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journal homepage: www.elsevier.com/locate/ejcFlagged $(\mathcal{P}, \rho)$ -partitions[☆]Sami Assaf^{a,1}, Nantel Bergeron^{b,2}^a Department of Mathematics, University of Southern California, 3620 S. Vermont Ave, Los Angeles, CA 90089-2532, USA^b Department of Mathematics and Statistics, York University, 4700 Keele St, Toronto, Ontario M3J 1P3, Canada

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ABSTRACT

We introduce the theory of $(\mathcal{P}, \rho)$ -partitions, depending on a poset $\mathcal{P}$ and a map ρ from $\mathcal{P}$ to positive integers. The generating function $\mathfrak{F}_{\mathcal{P}, \rho}$ of $(\mathcal{P}, \rho)$ -partitions is a polynomial that, when the images of ρ tend to infinity, tends to Stanley's generating function of $\mathcal{P}$ -partitions. Analogous to Stanley's fundamental theorem for $\mathcal{P}$ -partitions, we show the set of $(\mathcal{P}, \rho)$ -partitions decomposes as a disjoint union of $(\mathcal{L}, \rho)$ -partitions where $\mathcal{L}$ runs over the set of linear extensions of $\mathcal{P}$. In this more general context, the set of all $\mathfrak{F}_{\mathcal{L}, \rho}$ for linear orders $\mathcal{L}$ over $\mathcal{P}$ determines a basis of polynomials. We thus introduce the notion of flagged $(\mathcal{P}, \rho)$ -partitions, and we prove the set of all $\mathfrak{F}_{\mathcal{L}, \rho}$ for flagged $(\mathcal{L}, \rho)$ -partitions for linear orders $\mathcal{L}$ is precisely the fundamental slide basis of the polynomial ring, introduced by the first author and Searles. Our main theorem shows that any generating function $\mathfrak{F}_{\mathcal{P}, \rho}$ of flagged $(\mathcal{P}, \rho)$ -partitions is a positive integer linear combination of slide polynomials. As applications, we give a new proof of positivity of the slide product and, motivating our nomenclature, we also prove flagged Schur functions are slide positive.

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[☆] Both authors contributed equally in all aspects.E-mail addresses: shassaf@usc.edu (S. Assaf), bergeron@yorku.ca (N. Bergeron).¹ S.A. supported in part by National Science Foundation, USA DMS-1763336.² N.B. supported in part by York Research Chair in Applied Algebra, Canada and NSERC, Canada.

1. Introduction

The theory of $\mathcal{P}$ -partitions has its origins with MacMahon [6] and was developed in depth by Stanley [8]. There have been several applications of this theory. Notably, the theory of quasisymmetric functions developed by Gessel [3] originated from the study of $\mathcal{P}$ -partitions. Symmetric functions are a special case of quasisymmetric functions as can be seen by realizing Schur functions as certain $\mathcal{P}$ -partition generating functions.

Recently, Assaf and Searles [1] (Definition 3.6) introduced Fundamental slide polynomials to study Schubert polynomials [5]. The Fundamental slide polynomials have several interesting properties related to Schubert polynomials and certain limits related to quasisymmetric functions. Fundamental slide polynomials, indexed by weak compositions, form a basis for the full polynomial ring [1] (Theorem 3.9). Moreover, Assaf and Searles show the product of Fundamental slide polynomials expands positively in terms of Fundamental slide polynomials. We see in our present work that Fundamental slide polynomials arise naturally in a restricted version of $\mathcal{P}$ -partitions, and, moreover, the positivity of the slide product naturally follows. Here, we will say only *slide polynomials* as we do not use the other polynomials defined in [1].

A poset $\mathcal{P}$ of order p is a partial order on the set $[p] = \{1, 2, \dots, p\}$. We have several orders that come into play: $\prec$ denotes the partial order on $\mathcal{P}$, $\triangleleft$ denotes a cover relation in $\mathcal{P}$, and $<$ denotes the natural order of the integers. Given $\rho: [p] \rightarrow \mathbb{Z}$ any map, a function $f: \mathcal{P} \rightarrow \mathbb{N}$ is a $(\mathcal{P}, \rho)$ -partition if

- (1) if $i \prec j$, then $f(i) \leq f(j)$,
- (2) if $i \prec j$ and $i > j$, then $f(i) < f(j)$, and
- (3) $f(i) \leq \rho(i)$.

Here the first two conditions alone characterize classical $\mathcal{P}$ -partitions [8].

The generating function $\mathfrak{F}_{\mathcal{P}, \rho}$ of $(\mathcal{P}, \rho)$ -partitions is the polynomial

$$\mathfrak{F}_{\mathcal{P}, \rho} = \sum_{f \in \mathcal{A}_{\rho}(\mathcal{P})} x_{f(1)} \cdots x_{f(p)},$$

where $\mathcal{A}_{\rho}(\mathcal{P})$ denotes the set of all $(\mathcal{P}, \rho)$ -partitions. In Section 3.2 we show we have a decomposition

$$\mathcal{A}_{\rho}(\mathcal{P}) = \bigsqcup_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathcal{A}_{\rho}(\mathcal{L}), \quad (1.1)$$

where the disjoint union is over the set $\mathcal{L}(\mathcal{P})$ of linear extensions of $\mathcal{P}$. This implies $\mathfrak{F}_{\mathcal{P}, \rho}$ is a positive linear combination of $\mathfrak{F}_{\mathcal{L}, \rho}$ where $\mathcal{L}$ are linear orders. Unfortunately, the set $\{\mathfrak{F}_{\mathcal{L}, \rho}\}$ as $\mathcal{L}$ runs over all linear orders on $[p]$ and ρ is any restriction does not form a basis of polynomials. This leads us to restrict further the restriction maps ρ that we use.

We say a restriction map ρ is a $\mathcal{P}$ -flag if $\rho(i) > 0$ for all $i \in [p]$, and

- (i) if $i \triangleleft j$ and $i < j$, then $\rho(i) = \rho(j)$, and
- (ii) if $i \triangleleft j$ and $i > j$, then $\rho(i) \leq \rho(j)$.

When ρ is a $\mathcal{P}$ -flag, we say $\mathcal{A}_{\rho}(\mathcal{P})$ is the set of *flagged $(\mathcal{P}, \rho)$ -partitions*. In Section 3.1, we show the set $\{\mathfrak{F}_{\mathcal{L}, \rho}\}$ as $\mathcal{L}$ runs over all linear orders on $[p]$ and ρ over all $\mathcal{L}$ -flags is exactly the set of slide polynomials as defined in [1]. In particular this set is a basis for polynomials. The main theorem of this paper, proved in Section 4.1, states:

Theorem 1.1. *For any $\mathcal{P}$ on $[p]$ and $\mathcal{P}$ -flag ρ , the polynomials $\mathfrak{F}_{\mathcal{P}, \rho}$ expand positively in the basis of slide polynomials.*

We note if ρ is not a $\mathcal{P}$ -flag, then theorem above is false. Even for linear orders $\mathcal{L}$, if ρ is not an $\mathcal{L}$ -flag, then $\mathfrak{F}_{\mathcal{L}, \rho}$ will in general have negative coefficients.

There are several consequences of our main theorem; we mention two. First, an immediate application of the theorem is a new proof that the product of two slide polynomials is a positive

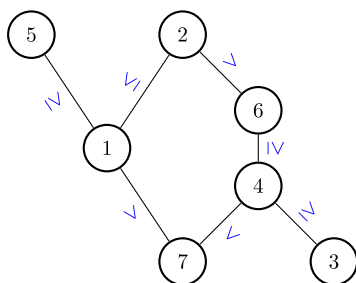


Fig. 1. An example of a labelled poset of order 7, with edges decorated by the conditions on a map to be a $\mathcal{P}$ -partition.

linear combination of slide polynomials. Second, the theorem shows the flagged Schur functions of Lascoux and Schützenberger [5,9] expand positively in terms of slides polynomials. The latter motivates for the term *flag* used for the subset of $(\mathcal{P}, \rho)$ partitions we study.

2. Stanley's theory of $\mathcal{P}$ -partitions

Before delving into the details of this new theory of $(\mathcal{P}, \rho)$ -partitions, we begin with a brief review of Stanley's theory of $\mathcal{P}$ -partitions and its consequences. For a comprehensive survey of $\mathcal{P}$ -partitions, see [4].

Definition 2.1. Given a poset $\mathcal{P}$ on $[p]$, a $\mathcal{P}$ -partition is a map $f: \mathcal{P} \rightarrow \mathbb{N}$ such that for all $i, j \in \mathcal{P}$ we have

- (1) if $i < j$, then $f(i) \leq f(j)$, and
- (2) if $i < j$ and $i > j$, then $f(i) < f(j)$.

If $\sum_{i \in \mathcal{P}} f(i) = n$, then f is a $\mathcal{P}$ -partition of n .

For example, the poset with Hasse diagram shown in Fig. 1 has edges indicating relations between $f(i)$ and $f(j)$ that any $\mathcal{P}$ -partition f must satisfy.

Let $\mathcal{A}(\mathcal{P})$ denote the set of $\mathcal{P}$ -partitions. Then we may define the generating function of a poset $\mathcal{P}$ by

$$F_{\mathcal{P}} = \sum_{f \in \mathcal{A}(\mathcal{P})} x_{f(1)} \cdots x_{f(p)}. \quad (2.1)$$

2.1. Linear $\mathcal{P}$ -partitions

As we will see below in the Fundamental Theorem of $\mathcal{P}$ -partitions, linear orders are of particular interest.

