (iv). For the Kohnert tableaux case, condition (iv) allowed for an *inversion* in a column, i.e. a pair $i < j$ with i above j in the same column, only if there is an i in the column immediately to the right of and strictly above j. Condition (iv) for lock tableaux is far simpler. Moreover, this simplification is forced in the following sense.

Proposition 6.6. *Given a weak composition a, let T be any filling of a diagram with entries* $1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$, *one per cell, satisfying the conditions (i), (ii), and (iii) of Definition 6.5 and the following*

(iv)' if $i < j$ appear in a column with i above j, then there is an i in the column immediately to the right of and strictly above j.

Then T is a lock tableau.

Proof. Suppose T satisfies conditions (i), (ii), and (iii) of Definition 6.5 and condition (iv)' above. Let c be the rightmost column of T such that there exist entries $i<j$ in column c with i above j. By condition (iv)', there is an i in column $c+1$. However, by condition (i), since there is an i in column $c+1$, there must also be a j in column $c+1$. By condition (iii), the j in column $c+1$ lies weakly below the j in column c which, by condition (iv'), lies strictly below the i in column $c+1$, contradicting the choice of c. Therefore, columns of T must be strictly decreasing top to bottom. □

We mirror the results of [Assaf and Searles 18] for Kohnert tableaux to prove that lock tableaux precisely characterize Kohnert diagrams of lock diagrams.

Lemma 6.7. *For $T \in \mathrm{LT}(a)$, the diagram of T is a Kohnert diagram for $\mathbb{L}(a)$.*

Proof. Fix $T \in \mathrm{LT}(a)$. We claim one can perform reverse Kohnert moves on T to obtain the lock diagram of a. Reading the cells of T left to right along rows, starting at the top row, find the first cell, say C, whose label is greater than its row index. Any cell above C must have had larger label by condition (iv) of Definition 6.5, so by choice of C there cannot be a cell immediately above it, so we may raise C to the row above. To show this reverse move is valid, we need to show there is no cell C' to the right of the position in which C lands. Any cell of the landing row must have entry equal to its row index by the choice of C, and by condition (ii) C has entry at least that large, so the entry of C is at least as great as the entry of C'. However, by condition (i), there must be a cell with entry the same as that of C in the column of C', and by condition (iii) that cell must be strictly lower. This creates a violation of condition (iv) in the column of C' in T, a contradiction.

This procedure preserves condition (i) of Definition 6.5. Since this procedure moves the top left cell having a given label greater than its row number, conditions (ii) and (iii) are preserved. Since cells do not change their order within a column, condition (iv) is preserved. Therefore, the result is in $\mathrm{LT}(a)$. Iterating this procedure, one eventually obtains the lock diagram with all entries in row i equal to i, and each move is a valid reverse Kohnert move. Hence T with its labels removed is in $\mathrm{KD}(\mathbb{L}(a))$. □

To establish the converse of Lemma 6.7, we give a canonical labeling of a Kohnert diagram. Once again, the algorithm for Kohnert diagrams coming from a lock diagram is far simpler than the analogous labeling algorithm for Kohnert diagrams coming from a key diagram.

Definition 6.8. Given $D \in \mathrm{KD}(\mathbb{L}(a))$, define the lock labeling of D with respect to a, denoted by $L_a(D)$, by placing the labels $\{i|a_i \geqslant j\}$ to cells of column $\max(a) - j + 1$ in increasing order from bottom to top.

The lock labeling algorithm is well-defined and establishes the following.

Theorem 6.9. *The labeling map L_a is a weight-preserving bijection between $\mathrm{KD}(\mathbb{L}(a))$ and $\mathrm{LT}(a)$. In particular, we have*

$$\mathfrak{L}_a = \sum_{T \in \mathrm{LT}(a)} x^{\mathrm{wt}(T)}, \tag{6–4}$$

where $\mathrm{wt}(T)$ is the weak composition whose ith part is the number of cells in row i of T.

