

CONFORMAL LIMITS AND THE BIALYNICKI-BIRULA STRATIFICATION OF THE SPACE OF λ -CONNECTIONS

BRIAN COLLIER AND RICHARD WENTWORTH

ABSTRACT. The Bialynicki-Birula decomposition of the space of λ -connections restricts to the Morse stratification on the moduli space of Higgs bundles and to the partial oper stratification on the de Rham moduli space of holomorphic connections. For both the Morse and partial oper stratifications, every stratum is a holomorphic Lagrangian fibration over a component of the space of complex variations of Hodge structure. In this paper, we generalize known results for the Hitchin section and the space of opers to arbitrary strata. These include the following: a biholomorphic identification of the fibers of the two strata over a stable variation of Hodge structure via the “ $\hbar$ -conformal limit” of Gaiotto, a proof that the fibers of the Morse and partial oper stratifications are transverse at the base point, and an explicit parametrization of the fibers as half-dimensional affine spaces.

1. INTRODUCTION

For a compact Riemann surface X , both Hitchin’s moduli space of Higgs bundles and the de Rham moduli space of holomorphic connections are interesting holomorphic symplectic manifolds. While the celebrated nonabelian Hodge correspondence provides a homeomorphism between the two, they are in fact quite different as complex analytic spaces. For example, the Higgs moduli space contains positive dimensional subvarieties (e.g. the moduli space of holomorphic bundles), whereas the de Rham moduli space is Stein. Nevertheless, the complex structures are related in the sense that their composition gives a third complex structure which in fact defines a hyperkähler structure. By a construction of Deligne [3], further elucidated by Simpson in [15], the associated twistor space may be interpreted as a moduli space of λ -connections. A further key property of the twistor space is the existence of a $\mathbb{C}^*$ -action extending the one defined by Hitchin on the Higgs moduli space.

The Higgs and de Rham moduli spaces each have distinguished holomorphic Lagrangian submanifolds which can be parametrized by a particular vector space of holomorphic differentials on X . On the Higgs bundle side, this space was defined by Hitchin in [8] as the image of a section of the Hitchin fibration, and it generalizes the harmonic maps parametrization of Teichmüller space (cf. [19]). On the de Rham side, this is the space of *opers*, which may be considered as a generalization of complex projective structures on X (cf. [1]).

Date: April 14, 2019.

2010 *Mathematics Subject Classification.* Primary: 58D27; Secondary: 14D20, 14D21, 32G13.

B.C. is supported by the NSF under Award No. 1604263. R.W. was supported in part by NSF grants DMS-1406513 and DMS-1564373. The authors also acknowledge support from NSF grants DMS-1107452, -1107263, -1107367 “RNMS: GEometric structures And Representation varieties” (the GEAR Network).

In fact, the Hitchin section and the set of opers each form a closed stratum of natural stratifications on the spaces of Higgs bundles and holomorphic connections coming from an application of Białynicki-Birula (BB) theory for the $\mathbb{C}^*$ -action mentioned above [16]. In both cases, each stratum is a holomorphic fibration by Lagrangians over a connected component of the set of complex variations of Hodge structure. In the language introduced by Kapustin and Witten [10], the fibers are submanifolds of type (BAA) and (ABA), respectively.

The goal of this paper is to extend the many relationships between the Hitchin section and the space of opers to all components of the BB-stratification on the space of λ -connections. The main results are:

- (1) a proof of the existence of an “ $\hbar$ -conformal limit” (in the sense of Gaiotto [6]) for every Higgs bundle whose $\mathbb{C}^*$ -limit is stable;
- (2) a proof that the de Rham fiber is transverse to the Higgs bundle fiber at each stable complex variation of Hodge structure;
- (3) a global parametrization of each of the fibers in the BB-stratification as a half dimensional affine space and a proof that their intersections with the Higgs and de Rham moduli spaces are submanifolds of type (BAA) and (ABA), respectively.

The following subsections contain more precise statements of these items.

1.1. Conformal limits. Let $E \rightarrow X$ be a complex vector bundle of rank n and degree zero. We denote by $\mathcal{M}_H$, $\mathcal{M}_{dR}$, and $\mathcal{M}_{Hod}$ the moduli spaces of Higgs bundles, holomorphic connections, and λ -connections, respectively, on the underlying smooth bundle E (see Section 2 for definitions). There is a holomorphic map $\lambda : \mathcal{M}_{Hod} \rightarrow \mathbb{C}$ with

$$\mathcal{M}_H = \lambda^{-1}(0) \quad \text{and} \quad \mathcal{M}_{dR} = \lambda^{-1}(1) .$$

In [16], Simpson showed that the λ -equivariant $\mathbb{C}^*$ -action on $\mathcal{M}_{Hod}$ given by $\nabla_\lambda \mapsto \xi \cdot \nabla_\lambda$, $\xi \in \mathbb{C}^*$, defines a Białynicki-Birula type decomposition. The strata $W_\alpha \subset \mathcal{M}_{Hod}$ are labeled by the components $VHS_\alpha \subset \mathcal{M}_H$ of the fixed point set. These are the Higgs bundles defining *complex variations of Hodge structure* (VHS). On each stratum there is a holomorphic map $W_\alpha \rightarrow VHS_\alpha$ with affine fibers. The restrictions W_α^0 , W_α^1 to $\lambda = 0, 1$ are respectively the *Morse strata* of $\mathcal{M}_H$ defined by Hitchin and the *partial oper strata* of $\mathcal{M}_{dR}$ defined by Simpson (see Section 4).

A choice of hermitian metric h on E associates hermitian adjoints $*_h$ to endomorphisms and Chern connections $\bar{\partial}_E + \partial_E^h$ to $\bar{\partial}$ -operators. The nonabelian Hodge correspondence (NHC) produces a special hermitian metric, known as the *harmonic metric*, to both a polystable Higgs bundle and a completely reducible flat connection. Through this association, the NHC defines a homeomorphism:

$$(1.1) \quad \mathcal{T} : \mathcal{M}_H \xrightarrow{\sim} \mathcal{M}_{dR} : [(\bar{\partial}_E, \Phi)] \mapsto [D = \bar{\partial}_E + \partial_E^h + \Phi + \Phi^*]$$

between Higgs bundles and flat connections (see Section 2.3 for more details).

For a stable Higgs bundle $(\bar{\partial}_E, \Phi)$ with harmonic metric h , there are two ways to scale the correspondence in (1.1) – one uses the $\mathbb{C}^*$ -action on $\mathcal{M}_H$ and the other a so-called *real twistor line* in $\mathcal{M}_{Hod}$. More precisely, if $R > 0$, then $(\bar{\partial}_E, R\Phi)$ is also stable, and so we obtain harmonic metrics h_R . Next, for $\xi \in \mathbb{C}^*$, the connection

$$\bar{\partial}_E + \partial_E^{h_R} + \xi^{-1} R\Phi + \xi R\Phi^{*h_R}$$

is also flat. If we keep $\hbar = R^{-1}\xi$ fixed, we obtain a family of flat connections parametrized by R :

$$(1.2) \quad D_{R,\hbar} = D_{R,\hbar}(\bar{\partial}_E, \Phi) = \bar{\partial}_E + \partial_E^{\hbar R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*\hbar R} .$$

The limiting flat connection $\lim_{R \rightarrow 0} D_{R,\hbar}$, when it exists, is called the $\hbar$ -conformal limit of the Higgs bundle $(\bar{\partial}_E, \Phi)$. The existence of a limit is a gauge invariant property. For example, if $\hbar = 1$ and $(\bar{\partial}_E, \Phi)$ is a fixed point of the $\mathbb{C}^*$ -action, then the conformal limit is just the associated flat connection from the NHC. More generally, in [6] Gaiotto conjectured that for $[(\bar{\partial}_E, \Phi)]$ in the Hitchin component, the $\hbar$ -conformal limit exists and lies in the space of opers, and moreover this gives a biholomorphism between these spaces. This conjecture was recently verified in [5].

In this paper we generalize the conformal limit correspondence. The first main result is the following.

Theorem 1.1 (CONFORMAL LIMIT). *Let $[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H$ be such that its limit $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_\alpha$ under the $\mathbb{C}^*$ -action is represented by a stable Higgs bundle. Then the $\hbar$ -conformal limit of $[(\bar{\partial}_E, \Phi)]$ exists. Moreover, the conformal limit gives a biholomorphism between the fibers $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ and $W_\alpha^1(\bar{\partial}_0, \Phi_0)$ of W_α^0 and W_α^1 over $[(\bar{\partial}_0, \Phi_0)]$.*

1.2. Transversality of fibers at the VHS. As mentioned above the moduli spaces $\mathcal{M}_H$ and $\mathcal{M}_{\text{dR}}$ are holomorphic symplectic manifolds. Let $(I, \omega_{\mathbb{C}}^I)$ and $(J, \omega_{\mathbb{C}}^J)$ denote the complex structures and holomorphic symplectic forms on $\mathcal{M}_H$ and $\mathcal{M}_{\text{dR}}$, respectively. These are not preserved by the homeomorphism (1.1), and their composition $K = I \circ J$ defines a third complex structure, giving $\mathcal{M}_H$ the structure of a hyperkähler manifold.

Proposition 1.2. *Let $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ be stable. Then the fiber $W_\alpha^0(\bar{\partial}_E, \Phi)$ of W_α^0 over $[(\bar{\partial}_E, \Phi)]$ is an $\omega_{\mathbb{C}}^I$ holomorphic Lagrangian submanifold of $\mathcal{M}_H$ (type (BAA)), and $W_\alpha^1(\bar{\partial}_E, \Phi)$ is an $\omega_{\mathbb{C}}^J$ holomorphic Lagrangian submanifold of $\mathcal{M}_{\text{dR}}$ (type (ABA)).¹*

In particular, the tangent space splits as

$$T_{\mathcal{T}(\bar{\partial}_E, \Phi)} \mathcal{M}_{\text{dR}} = T_{\mathcal{T}(\bar{\partial}_E, \Phi)} W_\alpha^1(\bar{\partial}_E, \Phi) \oplus K(T_{\mathcal{T}(\bar{\partial}_E, \Phi)} W_\alpha^1(\bar{\partial}_E, \Phi)) .$$

Moreover, $\mathcal{T}(W_\alpha^0(\bar{\partial}_E, \Phi))$ intersects $W_\alpha^1(\bar{\partial}_E, \Phi)$ at $\mathcal{T}(\bar{\partial}_E, \Phi)$. We shall prove that this intersection is transverse, generalizing the known result for the Hitchin component and opers (cf. [12]).

Theorem 1.3 (TRANSVERSALITY AT THE VHS). *For each stable $(\bar{\partial}_E, \Phi) \in \text{VHS}_\alpha$, the transformation $\mu \mapsto \mu - K(\mu)$ maps the tangent space $T_{\mathcal{T}(\bar{\partial}_E, \Phi)} W_\alpha^1(\bar{\partial}_E, \Phi)$ linearly isomorphically onto $d\mathcal{T}(T_{\mathcal{T}(\bar{\partial}_E, \Phi)} W_\alpha^0(\bar{\partial}_E, \Phi))$. In particular,*

$$\mathcal{T}(W_\alpha^0(\bar{\partial}_E, \Phi)) \pitchfork W_\alpha^1(\bar{\partial}_E, \Phi)$$

at $\mathcal{T}(\bar{\partial}_E, \Phi)$.

¹The second statement in Proposition 1.2 was already proved in [16, Lemma 7.3].

1.3. Global slices. Theorems 1.1 and 1.3 are consequences of the existence of a *global slice* in the deformation theory about a VHS. More precisely, for a stable Higgs bundle $(\bar{\partial}_E, \Phi)$, the NHC allows us to define a *Hodge slice* $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ in the space of Higgs bundles (see Definition 3.1). Via the Kuranishi map, this gives a local coordinate chart of $\mathcal{M}_H$ about the point $[(\bar{\partial}_E, \Phi)]$. If furthermore $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$, then there is a holomorphic splitting of $(E, \bar{\partial}_E)$. Since the deformation theory of $\mathcal{M}_H$ is compatible with this splitting, this extra structure defines a natural subvariety $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \subset \mathcal{S}_{(\bar{\partial}_E, \Phi)}$ which we call the *Białynicki-Birula slice* (see Sections 3.1 and 3.2).

We will show that the Białynicki-Birula slice $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ is biholomorphic to the fibers $W_\alpha^0(\bar{\partial}_E, \Phi)$ and $W_\alpha^1(\bar{\partial}_E, \Phi)$, generalizing the parameterizations of the Hitchin section and opers by the Hitchin base (see Section 2.7).

