FLAT LINE BUNDLES AND THE CAPPELL-MILLER TORSION IN ARAKELOV GEOMETRY

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Abstract. In this paper, we extend Deligne's functorial Riemann-Roch isomorphism for Hermitian holomorphic line bundles on Riemann surfaces to the case of flat, not necessarily unitary connections. The Quillen metric and $\star$-product of Gillet-Soulé are replaced with complex valued logarithms. On the determinant of cohomology side, we show that the Cappell-Miller torsion is the appropriate counterpart of the Quillen metric. On the Deligne pairing side, the logarithm is a refinement of the intersection connections considered in a previous work. The construction naturally leads to an Arakelov theory for flat line bundles on arithmetic surfaces and produces arithmetic intersection numbers valued in $\mathbb{C}/\pi i\mathbb{Z}$. In this context we prove an arithmetic Riemann-Roch theorem. This realizes a program proposed by Cappell-Miller to show that their holomorphic torsion exhibits properties similar to those of the Quillen metric proved by Bismut, Gillet and Soulé. Finally, we give examples that clarify the kind of invariants that the formalism captures; namely, periods of differential forms.

Résumé. Dans cet article nous étendons l'isomorphisme de Riemann-Roch fonctoriel pour les fibrés en droites holomorphes Hermitiens, dû à Deligne, au cas des fibrés plats non nécessairement unitaires. La métrique de Quillen et le produit $\star$ de Gillet-Soulé sont remplacés par des logarithmes à valeurs complexes. Sur le déterminant de la cohomologie, nous montrons que la torsion de Cappell-Miller est l'analogue approprié de la métrique de Quillen. Sur les accouplements de Deligne, les logarithmes raffinent les connexions d'intersection introduites dans un travail précédent. La construction conduit naturellement à une théorie d'Arakelov pour les fibrés plats sur les surfaces arithmétiques, et produit des nombres d'intersection arithmétique à valeurs dans $\mathbb{C}/\pi i\mathbb{Z}$. Dans ce contexte, nous démontrons une formule de Riemann-Roch arithmétique. On réalise ainsi un programme proposé par Cappell-Miller visant à montrer que leur torsion holomorphe possède des propriétés analogues à celles de la métrique de Quillen établies par Bismut, Gillet et Soulé. Finalement, nous donnons des exemples qui clarifient le type d'invariants que ce formalisme encode: des périodes de formes différentielles.

À Jean-Michel Bismut, à l'occasion de son 70ème anniversaire, avec admiration.

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1. Introduction

Arithmetic intersection theory was initiated by Arakelov [1] in an attempt to approach the Mordell conjecture on rational points of projective curves over number fields by mimicking the successful arguments of the function field case. The new insight was the realization that an intersection theory on arithmetic surfaces could be defined by adding some archimedean information to divisors. This archimedean datum consists of the Green's functions that arise from smooth Hermitian metrics on holomorphic line bundles. The use of a metric structure is also natural for diophantine purposes, as one may want to measure the size of integral sections of a line bundle on an arithmetic surface.

Arakelov's foundational work was complemented by Faltings, who proved among other things, the first version of an arithmetric Riemann-Roch type formula [13]. Later, in a long collaboration starting with [17], Gillet and Soulé vastly extended the theory both to higher dimensions and to more general structures on the archimedean side. Their point of view is an elaboration of the ideas of Arakelov and is cast as a suitable "completion" of the usual Chow groups of classical intersection theory over a Dedekind domain. Their formalism includes arithmetic analogues of characteristic classes of Hermitian holomorphic vector bundles [18, 19]. This led them to develop and prove a general Grothendieck-Riemann-Roch type theorem in this setting [20]. A key ingredient is the *analytic torsion* of the Dolbeault complex associated to a Hermitian holomorphic vector bundle over a compact Kähler manifold. Their proof requires deep properties of the analytic torsion due to Bismut and collaborators [2, 3, 4, 5, 6, 7]. In [12], Deligne proposed a program to lift the Grothendieck-Riemann-Roch theorem to a functorial isomorphism between line bundles that becomes an isometry when the vector bundles are endowed with suitable metrics. This goal was achieved in the case of families of curves. He established a canonical isometry between the determinant of cohomology of a Hermitian vector bundle with the Quillen metric and some Hermitian intersection bundles involving, in particular, the *Deligne pairings* of line bundles.

In our previous work [16], we produced natural connections on Deligne pairings of line bundles with flat relative connections on families of compact Riemann surfaces. These were called *intersection connections*, and they reduce to Deligne's constructions in the case where the relative connections are the Chern connections for a Hermitian structure. As in the case of Deligne's formulation, intersection connections are functorial, and via the Chern-Weil expression they realize a natural cohomological relationship for Deligne pairings. Moreover, we showed that in the case of a trivial family of curves, *i.e.* a single Riemann surface and a holomorphic family of flat line bundles on it, we could interpret Fay's holomorphic extension of analytic torsion for flat unitary line bundles [14] as the construction of a Quillen type holomorphic connection on the determinant of cohomology. This can be recast as a statement that the Deligne-Riemann-Roch type isomorphism is flat with respect to these connections. The relevant contents of [16] are summarized in Section 2 below.

The results in [16] on intersection and Quillen type connections are vacuous for a single Riemann surface and a single flat holomorphic line bundle, since there are no interesting connections over points! To proceed further, and especially with applications to Arakelov theory in mind, we establish "integrated" versions of the aforementioned connections. The nature of such an object is what we have referred to above as a logarithm of a line bundle $\mathcal{L} \to S$ over a smooth variety S. This takes the place of the logarithm of a Hermitian metric in the classical situation. More precisely, a logarithm is an equivariant map $\mathrm{LOG} : \mathcal{L}^\times \to \mathbb{C}/2\pi i\,\mathbb{Z}$. It has an associated connection which generalizes the Chern connection of a Hermitian metric, but which is not necessarily unitary for some Hermitian structure. Although the notion of a logarithm is equivalent simply to a trivialization of the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$, it nevertheless plays an important role in the archimedean part of the arithmetic intersection product, as we explain below.

1.1. **Quillen-Cappell-Miller and intersection logarithms.** Let (X, p) be a compact Riemann surface with a base point, $\overline{X}$ the conjugate Riemann surface, and $\mathcal{L}_\chi \to X$, $\mathcal{L}_\chi^c \to \overline{X}$ rigidified (at p) flat complex line bundles with respective holonomies χ^{-1} and χ, for some character $\chi\colon \pi_1(X, p) \to \mathbb{C}^\times$. Applied to these data, Deligne's canonical (up to sign) isomorphism for $\mathcal{L}_\chi$ and $\mathcal{L}_\chi^c$ gives

$$\mathcal{D}\colon \left\{\lambda(\mathcal{L}_\chi - \mathcal{O}_X) \otimes_{\mathbb{C}} \lambda(\mathcal{L}_\chi^c - \mathcal{O}_{\overline{X}})\right\}^{\otimes 2} \xrightarrow{\sim} \langle \mathcal{L}_\chi, \mathcal{L}_\chi \otimes \omega_X^{-1}\rangle \otimes_{\mathbb{C}} \langle \mathcal{L}_\chi^c, \mathcal{L}_\chi^c \otimes \omega_{\overline{X}}^{-1}\rangle \tag{1}$$

where λ denotes the determinant of coherent cohomology and $\langle\ ,\ \rangle$ denotes the Deligne pairing (see Section 2 below for a review of Deligne's isomorphism). After choosing a metric on T_X, a construction of Cappell-Miller [11] produces a trivialization of the product of determinants of cohomologies, and hence gives rise to a logarithm denoted LOG_Q. For unitary characters, the Cappell-Miller trivialization is equivalent to the Quillen metric. We call LOG_Q the *Quillen-Cappell-Miller logarithm*. Regarding the right hand side of (1), we shall show in Section 4 that the intersection connection of [16] can be integrated to an *intersection logarithm* LOG_{int}. The first main result is the following (see Theorem 5.10):

Theorem 1.1 (Deligne Isomorphism). *The map (1) is compatible with* LOG_Q *and* LOG_{int}, *modulo* $\pi i\,\mathbb{Z}$. *That is,*

$$\mathrm{LOG}_Q = \mathrm{LOG}_{int} \circ \mathcal{D} \tag{2}$$

in $\mathbb{C}/\pi i\mathbb{Z}$.

The idea of the proof is to deform the line bundles to the universal family over the Betti moduli space $M_B(X) = \mathrm{Hom}(\pi_1(X, p), \mathbb{C}^\times)$. Over $M_B(X)$, both LOG_Q and LOG_{int} turn out to be holomorphic. Moreover, through Deligne's isomorphism, they agree along the totally real subvariety consisting of unitary characters. This forces the coincidence everywhere. There is however a sign ambiguity, due to the sign ambiguity of Deligne's isomorphism. This explains the equality modulo $i\pi\mathbb{Z}$ instead of $2\pi i\mathbb{Z}$.

The holomorphic behavior of LOG_Q over $M_B(X)$ is established in Section 5. We follow the method of Bismut-Gillet-Soulé [6], which shows that Bismut-Freed's connection [2, 3] is the Chern connection of the Quillen metric with respect to the holomorphic structure on the determinant of cohomology given by the Knudsen-Mumford construction [25]. However, these authors work with Hermitian vector bundles and self-adjoint Laplace type operators. Since the operators here are not self-adjoint their arguments do not directly apply. The presentation below exhibits a holomorphic dependence with respect to parameters in

$M_B(X)$. In this context, Kato's theory of analytic perturbations of closed operators [23, Chap. VII] turns out to be well-suited and provides the necessary alternative arguments to those in [6].

Theorem 1.1 gives a positive answer to the question of Cappell-Miller as to whether their torsion element plays an analogous role to the Quillen metric in the work of Bismut-Gillet-Soulé. It also shows the relevance of Deligne's functorial formalism adopted here.

Finally, we emphasize that Fay's holomorphic extension of analytic torsion is replaced with the approach of Cappell-Miller. While for flat line bundles on Riemann surfaces the torsion in both cases may be compared, the latter is defined in any dimension and any rank and is therefore more suitable to generalizations of the work presented here.

1.2. **The Arithmetic-Riemann-Roch theorem.** The second aim of this paper is to use the results above to initiate an Arakelov theory for flat line bundles on arithmetic surfaces (Section 6). The quest for such a theory was made more conceivable by Burgos' cohomological approach to Arakelov geometry, which interprets Green's currents as objects in some truncated Deligne *real* cohomology [9]. This evolved into the abstract formalism of Burgos-Kramer-Kühn [10], allowing one to introduce *integral* Deligne cohomology instead. Despite these developments, to our knowledge, the attempts so far have been unsuccessful. It turns out that the intersection logarithm is the key in the construction of an arithmetic intersection pairing for flat line bundles. At the archimedean places, the nature of our tools forces us to work simultaneously with a Riemann surface and its conjugate, and pairs of flat line bundles with opposite holonomies. We propose an analogue of this apparatus in the arithmetic setting which we call a *conjugate pair* $\mathcal{L}^\sharp$ of line bundles with connections (see Definition 6.6). Through Deligne's pairing and the intersection logarithm, we attach to conjugate pairs $\mathcal{L}^\sharp$ and $\mathcal{M}^\sharp$ an object $\langle\mathcal{L}^\sharp,\mathcal{M}^\sharp\rangle$, which consists of a line bundle over $\operatorname{Spec}\mathcal{O}_K$ together with the data of intersection logarithms at the archimedean places. For such an object there is a variant of the arithmetic degree in classical Arakelov geometry, denoted $\deg^\sharp$, which takes values in $\mathbb{C}/\pi i\,\mathbb{Z}$ instead of $\mathbb{R}$. The construction also applies to mixed situations; for instance, to a rigidified conjugate pair $\mathcal{L}^\sharp$ and a Hermitian line bundle $\overline{\mathcal{M}}$. When the dualizing sheaf $\omega_{\mathcal{X}/S}$ is equipped with a smooth Hermitian metric, we can define $\lambda(\mathcal{L}^\sharp)_Q$, the determinant of cohomology of $\mathcal{L}^\sharp$ with the Quillen-Cappell-Miller logarithms at the archimedean places. Using this formalism, we prove an arithmetic Riemann-Roch type theorem for these enhanced line bundles (Theorem 6.12 below):

Theorem 1.2 (Arithmetic Riemann-Roch)**.** *Let $\mathcal{X}\to S=\operatorname{Spec}\mathcal{O}_K$ be an arithmetic surface with a section $\sigma: S\to\mathcal{X}$. Suppose the relative dualizing sheaf $\omega_{\mathcal{X}/S}$ is endowed with a smooth Hermitian metric. Let $\mathcal{L}^\sharp$ be a rigidified conjugate pair of line bundles with connections. Endow the determinant of cohomology of $\mathcal{L}^\sharp$ with the Quillen-Cappell-Miller logarithm. Then the following equality holds in $\mathbb{C}/\pi i\,\mathbb{Z}$.*

$$\begin{aligned}12\deg^\sharp\lambda(\mathcal{L}^\sharp)_Q-2\delta&=2(\overline{\omega}_{\mathcal{X}/S},\overline{\omega}_{\mathcal{X}/S})+6(\mathcal{L}^\sharp,\mathcal{L}^\sharp)-6(\mathcal{L}^\sharp,\overline{\omega}_{\mathcal{X}/S})\\&\quad-(4g-4)[K:\mathbb{Q}]\left(\frac{\zeta'(-1)}{\zeta(-1)}+\frac{1}{2}\right),\end{aligned}\tag{3}$$

where $\delta = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log(N\mathfrak{p})$ *is the "Artin conductor" measuring the bad reduction of* $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$*. If K does not admit any real embeddings then the equality canonically lifts to* $\mathbb{C}/2\pi i\,\mathbb{Z}$*.*

In the theorem it is possible to avoid the rigidification of $\mathcal{L}^\sharp$ along the section σ, at the cost of taking values in $\mathbb{C}/\pi i\,\mathbb{Z}[1/h_K]$, where h_K is the class number of K. However, the existence of a section is needed for the construction. A variant of the formalism (including an arithmetic Riemann-Roch formula) consists in introducing conjugate pairs of arithmetic surfaces and line bundles. This makes sense and can be useful when K is a CM field. The arithmetic intersection numbers are then valued in $\mathbb{C}/2\pi i\,\mathbb{Z}$.

In a future work [15] we extend these results to local systems of higher rank. With respect to the Deligne isomorphism, the new ingredient is the line bundle incarnation of the (direct image) of the second Chern class, expressed as IC_2. Analogues of the intersection connection and logarithm on IC_2 will be developed.

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2. Preliminaries

2.1. **Intersection connections.** Let $\pi\colon \mathcal{X} \to S$ be a smooth and proper morphism of quasi-projective and smooth complex varieties, with connected fibers of dimension 1. Let $\mathcal{L}$ and $\mathcal{M}$ be two holomorphic line bundles on $\mathcal{X}$. The Deligne pairing of $\mathcal{L}$ and $\mathcal{M}$ is a holomorphic line bundle $\langle \mathcal{L}, \mathcal{M}\rangle$ on S, that can be presented in terms of generators and relations. Locally on S (*i.e.* possibly after replacing S by an open subset), the line bundle is trivialized by symbols $\langle \ell, m\rangle$, where ℓ and m are meromorphic sections of $\mathcal{L}$ and $\mathcal{M}$, respectively, and their divisors $\operatorname{div}\ell$ and $\operatorname{div} m$ are disjoint, finite and étale[1] over an open subset of S (for simplicity, we say that ℓ and m are in *relative general position*). Relations, inducing the glueing and cocycle conditions, are given by $\langle f\ell, m\rangle = N_{\operatorname{div} m/S}(f)\langle \ell, m\rangle$, whenever f is a meromorphic function such that both symbols are defined, as well as a symmetric relation in the other "variable". Here, $N_{\operatorname{div} m/S}(f)$ denotes the norm of f along the divisor of m. It is multiplicative with respect to addition of divisors, and it is equal to the usual norm on functions for finite, flat divisors over the base. The construction is consistent, thanks to the Weil reciprocity law: for two meromorphic functions f and g whose divisors are in relative general position, we have $N_{\operatorname{div} f/S}(g) = N_{\operatorname{div} g/S}(f)$. The Deligne pairing can be constructed both in the analytic and the algebraic categories, and it is compatible with the analytification functor. This is why we omit specifying the topology. The Deligne pairing is compatible with base change and has natural functorial properties in $\mathcal{L}$ and $\mathcal{M}$.

Let $\nabla\colon \mathcal{L} \to \mathcal{L}\otimes\Omega^1_{\mathcal{X}/S}$ be a relative holomorphic connection, and assume for the time being that $\mathcal{M}$ has relative degree 0. We showed that there exists a $\mathcal{C}^\infty_{\mathcal{X}}$ connection $\widetilde{\nabla}\colon \mathcal{L} \to \mathcal{L}\otimes\mathcal{A}^1_{\mathcal{X}}$,

[1] Under the most general assumptions (l.c.i. flat morphisms between schemes), it only makes sense to require flatness of the divisors. In our setting (smooth morphisms of smooth varieties over $\mathbb{C}$), a Bertini type argument shows we can take them to be étale.

compatible with the holomorphic structure on $\mathcal{L}$ (that is $\widetilde{\nabla}^{0,1} = \overline{\partial}_{\mathcal{L}}$), such that the following rule determines a well-defined compatible connection on $\langle \mathcal{L}, \mathcal{M}\rangle$:

$$\nabla_{tr}\langle \ell, m\rangle = \langle \ell, m\rangle \otimes \mathrm{tr}_{\mathrm{div}\, m/S}\left(\frac{\widetilde{\nabla}\ell}{\ell}\right).$$

Notice that it makes sense to take the trace of the differential form $\widetilde{\nabla}\ell/\ell$ along $\mathrm{div}\, m$, since the latter is finite étale over the base, and the divisors of the sections are disjoint. The existence of $\widetilde{\nabla}$ is not obvious, since the rule just defined encodes a nontrivial reciprocity law, that we call (WR):

$$\mathrm{tr}_{\mathrm{div}\, f/S}\left(\frac{\widetilde{\nabla}\ell}{\ell}\right) = \mathrm{tr}_{\mathrm{div}\, \ell/S}\left(\frac{df}{f}\right),$$

whenever f is a meromorphic function and the divisors of f and ℓ are in relative general position. The construction of $\widetilde{\nabla}$ can be made to be compatible with base change, and then it is unique up to the additive action of $\Gamma(\mathcal{X}, \pi^{-1}\mathcal{A}_S^{1,0})$. Furthermore, if $\sigma\colon S \to \mathcal{X}$ is a section and $\mathcal{L}$ is trivialized along σ, one can isolate a particular extension $\widetilde{\nabla}$ that restricts to the exterior differentiation on S along σ (through the trivialization of $\mathcal{L}$). Then the connection ∇_{tr} can be extended to pairings with $\mathcal{M}$ of any relative degree, without ambiguity. We call $\widetilde{\nabla}$ a (or the) canonical extension of ∇, and ∇_{tr} a *trace connection*. We recall the construction of the canonical extension in §3.3.1 below.