Proof. Suppose that $D \in \mathrm{KD}(\mathbb{L}(a))$. The lock labeling map L_a is well-defined on D since the number of cells per column is preserved by Kohnert moves. No filling of D other than $L_a(D)$ can give an element of $\mathrm{LT}(a)$ by condition (iv). Therefore, by Lemma 6.7, removing the labels gives an inverse map provided $L_a(D)$ is a Kohnert tableau.

Condition (i) of Definition 6.5 is manifest from the selection of entries, and condition (iv) follows immediately from the lock labeling. For condition (ii), note that every cell in any given column of D must be weakly below where it started, since D is a Kohnert diagram. In particular, for any index i the number of boxes of D appearing below row i is weakly larger than the number of boxes of $\mathbb{L}(a)$ appearing below row i, so since the columns are labeled in increasing order from bottom to top with labels given by the row indices of the original positions of the cells, the label of every cell of D must be weakly larger than its row index.

Condition (iii) holds for the lock labeling of $\mathbb{L}(a)$, so it is enough to check this condition is preserved under Kohnert moves. Let D be a Kohnert diagram of D; assume that (iii) is satisfied under the lock labeling of D. When we make a Kohnert move on D and relabel according to the lock labeling, the overall effect is to move a cell C and the interval of cells immediately below C down one space, retaining their labels. Since all labels in the column of C move downwards, this does not introduce any violation of (iii) with entries to the left of the column of C. Since we started with a lock diagram, all labels appearing in the interval of cells below C must also appear in the column to the right of C, and since we performed a Kohnert move on C, the cell immediately right of C in D is empty. Hence C's label appears strictly lower than C in the column right of C, and by the lock labeling and the fact that the cells below C form an interval, the same must be true for all cells in the interval below C. Hence (iii) is preserved on moving all these cells (with their labels) down one space. □

Recall the quasi-Yamanouchi condition for Kohnert tableaux in [Assaf and Searles 18].

Definition 6.10. *A lock tableau is quasi-Yamanouchi if for each nonempty row i, one of the following holds:*

1. there is a cell in row i with entry equal to i, or
2. there is a cell in row $i+1$ that lies weakly right of a cell in row i.

Denote the set of quasi-Yamanouchi lock tableaux of content a by $\mathrm{QLT}(a)$.

For example, the quasi-Yamanouchi lock tableaux of content $(0,3,2,1)$ are shown in Figure 19.

Quasi-Yamanouchi lock tableaux allow for the following re-characterization of the fundamental slide expansion of lock polynomials.

Theorem 6.11. *Lock polynomials expand non-negatively into fundamental slide polynomials by*

$$\mathfrak{L}_a = \sum_{T \in \mathrm{QLT}(a)} \mathfrak{F}_{\mathrm{wt}(T)}. \tag{6–5}$$

Proof. We define a *de-standardization map* from $\mathrm{LT}(a)$ to $\mathrm{QLT}(a)$ by sending a lock tableau T to the quasi-Yamanouchi lock tableau $\mathrm{dst}(T)$ constructed as follows. For each row, say i, if every cell in row i lies strictly right of every cell in row $i+1$ and the leftmost cell of row i has label larger than i, then move every cell in row i up to row $i+1$. Repeat until no such row exists. To see that the de-standardization map maintains the lock tableau conditions, note that the labels within each column are maintained, proving (i). De-standardization does not move cells to a row higher than their label, so (ii) is maintained. No cell is moved from weakly below to strictly above any other, and no cell moves upward if there is a cell to its right in the row above, so conditions (iii) and (iv) are maintained. Finally, by definition de-standardization terminates if and only if the quasi-Yamanouchi condition is met.