Given a linear order $\mathcal{L}$, we say $i \prec j$ is a *descent* whenever $i > j$. A *strong composition* is a finite sequence of positive integers, and we use α, β, γ for strong compositions. We record descents of $\mathcal{L}$ with the descent composition as follows.

Definition 2.2. For $\mathcal{L}$ a linear order on $[p]$, the *descent composition* of $\mathcal{L}$, denoted by $\text{Des}(\mathcal{L})$, is formed by removing the edge in the Hasse diagram between i and j whenever a cover $i \prec j$ is such that $i > j$, calling the resulting chains $C_1, \dots, C_r$ taken in ascending order in $\mathcal{L}$, and setting $\text{Des}(\mathcal{L})_s = |C_s|$. We have $\text{Des}(\mathcal{L})$ is a strong composition of p .

Example 2.3. Let $\mathcal{L}$ be the linear order $2 < 5 < 1 < 4 < 6 < 7 < 3 < 8 < 9$. Then the chains after removing edges for descents become

$$\overbrace{2 < 5}^{c_1} \quad \overbrace{1 < 4 < 6 < 7}^{c_2} \quad \overbrace{3 < 8 < 9}^{c_3}.$$

Therefore $\text{Des}(\mathcal{L}) = (|C_1|, |C_2|, |C_3|) = (2, 4, 3)$.

Gessel [3] observed for $\mathcal{L}$ a linear order, the function $F_{\mathcal{L}}$ depends only on $\text{Des}(\mathcal{L})$. This led him to introduce the *fundamental quasisymmetric functions* [3], indexed by strong compositions, that form an important basis for quasisymmetric functions.

Given strong compositions α, β , we say β *refines* α if for all $1 \leq j \leq k$, where k is the length of α , there exist indices $i_1 < \dots < i_k$ such that

$$\beta_1 + \dots + \beta_{i_j} = \alpha_1 + \dots + \alpha_j.$$

For example, $(1, 2, 2)$ refines $(3, 2)$ since $1 + 2 = 3$ and $1 + 2 + 2 = 3 + 2$. However, $(1, 2, 2)$ does not refine $(2, 3)$ since $1 < 2$ and $1 + 2 > 2$.

A *weak composition* is a finite sequence of non-negative integers, and we use a, b, c for weak compositions. The flattening of a weak composition is the strong composition obtained by removing the zeros. For example, $\text{flat}(0, 3, 0, 2) = (3, 2)$. We will also use the *dominance order* on weak compositions defined by $b \supseteq a$ if and only if $b_1 + \dots + b_k \geq a_1 + \dots + a_k$ for all k . Note this is a partial order, and the linear order given by reverse lexicographic order extends dominance order.

Definition 2.4 ([3]). For α a strong composition, the *fundamental quasisymmetric function* F_{α} is given by

$$F_{\alpha}(X) = \sum_{\text{flat}(b) \text{ refines } \alpha} x_1^{b_1} x_2^{b_2} \dots, \quad (2.2)$$

where the sum is over weak compositions b whose flattening refines α .

For example, restricting to three variables to make the expansion finite, we have

$$F_{(3,2)}(x_1, x_2, x_3) = x_2^3 x_3^2 + x_1^3 x_3^2 + x_1^3 x_2^2 + x_1^3 x_2 x_3 + x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2.$$

Proposition 2.5 ([3], p291). For $\mathcal{L}$ a linear order on $[p]$, we have

$$F_{\mathcal{L}} = F_{\text{Des}(\mathcal{L})}, \quad (2.3)$$

where F_{α} denotes the fundamental basis for quasisymmetric functions.

2.2. Fundamental theorem for $\mathcal{P}$ -partitions

Given a poset $\mathcal{P}$, a *linear extension* of $\mathcal{P}$ is a linear order on $[p]$ that extends $\mathcal{P}$. For example, the linear order $7 < 1 < 3 < 5 < 4 < 6 < 2$ is a linear extension of the poset in Fig. 1.

Stanley proved the following result that has myriad consequences for $\mathcal{P}$ -partitions and their generating functions.

Theorem 2.6 (Fundamental Theorem of $\mathcal{P}$ -partitions [8]). Given a poset $\mathcal{P}$ on $[p]$, we have

$$\mathcal{A}(\mathcal{P}) = \bigsqcup_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathcal{A}(\mathcal{L}), \quad (2.4)$$

where the disjoint union is over the set $\mathcal{L}(\mathcal{P})$ of linear extensions of $\mathcal{P}$.

In particular, this gives a simple decomposition of the generating function of a poset as the sum of generating functions of all linear extensions of the poset, the latter of which are elements of Gessel's fundamental basis.

Corollary 2.7. Given a poset $\mathcal{P}$ on $[p]$, we have

$$F_{\mathcal{P}} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} F_{\mathcal{L}} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} F_{\text{Des}(\mathcal{L})}. \quad (2.5)$$

For example, there are 18 linear extensions of the poset in Fig. 1, and so the corresponding generating function is a sum of 18 terms of the fundamental basis.

An immediate and powerful consequence of Theorem 2.6 is a formula for the fundamental expansion of a product of elements of the fundamental basis.

Corollary 2.8. Let α, β be two strong compositions. Choose two linear orders A and B such that $\alpha = \text{Des}(A)$ and $\beta = \text{Des}(B)$. Then

$$F_{\alpha} F_{\beta} = F_A F_B = F_{A \oplus B} = \sum_{\mathcal{L} \in \mathcal{L}(A \oplus B)} F_{\text{Des}(\mathcal{L})}, \quad (2.6)$$

where $A \oplus B$ denotes the partial order given by the disjoint union of A and B and where no element of A is comparable to an element of B . The set $\mathcal{L}(A \oplus B)$ corresponds to the shuffle product of A and B .

Example 2.9. Let $\alpha = (2)$ and $\beta = (2)$, and set $A = (1 < 2)$ and $B = (3 < 4)$. We do have $\text{Des}(A) = \alpha$ and $\text{Des}(B) = \beta$. Since A and B are linear orders, let us represent them using words. That is $A = 12$ and $B = 34$. Then

$$A \sqcup B = \{1234, 1324, 1342, 3124, 3142, 3412\},$$

and the words we get are exactly the linear extensions of $\mathcal{L}(A \oplus B)$. They are distinct but could have the same descent composition. For example $\text{Des}(1324) = \text{Des}(3412) = (2, 2)$, hence the coefficient of $F_{(2,2)}$ in the product is 2. We have

$$F_{(2)} F_{(2)} = F_{A \oplus B} = F_{(4)} + 2F_{(2,2)} + F_{(3,1)} + F_{(1,3)} + F_{(1,2,1)}$$

Remark 2.10. At this point, it is important to notice if we order the monomials lexicographically, then the leading term of F_{α} is $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{\ell}^{\alpha_{\ell}}$ for any strong composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell})$. Hence the leading terms for the F_{α} are all distinct as α runs over all strong compositions. This shows the set $\{F_{\alpha}\}$ is linearly independent. Corollary 2.8 shows this set spans all possible products of F_{α} 's. Hence the set $\{F_{\alpha}\}$ is a basis of the algebra it generates.

For another compelling example, consider the labelled poset $\mathcal{P}_{\lambda}$ associated to a partition λ , as illustrated in Fig. 2. The generating function for $\mathcal{P}_{\lambda}$ precisely enumerates semistandard Young tableaux, and so

$$F_{\mathcal{P}_{\lambda}} = s_{\lambda}, \quad (2.7)$$

is the Schur function corresponding to λ . Linear extensions of $\mathcal{P}_{\lambda}$ are in bijection with standard Young tableaux, and if $\mathcal{L}$ corresponds to T , then $\text{Des}(\mathcal{L}) = \text{Des}(T)$. Therefore the Fundamental Theorem gives the following alternative expansion for Schur functions in terms of standard Young tableaux first shown in [3].

Corollary 2.11. For λ a partition, we have

$$s_{\lambda} = \sum_{T \in \text{STY}(\lambda)} F_{\text{Des}(T)}, \quad (2.8)$$

where $\text{STY}(\lambda)$ is the set of all Standard Young Tableaux of shape λ .

3. $(\mathcal{P}, \rho)$ -partitions

We introduce a new generalization of $\mathcal{P}$ -partitions by restricting the values of the images of the $\mathcal{P}$ -partition with an integer-valued map ρ on $\mathcal{P}$.

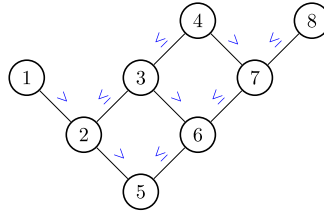


Fig. 2. The labelled poset corresponding to the partition (4, 3, 1).

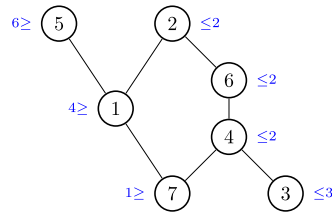


Fig. 3. An example of a poset $\mathcal{P}$ together with a restriction map ρ with restrictions indicated at each node.