Trace connections are manifestly not symmetric, since they do not require any connection on $\mathcal{M}$. Let $\widetilde{\nabla}'\colon \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}_{\mathcal{X}}^1$ be a smooth compatible connection on $\mathcal{M}$ and let ∇_{tr} be a trace connection on $\langle \mathcal{L}, \mathcal{M}\rangle$. If the relative degree of $\mathcal{M}$ is not zero, we tacitly assume that $\mathcal{L}$ is rigidified along a given section. The trace connection ∇_{tr} can then be completed to a connection that "sees" $\widetilde{\nabla}'$:

$$\frac{\nabla_{int}\langle \ell, m\rangle}{\langle \ell, m\rangle} = \frac{\nabla_{tr}\langle \ell, m\rangle}{\langle \ell, m\rangle} + \frac{i}{2\pi}\pi_*\left(\frac{\widetilde{\nabla}' m}{m} \wedge F_{\widetilde{\nabla}}\right),$$

where $F_{\widetilde{\nabla}}$ is the curvature of the canonical extension $\widetilde{\nabla}$ on $\mathcal{L}$. Assume now that $\widetilde{\nabla}'$ is a canonical extension of a relative holomorphic connection $\nabla'\colon \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{\mathcal{X}/S}$. Then the *intersection connection* ∇_{int} is compatible with the obvious symmetry of the Deligne pairing. These constructions carry over to the case when the relative connections only have a smooth dependence on the horizontal directions, but are still holomorphic on fibers. The intersection connection reduces to the trace connection if $\widetilde{\nabla}'$ is the Chern connection of a smooth Hermitian metric on $\mathcal{M}$ that is flat on fibers. Finally, the trace connection coincides with the Chern connection of the metrized Deligne pairing in case $\widetilde{\nabla}$ is a Chern connection, flat on fibers, as well.

2.2. **Deligne isomorphism.** Let us denote $\lambda(\mathcal{L})$ for the determinant of the cohomology of $\mathcal{L}$; that is: $\lambda(\mathcal{L}) = \det R\pi_*(\mathcal{L})$. The determinant of $R\pi_*(\mathcal{L})$ makes sense, since it is a perfect complex and so the theory of Knudsen-Mumford [25] applies. It can be extended, multiplicatively, to virtual objects, namely formal sums of line bundles with integer coefficients. Deligne [12] proves the existence of an isomorphism

$$\mathcal{D}\colon \lambda(\mathcal{L}-\mathcal{O})^{\otimes 2} \xrightarrow{\sim} \langle \mathcal{L}, \mathcal{L}\otimes\omega_{\mathcal{X}/S}^{-1}\rangle,$$

where $\omega_{\mathcal{X}/S}$ is the relative cotangent bundle of π. The isomorphism is compatible with base change and is functorial in $\mathcal{L}$. It is unique with these properties, up to sign. It can be combined with Mumford's canonical (up to sign) and functorial isomorphism [27], which in the language of Deligne pairings reads: $\lambda(\mathcal{O})^{\otimes 12} \xrightarrow{\sim} \langle\omega_{\mathcal{X}/S}, \omega_{\mathcal{X}/S}\rangle$. Hence, we have a canonical (up to sign) isomorphism

$$\mathcal{D}'\colon \lambda(\mathcal{L})^{\otimes 12} \xrightarrow{\sim} \langle\omega_{\mathcal{X}/S}, \omega_{\mathcal{X}/S}\rangle \otimes \langle \mathcal{L}, \mathcal{L}\otimes\omega_{\mathcal{X}/S}^{-1}\rangle^{\otimes 6},$$

which is again compatible with base change and functorial in $\mathcal{L}$. The latter is also usually called Deligne's isomorphism.

When the line bundles $\mathcal{L}$ and $\omega_{\mathcal{X}/S}$ are endowed with smooth Hermitian metrics, all the line bundles on S involved in Deligne's isomorphism inherit Hermitian metrics. On the Deligne pairings, the construction is the metrized counterpart of the intersection connection definition, and it will not be recalled here. It amounts to the $\star$-product of Green's currents introduced by Gillet-Soulé in arithmetic intersection theory. The determinant of cohomology can be equipped with the Quillen metric. When the metric on $\omega_{\mathcal{X}/S}$ is the restriction of a global Kähler metric on $\mathcal{X}$, then the Chern connection of the Quillen metric is given by Bismut-Freed's construction [2, 3]. The Deligne isomorphism is, up to an overall topological constant, an isometry for these metrics. The value of the constant can be pinned down, for instance by using the arithmetic Riemann-Roch theorem of Gillet-Soulé [20]. We refer the reader to the survey articles of Soulé [31] and Bost [8], where all these constructions and facts are summarized. Because the Deligne isomorphism is an isometry in the metrized case, it is in particular parallel for the corresponding Chern connections.

3. Logarithms and Deligne Pairings

3.1. **Logarithms and connections on holomorphic line bundles.** Let S be a connected complex analytic manifold and $\mathcal{P}\to S$ a smooth complex line bundle. To simplify the presentation, the same notation will be used when $\mathcal{P}$ is understood to have the structure of a holomorphic line bundle. Also, no notational distinction will be made between a holomorphic line bundle and the associated invertible sheaf of $\mathcal{O}_S$-modules. Finally, denote by $\mathcal{P}^\times$ the $\mathbb{G}_m$ torsor (or principal bundle) given by the complement of the zero section in the total space of $\mathcal{P}$.

Here we introduce the notion of smooth logarithm for $\mathcal{P}$, which is nothing but an additive reformulation of the notion of trivialization. For a holomorphic bundle there is also a notion of holomorphic logarithm, and whenever we talk about holomorphic logarithm it will be implicit that $\mathcal{P}$ has a holomorphic structure.

Definition 3.1. A *smooth (resp. holomorphic) logarithm* for $\mathcal{P}$ is a map: $\mathrm{LOG}:\mathcal{P}^\times \longrightarrow \mathbb{C}/2\pi i\,\mathbb{Z}$ satisfying: $\mathrm{LOG}(\lambda\cdot e) = \log\lambda + \mathrm{LOG}(e)$, for $\lambda\in\mathbb{G}_m$ and $e\in\mathcal{P}^\times$, and such that the well-defined $\mathbb{C}^\times$-valued function $\exp\circ\mathrm{LOG}$ is smooth (resp. holomorphic) with respect to the natural structure of a smooth (resp. complex analytic) manifold on $\mathcal{P}^\times$. Then $\mathrm{LOG}^{-1}(0)$ gives a trivialization of $\mathcal{P}$.

A logarithm LOG can be reduced modulo $\pi i\mathbb{Z}$. We will write $\overline{\mathrm{LOG}}$ for the reduction of LOG. By construction, the reduction of a logarithm modulo $\pi i\mathbb{Z}$ factors through $\mathcal{P}^\times/\{\pm 1\}$, and is equivalent to a trivialization up to sign:

$$\begin{tikzcd}
\mathcal{P}^\times \arrow[r, "\mathrm{LOG}"] \arrow[d, two heads] & \mathbb{C}/2\pi i\mathbb{Z} \arrow[d, two heads] \\
\mathcal{P}^\times/\{\pm 1\} \arrow[r, "\overline{\mathrm{LOG}}"] & \mathbb{C}/\pi i\mathbb{Z}\,.
\end{tikzcd}$$

Though perhaps not apparent at this moment, the necessity for this reduction will appear at several points below (notably because of the sign ambiguity in Deligne's isomorphism).

There is naturally a flat connection ∇_{LOG} on $\mathcal{P}$ associated to a smooth logarithm . If $\mathcal{P}$ is a holomorphic bundle, this connection is compatible with the holomorphic structure exactly when LOG is holomorphic.

3.2. **Construction of naive logarithms.** Let $\pi:\mathcal{X}\to S$ be a smooth proper morphism of smooth complex quasi-projective varieties with connected fibers of relative dimension one. The morphism π is in particular projective. Assume we are given a fixed holomorphic section $\sigma: S\to\mathcal{X}$ and $\mathcal{L}$, $\mathcal{M}$ holomorphic line bundles on $\mathcal{X}$. We require that $\mathcal{L}$ comes with a rigidification along σ, namely a fixed holomorphic trivialization $\sigma^*(\mathcal{L})\xrightarrow{\sim}\mathcal{O}_S$. Consider relative connections: $\nabla^{\mathcal{L}}_{\mathcal{X}/S}:\mathcal{L}\to\mathcal{L}\otimes\mathcal{A}^1_{\mathcal{X}/S}$ and $\nabla^{\mathcal{M}}_{\mathcal{X}/S}:\mathcal{M}\to\mathcal{M}\otimes\mathcal{A}^1_{\mathcal{X}/S}$, compatible with the holomorphic structures (here, $\mathcal{A}^1_{\mathcal{X}/S}=\mathcal{A}^1_{\mathcal{X}}/\pi^*\mathcal{A}^1_S$ denotes the sheaf of smooth relative 1-forms on $\mathcal{X}$). We suppose that $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$ is flat, but make no assumption on $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ for the time being. Note that this means that $\mathcal{L}$ has a smooth trivialization on each fiber. We would like to use this data to construct a smooth logarithm on the Deligne pairing $\langle\mathcal{L},\mathcal{M}\rangle$, whose associated connection is the intersection connection ∇_{int} (see Section 2). This, however, is not possible, since the intersection connection is in general not flat (see for instance the curvature computation in the universal case [16, Sec. 5.3]). We will thus try to construct a logarithm, which we call *naive*, whose connection looks as close as possible to the intersection connection.

We proceed in several steps. Let $\nu_{\mathcal{L}}: S\to H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\mathbb{Z})$ be the smooth classifying map of $(\mathcal{L},\nabla^{\mathcal{L}}_{\mathcal{X}/S})$ (see [16]). Locally on contractible open subsets S° of S, we can lift $\nu_{\mathcal{L}}$ to a smooth section of $H^1_{dR}(\mathcal{X}/S)$.

Step 1. Fix S° a contractible open neighborhood of a fixed base point $0\in S$, and write $\mathcal{X}^\circ=\pi^{-1}(S^\circ)$. Choose a lift of $\nu_{\mathcal{L}}$ on S°: $\hat{\nu}\colon S^\circ\to H^1_{dR}(\mathcal{X}/S)\big|_{S^\circ}$ such that $\hat{\nu}\equiv\nu_{\mathcal{L}}$ mod $R^1\pi_*(2\pi i\mathbb{Z})\big|_{S^\circ}$. Fix as well a relative differential form $\eta\in\Gamma(\mathcal{X}^\circ,\mathcal{A}^1_{\mathcal{X}^\circ/S^\circ})$, which is harmonic on fibers and represents $\hat{\nu}$ fiberwise.

Step 2. Fix a smooth trivialization $\mathcal{X}^\circ\simeq X\times S^\circ$, where X is the fiber $\pi^{-1}(0)$. Write $\widetilde{\mathcal{X}}\simeq\widetilde{X}\times S^\circ$ for the universal cover based at $\sigma(0)$. The trivialization induces identifications of $\widetilde{X}$ with the universal covers $\widetilde{\mathcal{X}}_s$ based at $\sigma(s)$, as well as identifications $\pi_1(\mathcal{X},\sigma(0)) \simeq \pi_1(X,\sigma(0)) \simeq \pi_1(\mathcal{X}_s,\sigma(s))$. We shall implicitly invoke these identifications below. Using the rigidification of $\mathcal{L}$ along σ and parallel transport with respect to $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$, the section ℓ gives rise to a function $\tilde{\ell}\colon \widetilde{\mathcal{X}} \to \mathbb{C}\cup\{\infty\}$, which is fiberwise meromorphic, and transforms via some character under the action of the fundamental group (the character depends on the fiber). Precisely, if $\gamma \in \pi_1(\mathcal{X}_s,\sigma(s))$, the transformation law for $\tilde{\ell}$ on $\widetilde{\mathcal{X}}_s$ with respect to translation by γ is: $\tilde{\ell}(\gamma z) = \exp\left(\int_\gamma \hat{\nu}\big|_{\mathcal{X}_s}\right)\tilde{\ell}(z)$, $z \in \widetilde{\mathcal{X}}_s$.

Step 3. Choose $\widetilde{\operatorname{div} m}$ a lift of $\operatorname{div} m$ to $\widetilde{\mathcal{X}}$. If $\operatorname{div} m = \sum_i n_i P_i$ (finite sum), for sections P_i, then $\widetilde{\operatorname{div} m} = \sum_i n_i \widetilde{P}_i$, where the $\widetilde{P}_i$ are lifts of P_i in $\widetilde{\mathcal{X}}$. Similarly let $\tilde{\sigma}$ be a lift of the section σ to $\widetilde{\mathcal{X}}$.

Step 4. We set:

$$\mathrm{LOG}_{na}(\langle \ell, m\rangle) = \log(\tilde{\ell}(\widetilde{\operatorname{div} m})) - \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta} - \frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}}_{\mathcal{X}/S} m}{m} \wedge \eta\right) \mod 2\pi i\,\mathbb{Z} \tag{4}$$

where $\tilde{\eta} \in \Gamma(\widetilde{\mathcal{X}}, \mathcal{A}^1_{\widetilde{\mathcal{X}}/S^\circ})$ is the pull-back of η to the universal cover. The index *na* stands for *naive*. To simplify the presentation, we omit the choices made from the notation.

Let us clarify the construction:

(i) The integrals are computed fiberwise, and hence they are well defined. The path integral is taken along a path in a fiber, and it is independent of the path since $\tilde{\eta}$ is fiberwise closed. Finally, π_* is well defined on relative differential forms

(ii) If on a given fiber $\widetilde{\mathcal{X}}_s$ we have $\widetilde{\operatorname{div} m} = \sum_i n_i \tilde{P}_i$, then the first two terms in the definition of LOG_{na} expand (by definition) to: $\sum_i n_i \log(\tilde{\ell}(\tilde{P}_i)) - \sum_i n_i \int_{\tilde{\sigma}(s)}^{\tilde{P}_i} \tilde{\eta}$. The integration path from $\tilde{\sigma}$ to $\tilde{P}_i$ is taken in $\widetilde{\mathcal{X}}_s$. This expression does not depend on the choice of liftings $\tilde{\sigma}$ and $\tilde{P}_i$, modulo $2\pi i\,\mathbb{Z}$. For if P and γP are points in $\widetilde{\mathcal{X}}_s$ differing by the action of $\gamma \in \pi_1(\mathcal{X}_s,\sigma(s))$, then

$$\begin{aligned}\log(\tilde{\ell}(\gamma P)) - \int_{\tilde{\sigma}(s)}^{\gamma P}\tilde{\eta} &= \int_\gamma \hat{\nu} + \log(\tilde{\ell}(P)) - \int_{\tilde{\sigma}(s)}^{P}\tilde{\eta} - \int_P^{\gamma P}\tilde{\eta} \mod 2\pi i\,\mathbb{Z}\\ &= \log(\tilde{\ell}(P)) - \int_{\tilde{\sigma}(s)}^{P}\tilde{\eta} \mod 2\pi i\,\mathbb{Z}\,,\end{aligned}$$

where we used $\int_P^{\gamma P}\tilde{\eta} = \int_\gamma \hat{\nu}$. If we change the lifting $\tilde{\sigma}$ to $\sigma^* = \gamma\tilde{\sigma}$, then the new lifting of ℓ is ℓ^* with $\ell^*(z) = \exp\left(-\int_\gamma \hat{\nu}\right)\tilde{\ell}(z)$, while $-\int_{\gamma\tilde{\sigma}}^{P}\tilde{\eta} = -\int_{\gamma\tilde{\sigma}}^{\tilde{\sigma}}\tilde{\eta} - \int_{\tilde{\sigma}}^{P}\tilde{\eta} = \int_\gamma\hat{\nu} - \int_{\tilde{\sigma}}^{P}\tilde{\eta}$. From these relations we derive the independence of the lift $\tilde{\sigma}$ modulo $2\pi i\,\mathbb{Z}$.

(iii) There are several facts that can be checked similarly to our previous work [16, Sec. 3 and 4]. For instance, the compatibility to the relations defining the Deligne pairing, most notably under the change $f \mapsto fm$ (f a rational function), follows from various reciprocity laws for differential forms, plus the observation that $\pi_*((df/f)\wedge\eta) = \partial\pi_*(\log|f|^2\cdot\eta) = 0$, since η is fiberwise closed.

(iv) The last fibre integral in (4) defines a smooth function on S°. Indeed, let $\nabla^{\mathcal{M}}_{ch}$ be a Chern connection on $\mathcal{M}$, and write $\nabla^{\mathcal{M}}_{\mathcal{X}/S} = \nabla^{\mathcal{M}}_{ch} + \vartheta$, where ϑ is a smooth relative differential form. Then, as in the previous remark, one proves $\pi_*\left(\frac{\nabla^{\mathcal{M}}_{\mathcal{X}/S} m}{m}\wedge\eta\right) = \pi_*(\vartheta\wedge\eta)$. The last expression is clearly a smooth function.

With this understood, we conclude that LOG_{na} is a well-defined smooth logarithm for $\langle\mathcal{L},\mathcal{M}\rangle\big|_{S^\circ}$, that depends only on the choice of lifting $\hat{\nu}$ on S°.

Let us now focus on the case when $\mathcal{M}$ is endowed with a Chern connection.

Lemma 3.2. *Assume $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is the relative Chern connection of a smooth Hermitian metric on $\mathcal{M}$. Endow the line bundle $\mathcal{M}\otimes\mathcal{O}(-(\deg\mathcal{M})\sigma)$, of relative degree 0, with the relative flat unitary connection. Equip the Deligne pairings $\langle\mathcal{L},\mathcal{M}\rangle\big|_{S^\circ}$ and $\langle\mathcal{L},\mathcal{M}\otimes\mathcal{O}(-(\deg\mathcal{M})\sigma)\rangle\big|_{S^\circ}$ with the naive logarithms depending on the lift $\hat{\nu}$ of $\nu_{\mathcal{L}}$. Finally, equip $\sigma^*(\mathcal{L})$ with the logarithm induced by the rigidification $\sigma^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_S$. Then, the canonical isomorphism of Deligne pairings*

$$\langle\mathcal{L},\mathcal{M}\rangle \xrightarrow{\sim} \langle\mathcal{L},\mathcal{M}\otimes\mathcal{O}(-(\deg\mathcal{M})\sigma)\rangle\otimes\sigma^*(\mathcal{L})^{\otimes\deg\mathcal{M}}$$

is compatible with the respective logarithms defined on S°. In particular, the naive logarithm on $\langle\mathcal{L},\mathcal{M}\rangle\big|_{S^\circ}$ does not depend on the particular choice of Chern connection $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$.

Proof. For the Deligne pairing on the left hand side, we have $\mathrm{LOG}_{na}(\langle\ell,m\rangle) = \log(\tilde{\ell}(\widetilde{\operatorname{div} m})) - \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\eta}$, because $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is a Chern connection. Assume now that ℓ does not have a pole or a zero along σ. Then, by the very construction of $\tilde{\ell}$, we have on the one hand

$$\log(\tilde{\ell}(\widetilde{\operatorname{div} m})) = \log(\tilde{\ell}(\widetilde{\operatorname{div} m} - (\deg\mathcal{M})\tilde{\sigma})) + (\deg\mathcal{M})\log(\sigma^*\ell),$$

while on the other hand it is obvious that $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\eta} = \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}-(\deg\mathcal{M})\tilde{\sigma}}\tilde{\eta}$. The lemma follows from these observations. □

3.3. **The connection attached to a naive logarithm.** The previous setting and notations are still in force. We wish to compute the connection associated to LOG_{na}, this is $d\,\mathrm{LOG}_{na}$. This requires differentiation of functions of the form: $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\eta}$. For this, we first need to recall the construction of the canonical extension of the relative flat connexion $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$.