Let $T \in \mathrm{LT}(a)$ and suppose $\mathrm{dst}(T) = S \in \mathrm{QLT}(a)$. Since dst moves cells upwards we have $\mathrm{wt}(T) \geqslant \mathrm{wt}(S)$, and since dst moves *all* cells in row i to row $i+1$, we have $\mathrm{flat}(\mathrm{wt}(T))$ refines $\mathrm{flat}(\mathrm{wt}(S))$. Hence x^T is a monomial of $\mathfrak{F}_{\mathrm{wt}(S)}$. Conversely, let $S \in \mathrm{QLT}(a)$, and let b be a weak composition such that $b \geqslant \mathrm{wt}(S)$ and $\mathrm{flat}(b)$ refines $\mathrm{flat}(\mathrm{wt}(S))$. We show there is a unique $T \in \mathrm{LT}(a)$ with $\mathrm{wt}(T) = b$ and $\mathrm{dst}(T) = S$. To reconstruct T from b and U, for $j = 1, \ldots, n$, if $\mathrm{wt}(S)_j = b_{i_{j-1}+1} + \cdots + b_{i_j}$, then, from right to left, move the first $b_{i_{j-1}+1}$ cells down to row $i_{j-1}+1$, the next $b_{i_{j-1}+2}$ cells down to row $i_{j-1}+2$, and so on. By construction T is a Kohnert diagram of S (so $T \in \mathrm{LT}(a)$), $\mathrm{wt}(T) = b$, and $\mathrm{dst}(T) = S$. Uniqueness follows from the lack of choice at every step. □

For example, from Figure 19 we may quickly compute

$$\begin{aligned}\mathfrak{L}_{(0,3,2,1)} &= \mathfrak{F}_{(0,3,2,1)} + \mathfrak{F}_{(1,3,1,1)} + \mathfrak{F}_{(2,2,1,1)} + \mathfrak{F}_{(2,3,0,1)} \\ &\quad + \mathfrak{F}_{(1,3,2,0)} + \mathfrak{F}_{(2,2,2,0)} + \mathfrak{F}_{(3,1,2,0)}.\end{aligned}$$

While we noted that lock polynomials are not, in general, nonnegative on Demazure characters, there is a case where lock polynomials and Demazure characters coincide. Letting Conjecture 5.2 be our guide, we notice that a lock diagram $\mathbb{L}(a)$ is southwest if and only if $\mathrm{flat}(a)$ is weakly decreasing. Thus, the following result lends more weight to Conjecture 5.2.

Theorem 6.12. *Given a weak composition a such that* $\mathrm{flat}(a)$ *is weakly decreasing, we have*

$$\mathfrak{L}_a = \kappa_a. \tag{6–6}$$

Proof. We utilize the machinery of weak dual equivalence [Assaf 17] (Definition 3.20) to establish that $\mathfrak{L}_a$ expands nonnegatively into Demazure characters when $\mathrm{flat}(a)$ is weakly decreasing. For a a weak composition of n, we must define involutions $\psi_2, \ldots, \psi_{n-1}$ on $\mathrm{QLT}(a)$ such that $\psi_i\psi_j(T) = \psi_j\psi_i(T)$ whenever $|i-j| \geqslant 3$ and for $i-h \leqslant 3$, there exists a weak composition b of $i-h+3$ such that

$$\sum_{U \in [T]_{(h,i)}} \mathfrak{F}_{\mathrm{wt}_{(h-1,i+1)}(U)} = \kappa_b,$$

where $[T]_{(h,i)}$ is the equivalence class generated by $\psi_h, \ldots, \psi_i$, and $\mathrm{wt}_{(h,i)}(U)$ is the weak composition of $i-h+1$ obtained by deleting the first h – 1 and last n – i nonzero parts from $\mathrm{wt}(U)$. Once such involutions are defined, by [Assaf 17] (Theorem 3.29), the generating polynomial over each equivalence class under the involutions will be a nonnegative sum of Demazure characters.