Theorem 1.4 (GLOBAL SLICE). *Let $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ be a stable Higgs bundle with BB-slice $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$, and let $W_\alpha(\bar{\partial}_E, \Phi)$ (resp. $W_\alpha^\xi(\bar{\partial}_E, \Phi)$) denote the fiber of $W_\alpha \rightarrow \text{VHS}_\alpha$ (resp. $W_\alpha \cap \lambda^{-1}(\xi) \rightarrow \text{VHS}_\alpha$) over $[(\bar{\partial}_E, \Phi)]$.*

Then:

- (1) *the Kuranishi map gives a biholomorphism of $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ with a complex vector space of half the dimension of $\mathcal{M}_H$;*
- (2) *the natural projection $p_H : \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \rightarrow \mathcal{M}_H$ is a biholomorphic embedding onto the Morse fiber $W_\alpha^0(\bar{\partial}_E, \Phi)$;*
- (3) *there is a holomorphic map $p_{\text{Hod}} : \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \times \mathbb{C} \rightarrow \mathcal{M}_{\text{Hod}}$, extending p_H and making the following diagram commute*

$$\begin{array}{ccc} \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \times \mathbb{C} & \xrightarrow{p_{\text{Hod}}} & \mathcal{M}_{\text{Hod}} \\ & \searrow \text{pr}_2 \quad \swarrow \lambda & \\ & \mathbb{C} & \end{array}$$

where pr_2 denotes projection to the second factor. Moreover, p_{Hod} is a biholomorphic embedding onto the BB-fiber $W_\alpha(\bar{\partial}_E, \Phi)$. In particular, the fibers $W_\alpha^0(\bar{\partial}_E, \Phi)$ and $W_\alpha^1(\bar{\partial}_E, \Phi)$ are biholomorphic.

By standard BB-theory, the fibers $W_\alpha^0(\bar{\partial}_E, \Phi) \subset \mathcal{M}_H$ of the Morse strata are affine spaces. The theorem above immediately shows that this property generalizes to the fibers of the partial oper stratification in $\mathcal{M}_{\text{dR}}$.

Corollary 1.5. *For each stable $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ the fiber $W_\alpha^1(\bar{\partial}_E, \Phi) \subset \mathcal{M}_{\text{dR}}$ of $W_\alpha \rightarrow \text{VHS}_\alpha$ over $[(\bar{\partial}_E, \Phi)]$ is affine.*

In fact, the biholomorphism $W_\alpha^0(\bar{\partial}_0, \Phi_0) \simeq W_\alpha^1(\bar{\partial}_0, \Phi_0)$ in Theorem 1.4 (3) comes from the conformal limit.

Corollary 1.6. *Let $p_\hbar : \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \rightarrow W_\alpha^h(\bar{\partial}_E, \Phi)$ denote the biholomorphism obtained by restricting p_{Hod} from Theorem 1.4 to $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \times \{\hbar\}$. Then the $\hbar$ -conformal limit through the Higgs bundle $p_0(\mathbf{u})$ is $\hbar^{-1} \cdot p_\hbar(\mathbf{u})$.*

Hence, the process of taking conformal limits may be regarded as a deformation of $\mathcal{M}_{\text{dR}}$ which moves the Morse stratification to the partial oper stratification, preserving the VHS locus.

Remark 1.7. The comparison of the stratifications given by W_α^0 and W_α^1 is an interesting question. For example, the existence of Higgs bundles with nilpotent Higgs field implies that the fibers $W_\alpha^0(\bar{\partial}_E, \Phi)$ are not, in general, closed. On the other hand, as pointed out in [16], it is unknown whether the fibers $W_\alpha^1(\bar{\partial}_E, \Phi)$ are themselves closed (see also Remark 5.3 below). In [16] and [13], the structure of the closure of the strata W_α^1 is described for rank 2. The general case remains an open problem.

Remark 1.8. We comment on two restrictions made in this paper. The first is that we deal here only with Higgs vector bundles. However, analogous results should also hold for G -Higgs bundles and flat G -connections for any semisimple Lie group G . In particular, principal objects are considered in [16, Section 8]. The second assumption in all of the main results of this paper is that the limiting VHS is stable. It is unclear at this point what modifications might be necessary to the statements above if one relaxes this assumption.

Acknowledgments: It is a pleasure to thank Andy Neitzke and Carlos Simpson for enlightening conversations.

2. MODULI SPACES

2.1. The de Rham moduli space of flat connections. As in the Introduction, we fix a smooth rank n complex vector bundle $E \rightarrow X$ with a trivialization of the determinant line bundle $\det E$. Let $\mathcal{D} = \mathcal{D}(E, \mathrm{SL}(n, \mathbb{C}))$ denote the space of flat connections that induce the trivial connection on $\det E$. The space of all such connections is an infinite dimensional affine space modeled on the vector space $\Omega^1(X, \mathfrak{sl}(E))$,² where $\mathfrak{sl}(E) \subset \mathrm{End}(E)$ denotes the subbundle of traceless endomorphisms. In particular, the tangent space to $\mathcal{D}$ at a flat connection D is the set of $\mu \in \Omega^1(\mathfrak{sl}(E))$ so that $D + \mu$ is flat to first order:

$$T_D \mathcal{D} = \{\mu \in \Omega^1(\mathfrak{sl}(E)) \mid D\mu = 0\} .$$

The complex structure on $\mathrm{SL}(n, \mathbb{C})$ defines a complex structure on $\mathcal{D}$; namely,

$$J(\mu) = i\mu .$$

Moreover, the alternating form

$$(2.1) \quad \omega_J^{\mathbb{C}}(\mu, \nu) = \int_X \mathrm{Tr}(\mu \wedge \nu)$$

defines a (J -)holomorphic symplectic form on $\mathcal{D}$. This is the Atiyah-Bott-Goldman-Narasimhan symplectic form. Note that both J and $\omega_J^{\mathbb{C}}$ are independent of the Riemann surface structure X .

The gauge group $\mathcal{G}(E)$ of smooth automorphisms of E that act trivially on $\det E$ acts on the space of flat connections by $D \cdot g = g^{-1} \circ D \circ g$. A connection D is called *completely reducible* if every D -invariant subbundle $F \subset E$ has an D -invariant complement, and D is called *irreducible* if there are no nontrivial D -invariant subbundles. The orbits of the gauge group action on the set of completely reducible flat connections are closed. We define the *de Rham moduli space* of flat connections by

$$\mathcal{M}_{\mathrm{dR}} = \mathcal{D}^{\mathrm{cr}} / \mathcal{G}(E) ,$$

²Since X is fixed throughout, we will henceforth omit it from the notation and write $\Omega^i(*)$ instead of $\Omega^i(X, *)$.

where $\mathcal{D}^{\text{cr}} \subset \mathcal{D}$ denotes the set of completely reducible flat connections.

The complex structure J and the symplectic form $\omega_J^{\mathbb{C}}$ are clearly preserved by the gauge group action. In fact, J gives $\mathcal{M}_{\text{dR}}$ the structure of a (singular) complex analytic space which is Stein and has complex dimension $(n^2 - 1)(2g - 2)$ (see [18], Props. 6.1, 7.8, and Cor. 11.7). The smooth locus of $\mathcal{M}_{\text{dR}}$ consists of equivalence classes of irreducible representations and the symplectic form $\omega_J^{\mathbb{C}}$ defines a holomorphic symplectic form on the smooth locus.

2.2. The Higgs bundle moduli space. We now describe the Higgs bundle moduli space and its holomorphic symplectic structure. Throughout this paper we identify the set of holomorphic structures on $E \rightarrow X$ with the set $\mathcal{A} = \mathcal{A}(E)$ of $\bar{\partial}$ -operators $\bar{\partial}_E : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$. Holomorphic bundles (or their associated coherent sheaves) will sometimes be denoted by $\mathcal{E}$, and when we wish to emphasize this correspondence we will write $\mathcal{E} = (E, \bar{\partial}_E)$. Recall that we always assume that $\det E$ has a fixed trivialization and that $\bar{\partial}$ -operators induce on $\det E$ the $\bar{\partial}$ -operator compatible with the trivialization. In other notation, there is a fixed isomorphism $\Lambda^n \mathcal{E} \cong \mathcal{O}_X$.

Let $K \rightarrow X$ denote the canonical line bundle of X .

Definition 2.1. An $\text{SL}(n, \mathbb{C})$ -Higgs bundle on X is a pair $(\bar{\partial}_E, \Phi)$ where $\Phi : E \rightarrow E \otimes K$ is holomorphic: $\bar{\partial}_E \Phi = 0$, and traceless: $\text{Tr}(\Phi) = 0$.

The holomorphic section Φ is called the *Higgs field*. Let $\mathcal{H} = \mathcal{H}(E, \text{SL}(n, \mathbb{C}))$ denote the set of $\text{SL}(n, \mathbb{C})$ -Higgs bundles on E . Since $\mathcal{A}$ is an affine space modeled on the vector space $\Omega^{0,1}(\mathfrak{sl}(E))$, the tangent space to $\mathcal{H}$ at a Higgs bundle $(\bar{\partial}_E, \Phi)$ is the set of $(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$ such that, to first order, $\Phi + \varphi$ is holomorphic with respect to $\bar{\partial}_E + \beta$. That is,

$$T_{(\bar{\partial}_E, \Phi)} \mathcal{H} = \{(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \mid \bar{\partial}_E \varphi + [\Phi, \beta] = 0\}.$$

The space $\mathcal{H}$ carries a natural complex structure I defined by

$$I(\beta, \varphi) = (i\beta, i\varphi).$$

Moreover, the alternating form

$$(2.2) \quad \omega_I^{\mathbb{C}}((\beta_1, \varphi_1), (\beta_2, \varphi_2)) = i \int_X \text{Tr}(\varphi_2 \wedge \beta_1 - \varphi_1 \wedge \beta_2)$$

defines a (I-)holomorphic symplectic form on $\mathcal{H}$.

The gauge group $\mathcal{G}(E)$ acts on $\mathcal{H}$ by the adjoint action:

$$(\bar{\partial}_E, \Phi) \cdot g = (g^{-1} \circ \bar{\partial}_E \circ g, g^{-1} \circ \Phi \circ g).$$

To form the relevant moduli space of Higgs bundles we need the notion of stability.

Definition 2.2. A Higgs bundle $(\bar{\partial}_E, \Phi)$, $\mathcal{E} = (E, \bar{\partial}_E)$, is called

- *semistable* if for all Φ -invariant subbundles $\mathcal{F} \subset \mathcal{E}$ we have $\deg(\mathcal{F}) \leq 0$,
- *stable* if for all Φ -invariant subbundles $\mathcal{F} \subset \mathcal{E}$, $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$, we have $\deg(\mathcal{F}) < 0$, and
- *polystable* if it is semistable and a direct sum of Higgs bundles $(\mathcal{E}_j, \Phi_j)$ so that $\deg(\mathcal{E}_j) = 0$ for all j and each $(\mathcal{E}_j, \Phi_j)$ is stable.

The orbits of the gauge group action on the set of polystable Higgs bundles are closed. We define the *Higgs bundle moduli space* by

$$\mathcal{M}_H = \mathcal{H}^{\text{ps}} / \mathcal{G}(E) ,$$

where $\mathcal{H}^{\text{ps}} \subset \mathcal{H}$ is the set of polystable Higgs bundles.

The complex structure I and the symplectic form $\omega_I^{\mathbb{C}}$ are clearly preserved by the gauge group action. In fact, the complex structure I gives $\mathcal{M}_H$ the structure of a (singular) complex analytic space which is a normal, quasiprojective variety of dimension $(n^2 - 1)(2g - 2)$ [7, 14, 18]. The smooth locus of $\mathcal{M}_H$ consists of equivalence classes of stable Higgs bundles and the symplectic form $\omega_I^{\mathbb{C}}$ defines a holomorphic symplectic form on the smooth locus.

Another key feature of the Higgs bundle moduli space is the Hitchin fibration

$$(2.3) \quad \mathcal{M}_H \longrightarrow \bigoplus_{j=1}^{n-1} H^0(K^{j+1}) : [(\bar{\partial}_E, \Phi)] \mapsto (p_1(\Phi), \dots, p_{n-1}(\Phi)) .$$

Here $p_1, \dots, p_{n-1}$ is a basis of the invariant polynomials $\mathbb{C}[\mathfrak{sl}(n, \mathbb{C})]^{\text{SL}(n, \mathbb{C})}$ with $\deg(p_j) = j + 1$. In [9], Hitchin showed that the above map is proper and gives $\mathcal{M}_H$ the structure of a complex integrable system. The feature of the above fibration relevant us is the existence of a section $\bigoplus_{j=1}^{n-1} H^0(K^{j+1}) \rightarrow \mathcal{M}_H$ of the fibration above which is called the *Hitchin section* (see Section 2.7). Its image defines a half dimensional $\omega_I^{\mathbb{C}}$ -Lagrangian submanifold of $\mathcal{M}_H$ (cf. [8]).

2.3. Nonabelian Hodge correspondence. For reference, we state here the fundamental relationship between Higgs bundles and flat connections mentioned in the Introduction and used throughout the paper. It was established by Hitchin [7] and Donaldson [4] when $\text{rk}(E) = 2$, and Simpson [17] and Corlette [2] in general.

Theorem 2.3 (NHC). *A Higgs bundle $(\bar{\partial}_E, \Phi)$ is polystable if and only if there exists a hermitian metric h on E such that*

$$(2.4) \quad F_{(\bar{\partial}_E, h)} + [\Phi, \Phi^{*h}] = 0 ,$$

where $F_{(\bar{\partial}_E, h)}$ is the curvature of the Chern connection $\bar{\partial}_E + \partial_E^h$ associated to the pair $(\bar{\partial}_E, h)$, and Φ^{*h} is the h -adjoint of Φ . Moreover, the connection D defined by

$$(2.5) \quad D = \bar{\partial}_E + \partial_E^h + \Phi + \Phi^{*h} ,$$

is a completely reducible flat connection which is irreducible if and only if $(\bar{\partial}_E, \Phi)$ is stable. Conversely, a flat connection D is completely reducible if and only if there exists a hermitian metric h on E so that when we express D (uniquely) in the form (2.5), then $\bar{\partial}_E \Phi = 0$. In this case, $(\bar{\partial}_E, \Phi)$ is a polystable Higgs bundle, and it is stable if and only if D is irreducible.