3.3.1. *Preliminaries on the canonical extension.* For the lift $\tilde{\ell}$ of ℓ to $\widetilde{\mathcal{X}}$, the expression $d\tilde{\ell}/\tilde{\ell}$ does not descend to a differential form on $\mathcal{X}^\circ$. We need to correct it by taking into account the horizontal variation of the classifying map of $(\mathcal{L},\nabla^{\mathcal{L}}_{\mathcal{X}/S})$, which we denoted by $\nu_{\mathcal{L}}: S\to H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\,\mathbb{Z})$. If ∇_{GM} is the Gauss-Manin connection for the local system

$R^1\pi_*\mathbb{Z}$ on S, then $\nabla_{\mathrm{GM}}\nu_{\mathcal{L}}$ is a smooth section of $H^1_{dR}(\mathcal{X}/S)\otimes\mathcal{A}^1_S$. On the contractible open subset S° we can write

$$\nabla_{\mathrm{GM}}\nu_{\mathcal{L}} = \sum_{j=1}^{2g}[\eta_j]\otimes\theta_j, \tag{5}$$

where $\{[\eta_j]\}_{j=1}^{2g}$ defines a local flat frame of $H^1_{dR}(\mathcal{X}/S)\big|_{S^\circ}$, and the θ_j are smooth 1-forms on S°. As before, we assume that the η_j are fiberwise harmonic representatives of the classes $[\eta_j]$. Then we declare, for $s\in S^\circ$ and $z\in\widetilde{\mathcal{X}}_s$,

$$\int_{\tilde{\sigma}(s)}^{z}\nabla_{\mathrm{GM}}\nu_{\mathcal{L}} := \sum_{i=1}^{2g}\left(\int_{\tilde{\sigma}(s)}^{z}\tilde{\eta}_i\right)\theta_i.$$

Finally, we define the canonical extension $\widetilde{\nabla}^{\mathcal{L}}$ of $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$ (on S°), rigidified along σ, by

$$\frac{\widetilde{\nabla}^{\mathcal{L}}\ell}{\ell} = \frac{d\tilde{\ell}}{\tilde{\ell}} - \int_{\tilde{\sigma}(s)}^{z}\nabla_{\mathrm{GM}}\nu_{\mathcal{L}}. \tag{6}$$

In our previous work we showed that this rule determines a well-defined smooth global connection on $\mathcal{L}\mid_{\mathcal{X}^\circ}$, compatible with the holomorphic structure and compatible with the rigidification $\sigma^*(\mathcal{L})\xrightarrow{\sim}\mathcal{O}_S$ (in the sense that the restriction along σ corresponds to the trivial connection). We also proved that the construction patches together on intersecting contractible subsets, and hence it globalizes to the whole base S [16, Sec. 4.1].

Observe now that if the lift $\hat{\nu}$ of $\nu_{\mathcal{L}}$ on S° is written as

$$\hat{\nu} = \sum_{i=1}^{2g} f_i[\eta_i], \tag{7}$$

where the f_i are smooth functions on S°, then:

(i) for the fiberwise harmonic representative of $\hat{\nu}$ we may take $\eta=\sum_i f_i\eta_i$;

(ii) the Gauss-Manin connection applied to $\nu_{\mathcal{L}}$ is expressed as $\nabla_{\mathrm{GM}}\nu_{\mathcal{L}}=\sum_{i=1}^{2g}[\eta_i]\otimes df_i$.

In particular we have the decomposition (5) with $\theta_i = df_i$.

3.3.2. *Differentiation of naive logarithms.* We next find a relation between the naive logarithm LOG_{na} and the intersection connection (*cf.* Section 2). First, let us note that in the case where $\nabla^{\mathcal{M}}$ is a connection on $\mathcal{M}$ that is flat on the fibers the quantity $\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta)$ is well defined. Here $F_{\nabla^{\mathcal{M}}}$ is the curvature of the connection $\nabla^{\mathcal{M}}$. Indeed, consider a form $\sum h_k\pi^*\omega_k$, where h_k are functions on $\mathcal{X}$ and ω_k are forms on S. Then $\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\sum h_k\pi^*\omega_k)=\sum\pi_*(h_kF_{\nabla^{\mathcal{M}}})\omega_k$, which vanishes since $\nabla^{\mathcal{M}}$ is flat on fibers. With this understood, the aforementioned relationship is then given in the following.

Proposition 3.3. *Suppose that $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is flat on fibers, and let $\nabla^{\mathcal{M}}:\mathcal{M}\to\mathcal{M}\otimes\mathcal{A}^1_{\mathcal{X}}$ be a given global extension of $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$, compatible with the holomorphic structure. Then, on the open subset S°, we have*

$$d\,\mathrm{LOG}_{na}\langle\ell,m\rangle = \frac{\nabla^{int}_{\langle\mathcal{L},\mathcal{M}\rangle}\langle\ell,m\rangle}{\langle\ell,m\rangle} - \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta).$$

Proof. Recall the definition of the naive logarithm (4). To compute the differential of $\mathrm{LOG}_{na}\langle \ell, m\rangle$ and to compare to the intersection connection, we only need to describe the differentials of the functions

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta}, \quad \frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m} \wedge \eta\right).$$

For the argument, we need the auxiliary choice of an extension of the forms η_j to global differential forms on $\mathcal{X}^\circ$. We use the same notation for the extended forms. These extensions determine an extension of η by imposing the relation $\eta = \sum_i f_i \eta_i$.

The trivialization $\mathcal{X}^\circ \simeq X \times S^\circ$ induces a retraction $p\colon \mathcal{X} \to X$ (projection map), and we denote $\iota\colon X \hookrightarrow \mathcal{X}^\circ$ as the inclusion of X. Without loss of generality, we may assume that the components of $\operatorname{div} m$ are given by sections $S^\circ \to \mathcal{X}^\circ$ (this is true after base change by a finite étale cover, and this does not affect equalities of differential forms).

By the flatness condition on the $[\eta_j]$, there exist smooth functions g_j on $\mathcal{X}^\circ$ such that

$$\eta_j = p^*\iota^*(\eta_j) + dg_j \quad \text{on fibers.} \tag{8}$$

The functions g_j are uniquely determined by imposing the condition $\sigma^*(g_j) = 0$. It follows that we have an identity of differential forms on $\mathcal{X}^\circ$

$$\eta_j = p^*\iota^*(\eta_j) + dg_j - \vartheta_j, \tag{9}$$

where ϑ_j is a differential form vanishing on fibers. We also observe that $\iota^*(\eta_j)$ is closed, and hence so is $p^*\iota^*(\eta_j)$. Then (9) implies

$$d\eta_j = -d\vartheta_j. \tag{10}$$

Now from (8) the following relations hold:

$$\begin{aligned} \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta}_j &= \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \widetilde{p^*\iota^*(\eta_j)} + \mathrm{tr}_{\operatorname{div} m/S^\circ}(g_j) \\ d\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta}_j &= \mathrm{tr}_{\operatorname{div} m/S^\circ}(p^*\iota^*(\eta_j)) + \mathrm{tr}_{\operatorname{div} m/S^\circ}(dg_j). \end{aligned} \tag{11}$$

Combining (9) and (11), we obtain: $d\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta}_j = \mathrm{tr}_{\operatorname{div} m/S^\circ}(\eta_j) + \mathrm{tr}_{\operatorname{div} m/S^\circ}(\vartheta_j)$. Hence, we conclude with the first desired computation:

$$d\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta} = \mathrm{tr}_{\operatorname{div} m/S^\circ}(\eta) + \int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \nabla_{\mathrm{GM}}\nu + \sum_{j=1}^{2g} f_j\, \mathrm{tr}_{\operatorname{div} m/S^\circ}(\vartheta_j). \tag{12}$$

Let us now compute the differentials of the expressions

$$\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m} \wedge \eta_j\right).$$

For this, we recall the Poincaré-Lelong equation

$$\frac{i}{2\pi} d\left[\frac{\nabla^{\mathcal{M}} m}{m}\right] + \delta_{\operatorname{div} m} = \frac{i}{2\pi} F_{\nabla^{\mathcal{M}}},$$

where $[\nabla^{\mathcal{M}} m/m]$ stands for the current of integration against the locally integrable singular form $\nabla^{\mathcal{M}} m/m$. Taking into account the relation $d\eta_j = -d\vartheta_j$ (*cf.* (10)), the Poincaré-Lelong equation implies

$$d\left\{\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge\eta_j\right)\right\} = -\operatorname{tr}_{\operatorname{div} m/S^\circ}(\eta_j) + \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta_j) \tag{13}$$

$$+\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge d\vartheta_j\right). \tag{14}$$

For the last term (14), we claim

$$\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge d\vartheta_j\right) = -\operatorname{tr}_{\operatorname{div} m/S^\circ}(\vartheta_j). \tag{15}$$

Indeed, because ϑ_j vanishes on fibers, we have

$$\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge\vartheta_j\right) = 0.$$

Differentiating this equality and applying the Poincaré-Lelong equation, we derive

$$\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge d\vartheta_j\right) = -\operatorname{tr}_{\operatorname{div} m/S^\circ}(\vartheta_j) + \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\vartheta_j).$$

But now we argue as above: locally with respect to the base S°, we can write ϑ_j as a finite sum $\sum_k h_k\omega_k$, for smooth functions h_k on $\mathcal{X}^\circ$ and smooth differential forms ω_k on S°. Therefore

$$\frac{i}{2\pi}\pi_*\left(F_{\nabla^{\mathcal{M}}}\wedge\vartheta_j\right) = \sum_k\frac{i}{2\pi}\pi_*(h_kF_{\nabla^{\mathcal{M}}})\omega_k.$$

This expression vanishes, because $F_{\nabla^{\mathcal{M}}}$ is 0 on fibers by hypothesis. This proves the claim (15). From (13)–(15), we compute the second desired differentiation:

$$d\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge\eta\right) = -\operatorname{tr}_{\operatorname{div} m/S^\circ}(\eta) + \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta) - \sum_{j=1}^{2g} f_j\operatorname{tr}_{\operatorname{div} m/S^\circ}(\vartheta_j). \tag{16}$$

Putting (12) and (16) together we find

$$\begin{aligned} d\int_{\tilde\sigma}^{\widetilde{\operatorname{div} m}}\tilde\eta + d\frac{i}{2\pi}\pi_*\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge\eta\right) &= \int_{\tilde\sigma}^{\widetilde{\operatorname{div} m}}\nabla_{\mathrm{GM}}\nu + \frac{i}{2\pi}\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge\nabla_{\mathrm{GM}}\nu_{\mathcal{L}}\right) + \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta).\\ &= \int_{\tilde\sigma}^{\widetilde{\operatorname{div} m}}\nabla_{\mathrm{GM}}\nu - \frac{i}{2\pi}\left(\frac{\nabla^{\mathcal{M}} m}{m}\wedge F_{\widetilde{\nabla}^{\mathcal{L}}}\right) + \frac{i}{2\pi}\pi_*(F_{\nabla^{\mathcal{M}}}\wedge\eta). \end{aligned} \tag{17}$$

For the second equality we invoked the construction of the canonical extension $\widetilde{\nabla}^{\mathcal{L}}$ in terms of $\nabla_{\mathrm{GM}}\nu$ (see (6)). The statement of the proposition now follows from the definition of $\mathrm{LOG}_{na}\langle\ell, m\rangle$, equation (17) and the very definition of the intersection connection. □

Remark 3.4. (i) A formal computation could suggest that equations (12) and (16) hold without the right most terms. This is the case if the fibration is trivial, but not in general. Still, the deviation from the formal computation compensates when we add both equations.

(ii) The formula of Proposition 3.3 recovers the curvature formula of the intersection connection in [16, Prop. 3.16].

3.4. **Dependence of naive logarithms on liftings.** Continuing with the notation of Section 3.2, we now study the dependence of the construction of LOG_{na} for Deligne pairings $\langle \mathcal{L}, \mathcal{M}\rangle$ on the lifting $\hat{\nu}$ of $\nu_{\mathcal{L}}$. Because the Deligne pairing commutes with base change and the naive logarithm is defined pointwise, we reduce to the case when the base S is a point, and hence $\mathcal{X}$ is a single Riemann surface X.

Let $[\theta] \in H^1(X, 2\pi i\,\mathbb{Z})$, with harmonic representative θ. We wish to study the change of LOG_{na} under the transformation $\hat{\nu} \mapsto \hat{\nu} + [\theta]$. A first remark is that this transformation doesn't change given the lifting $\tilde{\ell}$ of a meromorphic section ℓ of $\mathcal{L}$. Therefore, we are led to study the change of the expression

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\eta} + \frac{i}{2\pi}\int_X \frac{\nabla^{\mathcal{M}} m}{m} \wedge \eta;$$

that is, the factor

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta} + \frac{i}{2\pi}\int_X \frac{\nabla^{\mathcal{M}} m}{m} \wedge \theta. \tag{18}$$

Observe that a change of representatives in $\tilde{\sigma}$ or $\widetilde{\operatorname{div} m}$ does not affect this factor modulo $2\pi i\,\mathbb{Z}$, because θ has periods in $2\pi i\,\mathbb{Z}$.

There is no general answer for the question posed above unless we make some additional assumptions on $\nabla^{\mathcal{M}}$. The first case to consider is when $\nabla^{\mathcal{M}}$ is the Chern connection of a smooth Hermitian metric on $\mathcal{M}$, not necessarily flat. Then

$$\int_X \frac{\nabla^{\mathcal{M}} m}{m} \wedge \eta = \int_X d(\log \|m\|^2 \eta) = 0.$$

We are then left with the term: $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta}$. This quantity does not vanish in general. In this case, the lack of invariance under the transformation $\hat{\nu} \mapsto \hat{\nu} + [\theta]$ will be addressed later in Section 4.1 by introducing the conjugate datum.

The second relevant case is when $\nabla^{\mathcal{M}}$ is flat. Let ϑ be a harmonic differential form whose class in $M_{dR}(X) := H^1(X, \mathbb{C})/H^1(X, 2\pi i\,\mathbb{Z})$ corresponds to the connection $\nabla^{\mathcal{M}}$. Then, the associated flat Chern connection corresponds to $\vartheta'' - \overline{\vartheta}''$, where ϑ'' denotes the $(0,1)$ part of ϑ (we will use similar notations for $(1,0)$ parts). We have the comparison

$$\nabla^{\mathcal{M}} = \nabla^{\mathcal{M}}_{ch} + \vartheta' + \overline{\vartheta''}\,. \tag{19}$$

Also, because θ has purely imaginary periods, $\theta = -\overline{\theta}$, and dividing into types we have a decomposition

$$\theta = \theta'' - \overline{\theta''}. \tag{20}$$

These relations will be used in the proof of the following statement.

Proposition 3.5 (Refined Poincaré-Lelong Formula)**.** *Assume $\nabla^{\mathcal{M}}$ is holomorphic and choose a harmonic one form ϑ representing the class of $\nabla^{\mathcal{M}}$ in $M_{dR}(X)$. Let θ be a harmonic one form with periods in $2\pi i\,\mathbb{Z}$. Then*

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta} + \frac{i}{2\pi}\int_X \frac{\nabla^{\mathcal{M}} m}{m} \wedge \theta = \frac{i}{2\pi}\int_X \vartheta \wedge \theta \qquad \operatorname{mod} 2\pi i\,\mathbb{Z}. \tag{21}$$

Proof. The statement is the conjunction of various reciprocity laws [22, Reciprocity Law I, p.230]. They involve the boundary of a fundamental domain delimited by (liftings of) simple closed curves α_i and β_i whose homology classes provide with a basis of $H_1(X,\mathbb{Z})$, symplectic with respect to the intersection pairing. We can take $\widetilde{\operatorname{div} m}$ in the chosen fundamental domain based at $\tilde{\sigma}$, because we already justified that (18) does not depend on representatives, modulo $2\pi i\mathbb{Z}$. Applying the reciprocity formula to θ' and the meromorphic differential form $(\nabla^{\mathcal{M}}_{ch} m)/m = \partial \log \|m\|^2$ (where $\|\cdot\|$ stands for a flat metric on $\mathcal{M}$), we have

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta}' = \frac{1}{2\pi i}\sum_j \int_{\alpha_j}\theta' \int_{\beta_j}\partial \log\|m\|^2 - \int_{\beta_j}\theta'\int_{\alpha_j}\partial\log\|m\|^2. \tag{22}$$

Now we take into account that $\theta' = -\overline{\theta''}$, and conjugate the previous expression to obtain

$$-\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta}'' = \frac{1}{2\pi i}\sum_j \int_{\alpha_j}\theta'' \int_{\beta_j}\bar{\partial} \log\|m\|^2 - \int_{\beta_j}\theta''\int_{\alpha_j}\bar{\partial}\log\|m\|^2.$$

But observe that for a closed curve γ disjoint from the divisor of m, we have by Stokes' theorem, $\int_\gamma d\log\|m\|^2 = 0$, and therefore $\int_\gamma \partial\log\|m\|^2 = -\int_\gamma \bar{\partial}\log\|m\|^2$. We thus derive

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta}'' = \frac{1}{2\pi i}\sum_j \int_{\alpha_j}\theta'' \int_{\beta_j}\partial \log\|m\|^2 - \int_{\beta_j}\theta''\int_{\alpha_j}\partial\log\|m\|^2. \tag{23}$$

Equations (22)–(23) together lead to

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta} = \frac{1}{2\pi i}\sum_j \int_{\alpha_j}\theta \int_{\beta_j}\partial \log\|m\|^2 - \int_{\beta_j}\theta\int_{\alpha_j}\partial\log\|m\|^2. \tag{24}$$

But now, modulo $2\pi i\,\mathbb{Z}$, we have

$$\int_\gamma \partial\log\|m\|^2 = \int_\gamma (\vartheta'' - \overline{\vartheta''}). \tag{25}$$

Because the periods of θ are in $2\pi i\,\mathbb{Z}$, eqs. (24)–(25) yield

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta} = \frac{1}{2\pi i}\sum_j \int_{\alpha_j}\theta \int_{\beta_j}(\vartheta'' - \overline{\vartheta''}) - \int_{\beta_j}\theta\int_{\alpha_j}(\vartheta'' - \overline{\vartheta''})$$

modulo $2\pi i\mathbb{Z}$. Using the Riemann bilinear relations,

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}} \tilde{\theta} = \frac{1}{2\pi i}\int_X \theta\wedge(\vartheta'' - \overline{\vartheta''}). \tag{26}$$

Let's now treat the second integral:

$$\int_X \frac{\nabla^{\mathcal{M}} m}{m}\wedge\theta = \int_X \frac{\nabla^{\mathcal{M}}_{ch} m}{m}\wedge\theta + \int_X (\vartheta' + \overline{\vartheta''})\wedge\theta.$$

The first integral on the right hand side is equal to $\int_X d(\log\|m\|^2\theta'') = 0$, where we use the fact that θ'' is closed. Hence, we arrive at

$$\frac{i}{2\pi}\int_X \frac{\nabla^{\mathcal{M}} m}{m}\wedge\theta = \frac{i}{2\pi}\int_X (\vartheta' + \overline{\vartheta''})\wedge\theta = \frac{1}{2\pi i}\int_X \theta\wedge(\vartheta' + \overline{\vartheta''}). \tag{27}$$

Sum (26) and (27) to obtain

$$\int_{\sigma}^{\widetilde{\operatorname{div} m}} \tilde{\theta} + \frac{i}{2\pi}\int_X \frac{\nabla^{\mathcal{M}} m}{m}\wedge\theta = \frac{1}{2\pi i}\int_X \theta\wedge\vartheta \mod 2\pi i\mathbb{Z}$$

as was to be shown. □

Remark 3.6. The integral: $(i/2\pi)\int_X \vartheta\wedge\theta$, depends mod $2\pi i\mathbb{Z}$ only on the class of ϑ modulo the lattice $H^1(X, 2\pi i\mathbb{Z})$, or equivalently on the point $[\nabla^{\mathcal{M}}]$ in $M_{dR}(X)$.

4. The Intersection Logarithm

4.1. **Intersection logarithms in conjugate pairs.** We retain the notation from the previous section, and we work on a single Riemann surface X. We denote by $\overline{X}$ the conjugate Riemann surface: the underlying $\mathcal{C}^\infty$ surface stays the same, but we reverse the complex structure. For notational coherence with later considerations, we denote $\overline{\sigma}$ for the base point σ seen as a point of $\overline{X}$. Let $(\mathcal{L}^c, \nabla^{\mathcal{L},c})$ be a rigidified (at $\overline{\sigma}$) holomorphic line bundle with connection attached to $-\hat{\nu}$, regarded as a cohomology class on $\overline{X}$. Hence the holonomy characters of $\nabla^{\mathcal{L}}$ and $\nabla^{\mathcal{L},c}$ are mutually inverse. We say that $(\mathcal{L}, \nabla^{\mathcal{L}})$ and $(\mathcal{L}^c, \nabla^{\mathcal{L},c})$ form a *conjugate pair*. As rank 1 local systems, these bundles are dual, and complex conjugate exactly when the flat connections are unitary.

