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Spectral Data for Spin Higgs Bundles

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In this paper we determine the spectral data parametrizing Higgs bundles in a generic fiber of the Hitchin map for the case where the structure group is the special Clifford group with fixed Clifford norm. These are spin and "twisted" spin Higgs bundles. The method used relates variations in spectral data with respect to the Hecke transformations for orthogonal bundles introduced by Abe. The explicit description also recovers a result from the geometric Langlands program, which states that the fibers of the Hitchin map are the dual abelian varieties to the corresponding fibers of the moduli spaces of projective orthogonal Higgs bundles (in the even case) and projective symplectic Higgs bundles (in the odd case).

1 Results

Let X be a smooth projective algebraic curve of genus $g \geq 2$ and $p \in X$. Let $\mathcal{M}^{\pm}_{\mathsf{Spin}(N)}(X)$ denote the coarse moduli spaces of semistable Higgs bundles on X with the special Clifford group $\mathsf{SC}(N)$ as structure group and fixed Clifford norm of even $(+)$ or odd $(-)$ degree, respectively. For concreteness and without loss of generality, we require the Clifford norms to be $\mathcal{O}_X$ and $\mathcal{O}_X(p)$, respectively. Then $\mathcal{M}^{+}_{\mathsf{Spin}(N)}$ is exactly the moduli space of $\mathsf{Spin}(N)$ Higgs bundles, whereas $\mathcal{M}^{-}_{\mathsf{Spin}(N)}$ is a moduli space of *twisted* $\mathsf{Spin}(N)$ Higgs bundles (see Section 2.1).

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The Hitchin fibration takes the form

$$h^{\pm}_{\mathsf{Spin}(N)} : \mathcal{M}^{\pm}_{\mathsf{Spin}(N)} \longrightarrow \mathcal{B}(N) := \begin{cases} \bigoplus_{i=1}^{m} H^0(X, K_X^{2i}) & N = 2m+1 \\ \bigoplus_{i=1}^{m-1} H^0(X, K_X^{2i}) \oplus H^0(X, K_X^m) & N = 2m. \end{cases}$$

The maps $h^{\pm}_{\mathsf{Spin}(N)}$ realize these moduli spaces as algebraically complete integrable systems whose generic fibers are torsors over abelian varieties. The main goal of this note is to describe these abelian varieties explicitly in terms of *spectral data*.

The problem is clearly related to the case of Higgs bundles for orthogonal groups. Here there is a complete description (see [10, 13]). Hitchin describes the spectral data in terms of line bundles in the *Prym variety* associated to the spectral curve defined by the point in $\mathcal{B}(N)$. The construction, which we briefly review in Sections 2.5 and 2.6 below, involves fixing a spin structure on X. In the end, this ancillary choice is irrelevant, but it gives a hint that hidden in the argument is actually a lift to $\mathsf{Spin}(N)$ (or $\mathsf{SC}(N)$). We shall show that these data indeed provide the extra structure of a Clifford bundle.

To be more precise, let $\vec{b} \in \mathcal{B}(N)$ be a generic point. By the *spectral curve* $\pi : S \to X$ we mean (somewhat unconventionally) the normalization of the branched cover of X defined by $\vec{b}$ (see Sections 2.5 and 2.6). Let $\overline{S} = S/\sigma$, where σ is the natural involution, and let $K(S,\overline{S})$ denote the kernel of the norm map $\mathrm{Nm}_{S/\overline{S}} : J(S) \to J(\overline{S})$. Then $K(S,\overline{S})$ is just the Prym variety $P(S,\overline{S})$ of the cover $p : S \to \overline{S}$ for N odd, whereas for N even, $P(S,\overline{S}) \subset K(S,\overline{S})$ is the connected component of the trivial bundle. In both cases, $J_2(\overline{S})$ acts additively on $K(S,\overline{S})$ by pulling back via p^*, and it acts on $J_2(X)$ via the norm map of the covering $\overline{S} \to X$. The main result may then be stated as follows.

Theorem 1.1 (Spectral Data). For generic points $\vec{b} \in \mathcal{B}(N)$, the fiber $(h^{\pm}_{\mathsf{Spin}(N)})^{-1}(\vec{b})$ is a torsor over the abelian variety:

$$A_{\mathsf{Spin}(N)}(X, \vec{b}) := K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X). \tag{1.1}$$

In terms of Prym varieties,

$$A_{\mathsf{Spin}(N)}(X, \vec{b}) = \begin{cases} P(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X) & N \text{ odd}; \\ P(S,\overline{S}) \times_{H_1/H_0} J_2(X) & N \text{ even}, \end{cases}$$

where in the even case $H_0 \simeq \mathbb{Z}/2$ is the subgroup of $J_2(\overline{S})$ generated by the line bundle defining the étale cover $S \to \overline{S}$ and H_1 is the annihilator of H_0 in $J_2(\overline{S})$ with respect to the Weil pairing.

The fact that the right-hand side of (1.1) is connected is not quite obvious (see Lemma 4.2). We have the following consequence.

Corollary 1.2. The fibers of the Hitchin map for $\mathcal{M}^{-}_{\mathsf{Spin}(N)}$ are connected.

In the case of $\mathcal{M}^{+}_{\mathsf{Spin}(N)}$, that is, $\mathsf{Spin}(N)$-Higgs bundles, the connectedness of the fibers follows from a general result of several authors (cf. [6, 8]), whereas the fact that the fiber structure is the same in the twisted case $\mathcal{M}^{-}_{\mathsf{Spin}(N)}$ (and hence also connected) is a consequence of Theorem 1.1. An application of this fact is the following: Hitchin's construction [11] of a projectively flat connection on the space of generalized $\mathsf{Spin}(N)$ theta functions works as well in the twisted case. This connection was used by the authors in [15] in their study of strange duality for odd orthogonal bundles.

For a general complex reductive Lie group G, works of Donagi–Pantev [6], Hitchin [13], and Hausel–Thaddeus [9] show that the Hitchin system associated to G is dual to the Hitchin system associated to ${}^{L}G$, the Langlands dual group to G. Another consequence of Theorem 1.1 is an explicit duality for spin bundles. Let $A_{\mathsf{PSO}(2m)}(X, \vec{b})$ and $A_{\mathsf{PSp}(2m)}(X, \vec{b})$ denote the fibers over $\vec{b} \in \mathcal{B}(N)$ of the Hitchin map for the moduli spaces of projective special orthogonal and projective symplectic Higgs bundles, where $N = 2m$ or $N = 2m+1$, respectively. Then we have the following theorem (see Section 4):

Theorem 1.3 (Langlands Duality). For generic points $\vec{b}$, we have the following dualities of abelian varieties:

(1) $\left(A_{\mathsf{Spin}(2m)}(X, \vec{b})\right)^{\vee} \simeq A_{\mathsf{PSO}(2m)}(X, \vec{b})$;

(2) $\left(A_{\mathsf{Spin}(2m+1)}(X, \vec{b})\right)^{\vee} \simeq A_{\mathsf{PSp}(2m)}(X, \vec{b})$.