As we have already done in the Introduction, the hermitian metric h solving (2.4) is referred to as the *harmonic metric* associated to the Higgs bundle $(\bar{\partial}_E, \Phi)$.

2.4. The hyperkähler structure. Given a hermitian metric h on the bundle E , the Chern connection identifies the space of $\bar{\partial}$ -operators with the space of unitary connections $\mathcal{A} = \mathcal{A}(E, h)$ on E . Using this identification, both the space $\mathcal{D}$ of flat connections and the space $\mathcal{H}$ of Higgs bundles may be viewed as subsets of the total space of the cotangent bundle $T^*\mathcal{A}$. Since $\mathcal{A}$ is Kähler and affine,

$T^*\mathcal{A}$ has a natural hyperkähler structure, and $\mathcal{D}$ and $\mathcal{H}$ are zeros of (different) complex moment maps for actions by the group $\mathcal{K} = \mathcal{K}(E, h)$ of unitary gauge transformations.

To be more precise, we may view $T^*\mathcal{A}$ as the set of pairs (d_E, Ψ) , where $d_E \in \mathcal{A}$ and $\Psi \in \Omega^1(i\mathfrak{su}(E))$, and $i\mathfrak{su}(E)$ is the bundle of hermitian endomorphisms of E . Any connection D on E decomposes uniquely as $D = d_E + \Psi$, where d_E is a unitary connection (with respect to h) and $\Psi \in \Omega^1(i\mathfrak{su}(E))$. This realizes $\mathcal{D} \subset T^*\mathcal{A}$. Similarly, using the real isomorphism $\Omega^1(\mathfrak{sl}(E)) = \Omega^{0,1}(\mathfrak{su}(E))$, we can associate to a Higgs bundle $(\bar{\partial}_E, \Phi)$ the pair (d_E, Ψ) , where $d_E = \bar{\partial}_E + \partial_E^h$ is the Chern connection of $(E, \bar{\partial}_E)$, and $\Psi = \Phi + \Phi^{*h}$. This realizes $\mathcal{H} \subset T^*\mathcal{A}$.

At the level of tangent vectors,

$$\begin{aligned} T(T^*\mathcal{A}) &= \Omega^1(\mathfrak{su}(E)) \oplus \Omega^1(i\mathfrak{su}(E)) \\ (2.6) \qquad &= \Omega^1(\mathfrak{sl}(E)) \end{aligned}$$

$$(2.7) \qquad = \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$$

Under (2.6), $\mu \in \Omega^1(\mathfrak{sl}(E))$ maps to (α, ψ) , where

$$\alpha = \frac{1}{2}(\mu - \mu^*) \qquad \text{and} \qquad \psi = \frac{1}{2}(\mu + \mu^*) .$$

The complex structure on X induces the identification (2.7). For $(\beta, \varphi) \in T_{(\bar{\partial}_E, \Phi)}\mathcal{H}$ the identification is given by $\mu \mapsto (\beta, \varphi)$, where

$$\beta = \alpha^{0,1} \qquad \text{and} \qquad \varphi = \psi^{1,0} .$$

With this understood, we may write down the hyperkähler structure: J is multiplication by i on $T(T^*\mathcal{A})$ using (2.6), whereas I is multiplication by i on $T(T^*\mathcal{A})$ using (2.7). The composition $K = IJ$ defines another complex structure. Explicitly, extend the Hodge star operator $\star$ on X to $\Omega^1(\mathfrak{sl}(E))$ as follows: in a local holomorphic coordinate z , write $\mu \in \Omega^1(\mathfrak{sl}(E))$ as $\mu = Adz + Bd\bar{z}$ and set

$$\bar{\star}(\mu) = -iA^*d\bar{z} + iB^*dz .$$

Then I , J , and K are given by:

$$(2.8) \qquad \begin{aligned} I(\mu) &= \bar{\star}\mu \\ J(\mu) &= i\mu \\ K(\mu) &= -i\bar{\star}\mu . \end{aligned}$$

In terms of (β, φ) , we have

$$\begin{aligned} I(\beta, \varphi) &= (i\beta, i\varphi) \\ J(\beta, \varphi) &= (i\varphi^*, -i\beta^*) \\ K(\beta, \varphi) &= (-\varphi^*, \beta^*) . \end{aligned}$$

Remark 2.4. The hyperkähler structure discussed above descends to the moduli space $\mathcal{M}_H$. Indeed, the NHC, Theorem 2.3, may be recast as the statement that

$$\mathcal{M}_H \simeq (\mathcal{H}^{\text{ps}} \cap \mathcal{D})/\mathcal{K} ,$$

and $\mathcal{K}$ preserves all three complex structures.

Remark 2.5. Following the terminology introduced in [10], the letter “A” will be attached to Lagrangian submanifolds, and “B” to holomorphic submanifolds. Three letters together refer to the three complex structures I , J , and K , in that order. For example, a submanifold of $\mathcal{M}_H$ is of type (BAA) if it is holomorphic with

respect to I , and Lagrangian for $\omega_I^{\mathbb{C}}$, whereas it is of type (ABA) if it is holomorphic with respect to J and Lagrangian with respect to $\omega_J^{\mathbb{C}}$.

2.5. Fixed points of the $\mathbb{C}^*$ -action. The Higgs bundle moduli space $\mathcal{M}_H$ carries a holomorphic $\mathbb{C}^*$ -action which scales the Higgs field:

$$\xi \cdot [(\bar{\partial}_E, \Phi)] = [(\bar{\partial}_E, \xi\Phi)] .$$

Definition 2.6. A Higgs bundle $(\bar{\partial}_E, \Phi)$ is said to be a *complex variation of Hodge structure* (VHS) if $[(\bar{\partial}_E, \Phi)]$ is a fixed point of the $\mathbb{C}^*$ -action on $\mathcal{M}_H$. The notation VHS_α will mean a connected component of the fixed point set, labeled by some index α .

By properness of the Hitchin fibration (2.3), the limit $\lim_{\xi \rightarrow 0} [(\bar{\partial}_E, \xi\Phi)]$ always exists in $\mathcal{M}_H$ and is a VHS. Note that $[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H$ is a VHS if and only if there exists a $\mathbb{C}^*$ -family of gauge transformations $g(\xi)$ so that

$$(\bar{\partial}_E, \Phi) \cdot g(\xi) = (\bar{\partial}_E, \xi\Phi) .$$

In particular, $\lim_{\xi \rightarrow 0} [(\bar{\partial}_E, \xi\Phi)] = [(\bar{\partial}_E, 0)]$ if and only if the holomorphic bundle $(E, \bar{\partial}_E)$ is polystable. The following characterizes general fixed points (see [7, Sec. 7]).

Proposition 2.7. *A point $[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H$ is a fixed point of the $\mathbb{C}^*$ -action if and only if there is a splitting $E = E_1 \oplus \cdots \oplus E_\ell$ with respect to which:*

$$(2.9) \quad \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E_1} & & & \\ & \ddots & & \\ & & \bar{\partial}_{E_{\ell-1}} & \\ & & & \bar{\partial}_{E_\ell} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & & & & \\ \Phi_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \Phi_{\ell-1} & 0 \end{pmatrix} ,$$

where $\Phi_j : E_j \rightarrow E_{j+1} \otimes K$ is holomorphic. Note that we allow $\ell = 1$, in which case $\Phi = 0$.

Example 2.8. The *Fuchsian* or *uniformizing* Higgs bundle is defined by setting $E_j = K^{\frac{n+1}{2}-j}$, and (since $E_{j+1} \otimes K = E_j$) taking $\Phi_j = 1$.

The next result characterizes the flat connections associated to polystable Higgs bundles of the form (2.9).

Proposition 2.9. *If $(\bar{\partial}_E, \Phi)$ is a polystable Higgs bundle of the form (2.9), then the splitting is orthogonal with respect to the harmonic metric h . In particular, the associated flat connection $D = \bar{\partial}_E + \partial_E^h + \Phi + \Phi^{*h}$ is given by*

$$\begin{pmatrix} \bar{\partial}_{E_1} & \Phi_1^{*h} & & & \\ & \ddots & \ddots & & \\ & & \bar{\partial}_{E_{\ell-1}} & \Phi_{\ell-1}^{*h} & \\ & & & \bar{\partial}_{E_\ell} & \end{pmatrix} + \begin{pmatrix} \partial_{E_1}^h & & & & \\ \Phi_1 & \partial_{E_2}^h & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \Phi_{\ell-1} & \partial_{E_\ell}^h \end{pmatrix} ,$$

where $\Phi_j^{*h} : E_{j+1} \rightarrow E_j \otimes \bar{K}$ is the adjoint of Φ_j with respect to the induced metrics on E_j and E_{j+1} .

The splitting of E in Proposition (2.7) gives a $\mathbb{Z}$ -grading on $\text{End}(E)$, where

$$(2.10) \quad \text{End}_j(E) := \bigoplus_{i-k=j} \text{Hom}(E_i, E_k) ,$$

for $1 - \ell \leq j \leq \ell - 1$, and we set $\text{End}_j(E) = \{0\}$ otherwise. We also introduce the following notation:

$$(2.11) \quad N_+ = \bigoplus_{j>0} \text{End}_j(E), \quad N_- = \bigoplus_{j<0} \text{End}_j(E) \quad \text{and} \quad L = \text{End}_0(E) \cap \mathfrak{sl}(E).$$

2.6. λ -connections. Since $\mathcal{M}_{\text{dR}}$ is hyperkähler, its twistor space $\mathcal{M}_{\text{dR}} \times \mathbb{P}^1$ is a complex manifold. The restriction to $\mathcal{M}_{\text{dR}} \times \mathbb{C}$ can equivalently be described as the moduli space of λ -connections, which we briefly describe here (see [15] for details).

Definition 2.10. A λ -connection on X is a triple $(\lambda, \bar{\partial}_E, \nabla_\lambda)$, where $\lambda \in \mathbb{C}$, $\bar{\partial}_E$ is a $\bar{\partial}$ -operator on E , and $\nabla_\lambda : \Omega^0(E) \rightarrow \Omega^{1,0}(E)$ is a differential operator satisfying

- a λ -scaled Leibniz rule:

$$\nabla_\lambda(fs) = \lambda \cdot \partial f \otimes s + f \cdot \nabla_\lambda s,$$

for all smooth functions f and $s \in \Omega^0(E)$, and

- compatibility with the holomorphic structure: $[\bar{\partial}_E, \nabla_\lambda] = 0$.

Thus, when $\lambda = 1$, a λ -connection is just a holomorphic connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes K$ on the holomorphic bundle $\mathcal{E} = (E, \bar{\partial}_E)$ defined by $\bar{\partial}_E$, which is equivalent to a flat connection $D = \bar{\partial}_E + \nabla$ on E . When $\lambda = 0$, a λ -connection is an endomorphism $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes K$, and the data $(\bar{\partial}_E, \Phi)$ gives a Higgs bundle. Isomorphisms of λ -connections are defined via complex gauge transformations just as with holomorphic connections. The moduli space of (polystable) λ -connections $[(\lambda, \bar{\partial}_E, \nabla_\lambda)]$ is denoted by $\mathcal{M}_{\text{Hod}}$. The tautological map

$$\mathcal{M}_{\text{Hod}} \longrightarrow \mathbb{C} : [(\lambda, \bar{\partial}_E, \nabla_\lambda)] \mapsto \lambda$$

will (somewhat abusively) be called λ . From the comments above, $\lambda^{-1}(0) = \mathcal{M}_{\text{H}}$ and $\lambda^{-1}(1) = \mathcal{M}_{\text{dR}}$.

There is a $\mathbb{C}^*$ -action on $\mathcal{M}_{\text{Hod}}$ given by

$$(2.12) \quad \xi \cdot [(\lambda, \bar{\partial}_E, \nabla_\lambda)] = [(\xi\lambda, \bar{\partial}_E, \xi\nabla_\lambda)].$$

The only fiber preserved by this action is $\lambda^{-1}(0)$, and the restriction to $\lambda^{-1}(0)$ is the $\mathbb{C}^*$ -action on Higgs bundles of the previous section. Hence, the fixed points on $\mathcal{M}_{\text{Hod}}$ are exactly the locus of complex variations of Hodge structures.