For the connection $\nabla^{\mathcal{M}}$, from now on we focus on two cases:

- $\nabla^{\mathcal{M}}$ **is a Chern connection (not necessarily flat).** In this case, $\mathcal{M}^c$ denotes the complex conjugate line bundle to $\mathcal{M}$ on $\overline{X}$. We let $\nabla^{\mathcal{M},c}$ be the conjugate of the connection $\nabla^{\mathcal{M}}$.
- $\nabla^{\mathcal{M}}$ **is flat.** We assume that $\mathcal{M}$ is rigidified at σ. Then $(\mathcal{M}^c, \nabla^{\mathcal{M},c})$ is a flat holomorphic line bundle on $\overline{X}$, rigidified at $\overline{\sigma}$, with inverse holonomy to $(\mathcal{M}, \nabla^{\mathcal{M}})$.

There is an intersection between these two situations: the flat unitary case. The conventions defining $(\mathcal{M}^c, \nabla^{\mathcal{M},c})$ are consistent. By this we mean both are mutually isomorphic: there is a unique isomorphism respecting the connections and rigidifications. In either case, we write LOG^c_{na} for the corresponding naive logarithm for $\langle \mathcal{L}^c, \mathcal{M}^c\rangle$.

Proposition 4.1. *The sum of logarithms* LOG_{na} *and* LOG^c_{na}*, for* $\langle\mathcal{L},\mathcal{M}\rangle$ *and* $\langle\mathcal{L}^c,\mathcal{M}^c\rangle$*, defines a logarithm for* $\langle\mathcal{L},\mathcal{M}\rangle\otimes_{\mathbb{C}}\langle\mathcal{L}^c,\mathcal{M}^c\rangle$*, that only depends on the point* $[\nabla^{\mathcal{L}}]$ *in* $M_{dR}(X)$*, the rigidifications, and on* $\nabla^{\mathcal{M}}$*. If* $\nabla^{\mathcal{M}}$ *is flat, then the dependence on* $\nabla^{\mathcal{M}}$ *factors through* $M_{dR}(X)$ *as well.*

Proof. Let θ be a harmonic 1-form with periods in $2\pi i\mathbb{Z}$. We consider the change of LOG_{na} and LOG^c_{na} under the transformation $\hat{\nu}\mapsto\hat{\nu}+[\theta]$, and observe that they compensate each other.

We start with the Chern connection case on $\mathcal{M}$. Let m be a meromorphic section of $\mathcal{M}$. It defines a complex conjugate meromorphic section m^c of $\mathcal{M}^c$, and the divisors $\operatorname{div} m$ and $\operatorname{div} m^c$ are equal. We saw that the change in $\mathrm{LOG}_{na}(\langle\ell,m\rangle)$ under $\hat{\nu}\mapsto\hat{\nu}+[\theta]$ is reduced to $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\theta}$. The change in $\mathrm{LOG}^c_{na}(\langle\ell',m^c\rangle)$ will be $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m^c}}(-\tilde{\theta})$. But now, independently of the liftings $\widetilde{\operatorname{div} m}$ and $\widetilde{\operatorname{div} m^c}$ in to the universal cover, we have $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\theta}+\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m^c}}(-\tilde{\theta})=0$ mod $2\pi i\mathbb{Z}$. More generally, we can change m^c by a meromorphic function. For if f is meromorphic on $\overline{X}$, we have $\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} f}}\tilde{\theta}\in 2\pi i\mathbb{Z}$, precisely by Proposition 3.5 applied to the trivial line bundle in place of $\mathcal{M}$. Hence, m^c may be taken to be any meromorphic section of $\mathcal{M}^c$. In summary, we see that $\mathrm{LOG}_{na}+\mathrm{LOG}^c_{na}$ is invariant under $\hat{\nu}\mapsto\hat{\nu}+[\theta]$.

Now for the flat connection case on $\mathcal{M}$. We introduce a harmonic representative ϑ of the class of $\nabla^{\mathcal{M}}$ in $M_{dR}(X)$. Then $\nabla^{\mathcal{M},c}$ admits $-\vartheta$ as a harmonic representative in $M_{dR}(\overline{X})$. After Proposition 3.5, for any meromorphic section m of $\mathcal{M}$ on X, we have

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m}}\tilde{\theta}+\frac{i}{2\pi}\int_X\frac{\nabla^{\mathcal{M}}m}{m}\wedge\theta=\frac{i}{2\pi}\int_X\vartheta\wedge\theta\qquad\text{mod }2\pi i\mathbb{Z}.\tag{28}$$

Similarly, if m^c is a meromorphic section of $\mathcal{M}^c$ on $\overline{X}$, we have

$$\int_{\tilde{\sigma}}^{\widetilde{\operatorname{div} m^c}}(-\tilde{\theta})+\frac{i}{2\pi}\int_{\overline{X}}\frac{\nabla^{\mathcal{M}}m^c}{m^c}\wedge\theta=\frac{i}{2\pi}\int_{\overline{X}}(-\vartheta)\wedge(-\theta)\qquad\text{mod }2\pi i\mathbb{Z}.\tag{29}$$

Since $\overline{X}$ is oppositely oriented, $(i/2\pi)\int_{\overline{X}}(-\vartheta)\wedge(-\theta)=-(i/2\pi)\int_X\vartheta\wedge\theta$. Hence, the change in the sum of logarithms is (28)+(29)=0. Notice that from the formulas defining the logarithms the dependence on $\nabla^{\mathcal{M}}$ trivially factors through $M_{dR}(X)$. The statement follows. □

Remark 4.2. An observation on the change of complex structure is now in order. A complex manifold can be seen as a pair $\mathcal{Y}=(Y,\mathcal{O}_{\mathcal{Y}})$ formed by a differentiable manifold Y together with a sheaf of holomorphic functions $\mathcal{O}_{\mathcal{Y}}$, locally isomorphic to the sheaf of holomorphic functions on some $\mathbb{C}^n$. The conjugate complex manifold $\overline{\mathcal{Y}}$ is then the pair $(Y,\overline{\mathcal{O}}_{\mathcal{Y}})$, where the sheaf $\overline{\mathcal{O}}_{\mathcal{Y}}$ is constructed from $\mathcal{O}_{\mathcal{Y}}$ by complex conjugating its local sections. From this perspective, a holomorphic morphism of complex manifolds $f\colon\mathcal{Y}\to\mathcal{Z}$ clearly induces a holomorphic morphism $\overline{f}\colon\overline{\mathcal{Y}}\to\overline{\mathcal{Z}}$. Abusing notations, we may sometimes say that in the $\mathcal{C}^\infty$ category, we have $\mathcal{Y}=\overline{\mathcal{Y}}$ and $f=\overline{f}$. This construction is applied below to a family of compact Riemann surfaces $\pi\colon\mathcal{X}\to S$ and to a section $\sigma\colon S\to\mathcal{X}$, thus producing a corresponding conjugate family of Riemann surfaces with section $\overline{\pi}\colon\overline{\mathcal{X}}\to\overline{S}$, $\overline{\sigma}\colon\overline{S}\to\overline{\mathcal{X}}$.

4.2. **Smooth variation in non-trivial families.** Let us examine the variation in families of $\mathrm{LOG}_{na}+\mathrm{LOG}_{na}^{c}$. Because the construction we did of logarithms is a pointwise one, Proposition 4.1 extends to the family situation. We consider $(\pi\colon \mathcal{X}\to S, \sigma\colon S\to\mathcal{X})$, and the conjugate family $(\overline{\pi}\colon \overline{\mathcal{X}}\to\overline{S}, \overline{\sigma}\colon \overline{S}\to\overline{\mathcal{X}})$, cf. Remark 4.2 for details. Let $(\mathcal{L},\nabla^{\mathcal{L}}_{\mathcal{X}/S})$ and $(\mathcal{M},\nabla^{\mathcal{M}}_{\mathcal{X}/S})$ be line bundles with relative compatible connections on $\mathcal{X}$. We suppose $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$ is flat, and $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is either flat or the Chern connection associated to a smooth Hermitian metric on $\mathcal{M}$.

When both connections are flat, we have the smooth classifying sections $\nu_{\mathcal{L}}$ and $\nu_{\mathcal{M}}$ of $H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\,\mathbb{Z})$. We then *assume* that on $\overline{\mathcal{X}}$ we have rigidified line bundles with relative flat connections $(\mathcal{L}^c,\nabla^{\mathcal{L},c}_{\overline{\mathcal{X}}/\overline{S}})$ and $(\mathcal{M}^c,\nabla^{\mathcal{M},c}_{\overline{\mathcal{X}}/\overline{S}})$, corresponding to the smooth sections $-\nu_{\mathcal{L}}$ and $-\nu_{\mathcal{M}}$ of $H^1_{dR}(\overline{\mathcal{X}}/\overline{S})/R^1\overline{\pi}_*(2\pi i\,\mathbb{Z}) = H^1_{dR}(\mathcal{X}/S)/R^1\pi_*(2\pi i\,\mathbb{Z})$ (as differentiable manifolds). The existence is not always guaranteed, but below we deal with relevant situations when it is. The local construction of Section 3.2 produces local naive logarithms LOG_{na} and LOG^c_{na}, by taking local liftings $\hat{\nu}$ and $-\hat{\nu}$ for $\nu_{\mathcal{L}}$ and $-\nu_{\mathcal{L}}$, and using the canonical extensions of $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ and $\nabla^{\mathcal{M},c}_{\mathcal{X}/S}$. Proposition 4.1 ensures that the *a priori* locally defined combination $\mathrm{LOG}_{an}+\mathrm{LOG}^c_{an}$ on the $\mathcal{C}^\infty$ line bundle $\langle\mathcal{L},\mathcal{M}\rangle\otimes_{\mathcal{C}^\infty_S}\langle\mathcal{L}^c,\mathcal{M}^c\rangle$, actually globalizes to a well-defined logarithm, that we call *intersection logarithm*: $\mathrm{LOG}_{int} := \mathrm{LOG}_{na}+\mathrm{LOG}^c_{na}$.

When $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is the relative Chern connection attached to a smooth Hermitian metric on $\mathcal{M}$, we take $\mathcal{M}^c$ to be the conjugate line bundle $\overline{\mathcal{M}}$ on $\overline{\mathcal{X}}$, with its conjugate Chern connection $\nabla^{\mathcal{M},c}_{\mathcal{X}/S}$. For $\mathcal{L}$, as above we assume the existence of a rigidified $(\mathcal{L}^c,\nabla^{\mathcal{L},c})$, with classifying map $-\nu_{\mathcal{L}}$. Again, by Proposition 4.1 the locally defined $\mathrm{LOG}_{an}+\mathrm{LOG}^c_{an}$ extends to a global logarithm that we also denote LOG_{int}.

We summarize the main features of LOG_{int}.

Proposition 4.3. *(i) When all connections are flat, the construction of* LOG_{int} *does not depend on the section σ or rigidifications.*

(ii) In general, the connection defined by LOG_{int} *is the tensor product of intersection connections.*

Proof. We begin with the case when both connections are flat. The first item can be checked pointwise. Let us examine the terms in the definition of LOG_{na}, LOG^c_{na} and LOG_{int}. Suppose we fix another base point σ' (and lifting $\tilde{\sigma}'$) and another rigidification. Let $\tilde{\ell}$ and $\tilde{\ell}'$ be equivariant meromorphic functions with character $\gamma\mapsto\exp(\int_\gamma\hat{\nu})$, lifting the same meromorphic section of $\mathcal{L}$. Then, for some $\lambda\in\mathbb{C}^\times$, we have $\tilde{\ell}'=\lambda\tilde{\ell}$. Therefore, evaluating multiplicatively over a degree 0 divisor D in $\widetilde{\mathcal{X}}$, we see that $\tilde{\ell}'(D)=\tilde{\ell}(D)$. The same happens for $\mathcal{L}^c$. Also, in LOG_{na} we have the change: $\int_{\tilde{\sigma}}^{z}\tilde{\eta}=\int_{\tilde{\sigma}}^{\tilde{\sigma}'}\tilde{\eta}+\int_{\sigma'}^{z}\tilde{\eta}$. The evaluation at a divisor is defined to be additive. Therefore, for a divisor D we find: $\int_{\tilde{\sigma}}^{D}\tilde{\eta}=(\deg D)\int_{\tilde{\sigma}}^{\tilde{\sigma}'}\tilde{\eta}+\int_{\tilde{\sigma}'}^{D}\tilde{\eta}$. We are concerned with the case $D=\widetilde{\operatorname{div} m}$, when $\deg D=0$. This shows the independence of this term of the base point. The same argument applies to $\mathcal{L}^c$. Finally, there is nothing to say about the remaining terms in the definition of LOG_{na} and LOG^c_{na}, since they only depend on the vertical connections $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{M},c}$ (as we see pointwise) and $\hat{\nu}$, and hence do not depend on base points nor rigidifications. The dependence on the choice of $\hat{\nu}$ modulo $R^1\pi_*(2\pi i\,\mathbb{Z})$ was already addressed (Proposition 4.1). We conclude that LOG_{int} does not depend on σ and the rigidifications.

For the second item, it is enough to observe that: $\overline{\pi}_*(F_{\nabla^{\mathcal{M},c}} \wedge (-\eta)) = -\pi_*((-F_{\nabla^{\mathcal{M}}}) \wedge (-\eta)) = -\pi_*(F_{\nabla^{\mathcal{M}}} \wedge \eta)$ (opposite orientation on fibers) and apply Proposition 3.3. We obtain:

$$d\,\mathrm{LOG}_{int}(\langle \ell, m\rangle \otimes \langle \ell', m'\rangle) = \frac{\nabla^{int}_{\langle \mathcal{L},\mathcal{M}\rangle}\langle \ell, m\rangle}{\langle \ell, m\rangle} + \frac{\nabla^{int}_{\langle \mathcal{L}^c,\mathcal{M}^c\rangle}\langle \ell', m'\rangle}{\langle \ell', m'\rangle}.$$

For the second item when $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$ is a Chern connection, we reduce to the flat case by Lemma 3.2 and [16, Thm. 3.14] (when the connection on $\mathcal{M}$ is a Chern connection, the intersection and trace connections coincide). □

Corollary 4.4. *Given $(\mathcal{L}, \nabla^{\mathcal{L}}_{\mathcal{X}/S})$, $(\mathcal{M}, \nabla^{\mathcal{M}}_{\mathcal{X}/S})$,$(\mathcal{L}^c, \nabla^{\mathcal{L},c}_{\overline{\mathcal{X}}/S})$, $(\mathcal{M}^c, \nabla^{\mathcal{M},c}_{\overline{\mathcal{X}}/S})$ with flat connections and no assumption on rigidifications, the smooth line bundle $\langle \mathcal{L}, \mathcal{M}\rangle \otimes_{\mathcal{C}^\infty_S} \langle \mathcal{L}^c, \mathcal{M}^c\rangle$ has a canonically defined smooth logarithm, LOG_{int}, that coincides with the previous construction in presence of a rigidification. Its attached connection is the tensor product of intersection connections.*

Proof. Locally over S, we can find sections and rigidify our line bundles. We conclude by Proposition 4.3. □

Remark 4.5. Examples of the family setting above naturally arise from character varieties of quasi-fuchsian groups. See Remark 5.13 below for further details.

4.3. **Intersection logarithm over character varieties.** An important geometric setting when an intersection logarithm can be defined is the "universal" product situation: we fix the Riemann surface and we let the flat line bundle holomorphically change, parametrized by the affine space of holonomy characters. A study of this case will lead below to the proof of the symmetry of intersection logarithms.

Let (X, σ) be a pointed Riemann surface and $M_B(X)$ the Betti moduli space of complex characters of $\pi_1(X, \sigma)$. Observe that $M_B(X) = M_B(\overline{X})$. We have relative curves $X \times M_B(X) \to M_B(X)$ and similarly for $\overline{X}$. There are universal rigidified holomorphic line bundles with relative flat connections $(\mathcal{L}_\chi, \nabla_\chi)$ and $(\mathcal{L}^c_\chi, \nabla^c_\chi)$, whose holonomy characters over a given $\chi \in M_B(X)$ are χ^{-1} and χ respectively. We take the tensor product of holomorphic line bundles on $M_B(X)$: $\langle \mathcal{L}_\chi, \mathcal{L}_\chi\rangle \otimes_{\mathcal{O}_{M_B(X)}} \langle \mathcal{L}^c_\chi, \mathcal{L}^c_\chi\rangle$. This is in contrast with the previous subsection, where the tensor product was only in the $\mathcal{C}^\infty$ category. A formal modification of the construction of LOG_{int} produces a well-defined logarithm, still denoted LOG_{int}. The only difference is that now we do not need to change the holomorphic structure on $M_B(X)$. More generally, we may work over $S = M_B(X) \times M_B(X)$. On $X \times S$ and $\overline{X} \times S$ we consider the pairs of universal bundles $(\mathcal{L}_{\chi_1}, \mathcal{M}_{\chi_2})$ and $(\mathcal{L}^c_{\chi_1}, \mathcal{M}^c_{\chi_2})$. We also have a universal intersection logarithm LOG_{int} on $\langle \mathcal{L}_{\chi_1}, \mathcal{M}_{\chi_2}\rangle \otimes_{\mathcal{O}_S} \langle \mathcal{L}^c_{\chi_1}, \mathcal{M}^c_{\chi_2}\rangle$, whose connection we now know is the sum of intersection connections.

It proves useful to establish the symmetry of general intersection logarithms:

Proposition 4.6. *The intersection logarithms for line bundles with relative flat connections are symmetric,* i.e. *compatible with the symmetry of Deligne pairings.*

Proof. This is a pointwise assertion. Deforming to $M_B(X)$, it is enough to deal with the universal situation parametrized by $S = M_B(X) \times M_B(X)$. Because the intersection connection is symmetric, and S is connected, we see that the intersection logarithm is symmetric up to a constant. Now it is enough to specialize to the pair of trivial characters, when the intersection logarithm is indeed symmetric. This concludes the proof. □

Corollary 4.7. *The intersection logarithm on the universal pairing* $\langle \mathcal{L}_{\chi_1}, \mathcal{M}_{\chi_2}\rangle \otimes_{\mathcal{O}_S} \langle \mathcal{L}^c_{\chi_1}, \mathcal{M}^c_{\chi_2}\rangle$, *parametrized by* $M_B(X) \times M_B(X)$, *is holomorphic.*

Proof. The holomorphy along the diagonal $\chi_1 = \chi_2$ holds, since the intersection connection is holomorphic there by [16, Sec. 5.3]. For the general case, we reduce to the diagonal. First, the multiplication map $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$ is holomorphic, and induces the canonical identification $\mathcal{L}_{\chi_1\chi_2} = \mathcal{L}_{\chi_1} \otimes \mathcal{L}_{\chi_2}$, and similarly for $\mathcal{M}_{\chi_1\chi_2}$, etc. Second, we have the "polarization formula",

$$\langle \mathcal{L} \otimes \mathcal{M}, \mathcal{L} \otimes \mathcal{M}\rangle = \langle \mathcal{L}, \mathcal{L}\rangle \otimes \langle \mathcal{L}, \mathcal{M}\rangle \otimes \langle \mathcal{M}, \mathcal{L}\rangle \otimes \langle \mathcal{M}, \mathcal{M}\rangle.$$

and the symmetry of intersection logarithms already proven. These observations and the proposition are enough to conclude the result. □

A variant concerns the pairing of the universal bundles with a fixed Hermitian line bundle $\mathcal{M}$ on X, trivially extended to $X \times M_B(X)$.