Here is a brief sketch of the main idea behind the proof of Theorem 1.1. First, spectral data describe an orthogonal bundle $V_L \to X$ in terms of a line bundle $L \in K(S, \overline{S})$. In Section 3.2 we show that if L is modified by a line bundle defined by a generic point $p \in S$ (and its reflection by σ), then the new orthogonal bundle obtained is exactly the *Hecke transformation* of V_L at the point $\pi(p)$ introduced by Abe [1]. The result, Corollary 3.4, means that we can move around in the spectral data for orthogonal bundles via elementary transformations on the bundle itself. This interpretation makes

it transparent that a choice of lift of V_L to a Clifford bundle induces a lift on the transformed bundle as well (see Corollary 3.5). In this way, a Clifford structure is naturally defined on V_L, $L = M^2$, given one on the orthogonal bundle with "trivial" spectral data. We then show that the dependence of this structure on M is exactly given by the action of $J_2(\bar{S})$ via the norm map. In Section 4, we prove directly that the abelian varieties appearing are dual to the ones for the projective symplectic and orthogonal cases.

2 Preliminaries

2.1 Clifford bundles

Let $(\mathbb{V}, q)$ be a complex orthogonal vector space, $C(\mathbb{V})$ the Clifford algebra of $(\mathbb{V}, q)$, and $C_+(\mathbb{V})$ the even part. The *special Clifford group* is defined as follows:

$$\mathsf{SC}(\mathbb{V}) = \left\{ g \in C_+^\times(\mathbb{V}) \mid gvg^{-1} \in \mathbb{V} \text{ for all } v \in \mathbb{V} \right\}.$$

The induced action of $\mathsf{SC}(\mathbb{V})$ on $\mathbb{V}$ is by orthogonal transformations and gives rise to an exact sequence

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \mathsf{SC}(\mathbb{V}) \longrightarrow \mathsf{SO}(\mathbb{V}) \longrightarrow 0. \tag{2.1}$$

The *Clifford (or spinor) norm* of an element $g \in \mathsf{SC}(\mathbb{V})$ is defined as

$$\mathrm{Q}(g) = q(v_1) \cdots q(v_k)$$

where $g = v_1 \cdots v_k$, for $v_j \in \mathbb{V}$ (any $g \in \mathsf{SC}(\mathbb{V})$ has such an expression). The *spin group* is then $\mathsf{Spin}(\mathbb{V}) := \mathrm{Q}^{-1}(1)$. The restriction of (2.1) to $\mathsf{Spin}(\mathbb{V})$ becomes

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathsf{Spin}(\mathbb{V}) \longrightarrow \mathsf{SO}(\mathbb{V}) \longrightarrow 0.$$

We set $\mathsf{SC}(N) = \mathsf{SC}(\mathbb{C}^N)$, where $\mathbb{C}^N$ has the standard orthogonal structure.

For a connected complex reductive Lie group G, let $\mathcal{M}_G$ denote the coarse moduli space of semistable G-Higgs bundles on X. In the case $G = \mathsf{SC}(N)$, the Clifford norm induces a morphism $\mathcal{M}_{\mathsf{SC}(N)} \to \mathrm{Pic}(X)$, which we also denote by Q. For an $\mathsf{SC}(N)$ bundle P and $L \in \mathrm{Pic}(X)$, we will denote by $P \otimes L$ the $\mathsf{SC}(N)$ bundle whose transition functions are obtained by multiplying those of P and L. It is then clear that

$$\mathrm{Q}(P \otimes L) = \mathrm{Q}(P) \otimes L^2. \tag{2.2}$$

Fix $p \in X$, and consider bundles $\mathcal{O}_X(dp)$, where $d \in \mathbb{Z}$. Then the preimage by Ω of the class of $[\mathcal{O}_X(dp)] \in \mathrm{Pic}(X)$ depends only on the parity of d. Let $\mathcal{M}^{\pm}_{\mathsf{Spin}(N)}$ be the inverse images of the bundles $[\mathcal{O}_C(dp)]$, for $d = 0, 1$. Therefore, while by definition $\mathcal{M}^{+}_{\mathsf{Spin}(N)} = \mathcal{M}_{\mathsf{Spin}(N)}$, the space $\mathcal{M}^{-}_{\mathsf{Spin}(N)}$ is a "twisted" component that does not correspond to a moduli space of G-bundles for any complex reductive G. The connected components $\mathcal{M}^{\pm}_{\mathsf{SO}(N)}$ of $\mathcal{M}_{\mathsf{SO}(N)}$ are labeled by the 2nd Stiefel–Whitney class: $V \in \mathcal{M}^{\pm}_{\mathsf{SO}(N)} \iff w_2(V) = \pm 1$ (cf. [3, Prop. 1.3]), and the projection (2.1) induces a morphism $\mathcal{M}^{\pm}_{\mathsf{Spin}(N)} \to \mathcal{M}^{\pm}_{\mathsf{SO}(N)}$. We refer to [17, Prop. 3.4] for more details.

In this paper, we mostly regard points in $\mathcal{M}^{\pm}_{\mathsf{SO}(N)}$ as equivalence classes of rank N semistable orthogonal Higgs bundles: that is, a holomorphic bundle $V \to X$ with nondegenerate symmetric bilinear pairing $(\,,\,) : V \otimes V \to \mathcal{O}_X$, and a fixed isomorphism $\det V \simeq \mathcal{O}_X$, equipped with a holomorphic map $\Phi : V \to V \otimes K_X$ satisfying $(\Phi v, w) + (v, \Phi w) = 0$ for all $v, w \in V$.

2.2 Hecke transformations of orthogonal bundles

We first recall Hecke transformations for orthogonal bundles following Abe [1]. Let $V \to X$ be an orthogonal bundle. Choose a point $p \in X$ and an isotropic line τ in the fiber V_p of V at p. Let $\tau^{\perp}$ denote the orthogonal subspace to τ in V_p, and set $\tau_1 = V_p/\tau^{\perp}$. We view τ and τ_1 as torsion sheaves on X supported at p. Then we may define a locally free sheaf $V^{\flat} \to X$ by the elementary transformation:

$$0 \longrightarrow V^{\flat} \to V \to \tau_1 \longrightarrow 0. \tag{2.3}$$

Next let $V^{\sharp} = (V^{\flat})^*$. Since the orthogonal structure gives an isomorphism $V^* \simeq V$, dualizing (2.3) yields an exact sequence:

$$0 \longrightarrow V \to V^{\sharp} \to \tau_2 \longrightarrow 0 \tag{2.4}$$

where τ_2 is a torsion sheaf supported at p of length 1. Now the orthogonal structure also induces maps

$$V^{\sharp} \otimes V^{\sharp} \longrightarrow \mathcal{O}_X(p) \ \text{ and } V^{\flat} \otimes V^{\sharp} \longrightarrow \mathcal{O}_X. \tag{2.5}$$

Consider the subsheaf $V^{\flat} \hookrightarrow V^{\sharp}$ obtained by composing $V^{\flat} \hookrightarrow V$ in (2.3) with $V \hookrightarrow V^{\sharp}$ in (2.4). Then $V^{\sharp}/V^{\flat}$ is a torsion sheaf supported at p, and the fiber at p is a rank 2 orthogonal space. Since $V/V^{\flat}$ is isotropic, there is a canonical splitting $V^{\sharp}/V^{\flat} \simeq V/V^{\flat} \oplus \tau_2$. Finally, we define $V^{\iota} \subset V^{\sharp}$ to be the kernel of the map $V^{\sharp} \to V/V^{\flat}$. Equivalently, there is

an exact sequence

$$0 \longrightarrow V^\flat \to V^\iota \to \tau_2 \longrightarrow 0. \tag{2.6}$$

Then V^ι inherits an orthogonal structure from (2.5). Moreover, the exact sequence (2.6) determines an isotropic line τ^ι in the fiber of V^ι at p. Finally, from (2.3) and (2.6), the trivialization of $\det V$ induces one for $\det V^\iota$.