Definition 3.1. Given a poset $\mathcal{P}$ on $[p]$, and a map $\rho: [p] \rightarrow \mathbb{Z}$, a $(\mathcal{P}, \rho)$ -partition is a $\mathcal{P}$ -partition such that $f(i) \leq \rho(i)$ for all $i \in \mathcal{P}$. We denote by $\mathcal{A}_\rho(\mathcal{P})$ the set of all $(\mathcal{P}, \rho)$ -partitions.

Note if $\rho(i) \leq 0$ for some $i \in [p]$, then $\mathcal{A}_\rho(\mathcal{P}) = \emptyset$, and so some restriction maps ρ are too restrictive. Nevertheless, these are still useful to consider.

For a fixed poset $\mathcal{P}$ on $[p]$, some restriction maps ρ are redundant. For example, if we consider the labelled poset in Fig. 3 and a restriction map ρ such that $\rho(3) = 3$ and $\rho(6) = 2$, then since any $\mathcal{P}$ -partition f must satisfy $f(3) \leq f(6)$, the restriction on 3 can never be attained. Such redundancy happens if for two elements $i, j \in \mathcal{P}$ we have $i < j$ and $\rho(i) > \rho(j)$.

Furthermore, looking at Fig. 3, we see if we impose the restriction $\rho(2) = 2$, then a $(\mathcal{P}, \rho)$ -partition for that $\mathcal{P}$ would need to satisfy

$$1 \leq f(7) < f(4) \leq f(6) < f(2) \leq \rho(2) = 2$$

which is not possible. The set $\mathcal{A}_\rho(\mathcal{P})$ is empty unless $\rho(i)$ is larger than the size of any decreasing chain in $\mathcal{P}$ ending at i . To make this more general, we introduce the following definitions.

For $i \leq j$ in $\mathcal{P}$, define the *maximum descent distance* $\delta(i, j)$ by

$$\delta(i, j) = \max \left\{ k \mid i = i_0 \prec i_1 \prec \cdots \prec i_\ell = j \text{ and } k = |\{j : i_j > i_{j+1}\}| \right\}. \quad (3.1)$$

For example, in Fig. 3 we have $\delta(7, 2) = 2$ from the chain $7 \prec 4 \prec 6 \prec 2$ with two descents $7 > 4$ and $6 > 2$.

Using this terminology, given a restriction map ρ for a poset $\mathcal{P}$, we may define a canonical $\mathcal{P}$ -weakly increasing restriction map as follows.

Definition 3.2. Given a poset $\mathcal{P}$ on $[p]$ and a map $\rho: [p] \rightarrow \mathbb{Z}$, the *maximal* $(\mathcal{P}, \rho)$ -partition $\bar{\rho}^\mathcal{P}$ is

$$\bar{\rho}^\mathcal{P}(i) = \min\{\rho(x) - \delta(i, x) \mid i \leq x\}. \quad (3.2)$$

This is largest possible value that $f(i)$ can take for any $(\mathcal{P}, \rho)$ -partition f .

Example 3.3. Let $\mathcal{P}$ be the poset in Fig. 3, and let $\rho = (4, 6, 2, 3, 2, 3, 3)$ taken in the natural order. Then since 2 and 5 are maximal elements, we have $\bar{\rho}^\mathcal{P}(2) = \rho(2) = 6$ and $\bar{\rho}^\mathcal{P}(5) = \rho(5) = 2$. The most interesting computation happens for the minimal element 7, where we have

$$\bar{\rho}^\mathcal{P}(7) = \min\{\rho(x) - \delta(7, x) \mid x = 7, 4, 1, 6, 5, 2\} = 1$$

achieved by taking $x = 5$. The complete example has $\bar{\rho}^{\mathcal{P}} = (2, 6, 2, 3, 2, 3, 1)$, which is the maximal $(\mathcal{P}, \rho)$ -partition.

Given a poset $\mathcal{P}$ and any map $f: \mathcal{P} \rightarrow \mathbb{N}$, let $\text{comp}(f)$ be the weak composition whose r th part is given by $c_r = \#\{i \in \mathcal{P} \mid f(i) = r\}$. Then for $f \in \mathcal{A}_{\rho}(\mathcal{P})$, $\text{comp}(f) \succeq \text{comp}(\bar{\rho}^{\mathcal{P}})$ in dominance order. We have the following.

Proposition 3.4. *Given any restriction $\rho: [p] \rightarrow \mathbb{N}$, we have $\mathcal{A}_{\rho}(\mathcal{P}) \neq \emptyset$ if and only if $\bar{\rho}^{\mathcal{P}}(i) \geq 1$ for all $i \in \mathcal{P}$. Furthermore, in this case, $\mathcal{A}_{\rho}(\mathcal{P}) = \mathcal{A}_{\bar{\rho}^{\mathcal{P}}}(\mathcal{P})$.*

Proof. By definition $\bar{\rho}^{\mathcal{P}}(i)$ is the largest possible value that $f(i)$ can take for any $(\mathcal{P}, \rho)$ -partition f , so if $\bar{\rho}^{\mathcal{P}}(i) \leq 0$ for some i , then there are no $(\mathcal{P}, \rho)$ -partitions. Conversely, if $\bar{\rho}^{\mathcal{P}}(i) \geq 1$ for all i , then $\bar{\rho}^{\mathcal{P}}$ itself is a $(\mathcal{P}, \rho)$ -partition. The equality $\mathcal{A}_{\rho}(\mathcal{P}) = \mathcal{A}_{\bar{\rho}^{\mathcal{P}}}(\mathcal{P})$ follows as $\bar{\rho}^{\mathcal{P}}$ is the maximal $(\mathcal{P}, \rho)$ -partition for ρ , and all other $(\mathcal{P}, \rho)$ -partitions must be bounded above by $\bar{\rho}^{\mathcal{P}}$. $\square$

We define the generating polynomial of $(\mathcal{P}, \rho)$ -partitions by

$$\mathfrak{F}_{(\mathcal{P}, \rho)} = \sum_{f \in \mathcal{A}_{\rho}(\mathcal{P})} x_{f(1)} \cdots x_{f(p)}. \quad (3.3)$$

By Proposition 3.4, we have $\mathfrak{F}_{(\mathcal{P}, \rho)} = \mathfrak{F}_{(\mathcal{P}, \bar{\rho}^{\mathcal{P}})}$ and $\mathfrak{F}_{(\mathcal{P}, \rho)} = 0$ unless $\bar{\rho}^{\mathcal{P}}(i) \geq 1$ for all minimal elements i .

3.1. Linear $(\mathcal{P}, \rho)$ -partitions

The case of linear orders is again of particular interest. To study it, we generalize the notion of descent compositions to account for the restriction map, giving the following two definitions.

Definition 3.5. For $\mathcal{L}$ a linear order on $[p]$ and ρ a restriction map satisfying $\bar{\rho}^{\mathcal{L}}(i) \geq 1$ for all $i \in \mathcal{L}$, we define two associated weak compositions.

- (1) The *reduced weak descent composition* of $(\mathcal{L}, \rho)$, denoted by $\text{rdes}(\mathcal{L}, \rho)$, is formed as follows. Remove edges between descents, i.e. whenever $i \prec j$ and $i > j$, and call the resulting chains $C_1, \dots, C_r$ in ascending order of $\mathcal{L}$. For $s = 1, \dots, r$, set $c_s = \bar{\rho}^{\mathcal{L}}(\min_{\mathcal{L}}\{C_s\})$, and define the part c_s of $\text{rdes}(\mathcal{L}, \rho)$ to be $\text{rdes}(\mathcal{L}, \rho)_{c_s} = |C_s|$ and set all other parts to 0.
- (2) The *weak descent composition* of $(\mathcal{L}, \rho)$, denoted by $\text{des}(\mathcal{L}, \rho)$, is formed as follows. Remove edges between i and j whenever $i \prec j$ and $\bar{\rho}^{\mathcal{L}}(i) < \bar{\rho}^{\mathcal{L}}(j)$, and call the resulting chains $C'_1, \dots, C'_\ell$ in ascending order of $\mathcal{L}$. For $s = 1, \dots, \ell$, set $c'_s = \bar{\rho}^{\mathcal{L}}(\min_{\mathcal{L}}\{C'_s\})$, and define the part c'_s of $\text{des}(\mathcal{L}, \rho)$ as $\text{des}(\mathcal{L}, \rho)_{c'_s} = |C'_s|$ and set all other parts to 0.

For Definition 3.5(1), we have the same chain decomposition as in Definition 2.2. In particular, for any restriction map ρ we have

$$\text{flat}(\text{rdes}(\mathcal{L}, \rho)) = \text{Des}(\mathcal{L}), \quad (3.4)$$

and so $\text{rdes}(\mathcal{L}, \rho)$ encodes $\text{Des}(\mathcal{L})$ along with information about the restriction ρ . In Lemma 3.7 we will see that $\text{des}(\mathcal{L}, \rho)$ is a refinement of the information in $\text{rdes}(\mathcal{L}, \rho)$.

For a linear order $\mathcal{L}$ on $[p]$, we now always list the values of ρ according to the order $\mathcal{L}$, not according to the natural order on $[p]$. With this convention, $\bar{\rho}^{\mathcal{L}}$ is always weakly increasing in the order of $\mathcal{L}$.