To define the desired involutions, we first relabel the cells of $T \in \mathrm{QLT}(a)$ with $1, 2, \ldots, n$ from the bottom row up, labeling each row right to left, then raise the cells to $\mathbb{L}(a)$ maintaining their relative order. The result is a bijective filling of $\mathbb{L}(a)$ with $1, 2, \ldots, n$ such that rows decrease left to right and columns decrease top to bottom. This process is reversible by lowering cells of a bijective filling of $\mathbb{L}(a)$ such that 1 is the lowest, 2 the next lowest, and so on, and then applying the de-standardization map from the proof of Theorem 6.11. In fact, if we allow cells to fall below the x-axis, then this establishes a bijection between $\mathrm{QLT}(a)$ and bijective fillings of $\mathbb{L}(a)$ decreasing rows and columns.

For $2 \leqslant i \leqslant n-1$, let ψ_i act on $T \in \mathrm{QLT}(a)$ by instead acting on bijective fillings of $\mathbb{L}(a)$ with decreasing rows and columns as follows. If, in reading entries right to left from the top row down, $i\pm1$ lies between i and $i\mp1$, then ψ_i exchanges i and $i\mp1$; otherwise ψ_i acts by the identity. To see that this is well-defined, there are two cases to check. If $i+1$ lies above i in the same column, then $i-1$ lies between them in the previous sense only if it lies right of i. Since $\mathbb{L}(a)$ is right justified, this forces an entry j right of $i+1$ and above $i-1$. The decreasing rows and columns forces $j=i$, which is a contradiction. Similarly, if i lies above $i-1$ in the same column, then $i+1$ lies between them in the previous sense only if it lies left of i. If $\mathrm{flat}(a)$ is weakly decreasing, then this forces an entry j below $i+1$ and left of $i-1$. The decreasing rows and columns forces $j=i$, which is again a contradiction. Therefore when $\mathrm{flat}(a)$ is weakly decreasing, ψ_i is well-defined.

Given the local nature of ψ_i, since $\{i-1, i, i+1\} \cap \{j-1, j, j+1\} = \emptyset$ whenever $|i-j| \geqslant 3$, we have the commutativity relation $\psi_i\psi_j(T) = \psi_j\psi_i(T)$. The second condition is local, requiring between three and six consecutively labeled cells that must fit inside a staircase diagram. Therefore, there are finitely many cases to check, which can be verified easily by direct (and tedious) enumeration or by computer. We have done both, so Demazure positivity follows by [Assaf 17] (Theorem 2.29).

To see that this is a single Demazure character, we note that bijective fillings of $\mathbb{L}(a)$ with decreasing rows and columns are in bijection with bijective fillings of partition shape $\mathrm{flat}(a)$ with increasing rows and columns, the latter of which are standard Young tableaux that generate a Schur function. Therefore, in the stable limit we have a single term in the Schur expansion, so the non-negativity together with Proposition 2.7 implies the Demazure expansion must have a single term as well. By the unique leading term for lock polynomials and Demazure characters, we must have $\mathfrak{L}_a = \kappa_a$. □

6.3. The extended Schur basis

By Theorem 3.11, we may consider the stable limits of lock polynomials. These stable limits contain the Schur functions (Prop 6.15) and, moreover, form a basis for quasisymmetric functions (Theorem 6.20). Thus, we may regard them as extending the Schur functions to a full basis of quasisymmetric functions, and so call the stable limits the *extended Schur functions.*

Definition 6.13. Given a (strong) composition α, the extended Schur polynomial indexed by α is given by

$$\mathcal{E}_\alpha(x_1, \ldots, x_m) = \mathfrak{L}_{0^m\times\alpha}(x_1, \ldots, x_m, 0, \ldots, 0), \qquad (6\text{–}7)$$

and the *extended Schur function indexed by* α is given by

$$\mathcal{E}_\alpha(X) = \lim_{m\to\infty} \mathfrak{L}_{0^m\times\alpha} = \mathcal{K}_{\mathbb{L}(\alpha)}. \qquad (6\text{–}8)$$