Definition 2.1. The ι-transform is the map: $(V,\tau) \mapsto (V^\iota, \tau^\iota)$.

2.3 Hecke transformations of Clifford bundles

We wish to extend the previous discussion to Clifford bundles. For this it is useful to have a description of the ι-transform explicitly in terms of transition functions. Let P be a special Clifford bundle and V the associated orthogonal bundle. Let $\Delta \subset X$ be a disk centered at p, and $\sigma : \Delta \to P$ a section. Set $X^* = X \setminus \{p\}$. This gives a trivialization of P and a local frame $e_1, \dots, e_N$ for V on Δ with respect to which the quadratic structure is, say, of the form $(e_i, e_j) = \delta_{i+j-1,N}$. Similarly, we may choose a section of $P|_{X^*}$. Set $\Delta^* = X^* \cap \Delta$. Let $\widehat{\varphi} : \Delta^* \to \mathsf{SC}(N)$ denote the transition function gluing the bundles $P|_\Delta$ and $P|_{X^*}$, and let $\varphi : \Delta^* \to \mathsf{SO}(N)$ be the quotient transition function for $(\mathcal{E}, q)$. The transformed bundle V^ι is defined by modifying φ by $\zeta : \Delta^* \to \mathsf{SO}(N)$, where

$$\zeta = \begin{pmatrix} z & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z^{-1} \end{pmatrix}. \tag{2.7}$$

Write $z = \exp(2\pi i\xi)$, $\operatorname{Im}\xi > 0$, and set

$$\widehat{\zeta}(z) = \exp(\pi i\xi)\exp\left((\pi i\xi/2)(e_1 e_N - e_N e_1)\right). \tag{2.8}$$

One checks that $\widehat{\zeta}$ is well defined under $\xi \mapsto \xi + 1$, and so it yields a map $\widehat{\zeta} : \Delta^* \to \mathsf{SC}(N)$. Moreover, the projection (2.1) of $\widehat{\zeta}$ to $\mathsf{SO}(N)$ recovers ζ. Gluing the trivial $\mathsf{SC}(N)$-bundles over Δ and X^* via $\widehat{\varphi}(z)\widehat{\zeta}(z)$, we define a new Clifford bundle P^ι. The associated orthogonal bundle (with transition function $\varphi(z)\zeta(z)$) coincides with V^ι. With this understood, the main observation is the following:

Proposition 2.2. We have $\mathrm{Q}(P^\iota) \simeq \mathrm{Q}(P) \otimes \mathcal{O}_X(p)$. In particular, $w_2(V^\iota) = -w_2(V)$.

Proof. From (2.8), $\mathrm{Q}(P^\iota)$ is a line bundle with transition function on Δ^* given by

$$\mathrm{Q}(\widehat{\varphi\zeta}) = \exp(2\pi i\xi)\,\mathrm{Q}(\widehat{\varphi}(z))\,\mathrm{Q}\left(\exp\left((\pi i\xi/2)(e_1e_N - e_Ne_1)\right)\right) = z\cdot\mathrm{Q}(\widehat{\varphi}(z))$$

since $\exp\left((\pi i\xi/2)(e_1e_N - e_Ne_1)\right) \in \mathsf{Spin}(N)$; hence, the result . ■

Remark 2.3 We could just as well have chosen a prefactor of $\exp(-\pi i\xi)$ in (2.8) to obtain a Clifford bundle with norm $\mathrm{Q}(P)\otimes\mathcal{I}_p$, where $\mathcal{I}_p$ denotes the ideal sheaf of $p \in X$. The two Clifford bundles thus defined are isomorphic after tensoring by $\mathcal{O}_X(p)$. Later on, however, we shall consider multiple Hecke transformations at points of a reduced, not necessarily effective divisor D; hence, the points will have a sign. For convenience, at each point p in the support of D we shall choose $\exp(\pm\pi i\xi)$ in (2.8) so that the change of Clifford norm corresponds to the sign of p.

2.4 Spectral data for symplectic bundles

In this section, following [13] and [10], we briefly recall the explicit description of generic fibers of the Hitchin map for the groups $\mathsf{Sp}(2m)$ and $\mathsf{PSp}(2m)$ in terms of spectral data. Let $E \to X$ be a symplectic bundle of rank $2m$ with pairing $\langle\,,\,\rangle$, and $\Phi : E \to E\otimes K_X$ be a Higgs field such that $\langle v, \Phi w\rangle + \langle \Phi v, w\rangle = 0$, for all sections $v, w \in E$. The coefficients of the characteristic polynomial of Φ gives an element $\vec{b} \in \mathcal{B}(N)$, $N = 2m+1$. Let $\pi : \mathrm{tot}(K_X) \to X$ be the total space of the canonical bundle on X, and let $\lambda : \mathrm{tot}(K_X) \to \pi^*K_X$ denote the tautological section. The spectral curve associated to $\vec{b}$ is

$$S := \left\{ w \in \mathrm{tot}(K_X) \mid \lambda^{2m} + \pi^*(b_2)\lambda^{2m-2} + \cdots + \pi^*(b_{2m}) = 0 \text{ at } w \right\}. \tag{2.9}$$

We assume that S is smooth. This implies that the last coefficient b_{2m} of $\vec{b}$ has simple zeros at $Z(b_{2m}) \subset X$ and that b_{2m-2} is nonzero on $Z(b_{2m})$ (see [4]). Let σ denote the involution $w \mapsto -w$ on S, and $\overline{S} = S/\sigma$. In this case, the kernel $K(S,\overline{S})$ of the norm map $\mathrm{Nm}_{S/\overline{S}} : J(S) \to J(\overline{S})$ is connected and so coincides with the Prym variety $P(S,\overline{S})$. Let $L \in P(S,\overline{S})$, and consider $U = L \otimes \left(K_S \otimes \pi^*K_X^{-1}\right)^{1/2}$. Since $K_S = \pi^*K_X^{2m}$, a square root of $K_S \otimes \pi^*K_X^{-1}$ can be given by a choice of theta characteristic on X. We then have the following result of Hitchin.