Example 3.6. Let $\mathcal{L}$ be the linear order $2 \prec 5 \prec 1 \prec 4 \prec 6 \prec 7 \prec 3 \prec 8 \prec 9$ and let $\rho = (\rho(2), \rho(5), \rho(1), \rho(4), \rho(6), \rho(7), \rho(3), \rho(8), \rho(9)) = (2, 3, 4, 3, 6, 8, 6, 8, 8)$ be a restriction map. The chains after removing edges for descents become

$$\overbrace{2 \prec 5}^{c_1} \quad \overbrace{1 \prec 4 \prec 6 \prec 7}^{c_2} \quad \overbrace{3 \prec 8 \prec 9}^{c_3}.$$

Note this is the same ascending chain decomposition as in [Example 2.3](#) where we computed the descent composition for the same linear order $\mathcal{L}$. We compute $\bar{\rho}^{\mathcal{L}} = (2, 2, 3, 3, 5, 6, 8, 8)$, and so $c_1 = \bar{\rho}^{\mathcal{L}}(2) = 2$, $c_2 = \bar{\rho}^{\mathcal{L}}(1) = 3$, $c_3 = \bar{\rho}^{\mathcal{L}}(3) = 6$. Therefore, $\text{rdes}(\mathcal{L}, \rho) = (0, |C_1|, |C_2|, 0, 0, |C_3|) = (0, 2, 4, 0, 0, 3)$.

The chains after removing edges for $i \prec j$ and $\bar{\rho}^{\mathcal{L}}(i) < \bar{\rho}^{\mathcal{L}}(j)$ become

$$\underbrace{c'_1}_{2 \prec 5} \quad \underbrace{c'_2}_{1 \prec 4} \quad \underbrace{c'_3}_{6 \prec 7} \quad \underbrace{c'_4}_3 \quad \underbrace{c'_5}_{8 \prec 9}.$$

We have $c'_1 = \bar{\rho}(2) = 2$, $c'_2 = \bar{\rho}(1) = 3$, $c'_3 = \bar{\rho}(6) = 5$, $c'_4 = \bar{\rho}(3) = 6$, and $c'_5 = \bar{\rho}(8) = 8$. Therefore $\text{des}(\mathcal{L}, \rho) = (0, |C'_1|, |C'_2|, 0, |C'_3|, |C'_4|, 0, |C'_5|) = (0, 2, 2, 0, 2, 1, 0, 2)$.

Notice, when $i \prec j$ and $i > j$, we have $\bar{\rho}^{\mathcal{L}}(i) < \bar{\rho}^{\mathcal{L}}(j)$. Thus, the following lemma expresses precisely how $\text{des}(\mathcal{L}, \rho)$ refines $\text{rdes}(\mathcal{L}, \rho)$.

Lemma 3.7. For $\mathcal{L}$ a linear order on $[p]$ and $\rho: [p] \rightarrow \mathbb{N}$ a restriction map, we have $\text{flat}(\text{des}(\mathcal{L}, \rho))$ refines $\text{flat}(\text{rdes}(\mathcal{L}, \rho))$ and $\text{rdes}(\mathcal{L}, \rho) \succeq \text{des}(\mathcal{L}, \rho)$.

Proof. The chains $C_1, \dots, C_r$ are unions of consecutive chains $C'_1, \dots, C'_\ell$, and so $(|C'_1|, \dots, |C'_\ell|)$ refines $(|C_1|, \dots, |C_r|)$, proving the first statement. If for $s \leq t$, C'_s is a sub-chain of C_t , then $c'_s = \bar{\rho}^{\mathcal{L}}(\min_{\mathcal{L}}\{C'_s\}) \geq \bar{\rho}^{\mathcal{L}}(\min_{\mathcal{L}}\{C_t\}) = c_t$ since $\bar{\rho}^{\mathcal{L}}$ is weakly increasing with respect to $\mathcal{L}$. Thus each nonzero part of $\text{rdes}(\mathcal{L}, \rho)$ occurs weakly before all nonzero parts of $\text{des}(\mathcal{L}, \rho)$ that refine it, proving the second. $\square$

Under certain assumptions on $\mathcal{L}$ and ρ , the generating polynomial for $(\mathcal{L}, \rho)$ depends only on the reduced weak descent composition $\text{rdes}(\mathcal{L}, \rho)$.

Definition 3.8 ([1]). For a weak composition a of length n , the slide polynomial $\mathfrak{F}_a$ is given by

$$\mathfrak{F}_a(x_1, \dots, x_n) = \sum_{\substack{\text{flat}(b) \text{ refines } \text{flat}(a) \\ b \succeq a}} x_1^{b_1} \cdots x_n^{b_n}, \quad (3.5)$$

where the sum is over weak compositions b that dominate a and for which the flattening of b refines the flattening of a .

For example, we have

$$\mathfrak{F}_{(3,0,2)} = x_1^3 x_3^2 + x_1^3 x_2^2 + x_1^3 x_2 x_3.$$

Note if a is a weak composition of length n with the property $a_j > 0$ whenever $a_i > 0$ for some $i < j$, then $\mathfrak{F}_a = F_{\text{flat}(a)}(x_1, \dots, x_n)$ [1](Lemma 3.8). In particular, fundamental quasisymmetric polynomials are slide polynomials. Furthermore, we have the following result from [1](Theorem 4.5).

Theorem 3.9 ([1]). For a weak composition a , we have

$$\lim_{m \rightarrow \infty} \mathfrak{F}_{0^m \times a}(x_1, \dots, x_m, 0, \dots, 0) = F_{\text{flat}(a)}(x_1, x_2, \dots), \quad (3.6)$$

where $0^m \times a$ denotes the weak composition obtained by prepending m 0's to a .

With this in mind, we have the following generalization of [Proposition 2.5](#).

Proposition 3.10. Let $\mathcal{L}$ be a linear order on $[p]$ and $\rho: [p] \rightarrow \mathbb{N}$ a restriction map. Then $\text{rdes}(\mathcal{L}, \rho) = \text{des}(\mathcal{L}, \rho)$ if and only if for all $i \prec j$, whenever $i < j$ we have $\bar{\rho}^{\mathcal{L}}(i) = \bar{\rho}^{\mathcal{L}}(j)$. Moreover, when this is the case, we have

$$\mathfrak{F}_{(\mathcal{L}, \rho)} = \mathfrak{F}_{\text{rdes}(\rho)}. \quad (3.7)$$

Conversely, for any weak composition a , there exists ρ and $\mathcal{L}$ such that $\mathfrak{F}_{(\mathcal{L}, \rho)} = \mathfrak{F}_a$.

Proof. We may assume $\rho = \bar{\rho}^{\mathcal{L}}$ and let $\text{comp}(\rho) = \text{rdes}(\rho)$. The condition on ρ ensures the chains used to compute $\text{des}(\mathcal{L}, \rho)$ are broken only between descents, since we never have $i \prec j$, $i < j$, and $\bar{\rho}^{\mathcal{L}}(i) < \bar{\rho}^{\mathcal{L}}(j)$. Thus $\text{des}(\mathcal{L}, \rho) = \text{rdes}(\mathcal{L}, \rho)$.

Let $f \in \mathcal{A}_{\rho}(\mathcal{L})$ be a $(\mathcal{L}, \rho)$ -partition and let $b = \text{comp}(f)$. Then $b \supseteq \text{comp}(\rho)$. Letting $\mathcal{L} = \{\ell_1 \prec \ell_2 \prec \dots \prec \ell_p\}$ so that we have

$$\begin{array}{ccccccc} f(\ell_1) & \leq & f(\ell_2) & \leq & \dots & \leq & f(\ell_p) \\ \wedge! & & \wedge! & & & & \wedge! \\ \rho(\ell_1) & \leq & \rho(\ell_2) & \leq & \dots & \leq & \rho(\ell_p) \end{array}$$

notice $\ell_i < \ell_{i+1}$ implies $\rho(\ell_i) = \rho(\ell_{i+1})$, and $\ell_i > \ell_{i+1}$ implies both $f(\ell_i) < f(\ell_{i+1})$ and $\rho(\ell_i) < \rho(\ell_{i+1})$. Thus $\text{flat}(b)$ refines $\text{flat}(\text{comp}(\rho))$. In particular, every $(\mathcal{L}, \rho)$ -partition contributes a term of $\mathfrak{F}_{\text{comp}(\rho)} = \mathfrak{F}_{\text{rdes}(\rho)}$. Conversely, for any weak composition b such that $\text{flat}(b)$ refines $\text{flat}(\text{comp}(\rho))$ and $b \supseteq \text{comp}(\rho)$, we may construct a unique $(\mathcal{L}, \rho)$ -partition f such that $\text{comp}(f) = b$.