In [15], Simpson shows that the limit as $\xi \rightarrow 0$ in (2.12) defines a Białynicki-Birula-type stratification of $\mathcal{M}_{\text{Hod}}$. If $\coprod_{\alpha} \text{VHS}_{\alpha}$ is the decomposition of VHS into connected components, then define

$$W_{\alpha} = \{[(\lambda, \bar{\partial}_E, \nabla_\lambda)] \in \mathcal{M}_{\text{Hod}} \mid \lim_{\xi \rightarrow 0} [(\xi\lambda, \bar{\partial}_E, \xi\nabla_\lambda)] \in \text{VHS}_{\alpha}\}.$$

Each W_{α} is connected and foliated by $\mathbb{C}^*$ -orbits, and $\mathcal{M}_{\text{Hod}} = \coprod_{\alpha} W_{\alpha}$. The restrictions $W_{\alpha}^0 = W_{\alpha} \cap \lambda^{-1}(0)$ and $W_{\alpha}^1 = W_{\alpha} \cap \lambda^{-1}(1)$ define stratifications of $\mathcal{M}_{\text{H}}$ and $\mathcal{M}_{\text{dR}}$ respectively:

$$\mathcal{M}_{\text{H}} = \coprod_{\alpha} W_{\alpha}^0 \quad \text{and} \quad \mathcal{M}_{\text{dR}} = \coprod_{\alpha} W_{\alpha}^1.$$

Remark 2.11. The open stratum W_{open} fibers over the component of VHS associated to the moduli space of polystable bundles. Namely,

$$\lim_{\xi \rightarrow 0} [(\xi\lambda, \bar{\partial}_E, \xi\nabla_\lambda)] = [(\bar{\partial}_E, 0)]$$

if and only if the holomorphic bundle $(E, \bar{\partial}_E)$ is polystable. In this case, the open stratum $W_{\text{open}}^0 \subset \mathcal{M}_H$ is the cotangent sheaf of the moduli space of polystable bundles.

The map $\mathcal{T}$ from (1.1) actually produces a family of flat connections

$$\bar{\partial}_E + \partial_E^h + \xi^{-1}\Phi + \xi\Phi^{*h}$$

parametrized by $\xi \in \mathbb{C}^*$. If we furthermore use the $\mathbb{C}^*$ -action (2.12) on λ -connections we obtain

$$(2.13) \quad \xi \cdot [(1, \bar{\partial}_E + \xi\Phi^{*h}, \partial_E + \xi^{-1}\Phi)] = [(\xi, \bar{\partial}_E + \xi\Phi^{*h}, \xi\partial_E + \Phi)] .$$

Moreover,

$$\lim_{\xi \rightarrow 0} [(\xi, \bar{\partial}_E + \xi\Phi^{*h}, \xi\partial_E + \Phi)] = [(\bar{\partial}_E, \Phi)] .$$

The extension of the $\mathbb{C}^*$ -family (2.13) to $\mathbb{C}$ is an example of a *real twistor line* in $\mathcal{M}_{\text{Hod}}$. Thus, the above real twistor line through a Higgs bundle $[(\bar{\partial}_E, \Phi)]$ is a section of $\mathcal{M}_{\text{Hod}}$ which interpolates between the Higgs bundle and the flat connection $\mathcal{T}(\bar{\partial}_E, \Phi)$ from the nonabelian Hodge correspondence.

The real twistor line through a Higgs bundle $[(\bar{\partial}_E, \Phi)]$ and the $\mathbb{C}^*$ -orbit through $\mathcal{T}(\bar{\partial}_E, \Phi)$ coincide if and only if $(\bar{\partial}_E, \Phi)$ is a VHS. In particular,

$$\mathcal{T}(\bar{\partial}_E, \Phi) \in W_\alpha^1(\bar{\partial}_E, \Phi) \text{ if } [(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha .$$

However, the nonabelian Hodge correspondence does not preserve the stratification in general. That is, $\mathcal{T}$ does not map W_α^0 to W_α^1 .

2.7. The Hitchin section and opers. We now recall the explicit parameterizations of the Hitchin section and the space of opers by the vector space $\bigoplus_{j=1}^{n-1} H^0(K^{j+1})$.

The starting point is the uniformizing rank two VHS from Example 2.8

$$(2.14) \quad \left(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) .$$

Taking the $(n-1)^{\text{st}}$ -symmetric product, renormalizing and twisting by an n^{th} -root of the the trivial bundle defines the following VHS

$$(2.15) \quad (E, \bar{\partial}_0, \Phi_0) = \left(L \oplus (L \otimes K^{-1}) \oplus \cdots \oplus (L \otimes K^{1-n}), \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \right) ,$$

where $L^n \otimes K^{\frac{n(n-1)}{2}} = \mathcal{O}$. Note that the set of such VHS is finite of cardinality n^{2g} . These are called *Fuchsian points*, since they are obtained from the Higgs bundle (2.14) whose associated flat connection corresponds to the Fuchsian representation that uniformizes the Riemann surface X (cf. [7]).

By Proposition 2.9, the flat connection $\mathcal{T}(\bar{\partial}_0, \Phi_0) = \bar{\partial}_0 + \Phi_0^{*h} + \partial_0^h + \Phi_0$ is given by

$$\begin{pmatrix} \bar{\partial}_L & 1^{*h} & & \\ & \ddots & \ddots & \\ & & \bar{\partial}_{L \otimes K^{n-2}} & 1^{*h} \\ & & & \bar{\partial}_{L \otimes K^{n-1}} \end{pmatrix} + \begin{pmatrix} \partial_L^h & & & \\ 1 & \partial_{L \otimes K^{-1}}^h & & \\ & \ddots & \ddots & \\ & & 1 & \partial_{L \otimes K^{n-1}}^h \end{pmatrix} ,$$

As in (2.10), the holomorphic splitting of E gives a decomposition $\bigoplus_{j=1-n}^{n-1} \text{End}_j(E)$ of the endomorphism bundle. For the Higgs bundle (2.15) we have

$$\text{End}_j(E) \cong \underbrace{K^j \oplus \cdots \oplus K^j}_{n-|j|}.$$

With respect to this splitting we have

$$\text{ad}_{\Phi_0^{*h}} : \text{End}_j(E) \otimes K \rightarrow \text{End}_{j+1}(E) \otimes K \bar{K}.$$

Moreover, $\ker(\text{ad}_{\Phi_0^{*h}}) \simeq K^{j+1}$ for $1 \leq j \leq n-1$. Set

$$V_j \otimes K = \ker(\text{ad}_{\Phi_0^{*h}}) \simeq K^{j+1} \hookrightarrow \text{End}_j(E) \otimes K.$$

With the above notation and a fixed choice of L , consider the maps

$$(2.16) \quad \begin{aligned} p_{\text{Hit}} : \bigoplus_{j=1}^{n-1} H^0(K^{j+1}) &\longrightarrow \mathcal{M}_{\text{H}} : \\ (q_2, \dots, q_n) &\mapsto [(\bar{\partial}_0, \Phi_0 + q_2 + \cdots + q_n)] , \\ p_{\text{oper}} : \bigoplus_{j=1}^{n-1} H^0(K^{j+1}) &\longrightarrow \mathcal{M}_{\text{dR}} : \\ (q_2, \dots, q_n) &\mapsto [(\bar{\partial}_0 + \Phi_0^{*h}, \partial_0^h + \Phi_0 + q_2 + \cdots + q_n)] , \end{aligned}$$

where for $1 \leq j \leq n-1$ and $q_{j+1} \in H^0(V_j \otimes K) \subset H^0(\text{End}_j(E) \otimes K)$.

The following theorem was proven in [8] for p_{Hit} and in [1] for p_{oper} .

Theorem 2.12. *Let $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}$ be given by (2.15) and let p_{Hit} and p_{oper} be as in (2.16).*

- (1) *The map p_{Hit} is a holomorphic embedding onto $W^0(\bar{\partial}_0, \Phi_0)$, which is closed in $\mathcal{M}_{\text{H}}$.*
- (2) *The map p_{oper} is a holomorphic embedding onto $W^1(\bar{\partial}_0, \Phi_0)$, which is closed in $\mathcal{M}_{\text{dR}}$.*

Remark 2.13. For each choice of line bundle L , the image of p_{Hit} is called a *Hitchin component* and the image of the map p_{oper} is called a *connected component* of the space of *opers*.

3. DEFORMATION THEORY

3.1. The Hodge slice. In this section we describe local Kuranishi models for $\mathcal{M}_{\text{dR}}$ and $\mathcal{M}_{\text{H}}$. Let $(\bar{\partial}_E, \Phi)$ be a polystable Higgs bundle with harmonic metric h and Chern connection $\bar{\partial}_E + \partial_E^h$. Thus, the connection $D = \bar{\partial}_E + \partial_E^h + \Phi^{*h} + \Phi$ is flat. Since h will be fixed throughout this section we omit it from the notation.

Set

$$D = D' + D'' , \quad D'' := \bar{\partial}_E + \Phi \quad \text{and} \quad D' := \partial_E + \Phi^* .$$

A key fact is that these operators satisfy the usual Kähler identities:

$$(3.1) \quad (D'')^* = -i[\Lambda, D'] \quad \text{and} \quad (D')^* = +i[\Lambda, D''] .$$

The *deformation complex* is then given by:

$$(3.2) \quad C(\bar{\partial}_E, \Phi) : \Omega^0(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{sl}(E)) .$$

By (3.1), $\ker(D'')^* = \ker D'$, so the harmonic representation of $H^1(C(\bar{\partial}_E, \Phi))$ is

$$(3.3) \quad \mathcal{H}^1(\bar{\partial}_E, \Phi) := \{(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \mid D''(\beta, \varphi) = 0, D'(\beta, \varphi) = 0\}.$$

At a stable Higgs bundle $(\bar{\partial}_E, \Phi)$, we have $H^i(C(\bar{\partial}_E, \Phi)) = \{0\}$, $i = 0, 2$, and

$$T_{[(\bar{\partial}_E, \Phi)]} \mathcal{M}_H \simeq \mathcal{H}^1(\bar{\partial}_E, \Phi).$$

Note that:

$$D^* = (D')^* + (D'')^* = -i\Lambda D^c,$$

where $D^c = D'' - D'$. Via the identification $\mu = \beta + \varphi$, we have also

$$\mathcal{H}^1(\bar{\partial}_E, \Phi) = \{\mu \in \Omega^1(\mathfrak{sl}(E)) \mid D\mu = 0, D^*\mu = 0\},$$

which for irreducible D also gives the identification:

$$T_{[D]} \mathcal{M}_{\text{dR}} \simeq \mathcal{H}^1(\bar{\partial}_E, \Phi).$$

Definition 3.1. The *Hodge slice* (or simply *slice*) at $(\bar{\partial}_E, \Phi) \in \mathcal{H}$ is given by

$$\mathcal{S}_{(\bar{\partial}_E, \Phi)} = \{(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \mid D''(\beta, \varphi) + [\beta, \varphi] = 0, D'(\beta, \varphi) = 0\}.$$

Notice that $\mathcal{S}_{(\bar{\partial}_E, \Phi)} \subset \mathcal{H}$ under the identification $(\beta, \varphi) \mapsto (\bar{\partial}_E + \beta, \Phi + \varphi)$. Moreover, at points where $\mathcal{H}$ is smooth and $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ is a submanifold, then $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ is a holomorphic submanifold. By direct computation we have the following:

Lemma 3.2. *The map $(\beta, \varphi) \mapsto D + \beta + \varphi$ defines a locally closed, holomorphic embedding $\mathcal{S}_{(\bar{\partial}_E, \Phi)} \hookrightarrow \mathcal{D}$.*

We shall often view $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ as a subset of $\mathcal{D}$ via this embedding.

Remark 3.3. The subset $\mathcal{S}_{(\bar{\partial}_E, \Phi)} \subset \mathcal{D}$ does not coincide with the usual *de Rham slice* to $\mathcal{D}$ at D . The latter is defined by requiring $D + \beta + \varphi$ to be flat and $D^*(\beta + \varphi) = 0$. The first of these equations is satisfied by points in $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$. The second equation, however, is equivalent to $D^c(\beta, \varphi) = 0$, whereas the second equation defining $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ is $D'(\beta, \varphi) = 0$. Of course, the tangent spaces at the origin of the two slices agree.

Let

$$(3.4) \quad \begin{aligned} p_H : \mathcal{S}_{(\bar{\partial}_E, \Phi)} \cap \mathcal{H}^{\text{ps}} &\longrightarrow \mathcal{M}_H : (\beta, \varphi) \mapsto [(\bar{\partial}_E + \beta, \Phi + \varphi)], \\ p_{\text{dR}} : \mathcal{S}_{(\bar{\partial}_E, \Phi)} \cap \mathcal{D}^{\text{cr}} &\longrightarrow \mathcal{M}_{\text{dR}} : (\beta, \varphi) \mapsto [D + \beta + \varphi]. \end{aligned}$$

be the projection maps to moduli. Note that if $(\bar{\partial}_E, \Phi)$ is stable (or equivalently, D is irreducible), then the maps p_H and p_{dR} are well-defined in a neighborhood of the origin in $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$. For completeness, we state without proof the following result which shows that the slice defines a local coordinate chart for the moduli space (cf. [11, Thm. 7.3.17]). We shall prove a stronger statement for a restricted slice in the next section.