Theorem 2.4 ([10, Section 5.10]). Let $\vec{b} \in \mathcal{B}(N)$ be such that b_{2m} has simple zeros and b_{2m-2} is nonzero on $Z(b_{2m})$. Then $h^{-1}_{\mathsf{Sp}(2m)}(\vec{b})$ identifies with points on the Prym variety

$P(S,\overline{S})$. The correspondence sends $L \in P(S,\overline{S})$ to $E = \pi_*(U)$, where

$$U = L \otimes (K_S \otimes \pi^* K_X^{-1})^{-1/2} \in P(S,\overline{S}).$$

The Higgs field Φ is obtained by multiplication with λ. Conversely, given a symplectic Higgs bundle (E,Φ), let $U \to S$ be the line bundle

$$U := \ker(\pi^* E \xrightarrow{\Phi} \pi^*(E \otimes K_X)).$$

Then $L = U \otimes (K_S \otimes \pi^* K_X^{-1})^{1/2} \in P(S,\overline{S})$.

The symplectic structure on $E = \pi_* U$ is defined as follows. Let $\widehat{\sigma}$ denote a linearization of σ on U. Then $(\widehat{\sigma})^2$ acts as -1 on the fibers. It suffices to define a nondegenerate skew pairing on sections of U over open sets $A \subset X$. For two such sections u and v, let

$$\langle u, v\rangle = \operatorname{tr}_{S/X}\left(\frac{\hat{\sigma}(u)v}{d\pi}\right). \tag{2.10}$$

Since $\hat{\sigma}$ squares to -1, we have

$$\left(\frac{\hat{\sigma}(u)v}{d\pi}\right)(z) = -\left(\frac{\hat{\sigma}(v)u}{d\pi}\right)(\sigma(z)),$$

and hence since $\operatorname{tr}_{S/X}(\cdot) = \operatorname{tr}_{\overline{S}/X}\operatorname{tr}_{S/\overline{S}}(\cdot)$, the pairing $\langle\ ,\ \rangle$ is skew. The fact that it is nondegenerate follows as in [10].

Let $\mathcal{M}^0_{\mathsf{PSp}(2m)}$ be the moduli space of semistable Higgs bundles for $\mathsf{PSp}(2m)$ that lift to $\mathsf{Sp}(2m)$ bundles, and let $h_{\mathsf{PSp}(2m)} : \mathcal{M}^0_{\mathsf{PSp}(2m)} \to \mathcal{B}(N)$ be the Hitchin map. Since the Higgs field take values in the adjoint bundles, the natural projection of $\mathsf{Sp}(2m) \to \mathsf{PSp}(2m)$ give a natural projection $\mathcal{M}_{\mathsf{Sp}(2m)} \to \mathcal{M}^0_{\mathsf{PSp}(2m)}$. Now Theorem 2.4 has the following consequence:

Corollary 2.5. The fibers of the Hitchin map $h_{\mathsf{PSp}(2m)}$ are in one to one correspondence with points $P(S,\overline{S})/J_2(X)$, where $J_2(X)$ acts through pulling back by $\pi : S \to X$.

2.5 Spectral data for odd orthogonal bundles

Let V be a vector bundle of rank $2m+1$ with a nondegenerate symmetric bilinear $(\ ,\ )$ form along with a trivialization $\det V \simeq \mathcal{O}_X$. Let $\Phi : V \to V \otimes K_X$ be a Higgs field

satisfying $(v, \Phi w) + (\Phi v, w) = 0$. Then

$$\det(\lambda - \Phi) = \lambda(\lambda^{2m} + b_2\lambda^{2m-2} + \cdots + b_{2m}).$$

As in the case of the symplectic bundle, we assume that the zeros $Z(b_{2m})$ of b_{2m} are simple and that b_{2m-2} is nonvanishing on $Z(b_{2m})$. In [13, Section 4.1] Hitchin shows that the one-dimensional zero eigenspace of Φ generates the line bundle V_0 isomorphic to K_X^{-m}, and that the quotient $V_1 = V/V_0$ is of the form $E \otimes K_X^{1/2}$, where E is a symplectic bundle. The symplectic form $\langle\ ,\ \rangle$ on E is induced by the formula $\langle v_1, v_2\rangle = (v_1, \Phi v_2)$, and the Higgs field Φ restricted to $E \otimes K_X^{1/2}$ induces a symplectic Higgs field. We again define the spectral curve S by (2.9), and so the orthogonal bundle gives rise to $L \in P(S, \overline{S})$.

Going in the other direction, starting with a symplectic bundle (E, Φ') we define an orthogonal structure on $V = V_0 \oplus V_1$ by using b_{2m} on V_0 and

$$(u, v) = \operatorname{tr}_{S/X}\left(\frac{\hat{\sigma}(u)v \cdot \lambda^{-1}}{d\pi}\right) \tag{2.11}$$

[see (2.10)] on V_1. Because the section λ is odd, this becomes an even pairing. This only defines the orthogonal structure on V away from the ramification locus, however, and extending it to X requires more information. Since the details are not important for this paper, we simply state the result.

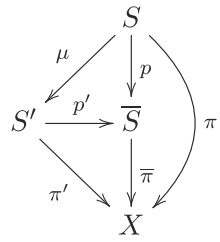
Theorem 2.6 ([13]). Let (E, Φ') be a generic symplectic Higgs bundle of rank $2m$. Then an associated $\mathsf{SO}(2m+1)$ Higgs bundle is determined by a vector $e_b \in E_b \otimes K_b^{m-1/2}$ for each point $b \in Z(b_{2m})$ satisfying a certain compatibility condition with Φ'. In terms of the spectral data L of E, this is equivalent to a choice of line $v_b \in L_b$, which squares to the canonical trivialization of L^2 at b. Moreover, the generic fibers of $h^{\pm}_{\mathsf{SO}(2m+1)}$ for each connected component $\mathcal{M}^{\pm}_{\mathsf{SO}(2m+1)}$ are isomorphic to $P(S, \overline{S})/J_2(\overline{S})$, where the $J_2(\overline{S})$-action is via the pullback by $p : S \to \overline{S}$.

2.6 Spectral data for even orthogonal bundles

In this case, a point in $\mathcal{B}(N)$ is of the form: $\vec{b} = (b_2, \ldots, b_{2m-2}, p_m)$, $N = 2m$. We assume p_m has simple zeros at $Z(p_m)$ and that b_{2m-2} is nonvanishing on $Z(p_m)$. The curve defined analogously to (2.9) is

$$S' := \left\{w \in \operatorname{tot}(K_X) \mid \lambda^{2m} + \pi^*(b_2)\lambda^{2m-2} + \cdots + \pi^*(b_{2m-2})\lambda^2 + \pi^*(p_m^2) = 0 \text{ at } w\right\}$$

The zeros $Z(p_m)$ (which we view also as points in S') are singularities of S'. With the assumption above these are the only singularities, and they consist of $2m(g-1)$ ordinary double points. Let σ be the involution on S' sending λ to $-\lambda$. The fixed points of σ are exactly the singularities. If S denotes the normalization of S', $\mu : S \to S'$, then since the singularities are double points, σ extends to an involution of S. The relevant diagram now is the following:



(2.12)

The double covering $p : S \to \overline{S}$ is unramified, and hence it is determined by a line bundle $\mathcal{L} \in J_2(\overline{S})$. We will need the following:

Lemma 2.7. The line bundle $\mathcal{L} \to \overline{S}$ is in the kernel of the norm map $\mathrm{Nm}_{\overline{S}/X} : J(\overline{S}) \to J(X)$.