Given a with $\text{flat}(a) = (\alpha_1, \dots, \alpha_k)$, we may take $\mathcal{L}$ to be the linear order

$$(p - \alpha_1 + 1) \prec \dots \prec p \prec (p - \alpha_1 - \alpha_2 + 1) \prec \dots \prec (p - \alpha_1) \prec \dots \prec 1 \prec \dots \prec \alpha_k.$$

Then clearly $\text{Des}(\mathcal{L}) = \alpha$. If the nonzero entries of a occur at indices $r_1 < \dots < r_k$, then we may take ρ to be the restriction map such that $\rho(i) = r_j$ whenever i is in the j th descent chain of $\mathcal{L}$. This ensures $a = \text{comp}(\rho) = \text{rdes}(\mathcal{L}, \rho) = \text{des}(\mathcal{L}, \rho)$, and so, by the first statement of the proposition, $\mathfrak{F}_{(\mathcal{L}, \rho)} = \mathfrak{F}_a$ as desired. $\square$

Example 3.11. Given the weak composition $a = (0, 2, 4, 0, 0, 3)$, we may realize the pair $(\mathcal{L}, \rho)$ for which $\mathfrak{F}_a = \mathfrak{F}_{(\mathcal{L}, \rho)}$ as follows. We have $\text{flat}(a) = (2, 4, 3)$ and so we will take $\mathcal{L}$ to be the linear order

$$\overbrace{8 \prec 9}^{c_1} \prec \overbrace{4 \prec 5 \prec 6 \prec 7}^{c_2} \prec \overbrace{1 \prec 2 \prec 3}^{c_3}.$$

The nonzero entries of a occur at indices 2, 3, 6, and so we set ρ to be

$$\rho(8) = \rho(9) = 2, \quad \rho(4) = \rho(5) = \rho(6) = \rho(7) = 3, \quad \rho(1) = \rho(2) = \rho(3) = 6.$$

Observe the conclusion of [Proposition 3.10](#) does not hold in general.

Example 3.12. Take $\mathcal{L}$ to be $2 \prec 3 \prec 1$ and $\rho = (2, 3, 4)$. Then we have

$$\begin{aligned} \mathfrak{F}_{(\mathcal{L}, \rho)} &= x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_1 x_2 x_3 + x_1 x_2 x_4 \\ &\quad + x_1 x_3 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_2 x_3 x_4 \\ &= \mathfrak{F}_{(0, 2, 0, 1)} + \mathfrak{F}_{(0, 1, 1, 1)} - \mathfrak{F}_{(1, 1, 0, 1)}. \end{aligned}$$

In this case, $\bar{\rho}^{\mathcal{L}} = \rho$ but it is not constant over the increasing chain $2 \prec 3$. Here, $\text{rdes}(\mathcal{L}, \rho) = (0, 2, 0, 1)$, and indeed we see $\mathfrak{F}_{\text{rdes}(\mathcal{L}, \rho)}$ appear as a term in $\mathfrak{F}_{(\mathcal{L}, \rho)}$, but there are other terms as well. Note as well the slide expansion is not positive.

Remark 3.13. In [Remark 2.10](#) we saw the linear extensions of $\mathcal{P}$ -partitions span an algebra containing all $\mathcal{P}$ -partition generating functions, and this algebra has a basis consisting of linear extensions with distinct descent compositions. [Corollary 2.7](#) gives us a beautiful positive expansion of any $\mathcal{P}$ -partition generating functions in terms of the fundamental basis. We aim to obtain similar results for $(\mathcal{P}, \rho)$ -partitions, but already [Example 3.12](#) shows we cannot expect such results for all $(\mathcal{P}, \rho)$. While [Corollary 3.15](#) gives a direct analogue of [Corollary 2.7](#) expanding positively any bounded $\mathcal{P}$ -partitions in terms of linear bounded $\mathcal{P}$ -partition, the example shows that, unfortunately, the linear bounded $\mathcal{P}$ -partitions are not linearly independent nor positive in terms of slide polynomials. In [Section 4](#) we will define flagged $(\mathcal{P}, \rho)$ -partitions for which all the desired properties hold.

3.2. Fundamental theorem for $(\mathcal{P}, \rho)$ -partitions

Stanley's Fundamental Theorem of $\mathcal{P}$ -partitions holds for $(\mathcal{P}, \rho)$ -partitions as well.

Theorem 3.14 (Fundamental Theorem of $(\mathcal{P}, \rho)$ -partitions). *Given a poset $\mathcal{P}$ on $[p]$ and a restriction ρ , we have*

$$\mathcal{A}_\rho(\mathcal{P}) = \bigsqcup_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathcal{A}_\rho(\mathcal{L}), \quad (3.8)$$

where the disjoint union is over the set $\mathcal{L}(\mathcal{P})$ of linear extensions of $\mathcal{P}$.

Proof. We proceed by induction on the number of incomparable pairs i, j in $\mathcal{P}$. If there is no such pair, then $\mathcal{P}$ must be a linear order, and so $\mathcal{L}(\mathcal{P}) = \{\mathcal{P}\}$ and the result follows trivially. Now assume $\mathcal{P}$ has some incomparable pair. Fix $i < j$ an incomparable pair in $\mathcal{P}$. We construct two new posets $\mathcal{P}_{i < j}$ and $\mathcal{P}_{i > j}$ that are the transitive closure obtained by adding $i < j$ or $i > j$ to $\mathcal{P}$, respectively. We will show that

$$\mathcal{A}_\rho(\mathcal{P}) = \mathcal{A}_\rho(\mathcal{P}_{i < j}) \sqcup \mathcal{A}_\rho(\mathcal{P}_{i > j}), \quad (3.9)$$

a disjoint union. The result will follow from the induction hypothesis.

Notice $\mathcal{A}_\rho(\mathcal{P}_{i < j}) \cap \mathcal{A}_\rho(\mathcal{P}_{i > j}) = \emptyset$ since a $(\mathcal{P}, \rho)_{i < j}$ -partition f satisfies $f(i) \leq f(j)$ whereas a $(\mathcal{P}, \rho)_{i > j}$ -partition f satisfies $f(i) > f(j)$. Since any relation of $\mathcal{P}$ is contained in both $\mathcal{P}_{i < j}$ and $\mathcal{P}_{i > j}$, if $f \in \mathcal{A}_\rho(\mathcal{P}_{i < j}) \sqcup \mathcal{A}_\rho(\mathcal{P}_{i > j})$, then $f \in \mathcal{A}_\rho(\mathcal{P})$. The restriction ρ is the same function across the three sets involved, so it has the same ρ -restrictions.

Now assume $f \in \mathcal{A}_\rho(\mathcal{P})$ and consider the values $f(i)$ and $f(j)$. If $f(i) \leq f(j)$, then $f \in \mathcal{A}_\rho(\mathcal{P}_{i < j})$. Indeed it satisfies all the conditions imposed by $\mathcal{P}$ plus the additional condition imposed by $i < j$. Moreover the restriction imposed by ρ is the same on both sets. Now if $f(i) > f(j)$, then $f \in \mathcal{A}_\rho(\mathcal{P}_{i > j})$. Such f satisfies all conditions imposed by $\mathcal{P}$ plus the condition imposed by $i > j$ and $i < j$. $\square$

In particular, this gives a simple decomposition of the generating function of a poset as the sum of generating functions of all linear extensions of the poset.

Corollary 3.15. *Given a poset $\mathcal{P}$ on $[p]$ and a restriction ρ , we have*

$$\mathfrak{F}_{(\mathcal{P}, \rho)} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathfrak{F}_{(\mathcal{L}, \rho)}. \quad (3.10)$$

Remark 3.16. The above corollary gives an efficient algorithm to program the polynomials $\mathfrak{F}_{(\mathcal{P}, \rho)}$, especially in packages like SageMath that contains modules for orders and linear extension. But again, be aware this does not, in general, give a decomposition into the slide polynomial basis.

4. Flagged $(\mathcal{P}, \rho)$ -partitions

Assaf and Searles [1, Theorem 3.9] showed (fundamental) slide polynomials form a basis of polynomials, and Proposition 3.10 shows slide polynomial are $(\mathcal{L}, \rho)$ -partitions for some ρ and $\mathcal{L}$. However, Example 3.12 shows for some restrictions ρ on linear orders we do not get single slide polynomials, and, in fact, the expansion in terms of slides polynomials may not be positive. In this section we are interested in important subfamilies of $(\mathcal{P}, \rho)$ -partitions that will give positivity results when expanded in the basis of slide polynomials.

4.1. Flagged and well-labelled $(\mathcal{P}, \rho)$ -partitions

The subfamily of $(\mathcal{P}, \rho)$ -partitions of interest includes the *flagged Schur functions* defined by Lascoux and Schützenberger [5] and studied further by Wachs [9]. This motivates calling the elements of this subfamily *flagged $(\mathcal{P}, \rho)$ -partitions*.

Definition 4.1. For a poset $\mathcal{P}$, a restriction ρ is a $\mathcal{P}$ -flag if $\rho(i) > 0$ for all $i \in [p]$,

- (i) if $i \triangleleft j$ and $i < j$, then $\rho(i) = \rho(j)$, and
- (ii) if $i \triangleleft j$ and $i > j$, then $\rho(i) \leq \rho(j)$.

We call $\mathcal{A}_\rho(\mathcal{P})$ the set of *flagged* $(\mathcal{P}, \rho)$ -partitions to emphasize ρ is a $\mathcal{P}$ -flag.

We see in [Proposition 3.10](#) the slide polynomials are exactly the generating functions of flagged $(\mathcal{L}, \rho)$ -partitions for $\mathcal{L}$ a linear order. To prove our main theorem, we introduce a second family of $(\mathcal{P}, \rho)$ -partitions that we call *well-labelled*.