Proposition 3.4. *If $(\bar{\partial}_E, \Phi)$ is stable, then p_H (resp. p_{dR}) is a homeomorphism from a neighborhood of the origin in $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ to a neighborhood of $[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H$ (resp. $[D] \in \mathcal{M}_{\text{dR}}$).*

The *Kuranishi map* is defined as follows:

$$(3.5) \quad k(\beta, \varphi) := (\beta, \varphi) + (D'')^* G([\beta, \varphi]) ,$$

where G is the Green's operator for the Laplacian $D''(D'')^* = iD''D'\Lambda$ (see (3.1)) acting on $\Omega^{1,1}(\mathfrak{sl}(E))$. Notice that if $(\beta, \varphi) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}$, then $[\beta, \varphi] \perp \mathcal{H}^2(C(\bar{\partial}, \Phi))$ and $k(\beta, \varphi) \in \mathcal{H}^1(C(\bar{\partial}, \Phi))$. By the implicit function theorem we have (cf. [11, Thm. 7.3.23]):

Proposition 3.5. *If $(\bar{\partial}_E, \Phi)$ is stable, then $k : \mathcal{S}_{(\bar{\partial}_E, \Phi)} \rightarrow \mathcal{H}^1(C(\bar{\partial}, \Phi))$ is a homeomorphism in a neighborhood of the origin.*

3.2. The BB-slice at a VHS. Suppose $(\bar{\partial}_E, \Phi)$ is a VHS as in Section 2.5, and recall the notation of (2.10) and (2.11). Since $\Phi \in H^0(\text{End}_{-1}(E) \otimes K)$, we have

$$\text{ad}_\Phi : \text{End}_j(E) \rightarrow \text{End}_{j-1}(E) \otimes K .$$

Since $\bar{\partial}_E$ preserves the grading, for each j we get a subcomplex $C_j(\bar{\partial}_E, \Phi)$ of (3.2) given by:

$$\Omega^0(\text{End}_j(E)) \xrightarrow{D''} \Omega^{0,1}(\text{End}_j(E)) \oplus \Omega^{1,0}(\text{End}_{j-1}(E)) \xrightarrow{D''} \Omega^{1,1}(\text{End}_{j-1}(E)) .$$

Let $\mathcal{H}_j^1(\bar{\partial}_E, \Phi) := \ker D'' \cap \ker D'$ for the middle term in $C_j(\bar{\partial}_E, \Phi)$. This gives a grading $\mathcal{H}^1(\bar{\partial}_E, \Phi) = \bigoplus_{1-\ell}^{\ell} \mathcal{H}_j^1(\bar{\partial}_E, \Phi)$. We will use the following notation:

$$(3.6) \quad \mathcal{H}_+^1(\bar{\partial}_E, \Phi) = \bigoplus_{j=1}^{\ell} \mathcal{H}_j^1(\bar{\partial}_E, \Phi) .$$

In particular, $\mathcal{H}_+^i(\bar{\partial}_E, \Phi) \simeq H^i(C_+(\bar{\partial}_E, \Phi))$ for the subcomplex $C_+(\bar{\partial}_E, \Phi)$ of (3.2) given by

$$(3.7) \quad \Omega^0(N_+) \xrightarrow{D''} \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+) \xrightarrow{D''} \Omega^{1,1}(L \oplus N_+) .$$

Lemma 3.6. *If $(\bar{\partial}_E, \Phi)$ is a stable VHS, then $\mathcal{H}_+^1(\bar{\partial}_E, \Phi)$ is half-dimensional.*

Proof. Roll up the complex (3.7) to obtain an operator

$$\mathcal{D} := (D'')^* + D'' : \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+) \longrightarrow \Omega^0(N_+) \oplus \Omega^{1,1}(L \oplus N_+) .$$

Since $(\bar{\partial}_E, \Phi)$ is stable, $H^0(C_+(\bar{\partial}_E, \Phi)) = H^2(C_+(\bar{\partial}_E, \Phi)) = \{0\}$, and so

$$\dim H^1(C_+(\bar{\partial}_E, \Phi)) = \text{index } \mathcal{D} .$$

Deforming the Higgs field to zero does not change the index but it does decouple the operators. We therefore have

$$\text{index } \mathcal{D} = \text{index } \bar{\partial}_0 - \text{index } \bar{\partial}_1 ,$$

where $\bar{\partial}_0$ is the $\bar{\partial}$ -operator on sections of $(L \oplus N_+) \otimes K$ induced by $\bar{\partial}_E$, and $\bar{\partial}_1$ is similarly the induced $\bar{\partial}$ -operator on sections of N_+ . Since $\deg L = 0$, we have by Riemann-Roch:

$$\begin{aligned} \text{index } \bar{\partial}_0 &= \deg N_+ + (\text{rank } L + \text{rank } N_+)(g-1) \\ \text{index } \bar{\partial}_1 &= \deg N_+ - (\text{rank } N_+)(g-1) . \end{aligned}$$

Notice that

$$n^2 - 1 = \text{rank}(\mathfrak{sl}(E)) = \text{rank } L + 2 \text{rank } N_+ ,$$

so that

$$\text{index } \mathcal{D} = (n^2 - 1)(g - 1) = \frac{1}{2} \dim \mathcal{M}_H .$$

□

Definition 3.7. The BB-slice at $(\bar{\partial}_E, \Phi)$ is defined as

$$\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ = \{(\beta, \varphi) \in \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+) \mid D''(\beta, \varphi) + [\beta, \varphi] = 0, D'(\beta, \varphi) = 0\} .$$

If $(\bar{\partial}_E, \Phi)$ is stable, then $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \subset \mathcal{S}_{(\bar{\partial}_E, \Phi)}$ is a half-dimensional complex submanifold in a neighborhood of the origin.

Example 3.8. Consider a Fuchsian point $(\bar{\partial}_0, \Phi_0)$ from (2.15) and recall the maps p_{Hit} and p_{oper} from (2.16). For $(q_2, \dots, q_n) \in \bigoplus_{j=1}^{n-1} H^0(K^{j+1})$, we have

$$p_{\text{Hit}}(q_2, \dots, q_n) - (\bar{\partial}_0, \Phi_0) = (0, q_2 + \dots + q_n) \in \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+) ,$$

$$p_{\text{oper}}(q_2, \dots, q_n) - (\bar{\partial}_0 + \Phi_0^*, \partial_0^h + \Phi_0) = (0, q_2 + \dots + q_n) \in \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+) .$$

Moreover, both $p_{\text{Hit}} - (\bar{\partial}_0, \Phi_0)$ and $p_{\text{oper}} - (\bar{\partial}_0 + \Phi_0^*, \partial_0^h + \Phi_0)$ map $\bigoplus_{j=1}^{n-1} H^0(K^{j+1})$ bijectively onto both $\mathcal{H}_+^1(\bar{\partial}_0, \Phi_0)$ and $\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+$. Indeed, $\bigoplus_{j=1}^{n-1} H^0(K^{j+1})$ and $\mathcal{H}_+^1(\bar{\partial}_0, \Phi_0)$ have the same dimension and, by definition of the maps p_{Hit} and p_{oper} , the image of each q_j is in the kernel of $\text{ad}_{\Phi_0^* h}$. Hence, we have

$$p_{\text{Hit}}(q_2, \dots, q_n) - (\bar{\partial}_0, \Phi_0) \in \ker(D') \cap \ker(D'') ,$$

$$p_{\text{oper}}(q_2, \dots, q_n) - (\bar{\partial}_0 + \Phi_0^*, \partial_0^h + \Phi_0) \in \ker(D') \cap \ker(D'') .$$

and since $\beta = 0$, these points are automatically in $\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+$ as well. Thus, for a Fuchsian point, the restrictions of the maps p_H and p_{dR} to $\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+$ recover the parameterization of the Hitchin sections and the components of opers by the affine space $\bigoplus_{j=1}^{n-1} H^0(K^{j+1})$.

While $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ gives only a local chart in $\mathcal{M}_H$, we shall see that $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ extends globally. First, we prove

Theorem 3.9. *The Kuranishi map (3.5) gives a biholomorphism*

$$k : \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \xrightarrow{\sim} \mathcal{H}_+^1(\bar{\partial}_E, \Phi) .$$

The proof requires the following

Lemma 3.10. *The $\mathbb{C}^*$ -action on $\Omega^{0,1}(\text{End}(E)) \oplus \Omega^{1,0}(\text{End}(E))$ defined by*

$$(3.8) \quad \xi \cdot \left(\sum_{j=1-\ell}^{\ell-1} \beta_j, \sum_{j=1-\ell}^{\ell-1} \varphi_j \right) = \left(\sum_{j=1-\ell}^{\ell-1} \xi^j \beta_j, \sum_{j=1-\ell}^{\ell-1} \xi^{j+1} \varphi_j \right)$$

preserves both the BB-slice $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ and $\mathcal{H}_+^1(\bar{\partial}_E, \Phi)$.

Proof. The j^{th} -graded piece of $D''(\beta, \varphi) + [\beta, \varphi]$ is given by

$$\bar{\partial}_E \varphi_j + [\beta_{j+1}, \Phi] + \sum_{a+b=j} [\beta_a, \varphi_b]$$

and the $\mathbb{C}^*$ -action (3.8) scales this equation by ξ^{j+1} . Similarly, the j^{th} -graded piece of $D'(\beta, \varphi)$ is given by $\partial_E \beta_j + [\Phi^*, \varphi_{j-1}]$, and the $\mathbb{C}^*$ -action scales this by ξ^j . $\square$

Proof of Theorem 3.9. The map (3.5) is clearly holomorphic. By Proposition 3.5, it is a biholomorphism in a neighborhood of the origin. The $\mathbb{C}^*$ -action defined in (3.8) preserves the slice and $\mathcal{H}_+^1(\bar{\partial}_E, \Phi)$, and the Kuranishi map is equivariant. It follows that the local biholomorphism extends to a global one. $\square$

We now come to an important technical result.

Proposition 3.11. *Assume $(\bar{\partial}_E, \Phi)$ is stable. For each*

$$(\beta, \varphi) \in \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+)$$

satisfying $D''(\beta, \varphi) + [\beta, \varphi] = 0$, there is a unique $f \in \Omega^0(N_+)$ such that the complex gauge transformation $g = \mathbf{I} + f$ takes $(\bar{\partial}_E + \beta, \Phi + \varphi)$ into the slice $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$.

Proof. Write f in terms of its graded pieces, $f = f_1 + \cdots + f_{\ell-1}$, where $f_j \in \text{End}_j(E)$. Then the grading is additive under multiplication. By the phrase “mod j ” we shall mean any expression involving only sections f_k for $k < j$. For example, if we write: $g^{-1} = \mathbf{I} + h_1 + \cdots + h_{\ell-1}$ in terms of graded pieces, then $h_j = -f_j \text{ mod } j$. In the following, suppose $R(f)$ is an algebraic expression in f and its derivatives $\bar{\partial}_E f$, and such that the j -th graded piece $R(f)_j$ depends only on f_k , $k \leq j$. We will say that $R(f)_j = S(f_j) \text{ mod } j$, where S is another such algebraic expression, if $R(f)_j - S(f_j)$ depends only on f_k , $k \leq j-1$.

The strategy is to solve recursively for the graded pieces of f_j of f , starting with $j = 1$. With the conventions of the previous paragraph understood, the effect of the gauge transformation $g = \mathbf{I} + f$ on the graded pieces of the data is:

$$\begin{aligned} \beta_j &\mapsto \beta_j + \bar{\partial}_E(f_j) \text{ mod } j \\ [\Phi^*, \varphi]_j &\mapsto [\Phi^*, \varphi]_j + ([\Phi^*, [\Phi, f]])_j \text{ mod } j. \end{aligned}$$

Hence, we wish to find f with components f_j solving the equation:

$$\partial_E \bar{\partial}_E(f_j) + ([\Phi^*, [\Phi, f]])_j = -(\partial_E \beta + [\Phi^*, \varphi])_j \text{ mod } j$$

But the left hand side is just $D' D''(f_j) \text{ mod } j$. Contracting with $-\sqrt{-1}\Lambda$, and using (3.1), this becomes

$$(3.9) \quad (D'')^* D''(f_j) = -2i\Lambda(\partial_E \beta + [\Phi^*, \varphi])_j \text{ mod } j.$$

By the assumption of stability, $\ker D'' = \{0\}$ on $\Omega^0(\mathfrak{sl}(E))$, and so the operator $(D'')^* D''$ is invertible. Now assuming we have solved for f_k , $k < j$, (3.9) gives a unique solution for f_j . This completes the proof. $\square$

3.3. First variation of the harmonic metric. Let $(\bar{\partial}_E, \Phi)$ be a stable Higgs bundle. For this section, we denote the harmonic metric on $(\bar{\partial}_E, \Phi)$ by h_0 . For each $(\beta, \varphi) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}$, the Higgs bundle $(\bar{\partial}_E + \beta, \Phi + \varphi)$ also admits a harmonic metric in a neighborhood of the origin. We view this as a function h from $\mathcal{S}_{(\bar{\partial}_E, \Phi)}$ to the space $\text{Met}(E)$ of hermitian metrics on E . A standard fact is that h is smooth in a neighborhood of the origin. The following is a generalization of [12, Thm. 3.5.1], and will be used in the proof Theorem 4.6.