Proof. We prove this by computing $\det \pi_* \mathcal{O}_S$ in two different ways. First, the normalization gives an exact sequence:

$$0 \longrightarrow \mathcal{O}_{S'} \longrightarrow \mu_* \mathcal{O}_S \longrightarrow \mathcal{O}_{Z(p_m)} \longrightarrow 0.$$

Since $\pi = \pi' \circ \mu$, this implies

$$0 \longrightarrow (\pi')_* \mathcal{O}_{S'} \longrightarrow \pi_* \mathcal{O}_S \longrightarrow \mathcal{O}_{Z(p_m)} \longrightarrow 0. \tag{2.13}$$

Now from general facts about spectral curves, we get $\det(\pi')_* \mathcal{O}_{S'} \simeq K_X^{-2m(m-1)}$ (cf. [4, Section 3]). So from (2.13) we have

$$\det \pi_* \mathcal{O}_S \simeq K_X^{-m(2m-1)} \otimes K_X^m = K_X^{-2m(m-1)}. \tag{2.14}$$

On the other hand, by the definition of $\mathcal{L}$, $p_* \mathcal{O}_S = \mathcal{O}_{\overline{S}} \oplus \mathcal{L}$. It follows that

$$\begin{aligned} \pi_* \mathcal{O}_S &= \overline{\pi}_* \left(p_* \mathcal{O}_S \right) = \overline{\pi}_* \mathcal{O}_{\overline{S}} \oplus \overline{\pi}_* \mathcal{L} \\ \det \pi_* \mathcal{O}_S &\simeq \left(\det \overline{\pi}_* \mathcal{O}_{\overline{S}} \right)^2 \otimes \mathrm{Nm}_{\overline{S}/X} \mathcal{L}. \end{aligned} \tag{2.15}$$

As before (except now note that $\overline{S} \subset \mathrm{tot}K_X^2$), we have $\overline{\pi}_* \mathcal{O}_{\overline{S}} \simeq \oplus_{i=0}^{m-1} K_X^{-2i}$. Plugging this into (2.15) we obtain

$$\det \pi_* \mathcal{O}_S \simeq K_X^{-2m(m-1)} \otimes \mathrm{Nm}_{\overline{S}/X} \mathcal{L}.$$

Now comparing this with (2.14), we conclude that $\mathrm{Nm}_{\overline{S}/X} \mathcal{L} \simeq \mathcal{O}_X$. ■

Returning to the spectral data, as in the symplectic and odd orthogonal cases, for $L \in K(S, \overline{S})$ we let $V = \pi_* U$, where $U = L \otimes (K_S \otimes \pi^* K_X^{-1})^{-1/2}$. The pairing is defined by

$$(u, v) = \mathrm{tr}_{S/X} \left(\frac{\hat{\sigma}(u) v}{d\pi} \right) \tag{2.16}$$

as in (2.10), except that now $\widehat{\sigma}$ squares to the identity so that the pairing is symmetric. Let $P(S, \overline{S}) \subset K(S, \overline{S})$ denote the connected component of the identity of $\ker \mathrm{Nm}_{S/\overline{S}}$. Then we have the following theorem due to Hitchin [10].

Theorem 2.8. The correspondence described above identifies a generic fiber of the Hitchin map $h^{\pm}_{\mathsf{SO}(2m)}$ with $P(S, \overline{S})$.

The moduli space of $\mathsf{PSO}(2m)$-Higgs bundles has four connected components. Let $\mathcal{M}^0_{\mathsf{PSO}(2m)}$ denote the neutral component consisting of those bundles that lift to $\mathsf{Spin}(2m)$-bundles. As an easy corollary of Theorem 2.8 we get a description of the spectral data:

Corollary 2.9. The generic fibers of the Hitchin map $h^0_{\mathsf{PSO}(2m)} : \mathcal{M}^0_{\mathsf{PSO}(2m)} \to \mathcal{B}(2m)$ are in one-to-one correspondence with elements of the abelian variety $P(S, \overline{S})/J_2(X)$.

3 Spin Structures From Spectral Data

3.1 Case of special spectral data

Lemma 3.1. Fix generic $\vec{b} \in \mathcal{B}(N)$. Consider the orthogonal bundles V_c:

- for $N = 2m + 1$:

$$V_c = K_X^{-m} \oplus K_X^{-m+1} \oplus \cdots \oplus \mathcal{O}_X \oplus \cdots \oplus K_X^{m-1} \oplus K^m$$

with orthogonal structure given by the pairing of K^{-j} with K^j, and $\mathcal{O}_X$ an orthogonal subbundle;

- for $N = 2m$:

$$V_c = K_X^{-m} \oplus K_X^{-m+1} \oplus \cdots \oplus \mathcal{O}_X \oplus \cdots \oplus K_X^{m-1} \oplus K^m \oplus \mathcal{O}_X$$

  where the last factor $\mathcal{O}_X$ is also an orthogonal subbundle.

Then V_c admits a Higgs field Φ_c such that (V_c, Φ_c) is a stable $\mathsf{SO}(N)$-Higgs bundle in the fiber over $\vec{b}$.

Proof. This follows from Hitchin's construction via the principal $\mathfrak{sl}(2)$-embedding for split real forms (see [12] and also [2, Sections 8.4 and 8.5]). ■

By Theorems 2.6 and 2.8, there is a line bundle $L_c \in P(S, \overline{S})$ such that L_c gives spectral data for (V_c, Φ_c).

Lemma 3.2. Recall that we have fixed a theta characteristic $K_X^{1/2}$. Then $K_X^{1/2}$ also determines a lift of (V_c, Φ_c) to $\mathsf{Spin}(N)$-Higgs bundles.

Proof. The bundles V_c admit quasi-isotropic decompositions $W^+ \oplus W^- \oplus \mathcal{O}_X$ (odd case) and $W^+ \oplus W^-$ (even case), where $\det W^+ = K_X^{m(m+1)/2}$. Indeed, in the N odd case we take $W^+ = \oplus_{i=1}^m K^i$. For N even, we add to this a choice of isotropic line in $\mathcal{O}_X \oplus \mathcal{O}_X$. It is well known that a choice of square root of $\det W^+$ determines a lift to Spin, and such a root is determined by $K_X^{1/2}$ (cf. [14]). ■

3.2 Application of the Hecke transformation for even orthogonal bundles

In Section 2.2 we described how an orthogonal bundle $V \to X$ with a choice of isotropic line τ at a point $p \in X$ gives rise to a new orthogonal bundle V^ι. In this section, we relate the spectral data of these orthogonal bundles under this transformation. We have the following.

Proposition 3.3. For $L \in K(S, \overline{S})$, let $V_L = \pi_* U$, $U = L \otimes (K_S \otimes \pi^* K_X^{-1})^{-1/2}$, be the orthogonal bundle associated to L by Theorem 2.8. Choose a point $p \in S$ outside the ramification locus, and let $\widetilde{L} = L \otimes \mathcal{O}_S(p) \otimes \mathcal{I}_{\sigma(p)}$, where $\mathcal{I}_{\sigma(p)}$ denote the ideal sheaf of the point $\sigma(p)$. The fiber of U at p corresponds to an isotropic line τ in the fiber of V_L at $\pi(p)$. Then $V_{\widetilde{L}}$ is isomorphic to the orthogonal bundle V_L^ι in Definition 2.1.