Corollary 4.8. *Let* $\mathcal{M}$ *be a line bundle on* X, $\overline{\mathcal{M}}$ *its conjugate line bundle on* $\overline{X}$, *and suppose that they are both endowed with a Chern connection. Extend trivially these data to* $X \times M_B(X)$ *and* $\overline{X} \times M_B(X)$ *by pull-back through the first projection. Then the intersection logarithm on* $\langle \mathcal{L}_\chi, \mathcal{M}\rangle \otimes \langle \mathcal{L}^c_\chi, \overline{\mathcal{M}}\rangle$, *parametrized by* $M_B(X)$, *is holomorphic and does not depend on the choices of Chern connections.*

Proof. By Lemma 3.2, we can suppose that (i) $\mathcal{M}$ is of relative degree 0 and rigidified along σ and (ii) its Chern connection is flat. Similarly, we can assume its conjugate line bundle comes with the conjugate connection. Therefore, there exists χ_0 a unitary character and an isomorphism of rigidified line bundles with connections: $(\mathcal{L}_{\chi_0}, \nabla_{\chi_0}) \xrightarrow{\sim} (\mathcal{M}, \nabla^{\mathcal{M}})$, $(\mathcal{L}^c_{\chi_0}, \nabla^c_{\chi_0}) \xrightarrow{\sim} (\overline{\mathcal{M}}, \nabla^{\overline{\mathcal{M}}})$. We conclude by Corollary 4.7 restricted to $\chi_2 = \chi_0$. □

4.4. **Explicit constructions.** For latter applications, it is important to exhibit natural geometric situations when the setting of Section 4.2 indeed obtains. With the notations therein, the difficulty is the existence of the invertible sheaf with connection $(\mathcal{L}^c, \nabla^{\mathcal{L},c}_{\overline{\mathcal{X}}/\overline{S}})$. Even when the existence is granted, it would be useful to have at our disposal a general *algebraic* procedure to build $(\mathcal{L}^c, \nabla^{\mathcal{L},c}_{\overline{\mathcal{X}}/\overline{S}})$ from $(\mathcal{L}, \nabla^{\mathcal{L}}_{\mathcal{X}/S})$. By algebraic procedure we mean a construction that can be adapted to the schematic (for instance the arithmetic) setting.

We distinguish three kinds of relative flat connections on the line bundles $\mathcal{L}$ and $\mathcal{M}$: real holonomies, imaginary holonomies, and the "mixed" case. When S is reduced to a point, the mixed case is actually the general one. Furthermore, it is then possible to give an explicit description of the intersection logarithm.

4.4.1. *Real holonomies.* We suppose here that the holonomies of the flat bundles $(\mathcal{L}, \nabla^{\mathcal{L}}_{\mathcal{X}/S})$ and $(\mathcal{M}, \nabla^{\mathcal{M}}_{\mathcal{X}/S})$ on fibers are real. On the conjugate variety $\overline{\mathcal{X}}$, the conjugate line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ admit the complex conjugate connections to $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$ and $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$. We see that: $(\mathcal{L}^c, \nabla^{\mathcal{L},c}_{\mathcal{X}/S}) = (\overline{\mathcal{L}}^{\vee}, \overline{\nabla}^{\mathcal{L},\vee}_{\mathcal{X}/S})$, $(\mathcal{M}^c, \nabla^{\mathcal{M},c}_{\mathcal{X}/S}) = (\overline{\mathcal{M}}^{\vee}, \overline{\nabla}^{\mathcal{M},\vee}_{\mathcal{X}/S})$. The bar on the connections stands for complex conjugation.

When the base is a point, we write X, p, L, M, ∇^L, ∇^M instead of $\mathcal{X}$, σ, $\mathcal{L}$, $\mathcal{M}$, $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$, $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$. The first important remark is that since the connections ∇^L and ∇^M have real holonomy characters χ_L and χ_M, they determine *unique* real harmonic differential forms η and ϑ (i.e. there is no ambiguity modulo a lattice as in the unitary case). The relation is then

$$\chi_L(\gamma) = \exp\left(-\int_\gamma \eta\right), \quad \chi_M(\gamma) = \exp\left(-\int_\gamma \vartheta\right), \quad \gamma \in \pi_1(X,p).$$

Because η and ϑ are real, we can write the decomposition into type as: $\eta = \eta' + \overline{\eta'}$, $\vartheta = \vartheta' + \overline{\vartheta'}$, where η' and ϑ' are holomorphic. The naive logarithm for the complex structure on X is determined by

$$\mathrm{LOG}_{na}(\langle \ell, m\rangle) = \log(\tilde{\ell}(\widetilde{\operatorname{div} m})) - \int_{\tilde{p}}^{\widetilde{\operatorname{div} m}} \tilde{\eta} - \frac{i}{2\pi}\int_X \frac{\nabla^M m}{m}\wedge \eta.$$

Recall that the first two terms together do not change under a transformation $\eta \mapsto \eta + \theta$, for θ holomorphic. Using the relation with the Chern connections $\nabla^L = \nabla^L_{ch} - 2\eta'$, $\nabla^M = \nabla^M_{ch} - 2\vartheta'$, we simplify the naive logarithm to

$$\mathrm{LOG}_{na}(\langle \ell, m\rangle) = \log(\tilde{\ell}_{ch}(\widetilde{\operatorname{div} m})) - \int_{\tilde{p}}^{\widetilde{\operatorname{div} m}} (\widetilde{\overline{\eta'} - \eta'}) + \frac{i}{\pi}\int_X \vartheta' \wedge \overline{\eta'}.$$

We denoted $\tilde{\ell}_{ch}$ the lift of ℓ using the Chern connection ∇^L_{ch}. Changing the holomorphic structure (and hence reversing the orientation in the last integral), the naive logarithm LOG^c_{na} computed with the conjugate sections $\overline{\ell}$ and $\overline{m}$ is

$$\mathrm{LOG}^c_{na}(\langle \overline{\ell}^{\vee}, \overline{m}^{\vee}\rangle) = \log\overline{(\tilde{\ell}_{ch}(\widetilde{\operatorname{div} m}))} - \int_{\tilde{p}}^{\widetilde{\operatorname{div} m}} (\widetilde{\eta' - \overline{\eta'}}) - \frac{i}{\pi}\int_X \overline{\vartheta'} \wedge \eta'.$$

All in all, we find: $\mathrm{LOG}_{int}(\langle \ell, m\rangle \otimes \langle \overline{\ell}^{\vee}, \overline{m}^{\vee}\rangle) = \log|\tilde{\ell}_{ch}(\widetilde{\operatorname{div} m})|^2 - (2/\pi)\operatorname{Im}\left(\int_X \vartheta' \wedge \overline{\eta'}\right)$. Notice that this expression is real valued.

4.4.2. *Unitary connections.* We suppose the holomorphic line bundles $\mathcal{L}$, $\mathcal{M}$ come with relative flat unitary connections $\nabla^{\mathcal{L}}_{\mathcal{X}/S}$, $\nabla^{\mathcal{M}}_{\mathcal{X}/S}$. For the complex conjugate family, it is therefore enough to take $(\mathcal{L}^c, \nabla^{\mathcal{L},c}_{\mathcal{X}/S}) = (\overline{\mathcal{L}}, \overline{\nabla}^{\mathcal{L}}_{\mathcal{X}/S})$, and $(\mathcal{M}^c, \nabla^{\mathcal{M},c}_{\mathcal{X}/S}) = (\overline{\mathcal{M}}, \overline{\nabla}^{\mathcal{M}}_{\mathcal{X}/S})$. Contrary to the real case, we do not need to dualize the complex conjugate line bundles.

In this case, the intersection logarithm LOG_{int} amounts to the logarithm of a smooth Hermitian metric: one easily sees that $\mathrm{LOG}_{int}(\langle \ell, m\rangle \otimes \langle \overline{\ell}, \overline{m}\rangle) = \log\|\langle \ell, m\rangle\|^2$. That is, the log of the square of the natural norm on the Deligne pairing. A similar formula holds more generally if $\mathcal{M}$ is endowed with a Chern connection.

4.4.3. *Mixed case.* Suppose that $\mathcal{L}$ is equipped with a flat relative connection with real holonomies, and $\mathcal{M}$ with a relative flat unitary connection. Then the tensor product of connections on $\mathcal{P} = \mathcal{L} \otimes \mathcal{M}$ is no longer real nor unitary. Nevertheless, we can still define $\mathcal{P}^c$ and $\nabla^{\mathcal{P},c}_{\mathcal{X}/S}$ on the conjugate family: $(\mathcal{P}^c, \nabla^{\mathcal{P},c}_{\mathcal{X}/S}) = (\overline{\mathcal{L}}^\vee \otimes \overline{\mathcal{M}}, \overline{\nabla}^{\mathcal{L},\vee}_{\mathcal{X}/S} \otimes \overline{\nabla}^{\mathcal{M}}_{\mathcal{X}/S})$.

Suppose now that we are dealing with a single compact Riemann surface X. We fix a base point $p \in X$. Let P be a line bundle over X with a connection: $\nabla^P : P \longrightarrow P \otimes \Omega^1_{X/\mathbb{C}}$. Let χ be the holonomy representation of ∇^P. The absolute value $|\chi|$ is the holonomy representation χ_L of a line bundle L on X endowed with a holomorphic connection ∇^L. We set $M := P \otimes L^\vee$, $\nabla^M = \nabla^P \otimes \nabla_{L,\vee}$. Then M is a line bundle with a flat unitary connection ∇^M, holonomy $\chi_M = \chi/|\chi|$, and $P = L \otimes M$, $\nabla^P = \nabla^L \otimes \nabla^M$. Hence, for a single Riemann surface, any flat line bundle fits the picture of the mixed case. General formulas for intersection logarithm then reduce to pairings between such (L, ∇^L) and (M, ∇^M).

The naive logarithm for the natural complex structure on X is determined by

$$\mathrm{LOG}_{na}(\langle \ell, m\rangle) = \log(\tilde{\ell}(\widetilde{\mathrm{div}\, m})) - \int_{\tilde{p}}^{\widetilde{\mathrm{div}\, m}} \tilde{\eta} - \frac{i}{2\pi}\int_X \frac{\nabla^M m}{m} \wedge \eta = \log(\tilde{\ell}(\widetilde{\mathrm{div}\, m})) - \int_{\tilde{p}}^{\widetilde{\mathrm{div}\, m}} \tilde{\eta} \quad (30)$$

The second equality uses that ∇^M is a Chern connection. Similarly

$$\mathrm{LOG}^c_{na}(\langle \overline{\ell}^\vee, \overline{m}\rangle) = -\log(\overline{\tilde{\ell}(\widetilde{\mathrm{div}\, m})}) - \int_{\tilde{p}}^{\widetilde{\mathrm{div}\, m}} (-\bar{\tilde{\eta}}). \quad (31)$$

Adding (30) and (31) and simplifying, we find for the intersection logarithm

$$\mathrm{LOG}_{int}(\langle \ell, m\rangle \otimes \langle \overline{\ell}^\vee, \overline{m}\rangle) = 2i \arg(\tilde{\ell}(\widetilde{\mathrm{div}\, m})). \quad (32)$$

This quantity is purely imaginary. The discussion is also valid if M has arbitrary degree and is endowed with a Hermitian metric. However, in this case the intersection logarithm depends on the rigidification of L.

5. The Quillen-Cappell-Miller Logarithm and Deligne's Isomorphism

5.1. **The Quillen-Cappell-Miller logarithm.** We review the definition of the Cappell-Miller torsion [11]. We aim at proving that the Cappell-Miller torsion behaves holomorphically in holomorphic families of flat line bundles on a fixed Riemann surface. Therefore, from the very beginning, we place ourselves in the universal family setting over $M_B(X)$.

Let X be a fixed compact Riemann surface with a fixed smooth Hermitian metric on T_X, p a base point, and $(\overline{X}, \overline{p})$ the conjugate datum. Let $M_B(X)$ be the space of characters of $\pi_1(X, p)$, and $\mathcal{L}$, $\mathcal{L}^c$ the holomorphic universal bundles on $\mathcal{X} := X \times M_B(X)$ and $\mathcal{X}^c := \overline{X} \times M_B(X)$. There are corresponding universal relative holomorphic connections. Write π and π^c for the projection maps onto $M_B(X)$.

Inspired by Quillen [28], Bismut-Freed [2, 3] and Bismut-Gillet-Soulé [4, 5, 6], we describe the determinant of cohomology $\lambda(\mathcal{L}) = \det R\pi_*(\mathcal{L})$ as the determinant of a truncated Dolbeault complex of finite dimensional *holomorphic* vector bundles. One can proceed similarly for $\mathcal{L}^c$ and $\lambda(\mathcal{L}^c) = \det R\pi_*^c(\mathcal{L}^c)$. The difference with the cited works lies in the holomorphicity of these vector bundles. In few words, Bismut-Gillet-Soulé proceed in three steps: 1) construction of a $\mathcal{C}^\infty$ determinant line bundle; 2) definition of a holomorphic structure on it; 3) comparison with the Knudsen-Mumford holomorphic structure, induced by the holomorphic structure of relative coherent cohomology. In our setting, 1) and 2) are merged in a single step where we directly produce a holomorphic determinant line bundle. That this is possible is a characteristic feature of the geometric setting of character varieties as parameter spaces, and the holomorphic dependence on parameters of our Laplace type operators. Still, the analog of step 3) is needed. For this we actually invoke the very description of the Knudsen-Mumford structure due to Bismut-Gillet-Soulé. Before embarking on these tasks, a final word on the strategy adopted. In [6] the authors propose two methods to compare $\mathcal{C}^\infty$ determinant lines with Knudsen-Mumford determinants: an analytic one (Chapter 2 in *loc. cit.*) and an sheaf theoretic one (Chapter 3 in *loc. cit.*). We closely follow the sheaf theoretic approach.

Introduce the relative Dolbeault complex of $\mathcal{L}$, considered as a smooth complex line bundle with a $\bar\partial$-operator. More precisely, this is the complex of sheaves of $\mathcal{C}^\infty_{M_B(X)}$-modules

$$\mathcal{D}_{\mathcal{X}/M_B(X)} = \mathcal{D}_{\mathcal{X}/M_B(X)}(\mathcal{L})\colon 0 \longrightarrow \mathcal{A}^{0,0}_{\mathcal{X}/M_B(X)}(\mathcal{L}) \xrightarrow{\overline{\partial}_X} \mathcal{A}^{0,1}_{\mathcal{X}/M_B(X)}(\mathcal{L}) \longrightarrow 0.$$

We have decorated the relative Dolbeault operator $\overline{\partial}_X$ with the index X to emphasize the fact that we are in a product situation, and we are only differentiating in the X direction. The cohomology sheaves of the complex $\pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}$ will be written $\mathcal{H}^{0,p}_{\overline{\partial}_X}(\mathcal{L})$. After [6, Thm.3.5], there are canonical isomorphisms of sheaves of $\mathcal{C}^\infty_{M_B(X)}$-modules: $\rho_p\colon R^p\pi_*(\mathcal{L})\otimes\mathcal{C}^\infty_{M_B(X)} \xrightarrow{\sim} \mathcal{H}^{0,p}_{\overline{\partial}_X}(\mathcal{L})$. By Proposition 3.10 of *loc. cit.*, there is a natural holomorphic structure on $\mathcal{H}^{0,p}_{\overline{\partial}_X}(\mathcal{L})$, defined in terms of both the relative and the global Dolbeault complexes of $\mathcal{L}$. For the sake of brevity, we refer to it as the *holomorphic structure of Bismut-Gillet-Soulé*. They prove that their structure coincides with the holomorphic structure on the coherent sheaves $R^p\pi_*(\mathcal{L})$, through the isomorphism ρ_p. Finally, in [6, Lemma 3.8] it is shown that $\pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}$ ($\mathscr{E}^\bullet$ in the notation of the cited paper) is a perfect complex in the category of sheaves of $\mathcal{C}^\infty_{M_B(X)}$-modules. As a result, to compute higher direct images and the determinant of cohomology, we can equivalently work with the complex $\pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}$ and the holomorphic structure of Bismut-Gillet-Soulé.

Associated to the relative connection on $\mathcal{L}$ and the Hermitian metric on T_X, there are non-self-adjoint Laplace operators $\Delta^{0,p} = (\overline{\partial}_X + \overline{\partial}^\sharp_X)^2$ on $\pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}$. Fiberwise, they restrict to the Laplace type operators of Cappell-Miller. We use the notation $\Delta^{0,p}_\chi$ for the restriction to the fiber above χ, and similarly for other operators. Let us explicitly describe them. Let $\widetilde{X}$ be the universal cover of X, with fundamental group $\Gamma = \pi_1(X,p)$ and the complex structure induced from X. The Dolbeault complex of $\mathcal{L}_\chi$ is isomorphic to the Dolbeault complex $A^{0,0}(\widetilde{X},\chi^{-1}) \xrightarrow{\overline{\partial}} A^{0,1}(\widetilde{X},\chi^{-1})$, where $A^{0,p}(\widetilde{X},\chi^{-1})$ indicates the smooth differential χ^{-1}-equivariant forms of type $(0,p)$, and $\overline{\partial}$ is the standard Dolbeault operator on functions on $\widetilde{X}$. In the identification, we are implicitly appealing to the canonical trivialization of $\mathcal{L}_\chi$ at the base point p. The metric on T_X induces a metric on $T_{\widetilde{X}}$ and a formal adjoint $\overline{\partial}^*$, defined as usual in terms of the Hodge $*$ operator. Let $D^{0,p} = (\overline{\partial} + \overline{\partial}^*)^2$. Then, the Dolbeault complex of $\mathcal{L}_\chi$ and $\Delta_\chi^{0,\bullet}$ are identified to $(A^{0,\bullet}(\widetilde{X}, \chi^{-1}), \overline{\partial}, D^{0,\bullet})$. To make the holomorphic dependence on χ explicit, we parametrize $M_B(X)$ by $H^1_{dR}(X, \mathbb{C})$, and further identify cohomology classes with harmonic representatives. In particular, let ν be a harmonic representative for χ, so that $\chi(\gamma) = \exp(-\int_\gamma \nu)$. Define the invertible function: $G_\nu(z) = \exp\left(\int_{\tilde{p}}^z \nu\right)$. We build the isomorphism of complexes

$$\begin{array}{ccc} A^{0,0}(\widetilde{X}, \chi^{-1}) & \xrightarrow{\overline{\partial}} & A^{0,1}(\widetilde{X}, \chi^{-1}) \\ \downarrow {\scriptstyle G_\nu^{-1}\cdot} & & \downarrow {\scriptstyle G_\nu^{-1}\cdot} \\ A^{0,0}(\widetilde{X})^\Gamma & \xrightarrow{\overline{\partial}+\nu''} & A^{0,1}(\widetilde{X})^\Gamma. \end{array}$$

Accordingly, the operators $\bar{\partial}^*$ and $D^{0,p}$ can be transported to the new complex, through conjugation by G_ν. We indicate with an index ν the new conjugated operators, so that for instance $\bar{\partial}_\nu = \bar{\partial} + \nu''$, and similarly for $\bar{\partial}_\nu^*$ and $D_\nu^{0,p}$. After all these identifications, we see that $\overline{\partial}_\chi^\sharp$ will correspond to $\bar{\partial}_\nu^*$ and $\Delta_\chi^{0,p}$ will correspond to $D_\nu^{0,p}$.

Lemma 5.1.

(i) The operators $D_\nu^{0,p}$ form a holomorphic family of type (A) in the sense of Kato [23, Chap. VII, Sec. 2]*: a) they all share the same domain $A^{0,p}(X)$ and are closed with respect to the L^2 structure induced by the choice of Hermitian metric on T_X and b) they depend holomorphically in ν.*

(ii) The operators $D_\nu^{0,p}$ have compact resolvent, and spectrum bounded below and contained in a "horizontal" parabola.