Definition 4.2. For a poset $\mathcal{P}$ on $[p]$ and a restriction ρ , we say $(\mathcal{P}, \rho)$ is *well-labelled* if

- (i) if $i \triangleleft j$ and $i < j$, then $\rho(i) \geq \rho(j)$,
- (ii) if $i \triangleleft j$ and $i > j$, then $\rho(i) \leq \rho(j)$, and
- (iii) for all incomparable $i, j \in \mathcal{P}$, if $i < j$, then $\rho(i) \geq \rho(j)$.

The inequality $\rho(i) \geq \rho(j)$ in (i) of [Definition 4.2](#) may appear strange, but in conjunction with (iii), it will be what we need in the inductive proof of [Theorem 1.1](#). As we inductively construct linear extension of $\mathcal{P}$, the property of being well-labelled is preserved (see [Lemma 4.5](#)). [Definition 4.1](#) is what we really want, but it is not preserved by the operation in [Lemma 4.5](#). Thankfully, the following result shows that every $\mathcal{P}$ -flag ρ generates a polynomial equal to a polynomial obtained by a well-labelled pair $(\mathcal{P}', \rho')$ where ρ' is a $\mathcal{P}'$ -flag.

Proposition 4.3. Given poset $\mathcal{P}$ on $[p]$ and a $\mathcal{P}$ -flag ρ , there exists a permutation $\pi: [p] \rightarrow [p]$ such that

- (1) $\rho \circ \pi$ is a $\pi^{-1}(\mathcal{P})$ -flag,
- (2) $(\pi^{-1}(\mathcal{P}), \rho \circ \pi)$ is well-labelled, and
- (3) $\mathcal{A}_\rho(\mathcal{P}) = \mathcal{A}_{\rho \circ \pi}(\pi^{-1}(\mathcal{P}))$.

Proof. The Hasse diagram $H(\mathcal{P})$ of $\mathcal{P}$ is the directed graph on $[p]$ where we have an edge $i \rightarrow j$ for each cover $i \triangleleft j$. Consider the connected components $M_1, M_2, \dots, M_\ell$ of the directed graph $H^+(\mathcal{P})$ obtained from $H(\mathcal{P})$ by removing every edge $i \triangleleft j$ where $i > j$. The order $\mathcal{P}$ induces a partial order on these connected components, and since ρ is a $\mathcal{P}$ -flag, it must be constant on each connected component. Therefore we may re-index the components as follows. Setting $m_i = \rho(x)$ for any $x \in M_i$, we may assume the M_i 's are indexed in such a way that $m_1 \geq m_2 \geq \dots \geq m_\ell$ and $M_\ell \cdots M_2 M_1$ is a linear extension of the $\mathcal{P}$ -induced order on the M_i 's.

Define a permutation π on $[p]$ such that for $1 \leq k \leq \ell$ we have

- (a) $\{\pi(|M_1| + \dots + |M_{k-1}| + 1), \dots, \pi(|M_1| + \dots + |M_k|)\} = M_k$, and
- (b) $\pi(|M_1| + \dots + |M_{k-1}| + 1) < \dots < \pi(|M_1| + \dots + |M_k|)$.

We claim π has the desired properties. To see $\rho \circ \pi$ is a $\pi^{-1}(\mathcal{P})$ -flag, suppose $i \triangleleft_{\pi^{-1}(\mathcal{P})} j$ or, equivalently, $\pi(i) \triangleleft_{\mathcal{P}} \pi(j)$. If $\pi(i) < \pi(j)$, then $\pi(i)$ and $\pi(j)$ are in the same component M_k , and so we must have $i < j$ as well by (a) and (b). Conversely, if $\pi(i) > \pi(j)$, then since ρ is a $\mathcal{P}$ -flag, we have $\rho(\pi(i)) \leq \rho(\pi(j))$ placing $\pi(i)$ in a later component than $\pi(j)$ by our indexing choice, and so $i > j$.

To see $\pi^{-1}(\mathcal{P})$ is well-labelled by $\rho \circ \pi$, notice conditions (i) and (ii) of [Definition 4.1](#) imply conditions (i) and (ii) of [Definition 4.2](#). For (iii), suppose i and j are incomparable in $\pi^{-1}(\mathcal{P})$ with $i < j$. If $\pi(i), \pi(j)$ are in the same component M_k , then $\rho \circ \pi(i) = \rho \circ \pi(j)$. Otherwise, since $i < j$, we must have $\pi(i)$ in an earlier, and thus greater, component than $\pi(j)$, and so $\rho \circ \pi(i) \geq \rho \circ \pi(j)$, as desired.

Finally, to see the partition generating functions coincide, regarding the nodes of $\mathcal{P}$ as fixed, as we permute their labels with π^{-1} , we correspondingly permute the restrictions so that the latter remain fixed with each node. Thus the set of $(\mathcal{P}, \rho)$ -partitions is identical to that of $(\pi^{-1}(\mathcal{P}), \rho \circ \pi)$ -partitions. $\square$

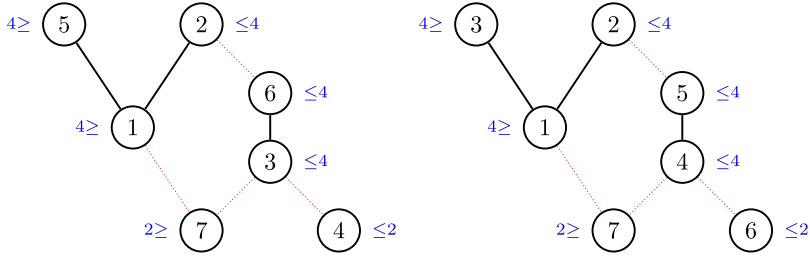


Fig. 4. An example of a poset $\mathcal{P}$ together with a $\mathcal{P}$ -flag ρ (left) and the poset $\pi^{-1}(\mathcal{P})$ well-labelled by $\rho \circ \pi$ (right).

Example 4.4. Consider the poset $\mathcal{P}$ in Fig. 4. The connected components of $H^+(\mathcal{P})$ are $M_1 = \{1, 2, 5\}$, $M_2 = \{3, 6\}$, $M_3 = \{4\}$ and $M_4 = \{7\}$. The $\mathcal{P}$ -flag ρ is constant on each component with values $m_1 = 4$, $m_2 = 4$, $m_3 = 2$ and $m_4 = 2$. The indexing is valid since $4 \geq 4 \geq 2 \geq 2$ and $M_4 M_3 M_2 M_1$ is a linear extension of the $\mathcal{P}$ -induced order on the components. The permutation π is given by

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 3 & 6 & 4 & 7 \end{pmatrix}$$

with result that $\pi^{-1}(\mathcal{P})$ is well-labelled by $\rho \circ \pi$, as seen in Fig. 4.

In view of Proposition 4.3, to prove our Main Theorem 1.1, we can always assume we have a well-labelled flagged $(\mathcal{P}, \rho)$. The next result is the heart of the proof; it relies on the Fundamental Theorem 3.14 for $(\mathcal{P}, \rho)$ -partitions.

Lemma 4.5. Let $\mathcal{P}$ be a poset on $[p]$ and ρ a map such that $(\mathcal{P}, \rho)$ is well-labelled. For any incomparable pair $i, j \in [p]$, let $\mathcal{P}_1 = \mathcal{P}_{i < j}$ and $\mathcal{P}_2 = \mathcal{P}_{i > j}$ be the transitive closure obtained by adding $i < j$ or $i > j$ to $\mathcal{P}$, respectively. Then $(\mathcal{P}_1, \rho)$ and $(\mathcal{P}_2, \rho)$ are both well-labelled.

Proof. Since i and j are incomparable in $\mathcal{P}$ with $i < j$, and $(\mathcal{P}, \rho)$ is well-labelled, we must have $\rho(i) \geq \rho(j)$. We claim $i \prec j$ in $\mathcal{P}_1 = \mathcal{P}_{i < j}$, since otherwise any intermediate k would already exist in $\mathcal{P}$, contradicting the incomparability of i and j . Therefore $i \prec j$, $i < j$ and $\rho(i) \geq \rho(j)$, hence condition (i) of Definition 4.2 is satisfied. For all other pairs of elements in $\mathcal{P}_1$ the conditions of being well labelled are already satisfied from $\mathcal{P}$. For $\mathcal{P}_2$, the situation is similar, we have $j \prec i$, $j > i$ and $\rho(j) \leq \rho(i)$, and condition (ii) of Definition 4.2 is satisfied. It is the only pair we need to check and the rest follows. $\square$

Lemma 4.6. Let $\mathcal{L}$ be a linear order on $[p]$ and ρ a map such that $(\mathcal{L}, \rho)$ is well-labelled. We have $\mathfrak{F}_{(\mathcal{L}, \rho)} = \mathfrak{F}_{(\mathcal{L}, \bar{\rho}^{\mathcal{L}})}$ is a slide polynomial or zero.