Proof. Let $\tau_1 = V_L / \tau^\perp$. It is a skyscraper sheaf supported at $\pi(p)$ of length 1. Denote the orthogonal structure on V_L by $(\ ,\ )$. By definition of the pairing (2.11) and (2.16),

τ_1 can be identified with the fiber of U at $\sigma(p)$. Under the direct image, the sheaf map $V_L \to \tau_1$ given by $e \mapsto (e, \tau)$ corresponds to evaluation $U \to U_{\sigma(p)}$. In other words, the direct image of the exact sequence

$$0 \longrightarrow U \otimes \mathcal{I}_{\sigma(p)} \longrightarrow U \longrightarrow U_{\sigma(p)} \longrightarrow 0$$

is

$$0 \longrightarrow V_L^\flat \longrightarrow V_L \longrightarrow \tau_1 \longrightarrow 0.$$

With respect to the orthogonal structure, $V_L^\sharp = (V_L^\flat)^*$ is the direct image of $U \otimes \mathcal{O}_S(p)$. Now the direct image of the exact sequence

$$0 \longrightarrow U \longrightarrow U \otimes \mathcal{O}_S(p) \longrightarrow U_p \longrightarrow 0$$

is

$$0 \longrightarrow V_L \longrightarrow V_L^\sharp \longrightarrow \tau_2 \longrightarrow 0.$$

By definition, V_L^ι is the kernel of the induced map $V_L^\sharp \to V_L/V_L^\flat \simeq \tau_1$, and it follows that $V_L^\iota = \pi_* \left(U \otimes \mathcal{O}_S(p) \otimes \mathcal{I}_{\sigma(p)} \right)$. ■

Corollary 3.4. Let $L \in K(S, \overline{S})$, $L = L_c \otimes M \otimes \sigma^*(M^*)$ for $M \in J(S)$. Then for any generic choice of divisor $\operatorname{div} M$, V_L is isomorphic to the Hecke transform V_c^ι at $\pi(\operatorname{div} M)$.

Proof. Write the divisor D of M as

$$D = p_1 + \cdots + p_r - q_1 - \cdots - q_s.$$

Then

$$-\sigma(D) = -\sigma(p_1) - \cdots - \sigma(p_r) + \sigma(q_1) + \cdots + \sigma(q_s)$$

is a divisor of $\sigma^*(M^*)$, and so $M \otimes \sigma^*(M^*)$ has divisor

$$\bigotimes_{i=1}^{r} \left(\mathcal{O}_S(p_i) \otimes \mathcal{I}_{\sigma(p_i)} \right) \otimes \bigotimes_{i=1}^{s} \left(\mathcal{O}_S(\sigma(q_i)) \otimes \mathcal{I}_{q_i} \right).$$

Now apply Proposition 3.3 repeatedly. ■

Using the results of Section 2.3 (see Proposition 3.3 and recall the convention in Remark 2.3), we also have the following:

Corollary 3.5. In addition to the hypothesis of Proposition 3.3, suppose that V_L has a lift to an SC-bundle P_L. Then this determines a lift $P_{\widetilde{L}}$ of $V_{\widetilde{L}}$ to an SC-bundle with

$$\Omega(P_{\widetilde{L}}) = \Omega(P_L) \otimes \mathcal{O}_X(p).$$

In particular, if $L = L_c \otimes M \otimes \sigma^*(M^*)$, $M \in J(S)$, a choice of spin structure on V_{L_c} determines a lift of V_L to an SC-bundle P_M with $\Omega(P_M) = \mathrm{Nm}_{S/X}(M)$.

Remark 3.6. Implicit in Corollary 3.4 is the following: if we modify the choice of divisor D of M by a generic meromorphic function f then there is a natural isomorphism of the orthogonal bundles obtained by Hecke transformations on $\pi(D)$ and $\pi(D+\mathrm{div}(f))$. Indeed, multiplication of sections of U (in the proof of Proposition 3.3) by $f/\sigma^*(f)$ is an isometry with respect to the pairing (2.11) or (2.16). Furthermore, in an appropriate local frame this isometry has the form of ζ in (2.7), and so the SC-bundles obtained in Corollary 3.5 are similarly isomorphic.

3.3 Application of the Hecke transformation for odd orthogonal bundles

With small modifications, the arguments of the previous section apply equally well to the case of odd-dimensional orthogonal bundles. We note the following:

(1) As in Section 2.5, the orthogonal bundle determined by $L \in \mathrm{Prym}(S, \overline{S})$ in the complement of the ramification locus is isometric to $V_L = V_0 \oplus V_1$, where $V_0 = K_X^{-m}$, $V_1 = E_L \otimes K_X^{1/2}$, and E_L is the symplectic bundle from Section 2.4; namely, $E_L = \pi_* U$, $U = L \otimes (K_S \otimes \pi^* K_X^{-1})^{1/2}$. Let $p \in S \setminus \pi^{-1}(Z(b_{2m}))$. Then the isotropic line for E_L at $\pi(p)$ determined by p gives a canonical isotropic line in V_1, and hence also V_L, at $\pi(p)$. Tensoring the exact sequences in the proof of Proposition 3.3 by $K_X^{1/2}$, we find as before that the orthogonal structure on V_L^ι coincides with that on $V_{\widetilde{L}}$, $\widetilde{L} = L \otimes \mathcal{O}_X(p) \otimes \mathcal{I}_{\sigma(p)}$, away from the ramification locus.

(2) Now apply the above to the case where L has the form $L_c \otimes M \otimes \sigma^*(M^*)$, $M \in J(S)$. The gluing data in Theorem 2.6 depends on a choice of vector $v_b \in L|_{\pi^{-1}(b)}$ at each $b \in Z(b_{2m})$, which squares to the canonical trivialization of L^2 at $\pi^{-1}(b)$. Since $M \otimes \sigma^*(M^*)$ is canonically trivial at these points, such

a choice for L_c gives one for L as well. As opposed to the even case, because of the gluing the extended orthogonal bundle V_L depends on the choice of M, even when $M \in \mathrm{Prym}(S, \overline{S})$.

Finally, in this case the procedure of the previous section gives a lift of V_L to an SC-bundle away from $Z(b_{2m})$, depending on a choice of square root $K_X^{1/2}$ on X. To extend this to X, we note the following result, which is certainly well known.

Proposition 3.7. Let $V \to X$ be an orthogonal bundle (odd or even rank) on a smooth curve X and $D \subset X$ finite. A lift of V to an SC-bundle on $X \setminus D$ determines a lift of V to an SC-bundle on X. The resulting SC-bundle is uniquely determined up to twisting by divisors supported on D.