Proof. For the first item, note that the $D_\nu^{0,p}$ are second order differential operators with the same principal symbol as $D^{0,p}$. Hence, they are elliptic, since the latter is. This also implies that the $D_\nu^{0,p}$ are closed as unbounded operators acting on $A^{0,p}(X)$ and with respect to the L^2 structure. We have thus checked the first condition in Kato's definition. For the holomorphicity, introduce a basis of holomorphic differentials $\{\omega_i\}$ of X and write: $\nu = \sum_i (s_i\omega_i + t_i\overline{\omega}_i)$. The holomorphic dependence on ν amounts to the holomorphic dependence on the parameter s_i, t_j, which is obvious from the construction of $D_\nu^{(0,p)}$ by conjugation by G_ν: given $\theta \in A^{0,p}(X)$, the differential form $D_\nu^{(0,p)}\theta$ is holomorphic in the parameters s_i, t_i. This establishes the second condition, and so the first claim.

For the compact resolvent property, we appeal to [23, Thm. 2.4]: for holomorphic families of type (A) in a parameter χ on a domain, compactness of the resolvent for all χ follows from the compactness of the resolvent at a given χ_0. Therefore, the compactness asserted by the lemma is automatic from the compactness in the unitary and self-adjoint case (for instance when $\nu = 0$), which is well-known.

The spectrum assertion is an observation of Cappell-Miller [11, Lemma 4.1]. □

Let $\chi_0 \in M_B(X)$, and choose $b > 0$ be such that no generalized eigenvalue of $\Delta^{0,p}_{\chi_0}$ has real part b. By Lemma 5.1 and [23, Chap. VII, Thm. 1.7], there exists a neighborhood U_{χ_0} of χ_0 such that the same property still holds for $\Delta^{0,p}_{\chi}$, if $\chi \in U_{\chi_0}$. Hence, the set U_b of those $\chi \in M_B(X)$ such that b is not the real part of any generalized eigenvalue of $\Delta^{0,p}_{\chi}$, forms an open set. Because $b > 0$, it is easy to see that this open set does not depend on whether we work with $\Delta^{0,0}_{\chi}$ or $\Delta^{0,1}_{\chi}$: it is the same for both. Such open subsets U_b form an open cover of $M_B(X)$. We define $\mathscr{V}^{0,p}_{b,\chi} \subset A^{0,p}(\mathcal{L}_\chi)$ the subspace spanned by generalized eigenfunctions of $\Delta^{0,p}_{\chi}$, of generalized eigenvalue λ with $\mathrm{Re}(\lambda) < b$. If $c > b > 0$ are not the real parts of the eigenvalues at some χ_0, we can similarly introduce $\mathscr{V}^{0,p}_{(b,c),\chi}$ on $U_b \cap U_c$, by consideration of generalized eigenfunctions with eigenvalues whose real part is in the open interval (b, c).

Proposition 5.2. *For $\chi \in U_b$ (resp. $U_b \cap U_c$), the vector spaces $\mathscr{V}^{0,p}_{b,\chi}$ (resp. $\mathscr{V}^{0,p}_{(b,c),\chi}$) define a holomorphic vector bundle on U_b (resp. $U_b \cap U_c$) with locally finite rank.*

Proof. In view of Lemma 5.1, this is a reformulation of [23, Chap. VII, Thm. 1.7]. □

Denote by $\mathscr{V}^{0,p}_{b} = \mathscr{V}^{0,p}_{b}(\mathcal{L}) \subset \pi_*\mathcal{A}^{0,p}_{\mathcal{X}/M_B(X)}(\mathcal{L})\big|_{U_b}$, the holomorphic bundle on U_b thus defined. The differential on the Dolbeault complex $\pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}$ induces a differential on $\mathscr{V}^{0,p}_{b}$, and $\bar\partial_X(\mathscr{V}^{0,0}_{b}) \subset \mathscr{V}^{0,1}_{b}$. Indeed, the relative $\bar\partial$ operator of $\mathcal{L}$ commutes with the operators $\Delta^{0,p}_{\chi}$. We introduce similar notation for eigenspaces with real parts in (b, c).

Proposition 5.3. *(i) The inclusion of complexes*

$$(\mathscr{V}^{0,\bullet}_{b} \otimes \mathcal{C}^\infty_{U_b}, \bar\partial_X) \hookrightarrow \pi_*\mathcal{D}_{\mathcal{X}/M_B(X)}\big|_{U_b} \tag{33}$$

is a quasi-isomorphism. Therefore, the complex $\mathscr{V}^{0,\bullet}_{b} \otimes \mathcal{C}^\infty_{U_b}$ computes $\mathcal{H}^{0,p}_{\bar\partial_X}(\mathcal{L})$ restricted to U_b.

(ii) The cohomology sheaves of $\mathscr{V}^{0,\bullet}_{b}$ have natural structures of coherent sheaves on U_b, compatible with the holomorphic structures of Bismut-Gillet-Soulé on $\mathcal{H}^{0,p}_{\bar\partial_X}(\mathcal{L})$. Therefore, the complex $\mathscr{V}^{0,\bullet}_{b}$ computes $R\pi_*(\mathcal{L})$ restricted to U_b.

(iii) The complex $\mathscr{V}^{0,\bullet}_{(b,c)}$ is acyclic.

Proof. First, by [6, Lemma 3.8] we know that the relative Dolbeault complex is perfect as a complex of $\mathcal{C}^\infty_{M_B(X)}$-modules, and its cohomology is bounded and finitely generated. Second, Cappell-Miller show that (33) is fiberwise a quasi-isomorphism [11, top of p. 151]. Finally, the $\mathscr{V}^{0,\bullet}_{b} \otimes \mathcal{C}^\infty_{U_b}$ are vector bundles, hence projective objects in the category of sheaves of $\mathcal{C}^\infty_{U_b}$-modules. The three assertions together are enough to conclude the first assertion.

That the cohomology of $\mathscr{V}^{0,\bullet}_{b}$ is formed by coherent sheaves is immediate, being the cohomology sheaves of a complex of finite rank holomorphic vector bundles. For the compatibility of holomorphic structures, taking into account the construction of Bismut-Gillet-Soulé, it is enough to observe the following. Assume θ is a local holomorphic section of $\mathscr{V}^{0,p}_{b}$. Hence, it depends holomorphically on χ and $\bar\partial_X\theta = 0$. Because $\mathcal{X} = X \times M_B(X)$ is a product, we can assume that θ is a global $(0, p)$ form, with $\bar\partial_X\theta = 0$ and depending holomorphically on χ. By the very construction of the universal bundle $\mathcal{L}$, this is tantamount to saying $\bar{\partial}_{\mathcal{L}}\theta = 0$. Here $\bar{\partial}_{\mathcal{L}}$ is the Dolbeault operator of $\mathcal{L}$ on $\mathcal{X}$. But now $\bar{\partial}_{\mathcal{L}}\theta = 0$ is exactly the condition defining the holomorphic structure of Bismut-Gillet-Soulé [6, p. 346] in our case. The last assertion is left as an easy exercise. □

Let us illustrate the proposition with a diagram:

$$\begin{tikzcd} & \mathcal{H}^p(\mathcal{V}_b^{0,\bullet}, \bar{\partial}_X) \otimes \mathcal{C}^\infty_{U_b} \arrow[dl, dashed, "\beta_{p,b}"'] \arrow[d, "\alpha_{p,b}", "\wr"'] \\ R^p\pi_*(\mathcal{L}) \otimes \mathcal{C}^\infty_{U_b} \arrow[r, "\sim", "\rho_p"'] & \mathcal{H}^{0,p}_{\bar{\partial}_X}(\mathcal{L})\big|_{U_b}. \end{tikzcd} \tag{34}$$

The complex structures on $\mathcal{H}^{0,p}_{\bar{\partial}_X}(\mathcal{L})\big|_{U_b}$ induced by ρ_p and $\alpha_{p,b}$ are compatible by Proposition 5.3, and hence $\beta_{p,b}$ is induced by an isomorphism of coherent sheaves. There are corresponding arrows between determinants of cohomologies, which we indicate ρ, α_b and β_b. In particular, the isomorphism β_b defines an isomorphism of holomorphic line bundles still denoted $\beta_b\colon \det(\mathcal{V}_b^{0,\bullet}) \xrightarrow{\sim} \det R\pi_*(\mathcal{L})\big|_{U_b}$. Here, we used the canonical isomorphism between the determinant of cohomology of $\mathcal{V}_b^{0,\bullet}$ and the determinant of its cohomology. A parallel digression applies to $\mathcal{L}^c$, and we use the index c for the corresponding objects. There is also a variant that applies to $\mathcal{L} \otimes \omega_X$ and $\mathcal{L}^c \otimes \omega_{\overline{X}}$, where we incorporate the Chern connections on ω_X and $\omega_{\overline{X}}$, with respect to the fixed Hermitian metric. We leave the details to the reader. We introduce the notation $\mathcal{V}_b^{0,p}(\mathcal{L} \otimes \omega_X)$, etc. when confusion can arise. We now have a fundamental duality phenomenon.

Proposition 5.4. *The operator $\bar{\partial}_X^\sharp$ induces a homological complex of holomorphic vector bundles on U_b: $\mathcal{V}_b^{0,1}(\mathcal{L}) \xrightarrow{\bar{\partial}_X^\sharp} \mathcal{V}_b^{0,0}(\mathcal{L})$. This complex is $\mathcal{O}_{U_b}$-isomorphic* (i.e. *holomorphically) to the cohomological complex: $\mathcal{V}_b^{0,0}((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}}) \xrightarrow{\bar{\partial}_{\overline{X}}^\sharp} \mathcal{V}_b^{0,1}((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}})$. Therefore, there is a canonical isomorphism of holomorphic line bundles* $\det(\mathcal{V}_b^{0,\bullet}) \xrightarrow{\beta_b^c} \det R\pi_*^c((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}})^\vee$.

Proof. The first assertion follows because $\bar{\partial}_X^\sharp$ commutes with $\Delta^{0,p}$. The equality: $\mathcal{V}_b^{0,1} = \mathcal{V}_b^{0,0}((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}})$, as holomorphic vector bundles is easily seen. Notice the natural appearance of $(\mathcal{L}^c)^\vee$, which has same holonomy characters as $\mathcal{L}$, but the opposite holomorphic structure fiberwise. Observe that the base point and the trivialization of the universal bundles at p is implicit in the identification. Moreover, there is an isomorphism of holomorphic vector bundles given by the Hodge star operator followed by conjugation, that following [11] we write: $\hat{\star}\colon \mathcal{V}_b^{0,0}(\mathcal{L}) \xrightarrow{\sim} \mathcal{V}_b^{1,0}((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}})$. Observe that $\hat{\star}$ is complex linear, and this is necessary if we want to preserve holomorphy. The compatibilities with the differentials are readily checked from the definitions. This concludes the first assertion. For the second, we just need to stress that the determinant of $(\mathcal{V}_b^{0,\bullet}, \bar{\partial})$ (cohomological complex) is dual to the determinant of $(\mathcal{V}_b^{0,\bullet}, \bar{\partial}_X^\sharp)$ (homological complex). □

Corollary 5.5. *(i) There is a commutative diagram of isomorphisms of holomorphic line bundles on U_b*

$$\begin{tikzcd} \det(\mathscr{V}_b^{0,\bullet}) \arrow[r, "id"] \arrow[d, "\beta_b"'] & \det(\mathscr{V}_b^{0,\bullet}) \arrow[d, "\beta_b^c"] \\ \det R\pi_*(\mathcal{L}) \arrow[r, "\sim"] & \det R\pi_*^c((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}})^\vee . \end{tikzcd}$$

By Serre duality it induces a holomorphic trivialization $\tau(b)$ of $\det R\pi_*(\mathcal{L}) \otimes \det R\pi_*^c(\mathcal{L}^c)$ on U_b.

(ii) Let $c > b > 0$. On $U_b \cap U_c$, the relation between $\tau(b)$ and $\tau(c)$ is given by

$$\tau(b) = \tau(c) \prod_{j=1}^{m} \det \Delta_{(b,c)}^{0,1} ,$$

where $\Delta_{(b,c)}^{0,1}$ is the endomorphism of the holomorphic vector bundle $\mathscr{V}_{(b,c)}^{0,1}$ defined by the Laplacians $\Delta_\chi^{0,1}$, $\chi \in U_b \cap U_c$.

Proof. The first item is a reformulation of the proposition, together with the canonical Serre duality $\det R\pi_*^c((\mathcal{L}^c)^\vee \otimes \omega_{\overline{X}}) \simeq \det R\pi_*^c(\mathcal{L}^c)$. For the second item, it is enough to check this equality pointwise and use that the determinant of a holomorphic bundle endomorphism is a holomorphic function. The pointwise relation follows from [11, Eq. (3.6)]. □

Remark 5.6. The holomorphic function $\det \Delta_{(b,c)}^{0,1}$ may be viewed as a trivialization of the holomorphic line bundle $\det \mathcal{H}^\bullet(\mathscr{V}_{(b,c)}^{0,\bullet})$.

For a given $\chi \in U_b$ and $b > 0$, let us denote P_b the spectral projector on generalized eigenfunctions of $\Delta_\chi^{0,1}$ of eigenvalues with real part $< b$. We put $Q_b = 1 - P_b$, and define the spectral zeta function of $Q_b\Delta_\chi^{0,1}$, as usual to be the Mellin transform of the heat operator $e^{-tQ_b\Delta_\chi^{0,1}}$. This depends on the auxiliary choice of an Agmon angle. Let this function be $\zeta_{b,\chi}(s)$. It is a meromorphic function on $\mathbb{C}$, regular at $s = 0$. The bases for these definitions and claims are due to Cappell-Miller, and rely on Seeley's methods [30]. Furthermore, the special value $\exp(\zeta_{b,\chi}'(0))$ does not depend on the choice of Agmon angle.

The following is a consequence of standard arguments and the asymptotic expansions of Seeley [30] and Greiner [21, Sec. 1].

Lemma 5.7. *The expression $\exp(\zeta_{b,\chi}'(0))$ defines a holomorphic function in $\chi \in U_b$.*

Next, we have

Proposition 5.8. *Let $c > b > 0$. We have an equality of holomorphic sections on $U_b \cap U_c$*

$$\tau(b) \exp(-\zeta_b'(0)) = \tau(c) \exp(-\zeta_c'(0)).$$

Proof. The proof is direct after Corollary 5.5 and the very definition of the spectral zeta functions. □

Hence, such expressions can be glued into a single holomorphic trivialization τ of $\det R\pi_*(\mathcal{L}) \otimes \det R\pi_*^c(\mathcal{L}^c)$ on $M_B(X)$. By construction, at a given χ, the section $\tau(\chi)$ coincides with the construction of Cappell-Miller. Hence, for χ unitary, $\tau(\chi)$ is the trivialization provided by the Quillen metric. This motivates the following terminology.

Definition 5.9. (i) The trivialization τ of $\lambda(\mathcal{L}) \otimes \lambda(\mathcal{L}^c)$ defined by Proposition 5.8 is called the holomorphic Cappell-Miller torsion.
(ii) The logarithm of $\lambda(\mathcal{L}) \otimes \lambda(\mathcal{L}^c)$ attached to the holomorphic Cappell-Miller torsion is called the Quillen-Cappell-Miller logarithm and is denoted LOG_Q.

5.2. **Deligne's isomorphism and compatibility with logarithms.** We now come to the proof of Theorem 1.1. The Deligne isomorphism (see Section 2) induces an isomorphism of holomorphic line bundles on $M_B(X)$:

$$\mathcal{D}\colon \left\{\lambda(\mathcal{L} - \mathcal{O}_X) \otimes \lambda(\mathcal{L}^c - \mathcal{O}_{\overline{X}})\right\}^{\otimes 2} \xrightarrow{\sim} \langle \mathcal{L}, \mathcal{L} \otimes \omega_X^{-1}\rangle \otimes \langle \mathcal{L}^c, \mathcal{L}^c \otimes \omega_{\overline{X}}^{-1}\rangle.$$

The left hand side is endowed with a combination of Quillen-Cappell-Miller logarithms, while the right hand side is endowed with a combination of intersection logarithms, which we continue to denote by LOG_Q and LOG_{int}, respectively.

Theorem 5.10 (*cf.* Theorem 1.1)**.** *On $M_B(X)$, $\mathrm{LOG}_Q = \mathrm{LOG}_{int} \circ \mathcal{D} \bmod \pi i\mathbb{Z}$.*

Proof. Both logarithms LOG_Q and $\mathrm{LOG}_{int} \circ \mathcal{D}$ are holomorphic on $M_B(X)$. Moreover, they coincide over the totally real subvariety of unitary characters, mod $\pi i\mathbb{Z}$. Indeed, along unitary characters LOG_Q is the logarithm of the Quillen metric by construction and LOG_{int} is the logarithm of the metric on the Deligne pairing (see §4.4.2). The Deligne isomorphism is an isometry for these metrics and is unique up to sign (hence, $\pi i\mathbb{Z}$ instead of $2\pi i\mathbb{Z}$). The result follows. □

Remark 5.11. (i) The logarithms in the theorem depend on the rigidifications of $\mathcal{L}$ and $\mathcal{L}^c$. For instance, for LOG_{int} the dependence is due to the pairings against ω_X and $\omega_{\overline{X}}$, which are in general non-zero degree line bundles.
(ii) Deligne's isomorphism is compatible with base change, and logarithms can be specialized at a given point. A pointwise version of the theorem follows. This is the actual form of the statement presented in the introduction.
(iii) The intersection logarithm in the theorem does not depend on the particular choice of metric on T_X (see Corollary 4.8 and §4.4.2). Therefore, the dependences on the Kähler metric of the Quillen-Cappell-Miller logarithms on $\lambda(\mathcal{L}) \otimes \lambda(\mathcal{L}^c)$ and $\lambda(\mathcal{O}_X) \otimes \lambda(\mathcal{O}_{\overline{X}})$ cancel out. From this, an anomaly formula for the change of Cappell-Miller torsion under a conformal change of metric can be derived. Such a formula already appears in the work of Cappell-Miller and [26, Thm 2.5].
(iv) The Cappell-Miller construction gives a holomorphic trivialization of

$$\lambda(\mathcal{L} \otimes \omega_X) \otimes \lambda(\mathcal{L}^c \otimes \omega_{\overline{X}})$$

which corresponds to τ via Serre duality. Indeed, this follows from compatibility of the Quillen metric with Serre duality and an argument similar to the one in the proof of Theorem 5.10 above.

Theorem 5.10 has consequences for general families, as we explain next. We place ourselves in the setting of §4.2. Let $(\pi\colon \mathcal{X} \to S, \sigma)$ be a smooth family of curves with section, and $(\overline{\pi}\colon \overline{\mathcal{X}} \to \overline{S}, \overline{\sigma})$ the conjugate family. Assume we are given a conjugate pair of relatively flat line bundles $(\mathcal{L}, \nabla_{\mathcal{X}/S})$ and $(\mathcal{L}^c, \nabla^c_{\mathcal{X}/S})$. Fix a smooth metric on $\omega_{\mathcal{X}/S}$. Recall, from Proposition 4.3, that LOG_{int} defines a smooth logarithm on the combination of Deligne pairings: $\langle \mathcal{L}, \mathcal{L} \otimes \omega^{-1}_{\mathcal{X}/S}\rangle \otimes_{\mathcal{C}^\infty_S} \langle \mathcal{L}^c, \mathcal{L}^c \otimes \omega^{-1}_{\overline{\mathcal{X}}/\overline{S}}\rangle$. The Cappell-Miller construction now provides an *a priori pointwise* trivialization of the product of determinants of cohomology $\lambda(\mathcal{L} - \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{C}^\infty_S} \lambda(\mathcal{L}^c - \mathcal{O}_{\overline{\mathcal{X}}})$.