For example, we can compute the extended Schur function $\mathcal{E}_{(2,1,2)}(X)$ by

$$\begin{aligned}
\mathfrak{L}_{(2,1,2)} &= \mathfrak{F}_{(2,1,2)},\\
\mathfrak{L}_{(0,2,1,2)} &= \mathfrak{F}_{(0,2,1,2)} + \mathfrak{F}_{(1,2,0,2)},\\
\mathfrak{L}_{(0,0,2,1,2)} &= \mathfrak{F}_{(0,0,2,1,2)} + \mathfrak{F}_{(0,1,2,0,2)} + \mathfrak{F}_{(1,1,2,0,1)},\\
&\vdots\\
\mathcal{E}_{(2,1,2)} &= F_{(2,1,2)} + F_{(1,2,2)} + F_{(1,1,2,1)}.
\end{aligned}$$

By Proposition 3.12, we have the following statement showing that the extended Schur functions include all Kohnert quasisymmetric functions for lock diagrams.

Corollary 6.14. *Given a weak composition a, we have*

$$\mathcal{K}_{\mathbb{L}(a)} = \mathcal{E}_{\mathrm{flat}(a)}.$$

To justify the name *extended Schur functions*, we have the following result.

Proposition 6.15. *For λ a partition, we have*

$$\mathcal{E}_\lambda(X) = s_\lambda(X). \qquad (6\text{–}9)$$

Proof. By Theorem 6.12, we have $\mathfrak{L}_{0^m\times\lambda} = \kappa_{0^m\times\lambda}$ since λ is weakly decreasing. By Proposition 2.7, we have $\lim_{m\to\infty} \kappa_{0^m\times\lambda} = s_\lambda(X)$. The result now follows from Definition 6.13. □

Note that Demazure characters do not expand non-negatively into lock polynomials. For example,

$$\kappa_{(0,2,1,2)} = \mathfrak{L}_{(0,2,1,2)} + \mathfrak{L}_{(0,2,2,1)} - \mathfrak{L}_{(1,2,2,0)}.$$

Conversely, Proposition 6.15 shows that the stable limits of Demazure characters do expand nonnegatively into the stable limits of lock polynomials.

We can use lock tableaux to give a tableaux model for extended Schur functions as follows.

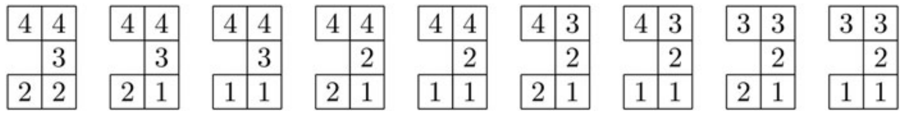

Figure 20. The set $\mathrm{SSET}_4(2,1,2)$ of semi-standard extended tableaux of shape (2, 1, 2) with entries in $\{1,2,3,4\}$.

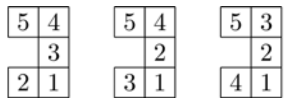

Figure 21. The set $\mathrm{SET}(2,1,2)$ of standard extended tableaux of shape (2, 1, 2).

Definition 6.16. Given a (strong) composition α, a *semi-standard extended tableau of shape* α is a filling of $\mathbb{D}(\alpha)$ with positive integers such that rows weakly decrease left to right and columns strictly decrease top to bottom. Denote the set of semi-standard extended tableaux of shape α by $\mathrm{SSET}(\alpha)$, or by $\mathrm{SSET}_n(\alpha)$ if we restrict the integers to $\{1,2,\ldots,n\}$.

For example, the semi-standard extended tableaux of shape (2, 1, 2) are shown in Figure 20. Compare these with the lock tableaux for $(0,2,1,2)$ shown in Figure 18.

For T a semi-standard extended tableau, let $\mathrm{wt}(T)$ be the weak composition whose ith part is the number of entries of T equal to i. Then extended Schur functions are the generating function for extended tableaux.