Proof. Let $P \to X$ denote the principal orthogonal bundle associated to V. We assume a lift $\widetilde{P}$ on $X \setminus D$ to a principal SC-bundle. By the uniformization theorem [7, Thm. 3] we can find a trivialization σ_- of P restricted to $X \setminus D$. Fix a lift $\widetilde{\sigma}_-$ of σ_- to $\widetilde{P}$, and a coordinate disk $z \in \Delta$ about $p \in D$. Let $\Delta^* = \Delta \setminus \{p\}$. Identify $P|_\Delta$ with $\Delta \times \mathsf{SO}$ via a trivialization σ_+. This also gives a lift of P to a spin bundle on Δ with trivialization $\widetilde{\sigma}_+$. The transition function for the bundle P identifies $\sigma_-|_{\Delta^*}$ with a section of $\Delta \times \mathsf{SO}$, and so on $\Delta^* \times \mathsf{SO}$ we may write $\sigma_- = g \cdot \sigma_+$ for some $g : \Delta^* \to \mathsf{SO}$.

Now since the group SO is not simply connected the element g may not admit a lift to $\widetilde{g} : \Delta^* \to \mathsf{Spin}$. If we set $z = \exp(2\pi i \xi)$, then $g(z)$ admits a lift $\widetilde{g}(\xi) \in \mathsf{Spin}$. If $\widetilde{g}(\xi + 1) = \widetilde{g}(\xi)$, then there is a well-defined map $\widetilde{g} : \Delta^* \to \mathsf{Spin}$. If $\widetilde{g}(\xi + 1) = -\widetilde{g}(\xi)$, then $\exp(\pi i \xi)\widetilde{g}(\xi)$ gives a well-defined map $\Delta^* \to \mathsf{SC}$ with Clifford norm z. Identifying $\widetilde{g}(z) \cdot \widetilde{\sigma}_+$ (*resp.* $\exp(\pi i \xi)\widetilde{g}(\xi) \cdot \widetilde{\sigma}_+$) with $\widetilde{\sigma}_-$ defines the transition function for an SC-bundle. This proves the existence of the extension, and the statement about uniqueness is then clear. ∎

In the case where $L = L_c \otimes M \otimes \sigma^*(M^*)$, $M \in J(S)$, then the lifts $\widetilde{g}$ will be to Spin at every point of $Z(b_{2m})$, since this is true when M is trivial. Hence, applying Proposition 3.7 and the preceding discussion, we see that the conclusion of Corollary 3.5 holds for odd orthogonal bundles as well.

3.4 Proof of Theorem 1.1

The argument can be made uniformly in the even and odd cases. We first consider the case of trivial Stiefel–Whitney class. Observe that

(1) If $S \to \overline{S}$ ramified, then the squaring map surjects $P(S,\overline{S})/J_2(\overline{S})$ onto $P(S,\overline{S})$;
(2) If $S \to \overline{S}$ is étale, then $\left(P(S,\overline{S})/J_2(\overline{S}) \cap P(S,\overline{S})\right) \simeq P(S,\overline{S})$ via squaring.

Let P be a spin Higgs bundle in the fiber over $\vec{b}$. According to Theorems 2.6 and 2.8, the underlying orthogonal bundle to P is of the form V_L for spectral data $L \in P(S,\overline{S})$ (in the odd case, the dependence is on a square root of L). Recall from Lemma 3.2 that a choice of theta characteristic determines a spin structure on V_c. By Corollary 3.5, if we write $L = L_c \otimes M^2$ for $M \in P(S,\overline{S})$, then there is a lift of V_L to a SC-bundle P_M. We must check the dependence of this lift on the choice of M. By Theorems 2.6 and 2.8 and (2) above, the ambiguity in the orthogonal bundle from the choice of M is the action of $J_2(\overline{S})$. So consider $M \otimes p^*N$, for $N \in J_2(\overline{S})$ such that $M \otimes p^*N$ is in $P(S,\overline{S})$. Let $N(t)$ be a family in $J(\overline{S})$, $N(0) = \mathcal{O}_{\overline{S}}$ and $N(1) = N$, and set $M(t) = M \otimes p^*N(t) \in J(S)$. Thus $L = L_c \otimes M(t) \otimes \sigma^*(M^*(t))$ for all t. By Corollary 3.5, we obtain a family of lifts of V_L to SC-bundles $P_{M(t)}$ with

$$\mathrm{Q}\left(P_{M(t)}\right) = \mathrm{Nm}_{S/X}(M) \otimes \mathrm{Nm}_{S/X}(p^*N(t)) = \left(\mathrm{Nm}_{\overline{S}/X} N(t)\right)^2. \tag{3.1}$$

In (3.1) we have used two facts: first, since $M \in \ker \mathrm{Nm}_{S/\overline{S}}$, and $\mathrm{Nm}_{S/X} = \mathrm{Nm}_{\overline{S}/X} \circ \mathrm{Nm}_{S/\overline{S}}$, we have $\mathrm{Nm}_{S/X}(M) = \mathcal{O}_X$; and second,

$$\mathrm{Nm}_{S/X}\left(p^*N(t)\right) = \mathrm{Nm}_{\overline{S}/X}\left(\mathrm{Nm}_{S/\overline{S}} p^*N(t)\right) = \mathrm{Nm}_{\overline{S}/X}\left(N^2(t)\right) = \left(\mathrm{Nm}_{\overline{S}/X}(N(t))\right)^2.$$

Now it follows from (3.1) that

$$\mathrm{Q}\left(P_{M(t)} \otimes \left(\mathrm{Nm}_{\overline{S}/X} N(t)\right)^*\right) = \mathcal{O}_X.$$

But then $P_{M(t)} \otimes \left(\mathrm{Nm}_{\overline{S}/X} N(t)\right)^*$ is a family of spin bundles that lift the fixed orthogonal bundle V_L. The set of such lifts is finite, so the family is necessarily constant. Evaluating at $t = 0$ and $t = 1$, we find

$$P_{M \otimes p^*N} \simeq P_M \otimes \mathrm{Nm}_{\overline{S}/X} N.$$

Given one lift of V_L to a spin bundle, the others are obtained by tensoring by elements of $J_2(X)$. From the above, a change of M is equivalent to the action of $J_2(\overline{S})$ on $J_2(X)$. Since $\mathrm{Nm}_{\overline{S}/X} : J_2(\overline{S}) \to J_2(X)$ is surjective (cf. the next section), $P = P_{M \otimes p^*N}$ for some choice of N. The proof of the theorem for spin bundles thus follows. The proof in the twisted case follows similarly by applying the first part of Corollary 3.5 and using the same argument as above.

4 Duality

We continue with the same notation as in the previous sections. We may regard $J_2(X)$ as a subgroup of $J_2(\overline{S})$ by pullback π^*. Using this fact, and dualizing the exact sequence

$$1 \longrightarrow J_2(X) \xrightarrow{\pi^*} J_2(\overline{S}) \longrightarrow J_2(\overline{S})/J_2(X) \longrightarrow 1 \tag{4.1}$$

we see that $\left[J_2(\overline{S})/J_2(X)\right]^\vee \subset J_2(\overline{S})^\vee$. Since the pullback π^* is injective it follows that $\mathrm{Nm}_{\overline{S}/X} : J_2(\overline{S}) \to J_2(X)$ is surjective. Recall that $K(S,\overline{S}) = \ker \mathrm{Nm}_{S/\overline{S}}$, whereas the Prym $P(S,\overline{S})$ is the connected component containing $\mathcal{O}_S$.