Corollary 5.12. *For a conjugate pair of line bundles as above, the pointwise defined Cappell-Miller torsion is actually a $\mathcal{C}^\infty$ trivialization of $\lambda(\mathcal{L} - \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{C}^\infty_S} \lambda(\mathcal{L}^c - \mathcal{O}_{\overline{\mathcal{X}}})$. The associated $\mathcal{C}^\infty$ logarithm corresponds to* LOG_{int} *through the tensor product of Deligne's isomorphisms in the $\mathcal{C}^\infty$ category, modulo $\pi i\mathbb{Z}$.*

Proof. The Deligne isomorphism commutes with base change. Specializing to a point of S and applying Theorem 5.10 (see Remark 5.11 (iii)), we derive a pointwise correspondence between the Cappell-Miller trivialization on $\lambda(\mathcal{L} - \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{C}^\infty_S} \lambda(\mathcal{L}^c - \mathcal{O}_{\overline{\mathcal{X}}})$ and the intersection logarithm LOG_{int} on $\langle \mathcal{L}, \mathcal{L} \otimes \omega^{-1}_{\mathcal{X}/S}\rangle \otimes_{\mathcal{C}^\infty_S} \langle \mathcal{L}^c, \mathcal{L}^c \otimes \omega^{-1}_{\overline{\mathcal{X}}/\overline{S}}\rangle$, mod $\pi i\mathbb{Z}$. Since LOG_{int} is already known to be $\mathcal{C}^\infty$, and Deligne's isomorphism induces an isomorphism in the $\mathcal{C}^\infty$ category, we deduce that the Cappell-Miller trivialization has to be $\mathcal{C}^\infty$. The proof is complete. □

Remark 5.13. As we anticipated in Remark 4.5, we expect that the previous statement will find applications in the context of quasi-fuchsian groups and their character varieties. After Bers, quasi-fuchsian groups allow for simultaneous uniformization of arbitrary pairs of compact Riemann surfaces (X, Y) of same genus. Fuchsian groups are particular cases, and uniformize conjugate pairs of Riemann surfaces $(X, \overline{X})$. There is a totally real embedding from the deformation space of a fuchsian group to its deformation space seen as a quasi-fuchsian group. It corresponds to the totally real embedding of moduli spaces of 1-pointed curves $\mathcal{M}_{g,1} \to \mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$, that sends a point defined by a pointed compact Riemann surface (X, p) to the point defined by $((X, p), (\overline{X}, \overline{p}))$. There is an analogous picture for universal curves and relative Betti moduli spaces, fibered over $\mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$. The setting in the corollary should arise by restriction of Deligne type isomorphisms defined over $\mathcal{M}_{g,1} \times \mathcal{M}_{g,1}$ (or rather the Betti space fibered over it) along the totally real embedding alluded to above. We plan to further explore this picture in future research, in connection with holomorphic extensions of determinants of Laplacians [24].

6. Arithmetic Intersection Theory for Flat Line Bundles

6.1. Conjugate pairs of line bundles with logarithms on $\operatorname{Spec} \mathcal{O}_K$.

Let K be a number field with ring of integers $\mathcal{O}_K$. We write $S = \operatorname{Spec} \mathcal{O}_K$. An invertible sheaf (or line bundle) $\mathcal{L}$ over S can be equivalently seen as a projective $\mathcal{O}_K$ module of rank 1. For simplicity we do make a distinction in the notation. This particularly concerns base change and tensor product.

Definition 6.1. A *conjugate pair of line bundles with logarithms*, or simply a conjugate pair, on S consists in the following data:

(i) a pair of line bundles $\mathcal{L}$ and $\mathcal{L}^c$ over S;
(ii) for every embedding $\tau\colon K\hookrightarrow\mathbb{C}$, a logarithm LOG_τ on the one dimensional complex vector space $\mathcal{L}_\tau\otimes_{\mathbb{C}}\mathcal{L}^c_{\overline{\tau}}$.

We introduce the notation $\mathcal{L}^\sharp$ for the data $(\mathcal{L},\mathcal{L}^c,\{\mathrm{LOG}_\tau\}_{\tau\colon K\hookrightarrow\mathbb{C}})$.

Given conjugate pairs $\mathcal{L}^\sharp$ and $\mathcal{M}^\sharp$, an isomorphism $\varphi^\sharp:\mathcal{L}^\sharp\to\mathcal{M}^\sharp$ is a pair (φ,φ^c) of isomorphisms, $\varphi\colon\mathcal{L}\to\mathcal{M}$ and $\varphi^c\colon\mathcal{L}^c\to\mathcal{M}^c$, such that for every $\tau\colon K\hookrightarrow\mathbb{C}$, $\varphi_\tau\otimes\varphi^c_{\overline{\tau}}$ preserves logarithms. There are standard constructions on conjugate pairs with logarithms, notably tensor product and duality.

Definition 6.2. The groupoid of conjugate pairs of line bundles with logarithms, denoted $\mathrm{PIC}^\sharp(S)$, is defined by:
- *objects*: conjugate pairs of line bundles with logarithms;
- *morphisms*: isomorphisms of pairs of line bundles with logarithms.

It has the structure of a Picard category. The group of isomorphisms classes of objects is denoted by $\mathrm{Pic}^\sharp(S)$ and is called the *arithmetic Picard group of conjugate pairs of line bundles with logarithms*.

Arithmetic degree. We proceed to construct an *arithmetic degree map* on $\mathrm{Pic}^\sharp(S)$: $\deg^\sharp:\mathrm{Pic}^\sharp(S)\longrightarrow\mathbb{C}/\pi i\,\mathbb{Z}$. We emphasize that the target group is not $\mathbb{C}/2\pi i\,\mathbb{Z}$, but $\mathbb{C}/\pi i\,\mathbb{Z}$. Let $\mathcal{L}^\sharp$ be a conjugate pair. Given nonvanishing elements $\ell\in\mathcal{L}_K$, $\ell^c\in\mathcal{L}^c_K$, the quantity

$$\sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(\ell\otimes\ell^c)\log(N\mathfrak{p})-\sum_{\tau:K\hookrightarrow\mathbb{C}}\mathrm{LOG}_\tau(\ell_\tau\otimes\ell^c_{\overline{\tau}})$$

taken in $\mathbb{C}/\pi i\,\mathbb{Z}$ does not depend on the choices ℓ, ℓ^c. Indeed, for $\lambda,\mu\in K^\times$, the following relations hold in $\mathbb{C}/\pi i\,\mathbb{Z}$:

$$\sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(\lambda\mu)\log(N\mathfrak{p})-\sum_{\tau:K\hookrightarrow\mathbb{C}}\log(\tau(\lambda)\overline{\tau}(\mu))=$$
$$-\log\left(\prod_{\mathfrak{p}}|\lambda|_{\mathfrak{p}}\prod_{\tau:K\hookrightarrow\mathbb{C}}\tau(\lambda)\right)-\log\left(\prod_{\mathfrak{p}}|\mu|_{\mathfrak{p}}\prod_{\tau:K\hookrightarrow\mathbb{C}}\overline{\tau}(\mu)\right)=-\log(\pm1)-\log(\pm1)=0\,.$$

We then conclude by the very definition of logarithm: modulo $2\pi i\,\mathbb{Z}$, and hence modulo $\pi i\,\mathbb{Z}$, LOG_τ satisfies

$$\mathrm{LOG}_\tau((\lambda\ell)_\tau\otimes(\mu\ell^c)_{\overline{\tau}})=\mathrm{LOG}_\tau(\tau(\lambda)\overline{\tau}(\mu)\ell_\tau\otimes\ell^c_{\overline{\tau}})=\log(\tau(\lambda)\overline{\tau}(\mu))+\mathrm{LOG}_\tau(\ell_\tau\otimes\ell^c_{\overline{\tau}}).$$

Remark 6.3. (i) When the field K cannot be embedded into $\mathbb{R}$, the arithmetic degree is well-defined in $\mathbb{C}/2\pi i\,\mathbb{Z}$, and the argument in $\mathbb{R}/2\pi\mathbb{Z}$.

(ii) In general, to obtain an arithmetic degree with values in $\mathbb{C}/2\pi i\,\mathbb{Z}$, one needs to add to conjugate pairs a positivity condition at real places (or equivalently, an orientation). However, our main goal is to prove an arithmetic Riemann-Roch formula, which relies on the Deligne isomorphism through Theorem 1.1. This introduces a $\log(\pm1)$ ambiguity and is why we do not impose any positivity conditions in this paper.

Example 6.4. Because a $\mathbb{Z}$ module of rank 1 admits a basis that is unique up to sign, one easily proves that the arithmetic degree on $\mathrm{Pic}^\sharp(\operatorname{Spec}\mathbb{Z})$ is an isomorphism:

$$\deg^\sharp\colon \mathrm{Pic}^\sharp(\operatorname{Spec}\mathbb{Z}) \xrightarrow{\sim} \mathbb{C}/\pi i\,\mathbb{Z}.$$

We will need the following functorialities for the Picard groups and the arithmetic degree, whose proof is elementary.

Proposition 6.5. *Let F be a finite extension of K and set $\mathcal{T} = \operatorname{Spec}\mathcal{O}_F$. With respect to the morphism $\pi\colon \mathcal{T} \to S$, the arithmetic Picard groups satisfy covariant and contravariant functorialities:*

(i) (Inverse images or pull-backs) Tensor product with $\mathcal{O}_F$ induces a morphism: $\pi^*\colon \mathrm{Pic}^\sharp(S) \longrightarrow \mathrm{Pic}^\sharp(\mathcal{T})$.

(ii) (Direct images or push-forwards) The norm down to $\mathcal{O}_K$ of a projective $\mathcal{O}_F$-module induces a morphism: $\pi_*\colon \mathrm{Pic}^\sharp(S) \longrightarrow \mathrm{Pic}^\sharp(\mathcal{T})$. The arithmetic degree on $\mathrm{Pic}^\sharp(\mathcal{O}_K)$ factors through the push-forward to $\mathrm{Pic}^\sharp(\mathbb{Z})$.

(iii) The composition $\pi_*\pi^*$ acts as multiplication by $[F:K]$.

6.2. **Conjugate pairs of line bundles with connections.** For the rest of this section, we fix a square root of -1, $i = \sqrt{-1} \in \mathbb{C}$. Let $\mathcal{X} \to S$ be an arithmetic surface. By this we mean a regular, irreducible and flat projective scheme over S, with geometrically connected generic fiber $\mathcal{X}_K$ of dimension 1. We fix some conventions on complex structures.

Conventions on complex structures.

(i) Given an embedding $\tau : K \hookrightarrow \mathbb{C}$, we write $\mathcal{X}_\tau$ for the base change of $\mathcal{X}$ to $\mathbb{C}$ through τ. After the choice we made of $\sqrt{-1}$, the set of complex points $\mathcal{X}_\tau(\mathbb{C})$ has a complex structure and is thus a Riemann surface. We call this complex structure the *natural* one. The other complex structure (corresponding to $-i$) is called the *reverse*, *opposite* or *conjugate* one, and as usual we indicate this with a bar: $\overline{\mathcal{X}_\tau(\mathbb{C})}$. With these notations, if τ is a complex, nonreal, embedding, then $\mathcal{X}_{\overline{\tau}}(\mathbb{C})$ is canonically biholomorphic to $\overline{\mathcal{X}_\tau(\mathbb{C})}$.

(ii) If τ is a real embedding, we put $\mathcal{X}_{\overline{\tau}}(\mathbb{C}) = \overline{\mathcal{X}_\tau(\mathbb{C})}$ (although $\tau = \overline{\tau}$!). For the natural complex structure on $\mathcal{X}_{\overline{\tau}}(\mathbb{C})$ we then mean the reverse structure on $\mathcal{X}_\tau(\mathbb{C})$.

(iii) The same conventions will apply to holomorphic line bundles, and sections of such, over $\mathcal{X}$. For instance, if $\mathcal{L}$ is a line bundle over $\mathcal{X}$ and τ is a complex, nonreal, embedding, the holomorphic line bundles $\mathcal{L}_{\overline{\tau}}$ on $\mathcal{X}_{\overline{\tau}}(\mathbb{C})$ and $\overline{\mathcal{L}}_\tau$ on $\overline{\mathcal{X}_\tau(\mathbb{C})}$ can be identified, after the identification of $\mathcal{X}_{\overline{\tau}}(\mathbb{C})$ with $\overline{\mathcal{X}_\tau(\mathbb{C})}$. If τ is real, then the convention is that $\mathcal{L}_{\overline{\tau}} = \overline{\mathcal{L}}_\tau$ on $\mathcal{X}_{\overline{\tau}}(\mathbb{C}) = \overline{\mathcal{X}_\tau(\mathbb{C})}$.

Definition 6.6. A *conjugate pair of line bundles with connections* on $\mathcal{X}$ consists in the following data:

(i) two line bundles $\mathcal{L}, \mathcal{L}^c$ on $\mathcal{X}$;

(ii) holomorphic connections ∇_τ on the holomorphic line bundles $\mathcal{L}_\tau$, with respect to the natural complex structure on $\mathcal{X}_\tau(\mathbb{C})$;

(iii) holomorphic connections $\nabla^c_{\overline{\tau}}$ on the holomorphic line bundles $\mathcal{L}_{\overline{\tau}}$, with respect to the natural complex structure on $\mathcal{X}_{\overline{\tau}}(\mathbb{C})$. Observe that by the previous conventions, if τ

is a real embedding, then $\nabla^{c}_{\overline{\tau}}$ is a holomorphic connection on the holomorphic line bundle $\overline{\mathcal{L}}^{c}_{\tau}$ on $\overline{\mathcal{X}_{\tau}(\mathbb{C})}$.

(iv) we impose the following relation: if χ_{τ} is the holonomy character of $\pi_1(\mathcal{X}_{\tau}(\mathbb{C}),*)$ associated to $(\mathcal{L}_{\tau},\nabla_{\tau})$, and $\chi^{c}_{\overline{\tau}}$ is the character associated to $(\mathcal{L}^{c}_{\overline{\tau}},\nabla^{c}_{\overline{\tau}})$, then $\chi^{c}_{\overline{\tau}}=\chi_{\tau}^{-1}$.

We introduce the notation $\mathcal{L}^{\sharp}=((\mathcal{L},\nabla),(\mathcal{L}^{c},\nabla^{c}))$, with $\nabla=\{\nabla_{\tau}\}_{\tau}$, $\nabla^{c}=\{\nabla^{c}_{\tau}\}_{\tau}$.

Remark 6.7. In the definition we do not impose any relationship between χ_{τ} and $\chi_{\overline{\tau}}$, in contrast to classical Arakelov geometry. Moreover, we require $\chi^{c}_{\overline{\tau}}=\chi_{\tau}^{-1}$, and not $\chi^{c}_{\overline{\tau}}=\overline{\chi}_{\tau}$. The conditions coincide only in the unitary case, which is the context of classical Arakelov geometry.

There is an obvious notion of isomorphism of conjugate pairs of line bundles with connections. There are also standard operations that can be performed, such as tensor products and duals. Base change is possible as well, for instance by unramified extensions of K (in order to preserve the regularity assumption for arithmetic surfaces).

Definition 6.8. We denote by $\mathrm{PIC}^{\sharp}(\mathcal{X})$ the groupoid of conjugate pairs of line bundles with connections. It is a Picard category. The group of isomorphism classes is denoted $\mathrm{Pic}^{\sharp}(\mathcal{X})$ and is called the *Picard group of conjugate pairs of line bundles with connections*.

Let us now suppose there is a section $\sigma: S\to\mathcal{X}$. A rigidification along σ of a conjugate pair of line bundles with connections $\mathcal{L}^{\sharp}$, is a choice of isomorphisms $\sigma^{*}\mathcal{L}\xrightarrow{\sim}\mathcal{O}_S$ and $\sigma^{*}\mathcal{L}^{c}\xrightarrow{\sim}\mathcal{O}_S$. The previous definitions have obvious counterparts in this setting.

Definition 6.9. Given a section $\sigma: S\to\mathcal{X}$, we denote by $\mathrm{PICRIG}^{\sharp}(\mathcal{X},\sigma)$ the groupoid of conjugate pairs of line bundles with connections, rigidified along σ.

Remark 6.10. (i) Observe that a rigidification of $\mathcal{L}^{\sharp}$ induces rigidifications of $\mathcal{L}_{\tau}$ at σ_{τ} and $\mathcal{L}^{c}_{\overline{\tau}}$ at $\sigma_{\overline{\tau}}$, for $\tau\colon K\hookrightarrow\mathbb{C}$.

(ii) A rigidification is unique up to $\mathcal{O}_K^{\times}$. Because the norm down to $\mathbb{Q}$ of a unit is ±1, the arithmetic degree is not sensitive to the particular choice of rigidification.

(iii) The Hilbert class field H of K is the maximal unramified abelian extension of K. It has the property that any invertible $\mathcal{O}_K$-module becomes trivial after base change to $\mathcal{O}_H$. Therefore, after possibly extending the base field to H, a rigidification always exists.

Arithmetic intersection product. The Deligne pairing and the intersection logarithm constructions define a symmetric bilinear pairing: $\mathrm{PIC}^{\sharp}(\mathcal{X})\times\mathrm{PIC}^{\sharp}(\mathcal{X})\longrightarrow\mathrm{PIC}^{\sharp}(S)$. The construction works as follows. Let $\mathcal{L}^{\sharp}$ and $\mathcal{M}^{\sharp}$ be conjugate pairs of line bundles with connections. We consider the Deligne pairings $\langle\mathcal{L},\mathcal{M}\rangle$, $\langle\mathcal{L}^{c},\mathcal{M}^{c}\rangle$. For every complex embedding $\tau\colon K\hookrightarrow\mathbb{C}$, $\langle\mathcal{L},\mathcal{M}\rangle_{\tau}\otimes_{\mathbb{C}}\langle\mathcal{L}^{c},\mathcal{M}^{c}\rangle_{\overline{\tau}}=\langle\mathcal{L}_{\tau},\mathcal{M}_{\tau}\rangle\otimes_{\mathbb{C}}\langle\mathcal{L}^{c}_{\overline{\tau}},\mathcal{M}^{c}_{\overline{\tau}}\rangle$, carries an intersection logarithm $\mathrm{LOG}_{int,\tau}$, build up from the connections defining $\mathcal{L}^{\sharp}$, $\mathcal{M}^{\sharp}$ and intermediate choices of rigidifications (we proved the construction is independent of these choices). In this way we obtain a conjugate pair of line bundle with logarithms on S, which we denote $\langle\mathcal{L}^{\sharp},\mathcal{M}^{\sharp}\rangle$. The bilinearity of this pairing is clear, and the symmetry is a consequence of Proposition 4.6. In terms of this pairing, the *arithmetic intersection product of* $\mathcal{L}^{\sharp}$ and $\mathcal{M}^{\sharp}$ is obtained by taking the arithmetic degree: $(\mathcal{L}^{\sharp},\mathcal{M}^{\sharp})=\deg^{\sharp}\langle\mathcal{L}^{\sharp},\mathcal{M}^{\sharp}\rangle\in\mathbb{C}/\pi i\mathbb{Z}$. One of the aims of this section is to prove an arithmetic Riemann-Roch formula that accounts for these arithmetic intersection numbers.

Argument of the Deligne pairing. Let $\mathcal{L}^\sharp$ and $\mathcal{M}^\sharp$ be conjugate pairs of line bundles with connections. By the *argument of the Deligne pairing of $\mathcal{L}^\sharp$ and $\mathcal{M}^\sharp$* we mean the imaginary part of the intersection product: $\arg^\sharp\langle\mathcal{L}^\sharp,\mathcal{M}^\sharp\rangle = \mathrm{Im}(\mathcal{L}^\sharp,\mathcal{M}^\sharp) \in \mathbb{R}/\pi\mathbb{Z}$.

6.3. **Mixed arithmetic intersection products.** The classical arithmetic Picard group in Arakelov geometry classifies smooth Hermitian line bundles, and is denoted $\widehat{\mathrm{Pic}}(\mathcal{X})$. There is an obvious groupoid version that we denote $\widehat{\mathrm{PIC}}(\mathcal{X})$. We constructed intersection logarithms between conjugate pairs of rigidified line bundles with connections and Hermitian line bundles. With this, we can define a pairing

$$\mathrm{PICRIG}^\sharp(\mathcal{X}) \times \widehat{\mathrm{PIC}}(\mathcal{X}) \longrightarrow \mathrm{PIC}^\sharp(S)$$

simply as follows. Given a conjugate pair of line bundles with connections $\mathcal{L}^\sharp$, rigidified along σ, and a Hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, we define the Deligne pairing

$$\langle\mathcal{L}^\sharp,\overline{\mathcal{M}}\rangle = (\langle\mathcal{L},\mathcal{M}\rangle, \langle\mathcal{L}^c,\mathcal{M}\rangle, \{\mathrm{LOG}_{int,\tau}\}_\tau).$$

Here, $\mathrm{LOG}_{int,\tau}$ is the intersection logarithm on the base change

$$\langle\mathcal{L},\mathcal{M}\rangle_\tau \otimes_{\mathbb{C}} \langle\mathcal{L}^c,\mathcal{M}\rangle_{\overline{\tau}} = \langle\mathcal{L}_\tau,\mathcal{M}_\tau\rangle \otimes \langle\mathcal{L}^c_{\overline{\tau}},\mathcal{M}_{\overline{\tau}}\rangle,$$

constructed using the connections defining $\mathcal{L}^\sharp$ at τ, the rigidifications, and the Hermitian metric on $\mathcal{M}$. In terms of this Deligne pairing, we define the mixed arithmetic intersection product: $(\mathcal{L}^\sharp,\overline{\mathcal{M}}) = \deg^\sharp\langle\mathcal{L}^\sharp,\overline{\mathcal{M}}\rangle \in \mathbb{C}/\pi i\mathbb{Z}$. Because a rigidification is unique up to $\mathcal{O}_K^\times$, this quantity does not depend on the particular choice of rigidification, but in general it depends on the section.

Variant in the absence of rigidification. When a section σ is given, but we do not have a rigidification, we may follow the observation made in Remark 6.10 and base change to the Hilbert class field H. Observe the base change $\mathcal{X}_{\mathcal{O}_H}$ is still an arithmetic surface: because the Hilbert class field H is unramified, the regularity of the scheme is preserved. Let us indicate base changed objects with a prime symbol. Given $\mathcal{L}^\sharp$, the base change $\mathcal{L}^{\sharp\prime}$ admits a rigidification, which is unique up to unit. Then, the arithmetic intersection number: $(\mathcal{L}^{\sharp\prime},\overline{\mathcal{M}}')\in\mathbb{C}/\pi i\mathbb{Z}$, is defined. Taking into account the functoriality properties of the arithmetic degree (Proposition 6.5), it is more natural to normalize this quantity by $[H:K]$, that is the class number h_K. We then write

$$(\mathcal{L}^\sharp,\overline{\mathcal{M}}) := \frac{1}{h_K}(\mathcal{L}^{\sharp\prime},\overline{\mathcal{M}}') \in \mathbb{C}/\pi i\mathbb{Z}[1/h_K].$$

In particular, when $K=\mathbb{Q}$, or more generally when $h_K=1$, the mixed arithmetic intersection number with values in $\mathbb{C}/\pi i\mathbb{Z}$ is always defined, without any reference to the rigidification (but always depending on the section).

6.4. **Variants over $\mathbb{R}$ and $\mathbb{C}$, argument and periods.** While classical Arakelov geometry over $\mathbb{R}$ or $\mathbb{C}$ cannot produce any interesting numerical invariants (only zero), the present theory has a nontrivial content over these fields. Let us discuss the case of the base field $\mathbb{C}$. We saw we can still define $\mathrm{Pic}^{\sharp}(\mathrm{Spec}\,\mathbb{C})$, and an arithmetic degree $\deg^{\sharp}$, now with values in $i\mathbb{R}/2\pi i\,\mathbb{Z}$. In the construction, one has to take into account the identity and conjugation embeddings $\mathbb{C}\to\mathbb{C}$. We denote the imaginary part of $\deg^{\sharp}$ by $\arg^{\sharp}\colon \mathrm{Pic}^{\sharp}(\mathrm{Spec}\,\mathbb{C})\to\mathbb{R}/2\pi\mathbb{Z}$. Let X be a smooth, proper and geometrically irreducible curve over $\mathbb{C}$. We can also define $\mathrm{PIC}^{\sharp}(X)$ and a Deligne pairing. The argument of the Deligne pairing is still defined: $\arg^{\sharp}\langle\mathcal{L}^{\sharp},\mathcal{M}^{\sharp}\rangle\in\mathbb{R}/2\pi\mathbb{Z}$. Similarly there is a well-defined argument of the mixed arithmetic intersection product, between $\mathrm{PICRIG}^{\sharp}(X)$ and $\widehat{\mathrm{PIC}}(X)$.

Interpretation of the argument. Let X be a smooth, projective and irreducible curve over $\mathbb{C}$. To apply the formalism above, we stress that $\mathbb{C}$ has to be considered with its identity and conjugation embeddings. Let L be a line bundle on X and $\overline{L}$ the conjugate line bundle on $\overline{X}$. Assume holomorphic connections $\nabla^{L}\colon L\to L\otimes\Omega^{1}_{X/\mathbb{C}}$ and $\nabla_{\overline{L}}\colon\overline{L}\to\overline{L}\otimes\Omega^{1}_{\overline{X}/\mathbb{C}}$ with *real* holonomy characters. We do not impose any further condition. We choose $L^{c}=L^{\vee}$, and we endow L^{c} and $\overline{L}^{c}$ with the dual connections to ∇^{L}, $\nabla_{\overline{L}}$. This provides an example of conjugate pair of line bundles with connections on X, that we write $L^{\sharp}$. Let M be a degree 0 line bundle on X, that we endow with its flat unitary connection. On $\overline{M}$ we put the conjugate connection. In this case we take $M^{c}=M$, with same connections. We proceed to describe $\arg^{\sharp}\langle L^{\sharp},M^{\sharp}\rangle\in\mathbb{R}/2\pi\mathbb{Z}$. We fix a base point $p\in X$ and a trivialization of L. Let ℓ and m be rational sections of L and M. Using the connection ∇^{L}, we lift as usual ℓ to $\tilde{\ell}$, on the universal covering. We also lift $\mathrm{div}\,m$ to $\widetilde{\mathrm{div}\,m}$. For the conjugate datum, we lift $\overline{\ell}$ to $\tilde{\overline{\ell}}$ and $\mathrm{div}\,\overline{m}$ to $\widetilde{\mathrm{div}\,\overline{m}}$. We will appeal to the explicit description of the intersection logarithm in Section 4.4.3, in particular formula (32). Because we didn't impose any relation between $\overline{\nabla}_{L}$ and $\nabla_{\overline{L}}$, we cannot conclude that: $\overline{\tilde{\ell}(\widetilde{\mathrm{div}\,m})}=\tilde{\overline{\ell}}(\widetilde{\mathrm{div}\,\overline{m}})$. In words, in general "conjugation does not commute with lifting". There exists a holomorphic differential form on $\overline{X}$, that we write as $\overline{\theta'}$ for some holomorphic form θ' on X, such that $\nabla_{\overline{L}}=\overline{\nabla}_{L}-\overline{\theta'}$. Because both connections are supposed to have real holonomy characters, we see that $\exp\left(\int_{\gamma}\overline{\theta'}\right)=\exp\left(\int_{\gamma}\theta'\right)$. Hence, the harmonic differential form $\theta=\theta'-\overline{\theta'}$ has periods in $2\pi i\,\mathbb{Z}$. Such differential forms are of course parametrized by $H^{1}(X,2\pi i\,\mathbb{Z})$, which is a rank $2g$ $\mathbb{Z}$-module. In terms of $\overline{\theta'}$ we have $\overline{\tilde{\ell}(\widetilde{\mathrm{div}\,m})}=\tilde{\overline{\ell}}(\widetilde{\mathrm{div}\,\overline{m}})\exp\left(\int_{\tilde{p}}^{\widetilde{\mathrm{div}\,m}}\widetilde{\overline{\theta'}}\right)$. From this and equation (32), we conclude that $\arg^{\sharp}\langle L^{\sharp},M^{\sharp}\rangle=-2\,\mathrm{Im}\left(\int_{\tilde{p}}^{\widetilde{\mathrm{div}\,m}}\widetilde{\overline{\theta'}}\right)=\mathrm{Im}\left(\int_{\tilde{p}}^{\widetilde{\mathrm{div}\,m}}\tilde{\theta}\right)$. Because θ has periods in $2\pi i\,\mathbb{Z}$, this quantity does not depend on the choice of lifting $\widetilde{\mathrm{div}\,m}$, modulo $2\pi\mathbb{Z}$. Moreover, modulo $2\pi\mathbb{Z}$ it only depends on the rational equivalence class of $\mathrm{div}\,m$, namely M itself. And this is again because θ has periods in $2\pi i\,\mathbb{Z}$. It is also independent of the base point, because M has degree 0. Finally, the connection on M played no role. This is of course in agreement with the properties of the intersection pairings. Therefore, given a degree 0 Weil divisor D on X, we have a well-defined argument

$$\arg^{\sharp}\langle L^{\sharp}, \mathcal{O}(D)\rangle = \mathrm{Im}\left(\int_{\tilde{p}}^{D} \theta\right) \in \mathbb{R}/2\pi\mathbb{Z}.$$

Let us write $\theta_{L^{\sharp}}$ for the harmonic differential form above. We thus have a pairing

$$\arg^{\sharp}\colon\ \mathrm{PIC}^{\sharp}(X)_{re} \times \mathrm{Pic}^{0}(X)(\mathbb{C}) \longrightarrow \mathbb{R}/2\pi\mathbb{Z} : (L^{\sharp}, \mathcal{O}(D)) \longmapsto \mathrm{Im}\left(\int_{\tilde{p}}^{D} \theta_{L^{\sharp}}\right),$$

where the subscript *re* indicates we restrict to conjugate pairs with real holonomy connections. The values of this pairing are imaginary parts of integer combinations of periods!

There is a variant of this pairing when $M = \mathcal{O}(D)$ has arbitrary degree. In this case one needs to equip $L^{\sharp}$ with a rigidification. Because $L^{c} = L^{\vee}$, it is enough to fix a rigidification for L. For the argument, one needs to fix a Hermitian metric on M and use the mixed intersection pairing. The final formula looks exactly the same. While the result will not depend on the metric on M, it depends on the base point (since $\deg D \neq 0$). If we had chosen unrelated rigidifications for L and L^{c}, the result would have depended on these choices, as well.

Remark 6.11. There is no simple formula for the general case of an arbitrary conjugate pair $L^{\sharp}$.

6.5. **Arithmetic Riemann-Roch theorem.** Let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface with a section $\sigma\colon S \to \mathcal{X}$. We fix a Hermitian metric on $\omega_{\mathcal{X}/S}$. Let $\mathcal{L}^{\sharp}$ be a rigidified pair of conjugate line bundles with connections. Recall the notation $\lambda(\mathcal{L})$ for $\det R\pi_{*}(\mathcal{L})$. It is compatible with base change. Following the construction of Section 5, for every τ there is a Quillen-Cappell-Miller logarithm $\mathrm{LOG}_{Q,\tau}$ on

$$\lambda(\mathcal{L}_{\tau}) \otimes_{\mathbb{C}} \lambda(\mathcal{L}^{c}_{\overline{\tau}}) = \det H^{\bullet}(\mathcal{X}_{\tau}(\mathbb{C}), \mathcal{L}_{\tau}) \otimes \det H^{\bullet}(\mathcal{X}_{\overline{\tau}}(\mathbb{C}), \mathcal{L}^{c}_{\overline{\tau}}).$$

Introduce the conjugate pair of line bundles with logarithms:

$$\lambda(\mathcal{L}^{\sharp})_{Q} = (\lambda(\mathcal{L}), \lambda(\mathcal{L}^{c}), \{\mathrm{LOG}_{Q,\tau}\}_{\tau})\ .$$

Notice that the construction of the Quillen-Cappell-Miller logarithm requires the rigidification, in order to identify $\mathcal{L}_{\tau}$ to $\mathcal{L}_{\chi_{\tau}}$ and $\mathcal{L}^{c}_{\overline{\tau}}$ to $\mathcal{L}^{c}_{\chi_{\tau}}$.

Theorem 6.12. *There is an equality in* $\mathbb{C}/\pi i\,\mathbb{Z}$

$$\begin{aligned}12\deg^{\sharp}\lambda(\mathcal{L}^{\sharp})_{Q} - 2\delta = {}& 2(\overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S}) + 6(\mathcal{L}^{\sharp}, \mathcal{L}^{\sharp}) - 6(\mathcal{L}^{\sharp}, \overline{\omega}_{\mathcal{X}/S})\\ & - (4g-4)[K:\mathbb{Q}]\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right),\end{aligned} \tag{35}$$

where $\delta = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log(N\mathfrak{p})$ *is the "Artin conductor" measuring the bad reduction of* $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$*. If* K *does not admit any real embeddings, then the equality already holds in* $\mathbb{C}/2\pi i\,\mathbb{Z}$*.*

Remark 6.13. The mixed arithmetic intersection product $(\mathcal{L}^\sharp, \overline{\omega}_{\mathcal{X}/S})$ involves the rigidification, and depends on it. This is in agreement with the dependence of the Quillen logarithm on the rigidification. Nevertheless, it does not depend on the choice of metric on $\omega_{\mathcal{X}/S}$, by Corollary 4.8. Therefore, on the right hand side of the formula, the dependence in the metric on $\omega_{\mathcal{X}/S}$ comes only from $(\overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S})$. See also Remark 5.11.

Proof of Theorem 6.12. The theorem is derived as a combination of the following statements:

(i) the Deligne isomorphism applied to $\mathcal{X} \to S$, $\mathcal{L}$, $\mathcal{L}^c$ and $\mathcal{O}_{\mathcal{X}}$, and its compatibility to base change under $\tau : K \hookrightarrow \mathbb{C}$;

(ii) the arithmetic Riemann-Roch theorem of Gillet-Soulé [20] applied twice to $\mathcal{O}_{\mathcal{X}}$ in Deligne's functorial formulation [12, 31], which guarantees a quasi-isometry

$$\lambda(\mathcal{O}_{\mathcal{X}})_Q^{\otimes 12} \otimes \mathcal{O}(-\Delta) \xrightarrow{\sim} \langle \overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S} \rangle,$$

with norm $\exp((2g-2)(\zeta'(-1)/\zeta(-1)+1/2))$. The index Q stands for the Quillen metric (for the trivial Hermitian line bundle in this case), Δ is the Deligne discriminant supported on finite primes, and $\mathcal{O}(\Delta)$ is endowed with the trivial metric (then δ is the arithmetic degree of $\mathcal{O}(\Delta)$). It is related to Artin's conductor through work of T. Saito [29];

(iii) the fact that for the trivial Hermitian line bundle, LOG_Q amounts to the Quillen metric for the trivial hermitian line bundle;

(iv) Theorem 5.10 specialized to $(\mathcal{X}_\tau(\mathbb{C}), \sigma_\tau)$, $\mathcal{L}_{\chi_\tau}$, $\mathcal{L}_{\chi_\tau}^c$ (fibers of universal objects over $M_B(X_\tau(\mathbb{C}))$);

(v) the use of the connections and rigidifications in order to identify $\mathcal{L}_\tau$ to $\mathcal{L}_{\chi_\tau}$ and $\mathcal{L}_{\overline{\tau}}^c$ to $\mathcal{L}_{\chi_\tau}^c$, plus the compatibility of Deligne's isomorphism to isomorphisms of line bundles.

This provides a statement in a finer form, at the level of $\mathrm{PIC}^\sharp(S)$. We conclude by applying the arithmetic degree $\deg^\sharp$. For the last claim, it is enough to observe first that the arithmetic intersection numbers are well-defined in $\mathbb{C}/2\pi i\,\mathbb{Z}$, and that the sign ambiguity in Deligne's isomorphism disappears, since there is an even number of different embeddings from K into $\mathbb{C}$. ◻

Variant in the absence of rigidification. In practical situations, while a section σ of $\pi \colon \mathcal{X} \to S$ may be given, a natural choice of rigidification may not. As we explained in Remark 6.10 and in Section 6.3, this can be remedied by base changing to the Hilbert class field of K. For instance, we justified that mixed intersection products $(\mathcal{L}^\sharp, \overline{\mathcal{M}})$ are naturally defined in $\mathbb{C}/\pi i\,\mathbb{Z}[1/h_K]$. For the determinant of cohomology $\lambda(\mathcal{L}^\sharp)$ it is even simpler, since the rigidification is only needed in the construction of the logarithms, which happen on the archimedean places. Clearly, $\lambda(\mathcal{L}^\sharp)$ can be defined over $\mathcal{O}_K$ if it is defined after base change to $\mathcal{O}_H$.

Corollary 6.14. *Let $\mathcal{X} \to S$ be an arithmetic surface with $\sigma \colon S \to \mathcal{X}$ a given section. Fix a Hermitian metric on $\omega_{\mathcal{X}/S}$. Let $\mathcal{L}^\sharp$ be a conjugate pair of line bundles with connections. Then, the formula* (35) *holds with values in $\mathbb{C}/\pi i\,\mathbb{Z}[1/h_K]$, where h_K is the class number of K.*

Proof. After Theorem 6.12, it is enough to base change to the Hilbert class field, and use the functoriality of the arithmetic degree and the compatibility of the determinant of cohomology with base change. □

Variant over $\operatorname{Spec}\mathbb{C}$. There is an interesting version of Theorem 6.12 when the base scheme $\operatorname{Spec}\mathbb{C}$, when the argument is still well-defined and with values in $\mathbb{R}/2\pi\mathbb{Z}$. The formula dramatically simplifies:

Theorem 6.15 (Argument of Arithmetic Riemann-Roch)**.** *When the base scheme is* $\operatorname{Spec}\mathbb{C}$*, there is the following equality of arguments in* $\mathbb{R}/2\pi\mathbb{Z}$*:*

$$12\arg^{\sharp}\lambda(\mathcal{L}^{\sharp})_{Q} = 6\arg^{\sharp}\langle\mathcal{L}^{\sharp},\mathcal{L}^{\sharp}\rangle - 6\arg^{\sharp}\langle\mathcal{L}^{\sharp},\overline{\omega}_{\mathcal{X}/S}\rangle.$$

Example 6.16. Let X be a compact Riemann surface with a fixed base point p. Let $L^{\sharp}$ be a conjugate pair of rigidified line bundles with connections. Assume the connections have real holonomies, that $L^{c} = L^{\vee}$ and the rigidification is induced by a trivialization of L alone. Because we are in the real holonomy case, the explicit description of the intersection logarithm in Section 4.4.1 shows that $\arg^{\sharp}\langle L^{\sharp},L^{\sharp}\rangle = 0$. For the other intersection product, recall we saw in Section 6.4 that $L^{\sharp}$ determines a harmonic differential form $\theta_{L^{\sharp}}$ with periods in $2\pi i\,\mathbb{Z}$. Then, if $\omega_{X/\mathbb{C}} = \mathcal{O}(K)$ for some canonical divisor K, we have $\arg^{\sharp}\langle L^{\sharp},\omega_{X/\mathbb{C}}\rangle = \operatorname{Im}\left(\int_{\tilde{p}}^{K}\theta_{L^{\sharp}}\right)$. Now the argument of the arithmetic Riemann-Roch theorem in this particular case specializes to $12\arg^{\sharp}\lambda(L^{\sharp})_{Q} = -6\operatorname{Im}\left(\int_{\tilde{p}}^{K}\theta_{L^{\sharp}}\right)$, in $\mathbb{R}/2\pi\mathbb{Z}$. This can be seen as an anomaly formula for the imaginary part of the Quillen-Cappell-Miller logarithm, under a change of connection (within the real holonomy assumption).

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