Theorem 6.17. *For α a (strong) composition, we have*

$$\mathcal{E}_\alpha(x_1,\ldots,x_n) = \sum_{T\in\mathrm{SSET}_n(\alpha)} x_1^{\mathrm{wt}(T)_1}\cdots x_n^{\mathrm{wt}(T)_n},$$

$$\mathcal{E}_\alpha(X) = \sum_{T\in\mathrm{SSET}(\alpha)} X^{\mathrm{wt}(T)},$$

where $X^{\mathrm{wt}(T)}$ is the monomial $x_1^{\mathrm{wt}(T)_1}\cdots x_n^{\mathrm{wt}(T)_n}$ when $\mathrm{wt}(T)$ has length n.

Proof. By Theorem 6.9 and Definition 6.13, we have that $\mathcal{E}_\alpha(x_1,\ldots,x_m)$ is the generating polynomial for lock tableaux of content $(0^m\times\alpha)$ with no cells above row m. Given such a lock tableau D, we may define a semi-standard extended tableau T as follows. For every cell x of D, place an entry equal to the row index of x into the cell of T in the same column as x and in the row given by m minus the entry of x. For example, the lock tableaux in Figure 18 map to the semi-standard extended tableau in Figure 20, respectively. To reverse the procedure, given a semi-standard extended tableau T, we may construct a lock tableau D by raising T up m rows then moving each cell of T down to the row equal to its entry.

Definition 6.5 condition (i) is equivalent to T having shape $\mathbb{D}(a)$, condition (ii) and the restriction to lock tableaux with all entries weakly below row m is equivalent to T having labels in $\{1,2,\ldots,m\}$, condition (iii) is equivalent to rows of T weakly decreasing, and condition (iv) is equivalent to columns of T strictly decreasing. Moreover, $\mathrm{wt}(D)=\mathrm{wt}(T)$. Therefore, this gives a weight-preserving bijection between $\mathrm{LT}(0^m\times\alpha)$ with all cells weakly below row m and $\mathrm{SSET}_m(\alpha)$, so the first formula follows. The second follows from the first by letting m go to infinity. □

Campbell, Feldman, Light, Shuldiner, and Xu [Campbell et al. 14] defined the same class of tableaux of composition shape, which they termed *shin-tableaux*, when they introduced the *shin functions*, which are a basis of symmetric functions in non-commuting variables that generalize the Schur functions. As these tableaux characterize the expansion of non-commutative homogeneous symmetric functions into shin functions, they also characterize the expansion of the dual basis, which are quasisymmetric functions, into monomial quasisymmetric functions. In other words, the extended Schur functions are the dual basis to the noncommutative shin functions. In [Campbell et al. 14], the authors observe this and state the positive expansion into monomial quasisymmetric functions. Below we develop further properties of this basis.

Just as the fundamental slide expansion of lock polynomials is a more compact formula, we may translate Theorem 6.17 into a fundamental quasisymmetric function expansion using the following.

Definition 6.18. A standard extended tableau of shape α is a semi-standard extended tableau of shape α that uses each of the integers $1,2,\ldots,n$ exactly once. Denote the set of standard extended tableaux of shape α by $\mathrm{SET}(\alpha)$, and call the element of $\mathrm{SET}(\alpha)$ whose entries in row $i+1$ are the first α_{i+1} integers larger than $\alpha_1+\ldots+\alpha_i$ the super-standard extended tableau of shape α.

For example, the standard extended tableaux of shape (2, 1, 2) are shown in Figure 21. The leftmost is the super-standard one.

Table 1. A table of the fundamental expansion of the extended Schur functions.