Remark 4.1. In the case where $S \to \overline{S}$ is étale, let H_0 denote the kernel of the map $J(\overline{S}) \to J(S)$, and recall that $\mathcal{L} \in J_2(\overline{S})$ defines the cover. Then $H_0 = \{\mathcal{O}_{\overline{S}}, \mathcal{L}\}$. Moreover, if $H_1 \subset J_2(\overline{S})$ denotes the annihilator of $\mathcal{L}$ with respect to the Weil pairing, then pulling back from $\overline{S}$ to S gives an identification of the two torsion points of the Prym $P_2(S,\overline{S}) \simeq H_1/H_0$ (cf. [16]).

Now consider the following variety:

$$K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X) := \{(a,b) \in K(S,\overline{S}) \times J_2(X)\}/\{(a\cdot s, b\cdot \mathrm{Nm}_{\overline{S}/X}(s)) \sim (a,b) \mid s \in J_2(\overline{S})\}$$

Observe that $K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X)$ can be realized as a group quotient of $K(S,\overline{S}) \times J_2(X)$ and hence has a natural group law. Now we define a map ι by

$$\iota : P(S,\overline{S}) \longrightarrow K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X) : a \mapsto [(a,1)].$$

Lemma 4.2. The map ι induces the following:

(1) If $S \to \overline{S}$ is ramified, then

$$\frac{P(S,\overline{S})}{\left[J_2(\overline{S})/J_2(X)\right]^\vee} \simeq K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X).$$

(2) If $S \to \overline{S}$ is étale, then

$$\frac{P(S,\overline{S})}{\left[(H_1/H_0)/J_2(X)\right]^\vee} \simeq K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X).$$

In particular, in both cases, $K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X)$ is an abelian variety.

Proof. First, suppose $S \to \overline{S}$ is ramified. Let $[(a,b)] \in K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X)$. Since $\mathrm{Nm}_{\overline{S}/X}$ is surjective we can rewrite any representative in the form $(a_1,1)$, where $a_1 = as$ and s is an element of $J_2(S)$ such that $\mathrm{Nm}_{S/X}(s) = b$. In particular, ι is surjective. On the other hand, $\ker\iota$ is clearly given by the kernel of $\mathrm{Nm}_{\overline{S}/X} : J_2(\overline{S})^\vee \to J_2(X)^\vee$. This is precisely $\left[J_2(\overline{S})/J_2(X)\right]^\vee$ by the exact sequence (4.1).

Now we consider the case where $S \to \overline{S}$ is étale. By [5, Prop. 11.4.3], the pullback $J(X) \to J(S)$ is injective. This implies that the line bundle $\mathcal{L}$ defining the étale cover is not in the image of $J_2(X)$ under pullback. We claim that the image of $J_2(X)$ lies in H_1. Indeed, with respect to the Weil pairing $\langle\ ,\ \rangle$, if $M \in J_2(X)$ then

$$\langle \overline{\pi}^* M, \mathcal{L}\rangle_{\overline{S}} = \langle M, \mathrm{Nm}_{\overline{S}/X}\,\mathcal{L}\rangle_X = 1,$$

since $\mathrm{Nm}_{\overline{S}/X}(\mathcal{L}) = \mathcal{O}_X$ by Lemma 2.7; hence, the claim. It follows that $J_2(X)$ injects into H_1/H_0 and also that H_1/H_0 surjects to $J_2(X)$ under the map $\mathrm{Nm}_{\overline{S}/X}$. If $[(a,b)] \in K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X)$ is such that $a \in P(S,\overline{S})$, then we can find an element $s \in P_2(S,\overline{S})$ such that $\mathrm{Nm}_{\overline{S}/X}(s) = b$. Then $[a,b] = [as,1]$, and clearly $as \in P(S,\overline{S})$. If, on the other hand, $a \in K(S,\overline{S})\backslash P(S,\overline{S})$, then there is ζ of $J_2(\overline{S})\backslash H_1$ such that $a\zeta \in P(S,\overline{S})$. By modifying ζ with elements in H_1 and using the surjectivity of $\mathrm{Nm}_{\overline{S}/X} : H_1/H_0 \to J_2(X)$, we can furthermore arrange that $\mathrm{Nm}_{\overline{S}/X}(\zeta) = \mathcal{O}_X$. Then we are done with the proof of surjectivity of ι, since in this case $[a,b] = [a\zeta, b]$. Finally, $a \in \ker\iota$ implies that $[a,1] = [1,1]$ in $K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X)$; hence, a is two torsion and $\mathrm{Nm}_{\overline{S}/X}(a) = 1$, and so by Remark 4.1 the kernel of ι is identified with $((H_1/H_0)/J_2(X))^\vee$. ∎

Lemma 4.3. We have the following isomorphism of abelian varieties.

$$\left[P(S,\overline{S})/J_2(X)\right]^\vee \simeq K(S,\overline{S}) \times_{J_2(\overline{S})} J_2(X).$$

Proof. Let $f : A \to B$ be an isogeny of abelian varieties, and $A^\vee$ and $B^\vee$ be the corresponding dual abelian varieties. Then there exists an isogeny of dual abelian varieties with the following exact sequence:

$$1 \to (\ker f)^\vee \to B^\vee \to A^\vee \to 1, \tag{4.2}$$

where $\ker f$ and $(\ker f)^\vee$ are Cartier dual of each other. Consider the case, when $S \to \overline{S}$ is not étale. Applying the above with $A = P(S,\overline{S})/J_2(X)$ and $B = P(S,\overline{S})/J_2(\overline{S})$, we obtain

an exact sequence

$$1 \to \left(J_2(\overline{S})/J_2(X)\right)^\vee \to \left(P(S,\overline{S})/J_2(\overline{S})\right)^\vee \to \left(P(S,\overline{S})/J_2(X)\right)^\vee \to 1.$$

It is well known that $\left(P(S,\overline{S})/J_2(\overline{S})\right)^\vee \simeq P(S,\overline{S})$. Hence, the result follows from Lemma 4.2.

If $S \to \overline{S}$ is étale, then we put $B = P(S,\overline{S})/(H_1/H_0)$ and we get an exact sequence

$$1 \to \left((H_1/H_0)/J_2(X)\right)^\vee \to \left(P(S,\overline{S})/(H_1/H_0)\right)^\vee \to \left(P(S,\overline{S})/J_2(X)\right)^\vee \to 1.$$

In this case, $P(S,\overline{S})$ is principally polarized and, moreover,$P_2(S,\overline{S}) \simeq H_1/H_0$. Hence

$$\left(P(S,\overline{S})/(H_1/H_0)\right) \simeq P(S,\overline{S}) \simeq P(S,\overline{S})^\vee.$$

Now using Lemma 4.2 completes the proof. ■

Proof of Corollary 1.2 Immediate from Theorem 1.1 and Lemma 4.2. ■

Proof of Theorem 1.3 Immediate from Theorem 1.1, Lemma 4.3, and Corollaries 2.5 and 2.9. ■

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