Proof. If $\bar{\rho}^{\mathcal{L}}$ has any values ≤ 0 , then we have $\mathfrak{F}_{(\mathcal{L}, \bar{\rho}^{\mathcal{L}})} = 0$. Now, since ρ is weakly decreasing on segments of $\mathcal{L}$ that do not have descents, $\bar{\rho}^{\mathcal{L}}$ will be constant on those same segments. The lemma then follows from Proposition 3.10. $\square$

Proof of Main Theorem 1.1. Given a poset $\mathcal{P}$ on $[p]$ and a $\mathcal{P}$ -flag ρ , by Proposition 4.3 we may transform this to a well-labelled and flagged $(\mathcal{P}', \rho')$ such that

$$\mathfrak{F}_{(\mathcal{P}, \rho)} = \mathfrak{F}_{(\mathcal{P}', \rho')}.$$

Following the proof of the Fundamental Theorem 3.14 for $(\mathcal{P}, \rho)$ -partitions, we recursively use Eq. (3.9). Lemma 4.5 guaranties at each step, we get well-labelled pair. Hence by induction on the number of incomparable pairs we have

$$\mathfrak{F}_{(\mathcal{P}, \rho)} = \mathfrak{F}_{(\mathcal{P}', \rho')} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathfrak{F}_{(\mathcal{L}, \rho)}$$

with each $(\mathcal{L}, \rho)$ in the sum well-labelled, so Theorem 1.1 follows by Lemma 4.6. $\square$

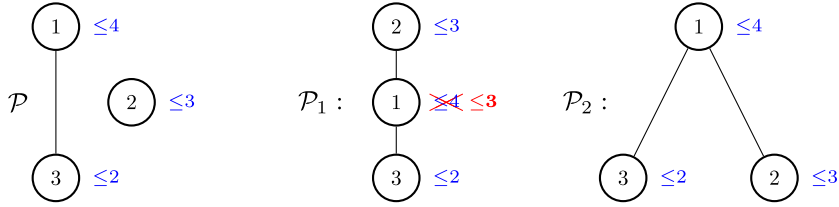


Fig. 5. An example of decomposition $\mathcal{A}_\rho(\mathcal{P}) = \mathcal{A}_\rho(\mathcal{P}_1) \sqcup \mathcal{A}_\rho(\mathcal{P}_2)$ for well-labelled flagged $(\mathcal{P}, \rho)$ -partition. The pairs $(\mathcal{P}_1, \rho)$ and $(\mathcal{P}_2, \rho)$ are well labelled.

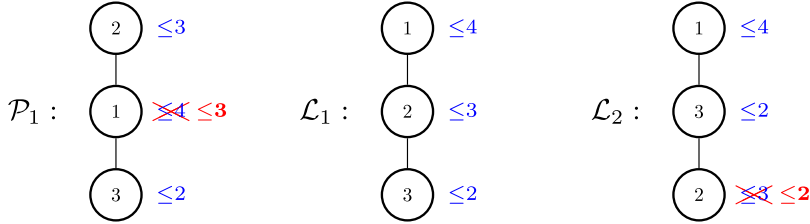


Fig. 6. Terminating the example, $\bar{\rho}^{\mathcal{L}}$ is computed (in red and bold). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Example 4.7. Let $\mathcal{P}$ be the poset depicted on the left side of Fig. 5, and let ρ be the restriction map $(\rho(1), \rho(2), \rho(3)) = (4, 3, 2)$. The pair $(\mathcal{P}, \rho)$ is well labelled and flagged. Take two incomparable elements in $\mathcal{P}$, say 1 and 2. We add the relation $1 < 2$ or $2 < 1$ obtaining the order $\mathcal{P}_1$ and $\mathcal{P}_2$ in Fig. 5. We have $(\mathcal{P}_1, \rho)$ and $(\mathcal{P}_2, \rho)$ are well labelled. We can repeat the process for $\mathcal{P}_2$ and get two well labelled pairs $(\mathcal{L}_1, \rho)$ and $(\mathcal{L}_2, \rho)$. The order $\mathcal{P}_1$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear order, we compute $\bar{\rho}^{\mathcal{L}}$ for each (see Fig. 6). We get that

$$\mathfrak{F}(\mathcal{P}, \rho) = \mathfrak{F}_{(0,1,2,0)} + \mathfrak{F}_{(0,1,1,1)} + \mathfrak{F}_{(0,2,0,1)},$$

where the right hand side are slide polynomials.

Remark 4.8. If we remove the hypothesis that $(\mathcal{P}, \rho)$ is well-labelled in Lemma 4.5, then the statement is wrong and cannot be fixed easily. For example in Fig. 7, $(\mathcal{P}'', \rho)$ is not well-labelled as 2,3 are incomparable in $\mathcal{P}''$, but $2 < 3$ and $\rho(2) = 2 < 3 = \rho(3)$. While we do have the decomposition

$$\mathfrak{F}(\mathcal{P}'', \rho) = \mathfrak{F}_{(\mathcal{L}_1'', \bar{\rho}^{\mathcal{L}_1''})} + \mathfrak{F}_{(\mathcal{L}_2'', \bar{\rho}^{\mathcal{L}_2''})},$$

$(\mathcal{L}_1'', \bar{\rho}^{\mathcal{L}_1''})$ is not well labelled and $\mathfrak{F}_{(\mathcal{L}_1'', \bar{\rho}^{\mathcal{L}_1''})}$ is not a single slide polynomial. Here $\mathfrak{F}_{(\mathcal{L}_2'', \bar{\rho}^{\mathcal{L}_2''})} = \mathfrak{F}_{(1,1,0,1)}$ cancels a term in $\mathfrak{F}_{(\mathcal{L}_1'', \bar{\rho}^{\mathcal{L}_1''})}$ to give the correct expansion

$$\mathfrak{F}(\mathcal{P}'', \rho) = \mathfrak{F}(\mathcal{P}_2, \rho) = \mathfrak{F}_{(0,1,1,1)} + \mathfrak{F}_{(0,2,0,1)}$$

as computed with the well-labelled flagged pair $(\mathcal{P}_2, \rho)$ seen in Fig. 6.

In particular, when ρ is a $\mathcal{P}$ -flag, we have the following analog of Corollary 2.7 giving a combinatorial formula for the nonnegative slide expansion.

Corollary 4.9. Given a poset $\mathcal{P}$ on $[p]$ and a (well labelled) $\mathcal{P}$ -flag ρ , we have

$$\mathfrak{F}(\mathcal{P}, \rho) = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathfrak{F}_{(\mathcal{L}, \rho)} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{P})} \mathfrak{F}_{\text{rdes}(\mathcal{L}, \rho)}. \quad (4.1)$$

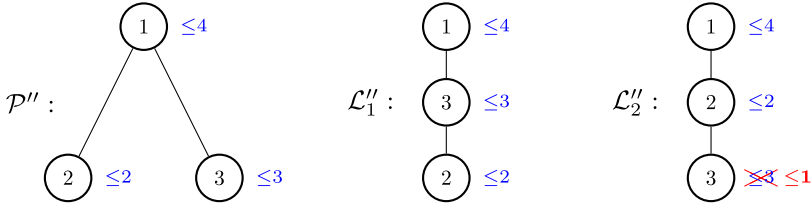


Fig. 7. An example where $\mathcal{P}''$ is not well-labelled with respect to ρ and Lemma 4.5 fails.

4.2. Two applications

We now give two important examples of positive expansions in terms of slide polynomials that are analogues of Corollaries 2.8 and 2.11, respectively.

4.2.1. Product of slide polynomials

In Corollary 2.8, the theory of $\mathcal{P}$ -partitions can be used to determine the structure constants for fundamental quasisymmetric functions in terms of the shuffle product on strong compositions. Assaf and Searles [1] (Definition 5.9) generalized the shuffle product on strong compositions to the slide product on weak compositions.

Definition 4.10 ([1]). Let a, b be weak compositions of length n . Let A and B be words defined by $A = (2n-1)^{a_1} \dots (3)^{a_{n-1}}(1)^{a_n}$ and $B = (2n)^{b_1} \dots (4)^{b_{n-1}}(2)^{b_n}$. Define the *shuffle set* of a and b , denoted by $\text{Sh}(a, b)$, by

$$\text{Sh}(a, b) = \{C \in A \sqcup B \mid \text{Des}_A(C) \geq a \text{ and } \text{Des}_B(B) \geq b\}, \quad (4.2)$$

where $\text{Des}_A(C)_i$ (respectively $\text{Des}_B(C)_i$) is the number of letters from A (respectively B) in the i th increasing run of C .

Define the *slide product* of a and b , denoted by $a \sqcup b$, to be the formal sum

$$a \sqcup b = \sum_{C \in \text{Sh}(a, b)} \text{Des}(\text{bump}_{(a, b)}(C)), \quad (4.3)$$

where $\text{bump}_{(a, b)}(C)$ is the unique element of the shuffle set $0^{n-\ell(\text{Des}(C))} \sqcup C$ such that $\text{Des}_A(\text{bump}_{(a, b)}(C)) \geq a$ and $\text{Des}_B(\text{bump}_{(a, b)}(C)) \geq b$ and if $D \in 0^{n-\ell} \sqcup C$ satisfies $\text{Des}_A(D) \geq a$ and $\text{Des}_B(D) \geq b$, then $\text{Des}(D) \geq \text{Des}(\text{bump}_{(a, b)}(C))$.

Assaf and Searles then used the slide product to characterize the structure constants for slide polynomials [1, Theorem 5.11] as follows.

Theorem 4.11 ([1] Thm 5.11). Let a, b be two weak compositions. Then

$$\mathfrak{F}_a \mathfrak{F}_b = \sum_c [c \mid a \sqcup b] \mathfrak{F}_c, \quad (4.4)$$

where $[c \mid a \sqcup b]$ denotes the coefficient of c in the slide product $a \sqcup b$.