$\mathcal{E}_{(1)}$	=	$F_{(1)}$
$\mathcal{E}_{(2)}$	=	$F_{(2)}$
$\mathcal{E}_{(11)}$	=	$F_{(11)}$
$\mathcal{E}_{(3)}$	=	$F_{(3)}$
$\mathcal{E}_{(21)}$	=	$F_{(12)}+F_{(21)}$
$\mathcal{E}_{(12)}$	=	$F_{(12)}$
$\mathcal{E}_{(111)}$	=	$F_{(111)}$
$\mathcal{E}_{(4)}$	=	$F_{(4)}$
$\mathcal{E}_{(31)}$	=	$F_{(13)}+F_{(22)}+F_{(31)}$
$\mathcal{E}_{(13)}$	=	$F_{(13)}$
$\mathcal{E}_{(22)}$	=	$F_{(121)}+F_{(22)}$
$\mathcal{E}_{(211)}$	=	$F_{(112)}+F_{(121)}+F_{(211)}$
$\mathcal{E}_{(121)}$	=	$F_{(112)}+F_{(121)}$
$\mathcal{E}_{(112)}$	=	$F_{(112)}$
$\mathcal{E}_{(1111)}$	=	$F_{(1111)}$
$\mathcal{E}_{(5)}$	=	$F_{(5)}$
$\mathcal{E}_{(41)}$	=	$F_{(14)}+F_{(23)}+F_{(32)}+F_{(41)}$
$\mathcal{E}_{(14)}$	=	$F_{(14)}$
$\mathcal{E}_{(32)}$	=	$F_{(23)}+F_{(122)}+F_{(131)}+F_{(221)}+F_{(32)}$
$\mathcal{E}_{(23)}$	=	$F_{(23)}+F_{(122)}$
$\mathcal{E}_{(311)}$	=	$F_{(113)}+F_{(122)}+F_{(131)}+F_{(212)}+F_{(221)}+F_{(311)}$
$\mathcal{E}_{(131)}$	=	$F_{(113)}+F_{(122)}+F_{(131)}$
$\mathcal{E}_{(113)}$	=	$F_{(113)}$
$\mathcal{E}_{(221)}$	=	$F_{(122)}+F_{(1121)}+F_{(212)}+F_{(1211)}+F_{(221)}$
$\mathcal{E}_{(212)}$	=	$F_{(122)}+F_{(1121)}+F_{(212)}$
$\mathcal{E}_{(122)}$	=	$F_{(122)}+F_{(1121)}$
$\mathcal{E}_{(2111)}$	=	$F_{(1112)}+F_{(1121)}+F_{(1211)}+F_{(2111)}$
$\mathcal{E}_{(1211)}$	=	$F_{(1112)}+F_{(1121)}+F_{(1211)}$
$\mathcal{E}_{(1121)}$	=	$F_{(1112)}+F_{(1121)}$
$\mathcal{E}_{(1112)}$	=	$F_{(1112)}$
$\mathcal{E}_{(11111)}$	=	$F_{(11111)}$

For T a standard extended tableau, define the *descent composition of* T, denoted by $\mathrm{Des}(T)$, to be the (strong) composition given by increasing runs of the entries $1,2,\ldots,n$ when read right to left in T. For example, the descent compositions for the standard extended tableaux in Figure 21 are (2, 1, 2), (1, 2, 2), and $(1,1,2,1)$, respectively; note the descent composition of the super-standard extended tableau is α. Compare this with the F-expansion of $\mathcal{E}_{(2,1,2)}$.

Theorem 6.19. *For α a (strong) composition, we have*

$$\mathcal{E}_\alpha(X)=\sum_{T\in \mathrm{SET}(\alpha)} F_{\mathrm{Des}(T)}(X). \qquad (6–10)$$

Proof. Similar to the proof of Theorem 6.11, we construct a *standardization map* from $\mathrm{SSET}(\alpha)$ to $\mathrm{SET}(\alpha)$ by reading entries from smallest to largest and reading cells with entry i from right to left, change the entries to $1,2,3,\ldots,n$. It is easy to see that this maintains the extended tableau conditions, so the result is a standard extended tableau.

Let $T\in\mathrm{SSET}(\alpha)$ and suppose that T standardizes to $S\in\mathrm{SET}(\alpha)$. By construction and the definition of the descent composition, we have $\mathrm{flat}(\mathrm{wt}(T))$ refines $\mathrm{Des}(S)$. Hence x^T is a monomial of $F_{\mathrm{Des}(S)}$. Conversely, let $S\in\mathrm{SET}(\alpha)$, and let b be a weak composition such that $\mathrm{flat}(b)$ refines $\mathrm{Des}(S)$. We show there is a unique $T\in SSET(\alpha)$ with $\mathrm{wt}(T)=b$ that standardizes to S. To reconstruct T from b and S, for $j=1,\ldots,n$, if $\mathrm{Des}(S)_j=b_{i_{j-1}+1}+\cdots+b_{i_j}$, then, from right to left, set the first $b_{i_{j-1}+1}$ cells to have entry $i_{j-1}+1$, the next $b_{i_{j-1}+2}$ cells to have entry $i_{j-1}+2$, and so on. This maintains the extended tableaux conditions, $\mathrm{wt}(T)=b$, and T standardizes to S. Uniqueness follows from the lack of choice at every step. □

Using this characterization, we now justify the terminology extended Schur *basis*.

Theorem 6.20. *The extended Schur functions form a basis for the ring of quasisymmetric functions.*

Proof. The descent composition of the super-standard extended tableau of shape α is larger in lexicographic order than the descent composition of any other element of $\mathrm{SET}(\alpha)$. Hence by Theorem 6.19, the extended Schur functions are upper uni-triangular with respect to lexicographic order on the fundamental quasisymmetric functions. □

The extended Schur basis exhibits many nice properties and should have interesting applications to symmetric and quasisymmetric functions. We close our introduction of this basis with two such properties.

Proposition 6.21. *Let α be a (strong) composition and let β be obtained from α by exchanging two adjacent parts $\alpha_i<\alpha_{i+1}$. Then the difference $\mathcal{E}_\beta-\mathcal{E}_\alpha$ is F-positive. In particular, the terms of the fundamental quasisymmetric expansion of $\mathcal{E}_\alpha$ are a sub(multi)set of the terms of s_λ where $\lambda=\mathrm{sort}(\alpha)$.*

Proof. Define a map from $\mathrm{SET}(\alpha)$ to $\mathrm{SET}(\beta)$ by dropping the leftmost $\alpha_{i+1}-\alpha_i$ cells of row $i+1$ of $\mathbb{D}(\alpha)$ down one row, retaining all entries. This map is well-defined since all cells retain their relative order within columns, and since in any element of $\mathrm{SET}(\alpha)$ the leftmost cell of row i has smaller entry than the cell immediately above it, which in turn has smaller entry than the cell immediately left of it (which is the rightmost cell to drop down). The map is injective, and moreover preserves Des since we only move entries within their original column. □

For example, taking $\alpha=(2,1,2)$ and exchanging α_2 and α_3 to get $\beta=(2,2,1)$, we have

$$\mathcal{E}_{(2,2,1)}-\mathcal{E}_{(2,1,2)}=F_{(2,2,1)}+F_{(1,2,1,1)}.$$

For further examples, compare entries for the extended Schur functions in Table 1.

Proposition 6.22. *The extended Schur function $\mathcal{E}_\alpha$ is equal to a single fundamental quasisymmetric function $\mathfrak{F}_\alpha$ if and only if $\mathbb{D}(\alpha)$ is a (reverse) hook shape, i.e. $\alpha = (1^k, \ell)$ for some k and ℓ.*

Proof. If α is a reverse hook shape, there is only one standard extended tableau of shape α, specifically the super-standard one. Conversely, suppose $\alpha_i > 1$ and $\alpha_{i+1} > 0$. Then a second element of $\mathrm{SET}(\alpha)$ may be obtained from the super-standard one by swapping the entry of the leftmost cell of row i and the right-most cell of row $i+1$. □

Acknowledgments

The authors thank Per Alexandersson, Nantel Bergeron, and Vic Reiner for helpful comments and illuminating discussions.

Funding

S. Assaf was support by the Simons Foundation Collaboration Grant for Mathematicians (Award 524477).

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