Proposition 3.12. *The derivative of the harmonic metric $h : \mathcal{S}_{(\bar{\partial}_E, \Phi)} \rightarrow \text{Met}(E)$ vanishes at the origin.*

Proof. We will consider one parameter variations (in the variable t) in the direction $(\beta, \varphi) \in \mathcal{H}^1(\bar{\partial}_E, \Phi)$ and use notation like $\dot{\mathbf{u}}$ to denote derivatives at $t = 0$. Let $\mathbf{u}(t) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}$ be a one parameter family such that $\mathbf{u}(0) = 0$ and $\dot{\mathbf{u}} = (\beta, \varphi)$. Let $h(t)$ denote the harmonic metric for the Higgs bundle defined by $\mathbf{u}(t)$. Write $h(t) = h_0 H(t)$ for an invertible hermitian endomorphism valued function H , with $\det H(t) = 1$ and $H(0) = \mathbf{I}$. It suffices to show that $\dot{H} = 0$, $H = H(t)$, for one parameter variations in the direction $(\beta, \varphi) \in \mathcal{H}^1(\bar{\partial}_E, \Phi)$. In the following, $*$ will denote adjoints with respect to h_0 , whereas $*_h$ will denote the same for h . Furthermore, $F_{(\bar{\partial}_E, h)}$ denotes the curvature of the Chern connection of $(E, \bar{\partial}_E)$ with respect to h , and $\partial_E := \partial_E^{h_0}$.

We first claim that

$$(3.10) \quad \widehat{\partial_E^h} = \partial_E(\dot{H}) - \beta^* .$$

Indeed, let $\{e_i\}$ be a local h_0 -unitary frame, and define

$$h_{ij} = \langle e_i, e_j \rangle_h = \langle H e_i, e_j \rangle_{h_0} .$$

Then $\dot{h}_{ij} = \langle \dot{H} e_i, e_j \rangle_{h_0}$, so that at $t = 0$,

$$\partial(\dot{h}_{ij}) = \langle \partial_E(\dot{H} e_i), e_j \rangle_{h_0} + \langle \dot{H} e_i, \bar{\partial}_E e_j \rangle_{h_0} .$$

On the other hand,

$$\begin{aligned} \partial h_{ij} &= \langle \partial_E^h e_i, e_j \rangle_h + \langle e_i, \bar{\partial}_E e_j \rangle_h \\ &= \langle H \partial_E^h e_i, e_j \rangle_{h_0} + \langle H e_i, \bar{\partial}_E e_j \rangle_{h_0} \\ \partial(\dot{h}_{ij}) &= \langle \dot{H} \partial_E e_i, e_j \rangle_{h_0} + \langle H \widehat{\partial_E^h} e_i, e_j \rangle_{h_0} + \langle \dot{H} e_i, \bar{\partial}_E e_j \rangle_{h_0} + \langle H e_i, \beta e_j \rangle_{h_0} . \end{aligned}$$

Evaluating at $t = 0$ and equating the two results, we have

$$\langle \partial_E(\dot{H}) e_i, e_j \rangle_{h_0} = \langle \widehat{\partial_E^h} e_i, e_j \rangle_{h_0} + \langle H e_i, \beta e_j \rangle_{h_0} .$$

This proves the claim.

From $\Phi^{*h} = H^{-1} \Phi^* H$ we also have (at $t = 0$)

$$(3.11) \quad \widehat{\Phi^{*h}} = (\dot{\Phi})^* + [\Phi^*, \dot{H}] .$$

Combining (3.10) and (3.11),

$$(3.12) \quad \begin{aligned} \widehat{F_{(\bar{\partial}_E, h)}} &= (\bar{\partial}_E + \partial_E)(\widehat{\bar{\partial}_E} + \widehat{\partial_E^h}) = \partial_E \beta - \bar{\partial}_E \beta^* + \bar{\partial}_E \partial_E \dot{H} , \\ \widehat{[\Phi, \Phi^{*h}]} &= [\varphi, \Phi^*] + [\Phi, \varphi^*] + [\Phi, [\Phi^*, \dot{H}]] . \end{aligned}$$

Since h is the harmonic metric for $(\bar{\partial}_E, \Phi)$, we differentiate (2.4) at $t = 0$. Plugging into (3.12), we obtain

$$0 = D'(\beta, \varphi) - \{D'(\beta, \varphi)\}^* + D'' D'(\dot{H}) .$$

Since $(\beta, \varphi) \in \mathcal{H}^1(\bar{\partial}_E, \Phi)$, the first two terms vanish, and so we have

$$D'' D'(\dot{H}) = 0 \implies D'(\dot{H}) = 0 \implies D''(\dot{H}) = 0 ,$$

since $\dot{H}$ is hermitian. But $(\bar{\partial}_E, \Phi)$ is stable and $\dot{H}$ is traceless, so this implies $\dot{H} = 0$. $\square$

4. STRATIFICATIONS

4.1. The stable manifold of a VHS in $\mathcal{M}_H$. For a fixed point $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_\alpha \subset \mathcal{M}_H$, the *stable manifold* is defined by

$$W_\alpha^0(\bar{\partial}_0, \Phi_0) = \left\{ [(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H \mid \lim_{\xi \rightarrow 0} [(\bar{\partial}_E, \xi \Phi)] = [(\bar{\partial}_0, \Phi_0)] \right\}.$$

Since stability is an open condition, $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ is a smooth submanifold in the nonsingular locus of $\mathcal{M}_H$ when $(\bar{\partial}_0, \Phi_0)$ is a stable Higgs bundle. Moreover, there is a relationship between the grading on the tangent space at a fixed point and its stable manifold via Morse theory. Namely, consider the function defined by the L^2 -norm of the Higgs field

$$f : \mathcal{M}_H \longrightarrow \mathbb{R}^{\geq 0},$$

$$[(\bar{\partial}_E, \Phi)] \longmapsto \|\Phi\|^2 = i \int_X \text{Tr}(\Phi \wedge \Phi^*)$$

where the adjoint is defined with respect to the harmonic metric solving the self duality equations. Using Uhlenbeck compactness, Hitchin showed that f is a proper function. Moreover, on the smooth locus, f is the moment map for the restriction of the $\mathbb{C}^*$ -action to $U(1)$, and hence a Morse-Bott function [7]. For smooth points, the Hessian of f has eigenvalue j on the subspace $\mathcal{H}_j^1(\bar{\partial}_0, \Phi_0)$ (see [8, Section 8]³). Using the notation of (3.6), this implies that

$$T_{[(\bar{\partial}_0, \Phi_0)]} W_\alpha^0(\bar{\partial}_0, \Phi_0) \cong \mathcal{H}_+^1(\bar{\partial}_0, \Phi_0),$$

and by Lemma 3.6 the latter is half dimensional. We record these results in the following

Lemma 4.1. *If $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_\alpha$ is a stable fixed point of the $\mathbb{C}^*$ -action, then $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ is a smooth submanifold of dimension $\frac{1}{2} \dim(\mathcal{M}_H)$.*

Proposition 4.2. *Let $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_\alpha$ be a stable variation of Hodge structure, and $[(\bar{\partial}_E, \Phi)] \in \mathcal{M}_H$. Then $[(\bar{\partial}_E, \Phi)] \in W_\alpha^0(\bar{\partial}_0, \Phi_0)$ if and only if after a gauge transformation*

$$\Phi - \Phi_0 \in \Omega^{1,0}(L \oplus N^+) \quad \text{and} \quad \bar{\partial}_E - \bar{\partial}_0 \in \Omega^{0,1}(N^+),$$

where N_+ and L are defined in (2.11).

Proof. First suppose $\varphi = \Phi - \Phi_0 \in \Omega^{1,0}(L \oplus N^+)$ and $\beta = \bar{\partial}_E - \bar{\partial}_0 \in \Omega^{0,1}(N^+)$, with respect to the grading we write

$$\varphi = \sum_{j=0}^{\ell-1} \varphi_j \quad \text{and} \quad \beta = \sum_{j=1}^{\ell-1} \beta_j,$$

³In [8], Hitchin proves that the Hessian of f has eigenvalue j on the subspace $\mathcal{H}_j^1(\bar{\partial}_0, \Phi_0)$. We are using the opposite grading as Hitchin, so the Hessian of f has eigenvalue j on the subspace $\mathcal{H}_j^1(\bar{\partial}_0, \Phi_0)$.

where $\varphi_j \in \Omega^{1,0}(\text{End}_j(E))$ and $\beta_j \in \Omega^{0,1}(\text{End}_j(E))$. Consider the gauge transformations

$$(4.1) \quad g_\xi = \begin{pmatrix} \xi^a \mathbf{I}_{E_1} & & & \\ & \xi^{a-1} \mathbf{I}_{E_2} & & \\ & & \ddots & \\ & & & \xi^{a-\ell+1} \mathbf{I}_{E_\ell} \end{pmatrix},$$

where $\sum_{j=0}^{\ell-1} \text{rk}(E_{j+1})(a-j) = 0$. The gauge transformation g_ξ has $\det(g_\xi) = 1$ and acts on $(\bar{\partial}_E, \xi\Phi)$ as

$$(\bar{\partial}_E, \xi\Phi) \cdot g_\xi = (\bar{\partial}_0 + \sum_{j=1}^{\ell-1} \xi^j \beta_j, \Phi_0 + \sum_{j=0}^{\ell-1} \xi^{j+1} \varphi_j).$$

Thus, we have $\lim_{\xi \rightarrow 0} [(\bar{\partial}_E, \xi\Phi)] = [(\bar{\partial}_0, \Phi_0)]$.

For the other direction, recall that the map p_H from (3.4) is a diffeomorphism $\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}$ onto an open neighborhood $\mathcal{U} \subset \mathcal{M}_H$ of $[(\bar{\partial}_0, \Phi_0)]$. Using the smooth splitting $E_1 \oplus \cdots \oplus E_\ell$ we can decompose $\beta \in \Omega^{0,1}(\text{End}(E))$ and $\varphi \in \Omega^{1,0}(\text{End}(E))$ as $\beta = \sum_{j=1-\ell}^{\ell-1} \beta_j$ and $\varphi = \sum_{j=1-\ell}^{\ell-1} \varphi_j$. The BB-slice $\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+ \subset \mathcal{S}_{(\bar{\partial}_0, \Phi_0)}$ is (see Definition 3.7):

$$\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+ = \left\{ (\beta, \varphi) \in \mathcal{S}_{(\bar{\partial}_0, \Phi_0)} \mid \beta = \sum_{j=1}^{\ell-1} \beta_j \text{ and } \varphi = \sum_{j=0}^{\ell-1} \varphi_j \right\}.$$

By the above argument the portion of $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ in $p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)})$ is contained in $p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+)$:

$$W_\alpha^0(\bar{\partial}_0, \Phi_0) \cap p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}) \subset p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+).$$

In fact, the above inclusion must be an equality, since by Lemmas 3.6 and 4.1, $\dim(\mathcal{S}_+) = \dim(W_\alpha^0(\bar{\partial}_0, \Phi_0))$.

We now extend this to all of $W_\alpha^0(\bar{\partial}_0, \Phi_0)$. For each point $[(\bar{\partial}_E, \Phi)] \in W_\alpha^0(\bar{\partial}_0, \Phi_0)$ there exists $\xi \in \mathbb{C}^*$ so that $[(\bar{\partial}_E, \xi\Phi)] \in \mathcal{U} = p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)})$. In fact, $[(\bar{\partial}_E, \xi\Phi)] \in p_H(\mathcal{S}_{(\bar{\partial}_0, \Phi_0)}^+)$, so there is a representative $(\bar{\partial}_E, \xi\Phi)$ of $[(\bar{\partial}_E, \xi\Phi)]$ such that

$$(\bar{\partial}_E, \xi\Phi) = (\bar{\partial}_0 + \sum_{j=1}^{\ell} \beta_j, \Phi_0 + \sum_{j=0}^{\ell} \varphi_j).$$

A representative of $[(\bar{\partial}_E, \Phi)]$ is given by

$$(\bar{\partial}_E, \Phi) = (\bar{\partial}_0 + \sum_{j=1}^{\ell} \beta_j, \xi^{-1} \Phi_0 + \sum_{j=0}^{\ell} \xi^{-1} \varphi_j).$$

Using the inverse of the holomorphic gauge transformation g_ξ from (4.1) we have

$$(\bar{\partial}_0 + \sum_{j=1}^{\ell} \beta_j, \xi^{-1} \Phi_0 + \sum_{j=0}^{\ell} \xi^{-1} \varphi_j) \cdot g_\xi^{-1} = (\bar{\partial}_0 + \sum_{j=1}^{\ell} \xi^{-j} \beta_j, \Phi_0 + \sum_{j=0}^{\ell} \xi^{-j-1} \varphi_j).$$

Thus, every $[(\bar{\partial}_E, \Phi)] \in W_\alpha^0(\bar{\partial}_0, \Phi_0)$ is gauge equivalent to a Higgs bundle $(\bar{\partial}_E, \Phi)$ with $\bar{\partial}_E - \bar{\partial}_0 \in \Omega^{0,1}(N_+)$ and $\Phi - \Phi_0 \in \Omega^{1,0}(L \oplus N_+)$. $\square$

Corollary 4.3. *Suppose $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ is stable. Then the map p_H in (3.4) gives a biholomorphism $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \xrightarrow{\sim} W_\alpha^0(\bar{\partial}_E, \Phi)$.*

Proof. We claim that p_H is injective. Indeed, this is true in a neighborhood of the origin by Proposition 3.4. In general, suppose $(\beta_1, \varphi_1), (\beta_2, \varphi_2) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ are such that

$$(\bar{\partial}_E + \beta_1, \Phi + \varphi_1) \cdot g = (\bar{\partial}_E + \beta_2, \Phi + \varphi_2)$$

for some $g \in \mathcal{G}(E)$. Let $\xi \in \mathbb{C}^*$ and consider g_ξ from (4.1). Then as in the proof of Proposition 4.2 we have

$$(\bar{\partial}_E + \sum_{j=1}^{\ell} \xi^j \beta_{1,j}, \Phi + \sum_{j=0}^{\ell} \xi^{j+1} \varphi_{1,j}) \cdot (g_\xi^{-1} \circ g \circ g_\xi) = (\bar{\partial}_E + \sum_{j=1}^{\ell} \xi^j \beta_{2,j}, \Phi + \sum_{j=0}^{\ell} \xi^{j+1} \varphi_{2,j}).$$

For ξ sufficiently small, this is a gauge equivalence between points in a neighborhood of the origin in $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$. Since p_H is injective there, we conclude that $(\beta_1, \varphi_1) = (\beta_2, \varphi_2)$, and this proves injectivity of p_H on all of $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$. Surjectivity of p_H follows from Propositions 4.2 and 3.11. Since p_H is in addition holomorphic, the result follows. $\square$

Remark 4.4. Note that Corollary 4.3 generalizes the parameterization of a Hitchin section from Example 3.8 to the fibers of every smooth strata.