Here we give a more direct interpretation with a greatly simplified proof using well-labelled, flagged $(\mathcal{P}, \rho)$ -partitions. We use Proposition 4.3 to prove the disjoint union of the two linear orders is well-labelled and flagged, and then use Lemma 4.5 to find the expression and recover Theorem 4.11.

Corollary 4.12. Let a, b be two weak compositions. Choose two linear posets $\mathcal{A}$ and $\mathcal{B}$ along with two well-labelled, flagged restriction maps α and β such that $\mathfrak{F}_{(\mathcal{A}, \alpha)} = \mathfrak{F}_a$ and $\mathfrak{F}_{(\mathcal{B}, \beta)} = \mathfrak{F}_b$. Then

$$\mathfrak{F}_a \mathfrak{F}_b = \mathfrak{F}_{(\mathcal{A}, \alpha)} \mathfrak{F}_{(\mathcal{B}, \beta)} = \mathfrak{F}_{(\mathcal{A} \oplus \mathcal{B}, \alpha \oplus \beta)} = \sum_{\mathcal{L} \in \mathcal{L}(\mathcal{A} \oplus \mathcal{B})} \mathfrak{F}_{\text{rdes}(\mathcal{L}, \alpha \oplus \beta)}, \quad (4.5)$$

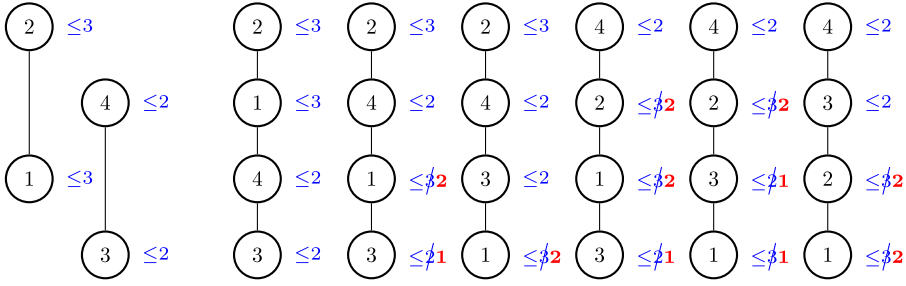


Fig. 8. The well-labelled flagged $(\mathcal{P}, \rho)$ for computing the slide product $\mathfrak{F}_{(0,0,2)}\mathfrak{F}_{(0,2,0)}$. We give here the six linear extensions together with $\bar{\rho}^{\mathcal{L}}$ for each.

where $A \oplus B$ denotes the partial order as the disjoint union of A and B , i.e. no element of A is comparable to an element of B . The set $\mathcal{L}(A \oplus B)$ corresponds to the slide product of A and B .

Example 4.13. Let $a = (0, 0, 2)$ and $b = (0, 2, 0)$, and take $(\mathcal{P}, \rho)$ as in Fig. 8. This gives the following expansion for slide polynomials as

$$\mathfrak{F}_{(0,0,2)}\mathfrak{F}_{(0,2,0)} = \mathfrak{F}_{(0,2,2)} + \mathfrak{F}_{(1,2,1)} + \mathfrak{F}_{(0,3,1)} + \mathfrak{F}_{(1,3,0)} + \mathfrak{F}_{(2,2,0)} + \mathfrak{F}_{(0,4,0)}.$$

Note this generalizes Example 2.9 for the shuffle product $F_{(2)}F_{(2)}$.

4.2.2. Flagged Schur polynomials

Also in [1, Theorem 3.13], the authors show the positive expansion of Schubert polynomials in terms of slide polynomials. Lascoux and Schützenberger [5] show when a permutation is vexillary, the Schubert polynomial is a flagged Schur polynomial (see also [9]). This shows many flagged Schur polynomials are slide positive. We claim all flagged Schur polynomials are slide positive, giving the restricted analogue to Corollary 2.11. Let us recall what are flagged Schur polynomials.

Given a partition λ and a flag $\mathbf{b} = b_1 \leq b_2 \leq \dots \leq b_{\ell_\lambda}$, the *flagged Schur function* $s_{\lambda, \mathbf{b}}$ is the sum of the monomials corresponding to semistandard tableau of shape λ and entries in row i bounded by b_i . As in the Schur case, we may express this in terms of $(\mathcal{P}, \rho)$ -partitions as

$$\mathfrak{F}_{(\mathcal{P}_\lambda, \rho_\mathbf{b})} = s_{\lambda, \mathbf{b}},$$

where $\mathcal{P}_\lambda$ is the poset constructed as in Fig. 2 and $\rho_\mathbf{b}$ is the restriction given by $\rho_\mathbf{b}(x) = b_i$ if $x \in \mathcal{P}$ is in row i .

For example, $s_{431, (2, 4, 4)}$ is given by the $(\mathcal{P}, \rho)$ -partitions with $\mathcal{P} = \mathcal{P}_\lambda$ and $\rho = \rho_\mathbf{b}$ as in Fig. 9. This is the flagged Schur polynomial equal to the Schubert polynomial indexed by the vexillary permutation $w = 1637245$ (see [9]).

Theorem 4.14. For λ a partition and $\mathbf{b} = b_1 \leq b_2 \leq \dots \leq b_{\ell_\lambda}$ a flag of integers, we have

$$s_{\lambda, \mathbf{b}} = \sum_{T \in \text{SYT}(\lambda)} \mathfrak{F}_{(\mathcal{L}_T, \rho_\mathbf{b})} = \sum_{T \in \text{SYT}(\lambda)} \mathfrak{F}_{\text{rdes}((\mathcal{L}_T, \rho_\mathbf{b}))}, \quad (4.6)$$

where a standard tableau T determines a unique linear extension $\mathcal{L}_T$ of $\mathcal{P}_\lambda$.

Proof. It is clear the pair $(\mathcal{P}_\lambda, \rho_\mathbf{b})$ is well-labelled and flagged, and so the formula follows from Corollary 4.9. $\square$

Any term $\mathfrak{F}_{\mathcal{L}_T, \bar{\rho}^{\mathcal{L}_T}}$ is zero if the first entry of $\bar{\rho}^{\mathcal{L}_T}$ is < 1 and can be removed from the sum. The remaining terms are all flagged, hence the summation is a sum of slide polynomials.

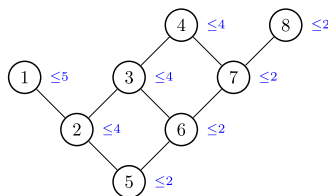


Fig. 9. The labelled poset corresponding to the partition (4, 3, 1) with ρ restriction given by the flag 2, 4, 4.

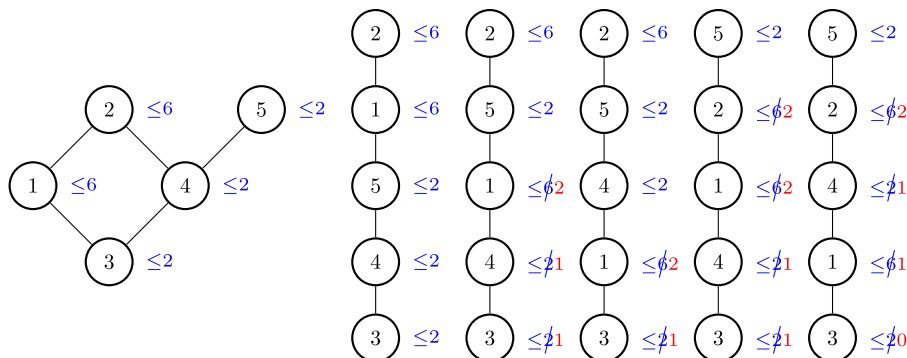


Fig. 10. The well-labelled flagged $(\mathcal{P}, \rho)$ partitions for $\lambda = (3, 2)$ and ρ restriction given by the flag 1,6. We give here the five linear extension together with $\bar{\rho}^{\mathcal{L}_T}$ for each standard tableau. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Remark 4.15. Reiner and Shimozono [7, Theorem 23] show any flagged Schur polynomial is a single key polynomial. More generally, they show any flagged skew Schur polynomial is a positive sum of key polynomials [7, Theorem 20]. More recently, Assaf and Searles [2, Theorem 2.13] show any key polynomial is a positive sum of slide polynomials. Combining those results yields any flagged skew Schur function expands positively in terms of slide polynomials. We offer here a more immediate approach to this result by using flagged $(\mathcal{P}, \rho)$ -partitions.

Example 4.16. Consider $\lambda = (3, 2)$ and $\mathbf{b} = (2, 6)$. Each standard tableau T gives rise to a linear extension $\mathcal{L}_T$ of $\mathcal{P}_\lambda$. We compute $\bar{\rho}^{\mathcal{L}_T}$ (in red in Fig. 10). If the first component of $\bar{\rho}^{\mathcal{L}_T}$ is < 1 , then we remove this linear order. The last linear order in Fig. 10 has component 0. Hence it will not contribute to the expansion. The four other linear orders are flagged $(\mathcal{L}_T, \bar{\rho}^{\mathcal{L}_T})$ -partitions and thus slide polynomials. Therefore taking terms as they appear from left to right in Fig. 10, we have

$$S_{(3,2),(2,6)} = \mathfrak{F}(0,3,0,0,0,2) + \mathfrak{F}(2,2,0,0,0,1) + \mathfrak{F}(1,3,0,0,0,1) + \mathfrak{F}(2,3).$$

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