4.2. The partial oper stratification of $\mathcal{M}_{\text{dR}}$. Let $\mathcal{E}$ be a bundle with holomorphic connection ∇ . We write $(\mathcal{E}, \nabla) = (E, \bar{\partial}_E, \nabla)$. Recall that $\mathcal{E}$ is polystable if and only if the $\mathbb{C}^*$ -action from (2.12) has limit

$$\lim_{\xi \rightarrow 0} [(\xi, \bar{\partial}_E, \xi \nabla)] = [(\bar{\partial}_E, 0)].$$

At the other extreme, suppose $\mathcal{E}$ is maximally unstable among bundles with holomorphic connections. Then (up to a twist) the successive quotients $\mathcal{E}_j$ in the Harder-Narasimhan filtration of $\mathcal{E}$ are line bundles of the form $\mathcal{E}_j = K^{\frac{n+1}{2}-j}$, and ∇ necessarily induces the tautological isomorphism $\mathcal{E}_j \simeq \mathcal{E}_{j+1} \otimes K$ (i.e. $(\mathcal{E}, \nabla)$ is an *oper*). In this case the limit on the left hand side above is a Fuchsian point (2.15).

In general, however, a Higgs bundle induced from the Harder-Narasimhan filtration of $(\mathcal{E}, \nabla)$ in this way will not be semistable. Nevertheless, in [16] an iterative process is described which produces a semistable Higgs bundle, and identifies the limit above.

A filtration $\mathcal{F}$ by holomorphic subbundles $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{E}$ is called *Griffiths transverse* if $\nabla(\mathcal{F}_j) \subset \mathcal{F}_{j+1} \otimes K$, for all $1 \leq j \leq \ell - 1$. Denote the associated graded of such a filtration by

$$\text{Gr } \mathcal{E} = (E, \bar{\partial}_{\text{Gr } E}) = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell,$$

where $\mathcal{E}_j = \mathcal{F}_j / \mathcal{F}_{j-1}$. The Griffiths transverse connection induces an $\mathcal{O}_X$ -linear map on the associated graded:

$$\nabla : \mathcal{E}_j = \mathcal{F}_j / \mathcal{F}_{j-1} \rightarrow \mathcal{E}_{j+1} \otimes K = (\mathcal{F}_{j+1} / \mathcal{F}_j) \otimes K$$

which we denote by $\Phi_{\mathcal{F}}$. Thus, associated to a Griffiths transverse filtration $\mathcal{F}$,

there is a Higgs bundle $(E, \bar{\partial}_{\text{Gr } E}, \Phi_{\mathcal{F}})$, where $\Phi_{\mathcal{F}} \in \bigoplus_{j=1}^{\ell-1} H^0(\text{Hom}(\mathcal{E}_j, \mathcal{E}_{j+1}) \otimes K)$.

Lemma 4.5. *If $(E, \bar{\partial}_{\text{Gr}E}, \Phi_{\mathcal{F}})$ is semistable then*

$$\lim_{\xi \rightarrow 0} [(\xi, \bar{\partial}_E, \xi \nabla)] = [(\bar{\partial}_{\text{Gr}E}, \Phi_{\mathcal{F}})] .$$

Proof. Choose a smooth identification of $\text{Gr } E$ with E compatible with the obvious filtrations. With this choice, the bundle $\text{End}(E)$ has a decomposition analogous to (2.10); namely,

$$\text{End}(E) = \bigoplus_{j=1-\ell}^{\ell-1} \text{End}_j(E) .$$

Note that the Higgs field $\Phi_{\mathcal{F}}$ is a holomorphic section of $\text{End}_{-1}(E) \otimes K$. Since $\mathcal{F}$ is Griffiths transverse there are $\beta_j \in \Omega^{0,1}(\text{End}_j(E))$ and $\varphi_j \in \Omega^{1,0}(\text{End}_j(E))$ so that

$$(\xi, \bar{\partial}_E, \xi \nabla) = \left(\xi, \bar{\partial}_{\text{Gr}E} + \sum_{j=1}^{\ell-1} \beta_j, \xi \Phi_{\mathcal{F}} + \xi D_{\text{Gr}E}^{1,0} + \sum_{j=1}^{\ell-1} \xi \varphi_j \right) ,$$

where is the $D_{\text{Gr}E}^{1,0}$ connection induced by ∇ . The gauge transformation g_{ξ} from (4.1) acts as

$$(\xi, \bar{\partial}_E, \xi \nabla) \cdot g_{\xi} = \left(\xi, \bar{\partial}_{\text{Gr}E} + \sum_{j=1}^{\ell-1} \xi^j \beta_j, \Phi_{\mathcal{F}} + \xi D_{\text{Gr}E}^{1,0} + \sum_{j=1}^{\ell-1} \xi^{j+1} \varphi_j \right) .$$

Now taking the limit $\xi \rightarrow 0$ yields $(\bar{\partial}_{\text{Gr}E}, \Phi_{\mathcal{F}})$. $\square$

The key result is the following.

Theorem 4.6 ([16, Theorem 2.5]). *Given a $(\mathcal{E}, \nabla)$ there exists a Griffiths transverse filtration $\mathcal{F}$ of $\mathcal{E}$ so that the associated Higgs bundle $(\bar{\partial}_{\text{Gr}E}, \Phi_{\mathcal{F}})$ is semistable. Moreover, if $(\bar{\partial}_{\text{Gr}E}, \Phi_{\mathcal{F}})$ is stable, then the filtration is unique.*

Recall that if $[(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_{\alpha}$, then a holomorphic flat bundle $[(\mathcal{E}, \nabla)] \in \mathcal{M}_{\text{dR}}$ is in $W_{\alpha}^1(\bar{\partial}_0, \Phi_0)$ if and only if

$$\lim_{\xi \rightarrow 0} [(\xi, \bar{\partial}_E, \xi \nabla)] = [(\bar{\partial}_0, \Phi_0)] .$$

The following is the de Rham analogue of Proposition 4.2. It follows easily from Theorem 4.6, Lemma 4.5, and Proposition 2.9.

Proposition 4.7. *Let $(\bar{\partial}_0, \Phi_0)$ be a stable complex variation of Hodge structure with harmonic metric h . Let $(\mathcal{E}, \nabla)$, $\mathcal{E} = (E, \bar{\partial}_0 + \Phi_0^{*h})$, $\nabla = \partial_0^h + \Phi_0$, be the associated holomorphic bundle with holomorphic connection. Then $[(\mathcal{E}, \nabla)] \in W_{\alpha}^1(\bar{\partial}_0, \Phi_0)$ if and only if, after a gauge transformation,*

$$\bar{\partial}_E - \bar{\partial}_0 - \Phi_0^{*h} \in \Omega^{0,1}(N^+) \quad \text{and} \quad \nabla - \partial_0^h - \Phi_0 \in \Omega^{1,0}(L \oplus N^+) ,$$

where N_+ and L are defined in (2.11).

We now prove the analogous result to Proposition 3.11 in the de Rham picture.

Proposition 4.8. *Let $(\bar{\partial}_E, \Phi)$ be a stable complex variation of Hodge structure with harmonic metric h and associated flat connection $D = \bar{\partial}_E + \partial_E^h + \Phi + \Phi^{*h}$. Suppose $(\beta, \varphi) \in \Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+)$ is such that $D + \beta + \varphi$ is flat. Then there is a unique smooth section $f \in \Omega^0(N_+)$ such that applying the complex gauge transformation $g = \mathbf{I} + f$,*

$$(D + \beta + \varphi) \cdot g = D + \tilde{\beta} + \tilde{\varphi} ,$$

then $(\tilde{\beta}, \tilde{\varphi}) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$. Moreover, if $(\beta_1, \varphi_1), (\beta_2, \varphi_2) \in \mathcal{S}_{(\bar{\partial}_E, \Phi)}^+$ are such that

$$(D + \beta_1 + \varphi_1) \cdot g = D + \beta_2 + \varphi_2$$

for some $g \in \mathcal{G}(E)$, then g is in the center and $(\beta_1, \varphi_1) = (\beta_2, \varphi_2)$.

Proof. For the existence part, note that the only difference with the proof of Proposition 3.11 is the addition of the term $g^{-1}\partial_0 g$ for the change of φ under a complex gauge transformation. But in terms of the graded pieces, $[\Phi^*, g^{-1}\partial_0 g]_j = 0 \pmod j$, and so the same proof applies in this case as well.

We move on to prove the second statement. Write $g = \mathbf{I} + f$, a calculation yields

$$(D'')^* D''(f) = -i\Lambda([\Phi^*, f] \wedge \varphi_2 + \varphi_1 \wedge [\Phi^*, f] + \partial_0 f \wedge \beta_2 + \beta_1 \wedge \partial_0 f) .$$

But since Φ^* and β_i raise the grading, we have $(D'')^* D''(f_j) = 0 \pmod j$. By an inductive argument and the stability of $(\bar{\partial}_E, \Phi)$, the endomorphism f , and hence also g , must lie in the center. $\square$

Corollary 4.9. *Suppose $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ is stable. Then the map p_{dR} in (3.4) gives a biholomorphism $\mathcal{S}_{(\bar{\partial}_E, \Phi)}^+ \xrightarrow{\sim} W_\alpha^1(\bar{\partial}_E, \Phi)$.*

Proof. Immediate from Propositions 4.7, 4.8 and the fact that p_{dR} is holomorphic. $\square$

Remark 4.10. Note that Corollary 4.9 generalizes the parameterization of the components of opers from Example 3.8 to the fibers of every smooth strata.

4.3. Proofs. We now put together the results from Sections 3 and 4 to prove Theorems 1.4 and 1.3, Proposition 1.2, and Corollary 1.6.

Proof of Theorem 1.4. Part (1) of Theorem 1.4 is contained in Lemma 3.6 and Theorem 3.9. Part (2) follows from Corollary 4.3. For part (3), for each $\lambda \in \mathbb{C}$ we extend the definition (3.4) to:

$$(4.2) \quad \begin{aligned} p_\lambda : \mathcal{S}_{(\bar{\partial}_E, \Phi)} &\longrightarrow \mathcal{M}_{\text{Hod}} \\ (\beta, \varphi) &\mapsto [(\lambda, \bar{\partial}_E + \lambda\Phi^* + \beta, \lambda\partial_E^h + \Phi + \varphi)] \end{aligned}$$

where h is the harmonic metric for $(\bar{\partial}_E, \Phi)$. The map p_{Hod} is then defined by $p_{\text{Hod}}(\beta, \varphi, \lambda) = p_\lambda(\beta, \varphi)$. Now p_λ is a biholomorphism onto the fiber at λ . Indeed, this is true for $\lambda = 0$ and $\lambda = 1$. The $\mathbb{C}^*$ -action identifies the fibers $\lambda \neq 0$ with $\mathcal{M}_{\text{dR}}$. The result then follows by a small modification of Propositions 4.7 and 4.8. $\square$

Proof of Proposition 1.2. For the first statement, since $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ is preserved by the complex structure I and is half-dimensional by Lemma 4.1, it suffices to prove that $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ is isotropic. But by Corollary 4.3, tangent vectors to $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ are represented by elements in $\Omega^{0,1}(N_+) \oplus \Omega^{1,0}(L \oplus N_+)$. The result now follows directly from the definition (2.2) of $\omega_I^{\mathbb{C}}$. The second statement is similar: we note that $W_\alpha^1(\bar{\partial}_E, \Phi)$ is J -holomorphic and half-dimensional, and the fact that it is isotropic follows from Corollary 4.9 and the expression (2.1). $\square$

Proof of Theorem 1.3. Let h be the harmonic metric for $(\bar{\partial}_E, \Phi)$. By Proposition 3.12, the derivative of h vanishes to first order for deformations in the Hodge slice. In

particular, if $\mathbf{u}(t) \in \mathcal{S}_{(\bar{\partial}_E, \varphi)}^+$ is a smooth one (real) parameter family with $\mathbf{u}(0) = 0$, $\dot{\mathbf{u}}(0) = (\beta, \varphi) \in \mathcal{H}_+^1(\bar{\partial}_E, \Phi)$, then

$$\mathcal{T}[\mathbf{u}(t)] = [\bar{\partial}_E + t\beta + \partial_E^h - t\beta^{*h} + \Phi + t\varphi + \Phi^{*h} + t\varphi^{*h} + O(t^2)] .$$

Hence, in these coordinates, $d\mathcal{T}(\beta, \varphi) = \beta + \varphi^* - \beta^* + \varphi$, where we have dropped h from the notation. But according to (2.8),

$$\beta + \varphi^* - \beta^* + \varphi = \mu - K(\mu)$$

for the tangent vector $\mu = \beta + \varphi \in T_{\mathcal{T}[(\bar{\partial}_E, \varphi)]} W_\alpha^1(\bar{\partial}_E, \Phi)$. This completes the proof. $\square$

5. CONFORMAL LIMITS

Let $[(\bar{\partial}_E, \Phi)]$ be a stable Higgs bundle with harmonic metric h , and

$$\lim_{\xi \rightarrow 0} [(\bar{\partial}_E, \xi\Phi)] = [(\bar{\partial}_0, \Phi_0)] \in \text{VHS}_\alpha .$$

Recall the real twistor line in $\mathcal{M}_{\text{Hod}}$:

$$\tau(\xi) = \bar{\partial}_E + \xi\partial_E^h + \Phi + \xi\Phi^*$$

which passes through $[(\bar{\partial}_E, \Phi)]$ at $\xi = 0$ and $\mathcal{T}[(\bar{\partial}_E, \Phi)]$ at $\xi = 1$. In general, $\tau(\xi)$ does not lie in the fiber $W_\alpha(\bar{\partial}_0, \Phi_0)$. In other words, nonabelian Hodge does not map $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ to $W_\alpha^1(\bar{\partial}_0, \Phi_0)$. For example, if $(\bar{\partial}_0, \Phi_0)$ is the Fuchsian point, then $\mathcal{T}[(\bar{\partial}_E, \Phi)]$ is a real representation for every $[(\bar{\partial}_E, \Phi)] \in W_\alpha^0(\bar{\partial}_0, \Phi_0)$, whereas the holonomy of an oper in $W_\alpha^1(\bar{\partial}_0, \Phi_0)$ is not real in general (see Example 3.8).

By contrast, the $\mathbb{C}^*$ -orbit of a point in $W_\alpha^1(\bar{\partial}_0, \Phi_0)$ does lie in $W_\alpha(\bar{\partial}_0, \Phi_0)$, but it collapses the entire stratum $W_\alpha^1(\bar{\partial}_0, \Phi_0)$ to the point $[(\bar{\partial}_0, \Phi_0)]$ at $\xi = 0$. Notice, however, that these two holomorphic sections of $\mathcal{M}_{\text{Hod}} \rightarrow \mathbb{C}$ – the twistor lines and the $\mathbb{C}^*$ -orbits – agree for the twistor line through $[(\bar{\partial}_0, \Phi_0)]$ itself. In Theorem 1.4, we have produced a third section, p_ξ from (4.2), which also agrees on the twistor lines through VHS , but which gives a biholomorphism $W_\alpha^0(\bar{\partial}_0, \Phi_0)$ and $W_\alpha^1(\bar{\partial}_0, \Phi_0)$.

In [6] Gaiotto introduced a rescaled version of these two sections to produce the family (1.2). In the case where $[(\bar{\partial}_0, \Phi_0)]$ is the Fuchsian point (Example 2.8), he conjectured the existence of a well-defined limit as $R \rightarrow 0$, and that it should be an oper. As evidence, he pointed to the analogy between the cyclic structure of the Higgs field and the jet bundle description ofopers. The conjecture was recently proven in [5].

In this section, we prove that in fact the limits exist in much greater generality. As an example, at the opposite extreme of the original conjecture is the case where $(E, \bar{\partial}_0)$ is a stable bundle. It is not hard to see that then the harmonic metric for $(\bar{\partial}_E, R\Phi)$ converges to the Hermitian-Einstein metric h_0 on $(E, \bar{\partial}_0)$. Hence, the flat connection in (1.2) simply converges to $\bar{\partial}_E + \partial_E^{h_0} + \hbar^{-1}\Phi$, and this in turn lies in the open stratum in $\mathcal{M}_{\text{dR}}$.

Let $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_\alpha$ be a stable Higgs bundle, and let h denote the harmonic metric for $(\bar{\partial}_E, \Phi)$. Throughout this section, an unannotated $*$ will refer to adjoints with respect to h , and we set $\partial_E := \partial_E^h$. Endomorphisms will be written with respect to the splitting in Proposition 2.7. By Corollary 4.3, any point in $W_\alpha^0(\bar{\partial}_E, \Phi)$ is of the form $[(\bar{\partial}_E + \beta, \Phi + \varphi)]$ for $\mathbf{u} = (\beta, \varphi) \in S_{(\bar{\partial}_E, \Phi)}^+$. Fix $\hbar > 0$. Recall from (4.2) that

$$p_\hbar(\mathbf{u}) = \bar{\partial}_E + \hbar\Phi^* + \beta + \hbar\partial_E^h + \Phi + \varphi .$$

For convenience, set $\bar{\partial}_{\mathbf{u}} := \bar{\partial}_E + \beta$ and $\Phi_{\mathbf{u}} := \Phi + \varphi$. For $R > 0$, the connection

$$D_{(\mathbf{u}, R)} := \hbar^{-1} \Phi_{\mathbf{u}} + \bar{\partial}_{\mathbf{u}} + \partial_{\mathbf{u}}^{*h(\mathbf{u}, R)} + \hbar R^2 \Phi_{\mathbf{u}}^{*h(\mathbf{u}, R)}$$

is flat for the harmonic metric $h(\mathbf{u}, R)$ for $(\bar{\partial}_{\mathbf{u}}, R\Phi_{\mathbf{u}})$.

The following is a restatement of Theorem 1.1 and Corollary 1.6.

Proposition 5.1. *As $R \rightarrow 0$, the flat connections $D_{(\mathbf{u}, R)}$ converge smoothly to*

$$\bar{\partial}_E + \partial_E + \beta + \hbar^{-1} \Phi_{\mathbf{u}} + \hbar \Phi^* .$$

In particular,

$$\lim_{R \rightarrow 0} [D_{(\mathbf{u}, R)}] = \hbar^{-1} \cdot p_{\hbar}(\mathbf{u}) .$$

Proof. Since the argument is modeled closely on the proof in [5], we shall be brief. Let $[(\bar{\partial}_E, \Phi)] \in \text{VHS}_{\alpha}$ be a stable point. First, some preliminaries. Let $g \in \mathcal{G}(E)$. Define a new metric $g(h)$ by the rule: $\langle u, v \rangle_{g(h)} = \langle gu, gv \rangle_h$. Let $k = g^*g$. Then we have for any Higgs bundle $(\bar{\partial}_E, \Phi)$,

$$(5.1) \quad \Phi^{*g(h)} = k^{-1} \Phi^* k$$

$$(5.2) \quad \partial_E^{g(h)} = k^{-1} \circ \partial_E \circ k$$

$$(5.3) \quad F_{(\bar{\partial}_E, g(h))} = F_{(\bar{\partial}_E, h)} + \bar{\partial}_E(k^{-1} \partial_E(k))$$

Recall that an expression of the form $F_{(\bar{\partial}_E, h')}$ denotes the curvature of the Chern connection associated to $\bar{\partial}_E$ and a metric h' .

With this understood, we first modify h to get a metric h'_R by using:

$$g = \begin{pmatrix} R^{m_1/2} & & 0 \\ & \ddots & \\ 0 & & R^{m_{\ell}/2} \end{pmatrix}, \quad k = \begin{pmatrix} R^{m_1} & & 0 \\ & \ddots & \\ 0 & & R^{m_{\ell}} \end{pmatrix}$$

where $m_j - m_{j+1} = 2$, $j = 1, \dots, \ell - 1$, and $\sum_{i=1}^{\ell} (\text{rk } E_j) m_j = 0$. Then using (5.1) and (5.2), we have:

$$\bar{\partial}_{\mathbf{u}} + \partial_{\mathbf{u}}^{h'_R} = \bar{\partial}_E + \partial_E + \beta - \sum_{j=1}^{\ell-1} R^{2j} (\beta_j)^*$$

$$\Phi_{\mathbf{u}}^{*h'_R} = R^{-2} \Phi^* + \sum_{j=0}^{\ell-1} R^{2j} (\varphi_j)^*$$

Taking the limit as $R \rightarrow 0$ we have

$$(5.4) \quad \begin{aligned} D_{(\mathbf{u}, 0)} &:= \lim_{R \rightarrow 0} \left\{ \hbar^{-1} \Phi_{\mathbf{u}} + \bar{\partial}_{\mathbf{u}} + \partial_{\mathbf{u}}^{h'_R} + \hbar R^2 \Phi_{\mathbf{u}}^{*h'_R} \right\} \\ &= \hbar^{-1} \Phi_{\mathbf{u}} + \bar{\partial}_E + \partial_E + \beta + \hbar \Phi^* . \end{aligned}$$

Let us record

$$(5.5) \quad R^2 \Phi_{\mathbf{u}}^{*h'_R} = \Phi^* + \sum_{j=0}^{\ell-1} R^{2j+2} (\varphi_j)^*$$

$$(5.6) \quad F_{(\bar{\partial}_{\mathbf{u}}, h'_R)} = F_{(\bar{\partial}_E, h)} + \left[\beta, \sum_{j=1}^{\ell-1} R^{2j} (\beta_j)^* \right] + \partial_E \beta - \sum_{j=1}^{\ell-1} R^{2j} \bar{\partial}_E (\beta_j)^*$$

The goal now is to find a solution of the form $h(\mathbf{u}, R) = g_R(h'_R)$, where $g_R = \exp(f_R)$, for f_R a (traceless) h'_R -hermitian endomorphism. Then $k_R = \exp(2f_R)$ in eqs. (5.1)–(5.3). This means we set $N_{(\mathbf{u}, R)}(f_R) = 0$, where

$$N_{(\mathbf{u}, R)}(f_R) := \bar{\partial}_{\mathbf{u}}(e^{-2f_R} \partial_{\mathbf{u}}^{h'_R}(e^{2f_R})) + F_{(\bar{\partial}_{\mathbf{u}}, h'_R)} + \left[\Phi_{\mathbf{u}}, e^{-2f_R} \left(R^2 \Phi_{\mathbf{u}}^{*h'_R} \right) e^{2f_R} \right]$$

Note that: $N_{(\mathbf{u}, 0)}(0) = 0$. Now consider the linearization in R at $R = 0$. We denote derivatives at $R = 0$ by $\dot{}$, for example. Now f_R is determined by a section $(f_L(R), f_+(R))$ of $L \oplus N_+$. Specifically,

$$f_R = f_L(R) + f_+(R) + \sum_{j=1}^{\ell-1} R^{2j} (f_{+,j}(R))^* .$$

Hence, we may consider $N_{(\mathbf{u}, R)}$ as a smooth map $C^{2,\alpha}(L \oplus N_+) \rightarrow C^\alpha(L \oplus N_+)$. By (5.5) and (5.6),

$$(5.7) \quad dN_{(\mathbf{u}, 0)}(0) \dot{f} = \bar{\partial}_{\mathbf{u}} \partial_E \dot{f} + \left[\Phi_{\mathbf{u}}, [\Phi^*, \dot{f}] \right] .$$

Lemma 5.2. $\ker dN_{(\mathbf{u}, 0)}(0) = \{0\}$.

Proof. Suppose $\dot{f} \in \ker dN_{(\mathbf{u}, 0)}(0)$. Write the decomposition into components as $\dot{f} = \dot{f}_L + \dot{f}_+$. Using (2.4),

$$\bar{\partial}_E \partial_E = -\partial_E \bar{\partial}_E + F_{(\bar{\partial}_E, h)} = -\partial_E \bar{\partial}_E - \Phi \Phi^* - \Phi^* \Phi .$$

Then a direct calculation from (5.7) yields $(D'')^* D''(\dot{f}_L) = 0$. As in the proof of Proposition 3.11, the operator on the left is invertible, so $\dot{f}_L = 0$. Using this, it follows from (5.7) that the first graded piece of $\dot{f}_+$ satisfies $(D'')^* D''(\dot{f}_{+,1}) = 0$, and therefore vanishes as well. More generally, if $\dot{f}_{+,k} = 0$ for $k < j$ then we have $(D'')^* D''(\dot{f}_{+,j}) = 0$. The result now follows. $\square$

By the lemma and the implicit function theorem, there is a unique f_R , $f_0 = 0$, such that $N_{(\mathbf{u}, R)}(f_R) = 0$ for R sufficiently small. As in [5], and by (5.4), this completes the proof. $\square$

Remark 5.3. Let us point out that the conformal limit, regarded as a map from an open dense set in $\mathcal{M}_H$ to $\mathcal{M}_{dR}$, is not continuous. For example, if it were then the image of the $\mathbb{C}^*$ -orbit of a Higgs bundle $(\bar{\partial}_E, \Phi)$ with $(E, \bar{\partial}_E)$ stable and $\Phi \neq 0$ nilpotent, would complete to a holomorphically embedded $\mathbb{P}^1 \subset \mathcal{M}_{dR}$ which, since the latter is an affine variety, cannot exist.

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E-mail address: briancollier01@gmail.com

E-mail address: raw@umd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA