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Finite-time singularity formation for an active scalar equation

Tarek Elgindi[1], Slim Ibrahim[2,*] and Shengyi Shen[3]

[1] Department of Mathematics, Duke University, Durham, NC 27710, United States of America

[2] Department of Mathematics and Statistics, University of Victoria, PO Box 3060 STN CSC, Victoria, BC, V8P 5C3, Canada

[3] NYUAD Research Institute, New York University Abu Dhabi, PO Box 129188, Abu Dhabi, United Arab Emirates

E-mail: tarek.elgindi@duke.edu, ibrahims@uvic.ca and ss14515@nyu.edu

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Abstract

We introduce an active scalar equation with a similar structure to the 3D Euler equations. Through studying the behavior of scale-invariant solutions, we show that compactly supported Lipschitz solutions belonging to $C^\infty(\mathbb{R}^2\backslash\{0\})$ can become singular in finite time. The interesting feature here is that we can achieve this in the absence of spatial boundaries.

Keywords: active scalar equations, blowup, Euler equations
Mathematics Subject Classification numbers: 35Q31, 35Q35.

(Some figures may appear in colour only in the online journal)

1. Introduction

Over the past few decades, there has been a growing interest in studying the problem of singularity formation in fluid equations through the lens of model problems that capture some, but not all, features of the original fluid equation. Our main interest here will be with the 3D Euler equation. As an example of this trend, we recall the analogy between the surface quasi-geostrophic (SQG) equation, inviscid porous medium (IPM) equation and 3D Euler systems. All three can be written in the following form:

$$\partial_t W + u \cdot \nabla W = W \cdot \nabla u \tag{1}$$

[*] Author to whom any correspondence should be addressed.
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with different relations between the unknown divergence-free vector fields W and u:

$$u_{\text{3DE}} = (-\Delta)^{-1}(\nabla \times W), \qquad u_{\text{SQG}} = (-\Delta)^{-1/2}W, \qquad u_{\text{IPM}} = (-\Delta)^{-1}\partial_1 W.$$

The analogy between the three systems is clear; the 'vorticity' vector field W satisfies the same equation and the velocity field is determined from W by a -1-homogeneous[4] Fourier multiplier. All that differs is the nature of Fourier multiplier in each case. For example, the Fourier multiplier in the SQG and 3D Euler systems is isotropic in general[5] while in the IPM equation it is anisotropic. On the other hand, in terms of parity, the Fourier multiplier is odd in the 3D Euler and IPM systems and even in the SQG system. The issue of parity seems to be one which has been, somehow, overlooked in previous works on the 3D Euler system, though it has now become clear that it is an issue of great importance [12]. In this work we will introduce and analyze a new model that highlights the importance of parity in the map $W \to u$. Toward this end, we consider the following 2D system:

$$\partial_t W + u \cdot \nabla W = W \cdot \nabla u, \tag{2}$$

$$u = (-\Delta)^{-1}\left(\frac{\partial_\theta W}{r}\right), \tag{3}$$

where r and θ are the radial and angular variables. The law $W \to u$ that we are introducing here is somehow in between the one of IPM and the SQG; it seems to have the correct 'parity' in relation to the 3D Euler equation and is also isotopic. It should be emphasized that this is simply a mathematical model to better understand the singularity problem in the 3D Euler equations and it is not clear whether there is a physical scenario that this system actually models. Our main result about this system is the existence of compactly supported bounded solutions on $\mathbb{R}^2$ that are smooth away from 0 and locally zero-homogeneous near 0 that become singular in finite time. To achieve this, we follow the methods introduced in [11] to focus our attention on a one dimensional equation satisfied by zero-homogeneous solutions. We then move to prove blow-up of such solutions.

It is important to emphasize that in the case of the IPM equation, this analysis can only be completed on corner domains (as in [9, 13]). Moreover, in the case of the SQG equation the parity of the mapping $W \to u$ seems to preclude blow-up of homogenous solutions altogether (though there is not yet a proof of global regularity in this case, there seems to be quite a bit of evidence in favour of global regularity, see for example [5–7, 10, 11, 14]). Additionally, one could compare this result with other results on blow-up of infinite energy solutions to fluid equations such as [2, 4, 18]. An important distinction between those works and the present one is that there is a clear way to localize the infinite-energy solutions we construct to get finite energy solutions in the end.

Unlike concavity techniques used to show finite singularity formation for Prandtl equations (see [8, 15]), or the primitive equation (see [1]), our proof is more based on shock type singularity through trajectories method, in the spirit of [16, 17].

[4] A function $f(x)$ is n-homogeneous if it satisfies $f(\alpha x) = \alpha^n f(x)$ for any real α.

[5] Though it ceases to be isotropic in the latter case when we look at axi-symmetric solutions, for example.

1.1. Statement of the main result

Observe that in 2D since W is divergence free in (2) and (3) we can write $W = \nabla^{\perp}\omega$ and then we observe that ω satisfies the following active scalar equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{4}$$

$$u = \nabla^{\perp}(-\Delta)^{-1}\left(\frac{\partial_\theta \omega}{r}\right), \tag{5}$$

where (r, θ) are the 2D polar coordinates. Let us recall a few definitions

Definition 1.1. We say that $f \in C(\mathbb{R}^2 \backslash 0)$ belongs to $\mathring{C}^{\alpha}$ for $0 \leqslant \alpha \leqslant 1$ if:

$$\sup_{x\neq 0} |f(x)| + \sup_{x\neq y} \frac{||x|^{\alpha} f(x) - |y|^{\alpha} f(y)|}{|x-y|^{\alpha}} < \infty.$$

We say that $f \in \mathring{C}^{\infty}$ if $|x|^k \nabla^k f \in \mathring{C}^1$ for all $k = 0, 1, 2, \cdots$.

Definition 1.2. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be m-fold symmetric for $m \in \mathbb{N}$ if $f(O_m x) = f(x)$ for all $x \in \mathbb{R}^2$, where O_m is the rotation matrix that rotates points by the angle $\frac{2\pi}{m}$ counter-clockwise.

Our first result concerns the local well-posedness for the 2D active scalar equations (4) and (5).

Theorem 1 (Local well-posedness). *Fix $0 < \alpha < 1$ and $m \geqslant 3$. If ω_0 is m-fold symmetric function such that $\nabla\omega_0 \in \mathring{C}^{\alpha}$, then there exists $T = T(\|\nabla\omega_0\|_{\mathring{C}^{\alpha}})$ and a unique m-fold symmetric solution ω of (4) and (5) on $[0, T)$ with $\nabla\omega \in C([0, T); \mathring{C}^{\alpha})$.*

Second, we have the following natural blow-up criterion in the spirit of Beale–Kato–Majda criterion.

Theorem 2 (Blow-up criterion). *A solution $\omega(t, x)$ of (4) and (5) as in theorem 1 cannot be extended past $T_* < \infty$ if and only if*

$$\lim_{t\to T_*} \int_0^t \|\nabla\omega(\tau, \cdot)\|_{L^\infty}\, \mathrm{d}\tau = +\infty.$$

Finally, in our last and main theorem, we construct compactly supported solutions covered under the above local well-posedness theorem 1 that become singular in finite time.

Theorem 3. *There exists a four-fold symmetric $\omega_0 \in C_c(\mathbb{R}^2)$ with $\nabla\omega_0 \in \mathring{C}^{\infty}$ so that the unique corresponding four-fold symmetric solution from theorem 1 becomes singular in finite time. Namely, there exists $T_* < \infty$ so that*

$$\lim_{t\to T_*} \int_0^t \|\nabla\omega\|_{L^\infty} = +\infty.$$

1.2. Discussion of previous works

1.2.1. The 1D model. One of the most interesting aspects of equations (4) and (5) is the structure of the 1D model satisfied by one-homogeneous solutions. We will focus our attention on the four-fold symmetric case, though this is not necessary. We search for solutions to (4) and (5) of the form:

$$\omega(x,t) = rg(\theta,t),$$

where g is $\frac{\pi}{2}$ periodic. Equation (5) then becomes:

$$u = \nabla^{\perp}(-\Delta)^{-1}\partial_\theta g.$$

Now let us introduce $F(\theta, T)$ with

$$4F + \partial_{\theta\theta}F = -\partial_\theta g,$$

note that we can solve uniquely for a $\frac{\pi}{2}$ periodic F since g is $\frac{\pi}{2}$ periodic. Then it is easy to see that:

$$u = \nabla^{\perp}(r^2 F) = 2x^{\perp}F - x\partial_\theta F.$$

Then we see that:

$$u\cdot\omega = (2x^{\perp}F - x\partial_\theta F)\cdot\nabla(rg) = 2F\partial_\theta g - \partial_\theta F g.$$

Thus we arrive at the following system on $\mathbb{S}^1$ for one-homogeneous solutions to (4) and (5):

$$\partial_t g + 2F\partial_\theta g = g\partial_\theta F, \tag{6}$$

$$4F + \partial_{\theta\theta}F = -\partial_\theta g. \tag{7}$$

One should compare this with the 1D model derived from the SQG equation in [11] where (7) is replaced (up to a lower order term) by:

$$\partial_\theta F = \mathcal{H}(g),$$

where $\mathcal{H}$ is the Hilbert transform. A very important distinction between these two systems is the difference in parity: in (7) g and F have the opposite parity while in the SQG case they have the same parity. This simple looking distinction is what allows us to prove finite time blow-up for (6) and (7), while it seems that there is global regularity in the SQG case (see [3, 6, 10, 14]). The important distinction is that the natural class of solutions to study to get growth in (6) and (7) is that of even solutions, rather than odd solutions.

1.2.2. Symmetry condition. The velocity in (5) can be recovered through Biot–Savart kernel:

$$u = \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\left(\frac{\partial_\theta \omega}{r}\right)(y)\mathrm{d}y. \tag{8}$$

Generally, the above Biot–Savart law does not converge at infinity. However we can introduce the symmetry and show that the Biot–Savart kernel actually gains the decay at infinity (see corollary 2.14, [11]). Indeed, assuming that $g(x)$ is m-fold symmetric with $m \geqslant 3$, then

$$K * g(x) = \frac{1}{m}\sum_{i=1}^{m}\int_{\mathbb{R}^2} K(x - O^i_{2\pi/m}y)g(y)\mathrm{d}y,$$

where K is the Biot–Savart kernel and $O^i_{2\pi/m}$ is the counter-clockwise rotation by angle $2\pi/m$ i-times. The new rotation kernel can be rewrite as

$$K^m(x,y) = \frac{1}{m}\sum_{i=1}^{m} K(x - O^i_{2\pi/m}y) = \frac{1}{m}\frac{\nabla_x^{\perp}\left(\Pi_{i=1}^m |x - O^i_{2\pi/m}y|^2\right)}{\Pi_{i=1}^m |x - O^i_{2\pi/m}y|^2}.$$

Noting that by fixing y, $\Pi_{i=1}^m |x - O^i_{2\pi/m}y|^2$ is also a m-fold symmetric polynomial in x, then the possible degrees in x are only m and $2m$. So that the components of the vector $\nabla_x^{\perp}\left(\Pi_{i=1}^m |x - O^i_{2\pi/m}y|^2\right)$ consists of x with degree $m-1, 2m-1$ and degree of y is at most m. Thus for $|y| \geqslant c|x|$ it holds that

$$K^m(x,y) \leqslant c\frac{|x|^{m-1}}{|y|^m} \tag{9}$$

and it is integrable for $m \geqslant 3$ but fails for $m \leqslant 2$.

1.3. Sketch of the proof

For theorems 1 and 2 the proofs go along the same lines as in [11]. The idea is to set up several self controlled inequalities including $\|\nabla\omega\|_{L^\infty}$, $\|\nabla\Phi\|_{L^\infty}$, and $Q(t)$ (given in below in (19)) where $\Phi(t,x)$ is the flow map defined by

$$\dot{\Phi}(t,x) = u(t,\Phi(t,x)), \qquad \Phi(0,x) = x$$

and $Q(t)$ is some semi-norm which, together with $\|\nabla\omega\|_{L^\infty}$ will control $\|\nabla\omega\|_{\mathring{C}^\alpha}$. Then *a priori* estimate on $\|\nabla\omega\|_{\mathring{C}^\alpha}$ is obtained to solve the flow map up to some time. The only difference is that the following inequality is needed in our case.

Lemma 1.1. *Assume ω is m-fold rotationally symmetric for some $m \geqslant 3$ and $u = \nabla^\perp(-\Delta)^{-1}\frac{\partial_\theta}{r}\omega$. Then it holds that*

$$\|\nabla u\|_{\mathring{C}^\alpha} \leqslant C_\alpha\|\nabla\omega\|_{\mathring{C}^\alpha}, \qquad \left\|\frac{u}{|\cdot|}\right\|_{\mathring{C}^\alpha} \leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}.$$

For theorem 3, we look for the blow-up solution to (4) and (5) via the blow-up of the 1D model (6) and (7). According to the previous discussion, a suitable assumption for blow-up solution to (6) and (7) is that

- $g(t,\theta)$ is even and non-positive on $[-\frac{\pi}{4},\frac{\pi}{4})$;
- $g(t,\theta)$ has only one maximum and one minimum on $[-\frac{\pi}{4},\frac{\pi}{4})$.

For example, we can choose the initial condition $g(0,\theta) = -1-\cos(4\theta)$. The solution before the blow-up time is shown in figure 1. Also figure 2 shows its trajectory $\gamma(t,\alpha)$ defined by $\dot{\gamma}(t,\alpha) = 2F(t,\gamma)$, $\gamma(0,\alpha) = \alpha$. The proof highly relies on the trajectory method (see [16, 17]).

2. Preliminary lemmas

We introduce a few notation and give some useful lemmas. For $k = 0, 1, 2, \cdot\cdot$, and $0 < \alpha \leqslant 1$, define the spaces $\mathring{C}^{k,\alpha}$ by

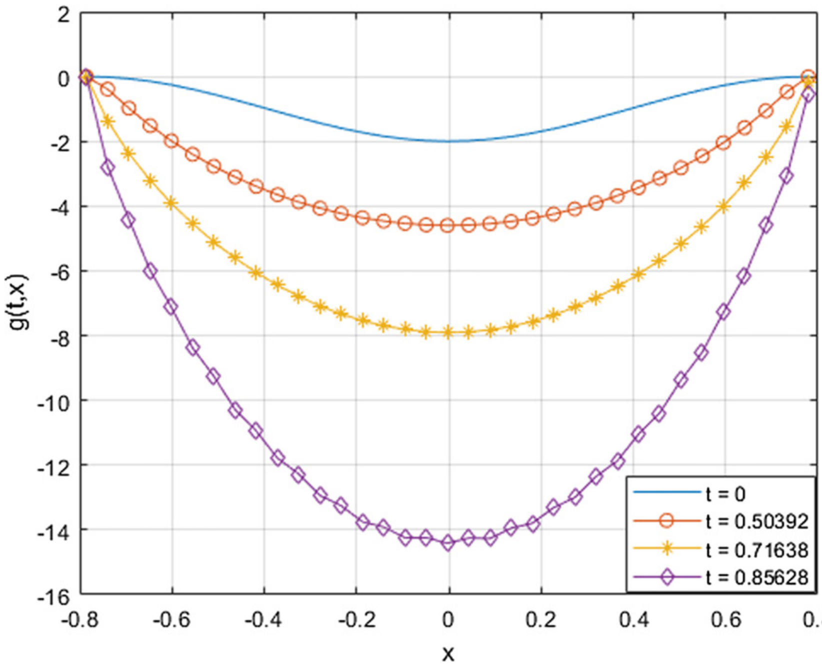

Figure 1. The blow-up time happens around $t = 1.03$, and is located at the maximum point.

$$\|u\|_{\mathring{C}^{0,\alpha}} = \|u\|_{\mathring{C}^{\alpha}} := \|u\|_{L^\infty} + \||x|^\alpha u\|_{C^\alpha}, \quad 0 < \alpha < 1,$$
$$\|u\|_{\mathring{C}^{0,1}} = \|u\|_{\mathring{C}^{1}} := \|u\|_{L^\infty} + \sup_{x\in\mathbb{R}^2\setminus\{0\}} (|x||\nabla u|);$$

and for $k \geqslant 1$,

$$\|u\|_{\mathring{C}^{k,\alpha}} := \|u\|_{\mathring{C}^{k-1,1}} + \||x|^{k+\alpha}\nabla^k u\|_{C^\alpha}, \quad 0 < \alpha < 1,$$
$$\|u\|_{\mathring{C}^{k,1}} := \|u\|_{L^\infty} + \sup_{x\in\mathbb{R}\setminus\{0\}} (|x|^{k+1}|\nabla^{k+1} u|).$$

The following lemma gives an equivalent definition of the $\mathring{C}^\alpha$ norm. See lemma 1.5 in Elgindi–Jeong [11].

Lemma 2.1. *For any* $0 < \alpha \leqslant 1$ *and* $f \in \mathring{C}^\alpha$, *we have*

$$\frac{1}{2}\|f\|_{\mathring{C}^\alpha} \leqslant \|f\|_{L^\infty} + \sup_{x\neq x'} \left[\min\{|x|^\alpha, |x'|^\alpha\} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}\right] \leqslant 2\|f\|_{\mathring{C}^\alpha},$$
$$\frac{1}{2}\|f\|_{\mathring{C}^\alpha} \leqslant \|f\|_{L^\infty} + \sup_{x\neq x'} \left[\max\{|x|^\alpha, |x'|^\alpha\} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}\right] \leqslant 2\|f\|_{\mathring{C}^\alpha}.$$

The following two lemmas are about the product rule in $\mathring{C}^\alpha$ spaces:

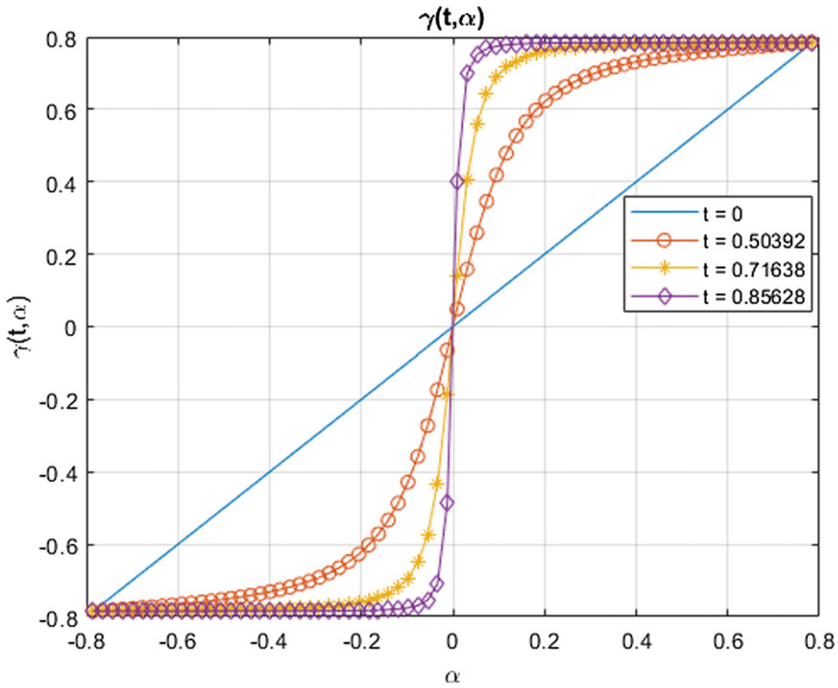

Figure 2. The trajectory shows the velocity fields is squeezing g near $\pm\frac{\pi}{4}$ and expanding it near 0.

Lemma 2.2. *For any $0 \leqslant \alpha \leqslant 1$, a positive constant C exists such if $f \in \mathring{C}^{\alpha}$ and $g \in C^{\alpha}$ with $g(0) = 0$, we have the product rule*

$$\|fg\|_{C^{\alpha}} \leqslant C\|f\|_{\mathring{C}^{\alpha}}\|g\|_{C^{\alpha}}.$$

Lemma 2.2 was proved in Elgindi–Jeong's paper [11].

Lemma 2.3. *If $f, g \in \mathring{C}^{\alpha}$ the following estimate holds*

$$\|fg\|_{\mathring{C}^{\alpha}} \lesssim \|f\|_{\mathring{C}^{\alpha}}\|g\|_{\mathring{C}^{\alpha}}.$$

Proof. To prove lemma 2.3, without loss of generality we may assume that $\min\{|x|^{\alpha}, |x'|^{\alpha}\} = |x|^{\alpha}$. Then,

$$\frac{\left||x|^{\alpha}f(x)g(x) - |x'|^{\alpha}f(x')g(x')\right|}{|x-x'|^{\alpha}} \leqslant \frac{|x|^{\alpha}|g(x)-g(x')|}{|x-x'|^{\alpha}}|f(x)| + |g(x')| \times \frac{\left||x|^{\alpha}f(x) - |x'|^{\alpha}f(x')\right|}{|x-x'|^{\alpha}}.$$

Thanks to lemma 2.1,

$$\frac{\left||x|^{\alpha}f(x)g(x) - |x'|^{\alpha}f(x')g(x')\right|}{|x-x'|^{\alpha}} \lesssim \|f\|_{L^{\infty}}\|g\|_{\mathring{C}^{\alpha}} + \|g\|_{L^{\infty}}\|f\|_{\mathring{C}^{\alpha}}$$

which proves the lemma. □

The next lemma is the key lemma (see [11], corollary 2.24) when proving propagation of the angular regularity of the 2D Euler equation. It will help us to prove the same result for our equation.

Lemma 2.4. *Let* $0 < \alpha < 1$. *If* g *is* m*-fold rotationally symmetric function for some* $m \geqslant 3$ *and* $\nabla g \in \mathring{C}^{\alpha}$, *then for*

$$u = \nabla^{\perp}(-\Delta)^{-1}g,$$

it holds that

$$\|\nabla u\|_{\mathring{C}^{\alpha}} \leqslant C_{\alpha}\|g\|_{\mathring{C}^{\alpha}}$$

for a positive constant C_{α}.

3. Proofs of the results

3.1. Sketch of the proof of theorems 1 and 2

The proof goes along the same line as in [11]. We restate the sketch of the proof here for completeness.

First we prove lemma 1.1 which is the only difference between our case and [11]:

Lemma 1.1. *Let* $0 < \alpha < 1$. *Assume that* ω *is* m*-fold rotationally symmetric function for some* $m \geqslant 3$ *and that* $u = \nabla^{\perp}(-\Delta)^{-1}\frac{\partial_{\theta}}{r}\omega$. *Then it holds that*

$$\|\nabla u\|_{\mathring{C}^{\alpha}} \leqslant C_{\alpha}\|\nabla\omega\|_{\mathring{C}^{\alpha}}, \qquad \left\|\frac{u}{|\cdot|}\right\|_{\mathring{C}^{\alpha}} \leqslant C_{\alpha}\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^{\alpha}}.$$

Proof. Let $g = \frac{\partial_{\theta}}{r}\omega$. It is clear that g is still m-fold symmetric. On the other hand, we have

$$g = \frac{x^{\perp}}{|x|}\cdot\nabla\omega.$$

Applying lemma 2.4 yields

$$\|\nabla u\|_{\mathring{C}^{\alpha}} \leqslant C_{\alpha}\|\frac{x^{\perp}}{|x|}\cdot\nabla\omega\|_{\mathring{C}^{\alpha}}.$$

Then the product rule lemma 2.3 gives

$$\|\nabla u\|_{\mathring{C}^{\alpha}} \leqslant C_{\alpha}\left\|\frac{x^{\perp}}{|x|}\right\|_{\mathring{C}^{\alpha}}\|\nabla\omega\|_{\mathring{C}^{\alpha}}.$$

Noticing that $\frac{x^{\perp}}{|x|}$ does belong to $\mathring{C}^{\alpha}$, the first inequality is proved.

The proof of the second inequality goes the similar way with the proof of lemma 2.4 in [11]. It is easy to see

$$
\begin{aligned}
\frac{u(x)}{|x|} &= \frac{1}{|x|}\int_{\mathbb{R}^2} K(x-y)\left(\frac{\partial_\theta \omega}{r}\right)(y)\mathrm{d}y = \frac{1}{|x|}\int_{\mathbb{R}^2} K(x-y)\frac{y^\perp \cdot \nabla\omega(y)}{|y|}\,\mathrm{d}y \\
&= \frac{1}{|x|}\int_{\mathbb{R}^2} K(x-y)\nabla_y \cdot \left(\frac{y^\perp \omega(y)}{|y|}\right)\mathrm{d}y \\
&= \frac{1}{|x|}\int_{\mathbb{R}^2} -\nabla_y K(x-y)\cdot y^\perp \frac{\omega(y)}{|y|}\,\mathrm{d}y. \qquad (10)
\end{aligned}
$$

The proof goes with three steps. During the proof we will frequently use lemma 2.1 when involving $\mathring{C}^\alpha$ norm.

(i) $\|\frac{u}{|\cdot|}\|_{L^\infty} \leqslant C\|\frac{\omega}{|\cdot|}\|_{\mathring{C}^\alpha}$. It is equivalent to prove

$$
\frac{1}{|x|}p.v.\int_{\mathbb{R}^2} -\nabla_y K(x-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|} - x^\perp\frac{\omega(x)}{|x|}\right)\mathrm{d}y \leqslant \left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}. \qquad (11)
$$

Splitting $\mathbb{R}^2$ into three regions, $A: |y| > 2|x|, B: |y| < \frac{1}{2}|x|, C$: the remainder set. In the remainder set C one has $|x-y| \leqslant 3|x|$. Since $|\nabla_y K(x-y)| \leqslant C\frac{1}{|x-y|^2}$, it holds that

$$
\begin{aligned}
&\frac{1}{|x|}p.v.\int_{|x-y|\leqslant 3|x|} -\nabla_y K(x-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|} - x^\perp\frac{\omega(x)}{|x|}\right)\mathrm{d}y \\
&\leqslant \frac{C}{|x|}\int_{|x-y|\leqslant 3|x|}\frac{1}{|x-y|^2}|(x-y)^\perp|\left|\frac{\omega(y)}{|y|}\right| + \frac{1}{|x-y|^2}|x^\perp| \\
&\quad\times\left|\frac{\omega(x)}{|x|} - \frac{\omega(y)}{|y|}\right|\mathrm{d}y \\
&\leqslant C\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty} + \int_{|x-y|\leqslant 3|x|}\frac{\|\frac{\omega}{|\cdot|}\|_{\mathring{C}^\alpha}}{|x|^\alpha|x-y|^{2-\alpha}}\,\mathrm{d}y \\
&\leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}.
\end{aligned}
$$

In region A and B, we shall use the rotationally symmetric property. The classic Biot–Savart kernel can be replaced by a new kernel $K^m(x-y) = \frac{1}{m}\frac{\nabla_x^\perp \Pi_{i=1}^m |x-O^i_{2\pi/m}y|^2}{\Pi_{i=1}^m |x-O^i_{2\pi/m}y|^2}$. Now (11) becomes

$$
\frac{1}{|x|}\int_{A\cup B} -\nabla_y K^m(x-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|}\right)\mathrm{d}y \leqslant \left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}. \qquad (12)
$$

The symmetric property brings extra decay for $\nabla_y K^m(x,y)$. Since fixing y the term $\Pi_{i=1}^m|x - O^i_{2\pi/m}y|^2$ is also a m-fold rotationally symmetric function in x, then $\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2$ only contains terms like $x^\alpha y^\beta$, where $(|\alpha|,|\beta|) = (0,2m),(m,m),(2m,0)$. Thus $\nabla_x^\perp\Pi_{i=1}^m|x - O^i_{2\pi/m}y|^2$ consists of terms like $x^{\alpha-1}y^\beta$ where $(|\alpha|,|\beta|) = (m,m),(2m,0)$. Now we apply the same argument to $\nabla_y K^m(x,y)$. Because

$$
\begin{aligned}
\nabla_y K^m(x,y) &= \frac{1}{m}\frac{\nabla_y\nabla_x^\perp\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2}{\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2} \\
&\quad - \frac{1}{m}\frac{\nabla_x^\perp\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2\nabla_y\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2}{\left(\Pi_{i=1}^m|x-O^i_{2\pi/m}y|^2\right)^2},
\end{aligned}
$$

it holds that in regions A and B

$$|\nabla_y K^m(x,y)| \leqslant C\left(\frac{|x|^{m-1}|y|^{m-1}}{|x|^{2m}+|y|^{2m}} + \frac{(|x|^{m-1}|y|^m+|x|^{2m-1})(|x|^m|y|^{m-1}+|y|^{2m-1})}{|x|^{4m}+|y|^{4m}}\right).$$

So that in A we have extra decay on y: $|\nabla_y K^m(x,y)| \leqslant C\frac{|x|^{m-1}}{|y|^{m+1}}$ meanwhile in B decay is on x: $|\nabla_y K^m(x,y)| \leqslant C\frac{|y|^{m-1}}{|x|^{m+1}}$.

We estimate (12) by applying the decay results respectively.

$$\begin{aligned}(12) &\leqslant C\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}\frac{1}{|x|}\int_{|y|\geqslant 2|x|}|y|\frac{|x|^{m-1}}{|y|^{m+1}}\,\mathrm{d}y \\ &\quad + C\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}\frac{1}{|x|}\int_{|y|\leqslant |x|/2}|y|\frac{|y|^{m-1}}{|x|^{m+1}}\,\mathrm{d}y \\ &\leqslant C\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}.\end{aligned}$$

(ii) For $|x|\leqslant|x'|$ and $|x-x'|\leqslant|x|$, $|\frac{u(x)}{|x|}-\frac{u(x')}{|x'|}|\leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}\frac{|x-x'|^\alpha}{|x|^\alpha}$. It is sufficient to prove the following bound

$$\begin{aligned}&\frac{1}{|x|}p.v.\int_{\mathbb{R}^2}-\nabla_y K(x-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|}-x^\perp\frac{\omega(x)}{|x|}\right)\mathrm{d}y \\ &\quad - \frac{1}{|x'|}p.v.\int_{\mathbb{R}^2}-\nabla_y K(x'-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|}-x'^\perp\frac{\omega(x')}{|x'|}\right)\mathrm{d}y \\ &\leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}\frac{|x-x'|^\alpha}{|x|^\alpha}.\end{aligned} \tag{13}$$

Splitting $\mathbb{R}^2$ into two regions $A : |x-y|\leqslant 2|x-x'|$ and $B : |x-y|>2|x-x'|$. In region A, it holds that

$$\begin{aligned}&\frac{1}{|x|}p.v.\int_{|x-y|\leqslant 2|x-x'|}-\nabla_y K(x-y)\cdot\left(y^\perp\frac{\omega(y)}{|y|}-x^\perp\frac{\omega(x)}{|x|}\right)\mathrm{d}y \\ &\leqslant \frac{1}{|x|}\int_{|x-y|\leqslant 2|x-x'|}\frac{1}{|x-y|^2}|(x-y)^\perp|\left|\frac{\omega(y)}{|y|}\right| \\ &\quad + \frac{1}{|x-y|^2}|x^\perp|\left|\frac{\omega(x)}{|x|}-\frac{\omega(y)}{|y|}\right|\mathrm{d}y \\ &\leqslant \frac{|x-x'|}{|x|}\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}+\int_{|x-y|\leqslant 2|x-x'|}\frac{\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}}{|x|^\alpha|x-y|^{2-\alpha}}\,\mathrm{d}y \\ &\leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}\frac{|x-x'|^\alpha}{|x|^\alpha}.\end{aligned}$$

It is similar to x'.

In region B, we write inequality as following

$$\begin{aligned}
&\frac{1}{|x|}p.v.\int_{|x-y|>2|x-x'|} -\nabla_y K(x-y)\cdot\left(y^{\perp}\frac{\omega(y)}{|y|}-x^{\perp}\frac{\omega(x)}{|x|}\right)\mathrm{d}y\\
&\quad -\frac{1}{|x'|}p.v.\int_{|x-y|>2|x-x'|} -\nabla_y K(x'-y)\cdot\left(y^{\perp}\frac{\omega(y)}{|y|}-x'^{\perp}\frac{\omega(x')}{|x'|}\right)\mathrm{d}y\\
&= p.v.\int_{|x-y|>2|x-x'|}\big(\nabla_y K(x-y)-\nabla_y K(x'-y)\big)\\
&\quad\cdot\left(\frac{x^{\perp}}{|x|}\frac{\omega(x)}{|x|}-\frac{y^{\perp}}{|x|}\frac{\omega(y)}{|y|}\right)\mathrm{d}y+\frac{|x'|-|x|}{|x|}\frac{1}{|x'|}\\
&\quad\times\int_{|x-y|>2|x-x'|}\nabla_y K(x'-y)\cdot\left(y^{\perp}\frac{\omega(y)}{|y|}\right)\mathrm{d}y\\
&\quad + p.v.\int_{|x-y|>2|x-x'|}\nabla_y K(x'-y)\cdot\left(\frac{x^{\perp}}{|x|}\frac{\omega(x)}{|x|}-\frac{x'^{\perp}}{|x'|}\frac{\omega(x')}{|x'|}\right)\mathrm{d}y
\end{aligned}.$$

The third term is zero since the integration is on a sphere. The second term can be bounded by $\frac{|x-x'|}{|x|}\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}$ therefore it is bounded by $\frac{|x-x'|^\alpha}{|x|^\alpha}\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}$. For the first term, by mean value theorem

$$|\nabla_y K(x-y)-\nabla_y K(x'-y)|\leqslant C\frac{|x-x'|}{|x-y|^3}.$$

So that

$$\begin{aligned}
&p.v.\int_{|x-y|>2|x-x'|}\big(\nabla_y K(x-y)-\nabla_y K(x'-y)\big)\cdot\left(\frac{x^{\perp}}{|x|}\frac{\omega(x)}{|x|}-\frac{y^{\perp}}{|x|}\frac{\omega(y)}{|y|}\right)\mathrm{d}y\\
&\leqslant\int_{|x-y|>2|x-x'|}\big(\nabla_y K(x-y)-\nabla_y K(x'-y)\big)\frac{1}{|x|}(x-y)^{\perp}\left(\frac{\omega(y)}{|y|}\right)\mathrm{d}y\\
&\quad + C\int_{|x-y|>2|x-x'|}\frac{|x-x'|}{|x-y|^3}\frac{x^{\perp}}{|x|}\left(\frac{\omega(x)}{|x|}-\frac{\omega(y)}{|y|}\right)\mathrm{d}y := I+II.
\end{aligned}$$

Involving $\mathring{C}^\alpha$ norm in II yields

$$II\leqslant C\int_{|x-y|>2|x-x'|}\frac{|x-x'|}{|x|^\alpha}\frac{\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}}{|x-y|^{3-\alpha}}\mathrm{d}y\leqslant C_\alpha\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}\frac{|x-x'|^\alpha}{|x|^\alpha}.$$

To estimate I, we furthermore split the region into three parts. Part 1: $|x-y|>2|x-x'|, |y|\geqslant 2|x|$, part 2: $|x-y|>2|x-x'|, |y|\leqslant |x|/2$, part 3 the remainder set. In part 3, $|x|, |y|$ are comparable, so that

$$\begin{aligned}
&\int_{\text{part 3}}\big(\nabla_y K(x-y)-\nabla_y K(x'-y)\big)\frac{1}{|x|}(x-y)^{\perp}\left(\frac{\omega(y)}{|y|}\right)\mathrm{d}y\\
&\leqslant C\int_{\text{part 3}}\frac{|x-x'|}{|x-y|^3}\frac{\omega(y)}{|y|}\mathrm{d}y\leqslant\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}\leqslant\frac{|x-x'|^\alpha}{|x|^\alpha}\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}.
\end{aligned}$$

In part 1 and part 2, we shall apply the symmetric property. Instead of using K, we replace K by K^m and study the decay of $\nabla_x\nabla_y K^m(x,y)$. Similar to the discussion in the proof of (i), we have in part 1, $|\nabla_x\nabla_y K^m(x,y)| \leqslant C\frac{|x|^{m-2}}{|y|^{m+1}}$ while in part 2, $|\nabla_x\nabla_y K^m(x,y)| \leqslant C\frac{|y|^{m-1}}{|x|^{m+2}}$. Thus by mean value theorem, it holds that

$$|\nabla_y K^m(x,y) - \nabla_y K^m(x',y)| \leqslant \begin{cases} C\dfrac{|x|^{m-2}}{|y|^{m+1}}|x-x'|, & y \text{ in part } 1; \\ C\dfrac{|y|^{m-1}}{|x|^{m+2}}|x-x'|, & y \text{ in part } 2. \end{cases}$$

Then integrating in both part 1 and part 2 gives

$$\int_{\text{part 1 and part 2}} \left(\nabla_y K(x-y) - \nabla_y K(x'-y)\right) \frac{1}{|x|}(x-y)^{\perp} \left(\frac{\omega(y)}{|y|}\right) \mathrm{d}y$$
$$\leqslant C\frac{|x-x'|}{|x|} \left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha} \leqslant \frac{|x-x'|^\alpha}{|x|^\alpha} \left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}.$$

Collecting above estimates proves (ii).

With (i) and (ii) we are ready to prove the second inequality in this lemma. It is sufficient to prove that for $|x| \leqslant |x'|$,

$$\frac{|x|^\alpha}{|x-x'|^\alpha}\left|\frac{u(x)}{|x|} - \frac{u(x')}{|x'|}\right| \leqslant C_\alpha \left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}.$$

If $|x-x'| \leqslant |x'|$, applying (ii) gives the result. Otherwise, we use L^∞ bound and (i) to get the result. □

Secondly, we start to prove theorems 1 and 2.

Existence and blow-up criterion. As we stated in the introduction, the idea is to control $\|\nabla\omega\|_{\mathring{C}^\alpha}$ up to some time. To do that we will set up several self controlled inequalities including $\|\nabla\omega\|_{L^\infty}, \|\nabla\Phi\|_{L^\infty}, Q(t)$, where $Q(t)$ is the semi-norm defined by (19) which, together with $\|\nabla\omega\|_{L^\infty}$, will control $\|\nabla\omega\|_{\mathring{C}^\alpha}$.

Recall that the system is

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = 0, \\ u = \nabla^{\perp}(-\Delta)^{-1}\left(\dfrac{\partial_\theta\omega}{r}\right). \end{cases} \tag{14}$$

Taking gradient and composing with the flow map Φ gives

$$\partial_t\nabla\omega(t,\Phi(t,x)) + \nabla u(t,\Phi(t,x))\cdot\nabla\omega(t,\Phi(t,x)) = 0. \tag{15}$$

Then L^∞ bounds comes immediately:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\nabla\omega\right| \leqslant C\|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^\infty} \leqslant C\|\nabla u\|_{\mathring{C}^\alpha}\|\nabla\omega\|_{L^\infty}. \tag{16}$$

Similarly, the L^∞ bounds for $\nabla\Phi$ and $\nabla\Phi^{-1}$ are

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\nabla\Phi(t,\cdot)\right| \leqslant C\|\nabla u\|_{L^\infty}\|\nabla\Phi(t,\cdot)\|_{L^\infty} \leqslant C\|\nabla u\|_{\mathring{C}^\alpha}\|\nabla\Phi(t,\cdot)\|_{L^\infty}. \tag{17}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\nabla\Phi^{-1}(t,\cdot)\right| \leqslant C\|\nabla u\|_{L^\infty}\|\nabla\Phi^{-1}(t,\cdot)\|_{L^\infty} \leqslant C\|\nabla u\|_{\mathring{C}^\alpha}\|\nabla\Phi^{-1}(t,\cdot)\|_{L^\infty}. \tag{18}$$

To estimate under $\mathring{C}^\alpha$ norm, we are interested in points with comparable distances from the origin. We write $x \simeq x'$ when $0 < |x'| < |x| < 2|x'|$ so that $\Phi(t,x), \Phi(t,x')$ are also in comparable distances from the origin. Note that the symmetry condition of ω forces the origin point as a fixed point, i.e. $u(t,0) = 0$. Additionally, by the mean value theorem we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x)\right| = |u(t,\Phi(t,x)) - u(t,0)| \leqslant C\|\nabla u(t)\|_{L^\infty}|\Phi(t,x)|.$$

Thus by Grönwall's inequality one has

$$|x|\ \exp\left(-\int_0^t C\|\nabla u(s)\|\mathrm{d}s\right) \leqslant |\Phi(t,x)| \leqslant |x|\ \exp\left(\int_0^t C\|\nabla u(s)\|_{L^\infty}\,\mathrm{d}s\right).$$

Similarly for $\Phi(t,x')$. On the other hand, given two comparable points $z \simeq z'$ we can find for each $t > 0$ such that $\Phi(t,x) = z, \Phi(t,x') = z'$ so that there exist a constant C depending on $\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty}$

$$\begin{aligned}
&\frac{\left||z|^\alpha\nabla\omega(t,z) - |z'|^\alpha\nabla\omega(t,z')\right|}{|z-z'|^\alpha}\\
&\quad\leqslant C(\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty})\\
&\qquad\times\ \frac{\left||\Phi(t,x)|^\alpha\nabla\omega(t,\Phi(t,x)) - |\Phi(t,x')|^\alpha\nabla\omega(t,\Phi(t,x'))\right|}{|x-x'|^\alpha}\\
&\quad\leqslant C(\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty})\\
&\qquad\times\ \left(\frac{\left||x|^\alpha\nabla\omega(t,\Phi(t,x)) - |x'|^\alpha\nabla\omega(t,\Phi(t,x'))\right|}{|x-x'|^\alpha} + \|\nabla\omega(t)\|_{L^\infty}\right).
\end{aligned}$$

So the following semi-norm:

$$Q(t) := \sup_{x\neq x', x\simeq x'} \frac{\left||x|^\alpha\nabla\omega(t,\Phi(t,x)) - |x'|^\alpha\nabla\omega(t,\Phi(t,x'))\right|}{|x-x'|^\alpha} \tag{19}$$

together with $\|\nabla\omega(t)\|_{L^\infty}$ control the full norm $\|\nabla\omega\|_{\mathring{C}^\alpha}$ up to the multiplicative constants depending on α and $\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty}$.

Next we try to estimate $|\frac{\mathrm{d}}{\mathrm{d}t}Q(t)|$. Simple calculation shows that

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}Q(t)\right| &\leqslant \frac{\left||x|^\alpha\nabla u(t,\Phi(t,x))\cdot\nabla\omega(t,\Phi(t,x)) - |x'|^\alpha\nabla u(t,\Phi(t,x'))\cdot\nabla\omega(t,\Phi(t,x'))\right|}{|x-x'|^\alpha}\\
&\leqslant C(\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty})\left(\|\nabla u\|_{\mathring{C}^\alpha}\|\nabla\omega\|_{L^\infty} + \|\nabla u\|_{L^\infty}\right.\\
&\quad\left.\times\ \frac{\left||x|^\alpha\nabla\omega(t,\Phi(t,x)) - |x'|^\alpha\nabla\omega(t,\Phi(t,x'))\right|}{|x-x'|^\alpha}\right).
\end{aligned} \tag{20}$$

Taking supremum over x, x' yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}Q(t)\right| \leqslant C(\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty})\|\nabla u\|_{\mathring{C}^\alpha}(\|\nabla\omega\|_{L^\infty} + Q(t)). \tag{21}$$

Applying lemma 1.1 and noting that $\|\nabla\omega\|_{\mathring{C}^\alpha} \leqslant C(\|\nabla\omega\|_{L^\infty} + Q(t))$, then (21) becomes

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}Q(t)\right| \leqslant C(\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty})(\|\nabla\omega\|_{L^\infty} + Q(t))^2. \tag{22}$$

Together with (16), we conclude that $\|\nabla\omega\|_{\mathring{C}^\alpha}$ is bounded by some constant depending on

$$\sup_{\tau\in[0,t]} \|\nabla\Phi(\tau)\|_{L^\infty}, \qquad \sup_{\tau\in[0,t]} \|\nabla\Phi^{-1}(\tau)\|_{L^\infty}.$$

To prove that we indeed bound $\|\nabla\omega\|_{\mathring{C}^\alpha}$, it is sufficient to prove $\|\nabla\Phi(t)\|_{L^\infty}, \|\nabla\Phi^{-1}(t)\|_{L^\infty}$ are bounded. Plugging this dependent $\|\nabla\omega\|_{\mathring{C}^\alpha}$ bound back into (17) and (18) yields two self controlled inequalities only about $\nabla\Phi$ and $\nabla\Phi^{-1}$ and both L^∞ norms are bounded as long as t is small. Therefore, $\|\nabla\omega\|_{\mathring{C}^\alpha}$ is bounded for small $t > 0$. Then according to lemma 1.1 the velocity is Lipschitz so that the flow map is well-defined and the time of existence can be extended as long as $\|\nabla\omega\|_{\mathring{C}^\alpha}$ is finite.

Uniqueness. Assuming that there are two solution pairs to system (14): $(\omega_1, u_1), (\omega_2, u_2)$ both with same initial condition. With out loss of generality, we can assume that $\omega_1(t, 0) = \omega_2(t, 0) = 0$, that the solutions exist up to some time $T > 0$ and that $\|\nabla\omega_i\|_{\mathring{C}^\alpha} < \infty$ for $i = 1, 2$. Denote the difference between the two solutions by $\omega := \omega_1 - \omega_2, u := u_1 - u_2$ so that $\omega(t = 0) = 0, u(t = 0) = 0$. Furthermore, notice that ω satisfies:

$$\partial_t\omega + u\cdot\nabla\omega_1 + u_2\cdot\nabla\omega = 0. \tag{23}$$

Dividing (23) by $|x|$ gives

$$\partial_t\left(\frac{\omega}{|x|}\right) + u_2\cdot\nabla\left(\frac{\omega}{|x|}\right) + \frac{u_2}{|x|}\cdot\frac{x}{|x|}\frac{\omega}{|x|} + \frac{u}{|x|}\cdot\nabla\omega_1 = 0. \tag{24}$$

Then composing with the flow map $\Phi_2(t, x)$ which satisfies

$$\dot{\Phi}_2(t, x) = u_2(t, \Phi_2(t, x)), \qquad \Phi_2(0, x) = x,$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega(t,\Phi_2)}{|\Phi_2|}\right) \leqslant \left|\frac{u_2(\Phi_2)}{|\Phi_2|}\right|\left|\frac{\omega(t,\Phi_2)}{|\Phi_2|}\right| + \left|\frac{u(t,\Phi_2)}{|\Phi_2|}\right||\nabla\omega_1(t,\Phi_2)|$$

and taking L^∞ norm on the right-hand side and applying lemma 1.1 yields:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\frac{\omega(x)}{|x|}\right| \leqslant C(\omega_1, \omega_2)\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}. \tag{25}$$

To control $\|\frac{\omega}{|\cdot|}\|_{\mathring{C}^\alpha}$, we follow the similar way that controlling $\|\nabla\omega\|_{\mathring{C}^\alpha}$ in the existence proof. Given two comparable points $z \simeq z'$, for each $t > 0$ there exist x, x' such that $\Phi_2(t, x) = z$,

$\Phi_2 t, x' = z'$. And the calculation shows that

$$\begin{aligned}\frac{\left||z|^{\alpha}\frac{\omega(t,z)}{|z|}-|z'|^{\alpha}\frac{\omega(t,z')}{|z'|}\right|}{|z-z'|^{\alpha}} &\leqslant C(\|\nabla\Phi_2(t)\|_{L^\infty},\|\nabla\Phi_2^{-1}(t)\|_{L^\infty})\\ &\quad\times\frac{\left||\Phi_2(t,x)|^{\alpha}\frac{\omega(t,\Phi_2(t,x))}{|\Phi_2(t,x)|}-|\Phi_2(t,x')|^{\alpha}\frac{\omega(t,\Phi_2(t,x'))}{|\Phi_2(t,x')|}\right|}{|x-x'|^{\alpha}}\\ &\leqslant C(\|\nabla\Phi_2(t)\|_{L^\infty},\|\nabla\Phi_2^{-1}(t)\|_{L^\infty})\\ &\quad\times\left(\frac{\left||x|^{\alpha}\frac{\omega(t,\Phi_2(t,x))}{|\Phi_2(t,x)|}-|x'|^{\alpha}\frac{\omega(t,\Phi_2(t,x'))}{|\Phi_2(t,x')|}\right|}{|x-x'|^{\alpha}}+\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}\right).\end{aligned}$$

Thus the semi-norm

$$R(t):=\sup_{x\neq x',x\simeq x'}\frac{\left||x|^{\alpha}\frac{\omega(t,\Phi_2(t,x))}{|\Phi_2(t,x)|}-|x'|^{\alpha}\frac{\omega(t,\Phi_2(t,x'))}{|\Phi_2(t,x')|}\right|}{|x-x'|^{\alpha}}$$

and $\|\frac{\omega}{|\cdot|}\|_{L^\infty}$ together control $\|\frac{\omega}{|\cdot|}\|_{\mathring{C}^\alpha}$. Calculating $\frac{\mathrm{d}R}{\mathrm{d}t}$ gives

$$\begin{aligned}\left|\frac{\mathrm{d}R}{\mathrm{d}t}\right| \leqslant \frac{1}{|x-x'|^{\alpha}}&\left||x|^{\alpha}\left(\frac{u_2(\Phi_2(x))}{|\Phi_2(x)|}\frac{\Phi_2(x)}{|\Phi_2(x)|}\frac{\omega(\Phi_2(x))}{|\Phi_2(x)|}+\frac{u(\Phi_2(x))}{|\Phi_2(x)|}\nabla\omega_1(\Phi_2(x))\right)\right.\\ &\left.-|x'|^{\alpha}\left(\frac{u_2(\Phi_2(x'))}{|\Phi_2(x')|}\frac{\Phi_2(x')}{|\Phi_2(x')|}\frac{\omega(\Phi_2(x'))}{|\Phi_2(x')|}+\frac{u(\Phi_2(x'))}{|\Phi_2(x')|}\nabla\omega_1(\Phi_2(x'))\right)\right|.\end{aligned}\tag{26}$$

Because

$$\begin{aligned}&|x|^{\alpha}\left(\frac{u_2(\Phi_2(x))}{|\Phi_2(x)|}\frac{\Phi_2(x)}{|\Phi_2(x)|}\frac{\omega(\Phi_2(x))}{|\Phi_2(x)|}\right)-|x'|^{\alpha}\left(\frac{u_2(\Phi_2(x'))}{|\Phi_2(x')|}\frac{\Phi_2(x')}{|\Phi_2(x')|}\frac{\omega(\Phi_2(x'))}{|\Phi_2(x')|}\right)\\ &\quad\leqslant\left(|x|^{\alpha}\frac{\omega(\Phi_2(x))}{|\Phi_2(x)|}-|x'|^{\alpha}\frac{\omega(\Phi_2(x'))}{|\Phi_2(x')|}\right)\frac{\Phi_2(x)}{|\Phi_2(x)|}\frac{u_2(\Phi_2(x))}{|\Phi_2(x)|}\\ &\qquad+\frac{\omega(\Phi_2(x'))}{|\Phi_2(x')|}|x'|^{\alpha}\left(\frac{\Phi_2(x)}{|\Phi_2(x)|}\frac{u_2(\Phi_2(x))}{|\Phi_2(x)|}-\frac{\Phi_2(x')}{|\Phi_2(x')|}\frac{u_2(\Phi_2(x'))}{|\Phi_2(x')|}\right),\end{aligned}$$

the first term in rhs of (26) can be bounded by $\|\frac{u_2}{|\cdot|}\|_{L^\infty}\|\frac{\omega}{|\cdot|}\|_{\mathring{C}^\alpha}+\|\frac{u_2}{|\cdot|}\|_{\mathring{C}^\alpha}\|\frac{\omega}{|\cdot|}\|_{L^\infty}$. Similarly, because

$$\begin{aligned}&|x|^{\alpha}\frac{u(\Phi_2(x))}{|\Phi_2(x)|}\nabla\omega_1(\Phi_2(x))-|x'|^{\alpha}\frac{u(\Phi_2(x'))}{|\Phi_2(x')|}\nabla\omega_1(\Phi_2(x'))\\ &\quad\leqslant\left(|x|^{\alpha}\frac{u(\Phi_2(x))}{|\Phi_2(x)|}-|x'|^{\alpha}\frac{u(\Phi_2(x'))}{|\Phi_2(x')|}\right)\nabla\omega_1(\Phi_2(x))\\ &\qquad+|x'|^{\alpha}\frac{u(\Phi_2(x'))}{|\Phi_2(x')|}(\nabla\omega_1(\Phi_2(x))-\nabla\omega_1(\Phi_2(x'))),\end{aligned}$$

with the help of lemma 1.1, the second term in rhs of (26) can be bounded by

$$\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}\|\nabla\omega_1\|_{L^\infty}+\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}\|\nabla\omega_1\|_{\mathring{C}^\alpha}.$$

Collecting these bounds yields

$$\left|\frac{\mathrm{d}R}{\mathrm{d}t}\right| \leqslant C(\omega_1,\omega_2)\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha}. \tag{27}$$

Therefore (25) and (27) together yields

$$\begin{aligned}\left\|\frac{\omega}{|\cdot|}\right\|_{\mathring{C}^\alpha} &\leqslant C(\|\nabla\Phi_2(t)\|_{L^\infty})\left(\left\|\frac{\omega}{|\cdot|}\right\|_{L^\infty}+R\right)\\ &\leqslant C(\|\nabla\Phi_2(t)\|_{L^\infty},\|\nabla\Phi_2^{-1}(t)\|_{L^\infty})\\ &\quad\times\left(\left\|\frac{\omega(t=0)}{|\cdot|}\right\|_{L^\infty}+R(t=0)\right)\mathrm{e}^{\int_0^t C(\omega_1,\omega_2)\mathrm{d}t}=0.\end{aligned}$$

To get the blow-up criterion, assuming that at blow-up time T^* it holds that

$$\lim_{t\to T^*}\int_0^{T^*}\|\nabla\omega\|_{\mathring{C}^\alpha}<\infty,$$

then (16)–(18) (using the first estimate) implies finiteness of $\|\nabla\omega\|_{L^\infty}$, $\|\Phi\|_{L^\infty}$ and $\|\nabla\Phi^{-1}(t)\|_{L^\infty}$. To control $Q(t)$, from (20) and lemma 1.1 we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}Q(t)\right| \leqslant C(\|\nabla\Phi(t)\|_{L^\infty},\|\nabla\Phi^{-1}(t)\|_{L^\infty})(\|\nabla\omega\|_{L^\infty}\|\nabla\omega\|_{\mathring{C}^\alpha}+\|\nabla\omega\|_{\mathring{C}^\alpha}Q(t)). \tag{28}$$

For simplicity we drop the dependence of $\|\nabla\Phi(t,\cdot)\|_{L^\infty},\|\nabla\Phi^{-1}(t)\|_{L^\infty}$ in $C(\|\nabla\Phi(t,\cdot)\|_{L^\infty}$, $\|\nabla\Phi^{-1}(t)\|_{L^\infty})$. Noting that $\|\nabla\omega\|_{\mathring{C}^\alpha}\leqslant C(Q(t)+\|\nabla\omega\|_{L^\infty})\leqslant C+CQ$, (28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t)\leqslant C+CQ+Q^2\leqslant C(1+Q^2).$$

Solving the inequality gives

$$Q(t)\leqslant\tan(CT^*+\arctan(Q_0)).$$

Thus $\|\nabla\omega\|_{\mathring{C}^\alpha}$ is finite and the existence of time to the solution passes T^* which is a contradiction.

3.2. *Proof of theorem 3*

Now we prove the last and main theorem. First we will rewrite the 1D model (6) and (7) as follows. Changing g to $-g$ and setting $G=\int F(t,\theta)\mathrm{d}\theta$. Then integrating (7) with respect to θ, the 1D model becomes

$$g_t+2G_\theta g_\theta=gG_{\theta\theta},\qquad 4G+G_{\theta\theta}=g. \tag{29}$$

Now it is sufficient to prove that there is a solution to (29) such that for some positive time T^*:

$$\lim_{t\to T^*}\|\partial_\theta g(t)\|_{L^\infty}=\infty.$$

Let us choose an initial data $g_0(\theta)$ to (29) satisfying

- $g_0(\theta)$ is smooth and nonnegative.
- $g_0(\theta)$ has a unique maximum value M at $\theta=\theta_M$, a unique minimum value at $\theta=\theta_m$ and g_0 is symmetric with respect to $\theta=\theta_M$ and $\theta=\theta_m$.

- Furthermore $g_0''(\theta_M) < 0$.

Remark 3.1. It is worth noticing here that since we changed g to $-g$ in the original 1D model (6) and (7), the properties of g_0 as we suggested in the introduction are all opposite.

Inspired by [16, 17], the system can be solved almost explicitly through the trajectory approach.

Without loss of generality we can always assume $\theta_M = 0$, and $\theta_m = \pm\frac{\pi}{4}$ so that g_0 is even. Indeed, one can always set the domain of θ by $[\theta_M - \frac{\pi}{4}, \theta_M + \frac{\pi}{4}]$.

Let $u = G_\theta, I = 4G$, then the 1D system (29) becomes

$$\begin{cases} g_t + 2ug_\theta = u_\theta g, \\ u_\theta = g - I. \end{cases} \tag{30}$$

Define the trajectory $\gamma(t, \alpha)$ by

$$\dot{\gamma} = 2u(t, \gamma), \qquad \gamma(0, \alpha) = \alpha.$$

The idea of solving equation along the trajectory is that if trajectory 'Jacobian' γ_α is known, the solution to (30) can be easily represented by

$$g(t, \gamma(t, \alpha)) = g_0(\alpha)\gamma_\alpha(t, \alpha)^{\frac{1}{2}}. \tag{31}$$

Indeed, differentiating $\dot{\gamma}$ with respect to α gives

$$\dot{\gamma_\alpha} = 2u_\theta\gamma_\alpha. \tag{32}$$

Then the first equation in (30) becomes

$$\frac{\mathrm{d}g}{\mathrm{d}t} = u_\theta g = g\frac{1}{2\gamma_\alpha}\frac{\mathrm{d}\gamma_\alpha}{\mathrm{d}t},$$

which gives (31).

Now we try to set up an equation of γ_α. Differentiating (32) with respect to time we obtain

$$\ddot{\gamma_\alpha} = 2(u_{\theta t} + u_{\theta\theta}\dot{\gamma})\gamma_\alpha + \dot{\gamma_\alpha}^2\gamma_\alpha^{-1}. \tag{33}$$

Next step is using (30) to eliminate the explicit dependence of u. Noticing that

$$\begin{aligned} u_{\theta t} + u_{\theta\theta}\dot{\gamma} &= g_t - I_t + 2u_{\theta\theta}u \\ &= gu_\theta - 2g_\theta u - I_t + 2u_{\theta\theta}u \\ &= (u_\theta + I)u_\theta - 2(u_{\theta\theta} + I_\theta)u + 2u_{\theta\theta}u - I_t \\ &= (u_\theta)^2 + Iu_\theta - 2I_\theta u - I_t \\ &= \frac{1}{4}(\dot{\gamma_\alpha}\gamma_\alpha^{-1})^2 + \frac{1}{2}I\dot{\gamma_\alpha}\gamma_\alpha^{-1} - \dot{I}, \end{aligned}$$

where in the last step we used

$$\dot{I}(t, \gamma(t, \alpha)) = I_t + I_\theta\dot{\gamma} = I_t + 2I_\theta u.$$

Substituting the above equality into (33) gives

$$\ddot{\gamma_\alpha} = \frac{3}{2}\dot{\gamma_\alpha}^2\gamma_\alpha^{-1} + I\dot{\gamma_\alpha} - 2\dot{I}\gamma_\alpha. \tag{34}$$

It is still hard to see how to solve γ_α from (34). Fortunately, we have

$$\begin{aligned}\left(\ddot{\gamma_\alpha^{-1/2}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}\gamma_\alpha^{-3/2}\dot{\gamma_\alpha}\right)\\ &= \frac{3}{4}\gamma_\alpha^{-\frac{5}{2}}\dot{\gamma_\alpha}^2 - \frac{1}{2}\gamma_\alpha^{-\frac{3}{2}}\ddot{\gamma_\alpha},\end{aligned}$$

this yields

$$\ddot{\gamma_\alpha} = \frac{3}{2}\gamma_\alpha^{-1}\dot{\gamma_\alpha}^2 - 2\gamma_\alpha^{\frac{3}{2}}\left(\ddot{\gamma_\alpha^{-\frac{1}{2}}}\right). \tag{35}$$

Plugging (35) into (34) gives

$$-2\gamma_\alpha^{\frac{3}{2}}\left(\ddot{\gamma_\alpha^{-\frac{1}{2}}}\right) = I\dot{\gamma_\alpha} - 2\dot{I}\gamma_\alpha. \tag{36}$$

By defining $\Omega = \gamma_\alpha^{-\frac{1}{2}}$, (36) becomes

$$\begin{aligned}-\ddot{\Omega} &= I\frac{1}{2}\frac{\dot{\gamma_\alpha}}{\gamma_\alpha^{\frac{3}{2}}} - \dot{I}\Omega\\ &= -I\dot{\Omega} - \dot{I}\Omega\\ &= -(\dot{I\Omega}).\end{aligned} \tag{37}$$

The above system is equivalent to

$$\begin{cases}\ddot{\Omega} = (\dot{I\Omega})\\ \Omega(0,\alpha) = 1\\ \dot{\Omega}(0,\alpha) = I_0(\alpha) - g_0(\alpha),\end{cases} \tag{38}$$

whose solution is

$$\Omega = \phi(t,\alpha)\left(1 - g_0(\alpha)\int_0^t \phi^{-1}(s,\alpha)\mathrm{d}s\right), \tag{39}$$

with

$$\phi(t,\alpha) = \exp(\int_0^t I(\tau,\gamma(\tau,\alpha))\mathrm{d}\tau).$$

By the periodic boundary condition, we have

$$\frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \gamma_\alpha(t,\alpha)\mathrm{d}\alpha = 1. \tag{40}$$

Indeed, G, g are even functions all the time since g_0 is even. Thus $u = G_\theta$ is odd. Because the solution is smooth in S^1 after periodic extension, we have $u(t, \frac{\pi}{4}) = u(t, -\frac{\pi}{4}) = 0$ which means that

$$\gamma\left(t, \frac{\pi}{4}\right) = \frac{\pi}{4}, \qquad \gamma\left(t, -\frac{\pi}{4}\right) = -\frac{\pi}{4}.$$

Then (40) immediately follows. Therefore, if we let

$$\eta(t,\alpha) = \int_0^t \phi^{-1}(s,\alpha)\mathrm{d}s,$$

it holds that

$$\frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \dot{\eta}^2(t,\alpha)(1 - g_0(\alpha)\eta(t,\alpha))^{-2}\mathrm{d}\alpha = 1. \tag{41}$$

Remark 3.2. A more general exponent in the definition of Ω (in our case is $-1/2$) can be found in [17]. However the function I in Sarria and Saxton's work only depends on time. In our case, even I depends on time and space, one can continue this procedure but can not solve γ_α completely. Furthermore, in the above argument the exact definition of I is irrelevant. The exact form of I will be used later, but the main thing we use is that $\frac{\partial I}{\partial \theta} \geqslant 0$ when $\theta \geqslant 0$.

The equation (41) actually gives us a hint about how to control the blow-up time. In Sarria's work [16], the above η is only a function of t so that $\dot{\eta}^2$ can be pulled out outside the integral and the equation becomes separable. In that case, integrating both sides of (41) yields that

$$t = \int_0^{\eta(t)} \sqrt{\frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha}\, \mathrm{d}\eta.$$

If the blowup happens, the above integral must be finite. It is clear that $\eta(t = 0) = 0$ and $\eta(T)$ is an increasing function. But the singular value in integral above shows that $\eta(T)$ has an upper bound $1/\max(g_0)$. Thus to prove the existence of the blowup, we need to prove that when $\eta(T) \to 1/\max(g_0)$, $t \to T < \infty$ which shows the finiteness of the existence of time T. Furthermore, if we can prove that $g(t,\theta)$ becomes unbound as $t \to T$, then T is exactly the blow-up time.

However, the difficulty in our case (41) is that η also depends on α which makes it hard to express t explicitly. Fortunately, since we will show that $\dot{\eta}$ is maximum at $\alpha = 0$ for all time, we can argue similarly. To do that, we introduce the following claim which will be proved later. This claim basically shows that all the functions will keep its original shape at $t = 0$ before blowing up.

Claim 3.1. Under the assumptions on g_0, it is true that $u(t,\gamma) \leqslant 0$ for $\alpha < 0$ and $u(t,\gamma) \geqslant 0$ for $\alpha > 0$ as long as the solution exists up to time t.

Now we fix time and consider the behavior of $\eta, \dot{\eta}$ and ϕ around $\alpha = 0$. A simple calculation shows that (recall that $I = 4G$)

$$\phi_\alpha(t,\alpha) = \phi(t,\alpha)\int_0^t 4u(s,\gamma(s,\alpha))\gamma_\alpha(s,\alpha)\mathrm{d}s. \tag{42}$$

Since g is even then G is even as well and $u(t,\theta) = G_\theta$ is odd so that $u(t,0) \equiv 0$ all the time which means the trajectory goes through $\alpha = 0$ if it is initially located at $\alpha = 0$, i.e. $\gamma(t,0) \equiv 0$. So $\phi_\alpha(t,0) = 0$. Noticing that by (32), $\gamma_\alpha = \exp(2\int_0^t u_\theta(s,\gamma(s,\alpha))\mathrm{d}s) > 0$. Therefore $\phi_\alpha(t,\alpha) \leqslant 0$ for $\alpha < 0$ and $\phi_\alpha(t,\alpha) \geqslant 0$ for $\alpha > 0$ which shows that $\phi(t,\alpha) \geqslant \phi(t,0)$. Consequently, we have

$$\eta_\alpha = -\int_0^t \phi^{-2}(s,\alpha)\phi_\alpha(s,\alpha)\mathrm{d}s,$$
$$\dot{\eta}_\alpha = -\phi^{-2}\phi_\alpha,$$

thus $\eta(t,\alpha) \leqslant \eta(t,0)$ and $\dot{\eta}(t,\alpha) \leqslant \dot{\eta}(t,0)$. By (41),

$$1 \leqslant \dot{\eta}^2(t,0)\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}(1 - g_0(\alpha)\eta(t,0))^{-2}\mathrm{d}\alpha. \tag{43}$$

Let $\eta^0(T) := \eta(t,0)$, then (43) leads to

$$\mathrm{d}t \leqslant \mathrm{d}\eta^0\left(\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}(1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha\right)^{\frac{1}{2}}. \tag{44}$$

Integrating (44) yields,

$$t \leqslant \int_0^{\eta^0(t)}\left(\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}(1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha\right)^{\frac{1}{2}}\mathrm{d}\eta. \tag{45}$$

Next we calculate $g(t,\gamma(t,\alpha))$ to indicate the maximum value of $\eta^0(T)$. By (31),

$$g(t,\gamma(t,\alpha)) = \frac{g_0(\alpha)}{\Omega(t,\alpha)} = \frac{g_0(\alpha)}{\phi(t,\alpha)(1 - g_0(\alpha)\eta(t,\alpha))}. \tag{46}$$

The above representation formulas for t and g (see (45) and (46)) show that the possible blowup may happen when $\eta^0(T)$ reaches $\frac{1}{g_0(0)}$ and the right-hand side of (45) is finite. So basically, we need to prove the following:

$$\int_0^{\frac{1}{g_0(0)}}\left(\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}(1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha\right)^{\frac{1}{2}}\mathrm{d}\eta < \infty, \tag{47}$$

then the blow-up comes with the blow-up criterion theorem 2. In the following argument, $A \lesssim B$ means that there is a universal constant C such that $A \leqslant CB$.

Since $g_0''(0) < 0$, Taylor expansion around $\alpha = 0$ gives that for all $\alpha \in J := [-r_0, r_0]$, there is $\theta \in J$ such that

$$g_0(\alpha) = M - C\alpha^2 + \frac{g_0^{(4)}(\theta)}{4!}\alpha^4,$$

where $C = -\frac{g_0''(0)}{2} > 0, M = g_0(0)$, and $0 < r_0 < 1$ is a constant which will be fixed later. Then for all $\alpha \in J$ and ε small, it holds that

$$\varepsilon + M - g_0(\alpha) = \varepsilon + C\alpha^2 - \frac{g_0^{(4)}(\theta)}{4!}\alpha^4.$$

Choosing η close enough to $\frac{1}{M}$, and let $\varepsilon = \frac{1}{\eta} - M$ and we have that for $\alpha \in J$,

$$\frac{1}{\eta} - g_0(\alpha) = \varepsilon + C\alpha^2 - \frac{g_0^{(4)}(\theta)}{4!}\alpha^4. \tag{48}$$

We choose r_0 small such that for all $\alpha \in J$, it holds that

$$\frac{|g_0^{(4)}(\theta)|}{4!}\alpha^4 \leqslant \frac{1}{2}C\alpha^2.$$

This choice of r_0 and (48) gives a lower bound of $1/\eta - g_0(\alpha)$ for $\alpha \in J$:

$$\frac{1}{\eta} - g_0(\alpha) > \frac{1}{2}(\varepsilon + C\alpha^2),$$

so that we can estimate the integral in (45) around the potential blow-up point $\alpha = 0$. While outside J, it is easy to see that $1 - g_0(\alpha)\eta > 1 - g_0(\alpha)/M$.

According to (43) (noting that $\dot{\eta} = \phi^{-1}$), with the help of above lower bounds, it holds that for small ε,

$$\begin{aligned}
\phi^2(t(\eta), 0) &\leqslant \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}(1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha \\
&= \frac{2}{\pi}\left(\int_{-\frac{\pi}{4}}^{-r_0} + \int_{r_0}^{\frac{\pi}{4}}(1 - g_0(\alpha)\eta)^{-2}\mathrm{d}\alpha\right) \\
&\quad + \frac{2}{\pi}(\eta)^{-2}\int_{-r_0}^{r_0}\left(\frac{1}{\eta} - g_0(\alpha)\right)^{-2}\mathrm{d}\alpha \\
&\leqslant C(r_0, g_0, M) + c(\eta)^{-2}\int_{-r_0}^{r_0}\left(\varepsilon + Cr^2\right)^{-2}\mathrm{d}r \\
&= C(r_0, g_0, M) + c\eta^{-2}\left(\frac{r_0}{\varepsilon^2 + C\varepsilon r_0^2} + \frac{\arctan\left(\frac{\sqrt{C}r_0}{\sqrt{\varepsilon}}\right)}{\sqrt{C}\varepsilon^{\frac{3}{2}}}\right) \\
&\lesssim (\eta\varepsilon)^{-2}\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

At last the finite blow up time $T^* < \infty$ comes from (47):

$$
\begin{aligned}
T^* &\leqslant \int_0^{\frac{1}{M}} \left(\frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - g_0(\alpha)\eta)^{-2} \mathrm{d}\alpha \right)^{\frac{1}{2}} \mathrm{d}\eta. \\
&= \int_0^{\frac{1}{M}} \phi(t(\eta), 0) \mathrm{d}\eta \\
&\lesssim \int_0^{\frac{1}{M}} (\eta\varepsilon)^{-1} \varepsilon^{\frac{1}{4}} \mathrm{d}\eta \\
&= \int_0^{\frac{1}{M}} (1 - M\eta)^{-1} \left(\frac{1}{\eta} - M \right)^{1/4} \mathrm{d}\eta \\
&= (M)^{-3/4} \int_0^1 \left(\frac{1}{1-y} \right)^{1/4} y^{-3/4} \mathrm{d}y \\
&< \infty,
\end{aligned}
$$

where we use the change of variable $y = 1 - M\eta$.

Construction of 2D solution. Now we can choose $\omega_0^{2\mathrm{D}}$ such that $\omega_0 = \omega_0^{2\mathrm{D}} + rg_0$ belongs to $C_c(\mathbb{R}^2)$ and is four-fold rotationally symmetric. So after some time, $\omega^{2\mathrm{D}} := \omega - rg$ is still four-fold rotationally symmetric. The symmetry property forces $\nabla\omega^{2\mathrm{D}}(x = 0) = 0$. Therefore

$$
\limsup_{x \to 0} |\nabla\omega| = \limsup_{x \to 0} |\nabla(rg)|.
$$

We also have

$$
|\nabla(rg)|^2 = |g\nabla r + r\nabla g|^2 = |g|^2 + |\partial_\theta g|^2 > |g|^2.
$$

Thus it holds that for $t \leqslant T^*$

$$
\|\nabla\omega\|_{L^\infty} \geqslant |g(t, 0)|.
$$

If ω does not blow up at T^*, it must have

$$
\sup_{t \in [0, T^*]} \|\nabla\omega\|_{L^\infty} < \infty.
$$

However according to our previous discussion, it holds that

$$
\sup_{t \in [0, T^*]} \|\nabla\omega\|_{L^\infty} > \sup_{t \in [0, T^*]} |g(t, 0)| = \infty.
$$

It is a contradiction!

We still have claim 3.1 remaining to prove. Recall that the claim is

Claim 3.1. Under the assumptions on g_0, it is true that $u(t, \gamma(t, \alpha)) \leqslant 0$ for $\alpha < 0$ and $u(t, \gamma(t, \alpha)) \geqslant 0$ for $\alpha > 0$ as long as the solution exists up to time t.

Proof. It is sufficient to prove that $u(t, \alpha) \geqslant 0$ for $\alpha \geqslant 0$. Indeed, by (32) we have

$$
\gamma_\alpha(t, \alpha) = \exp\left(2 \int_0^t u_\theta(s, \gamma(s, \alpha)) \mathrm{d}s \right) > 0, \tag{49}
$$

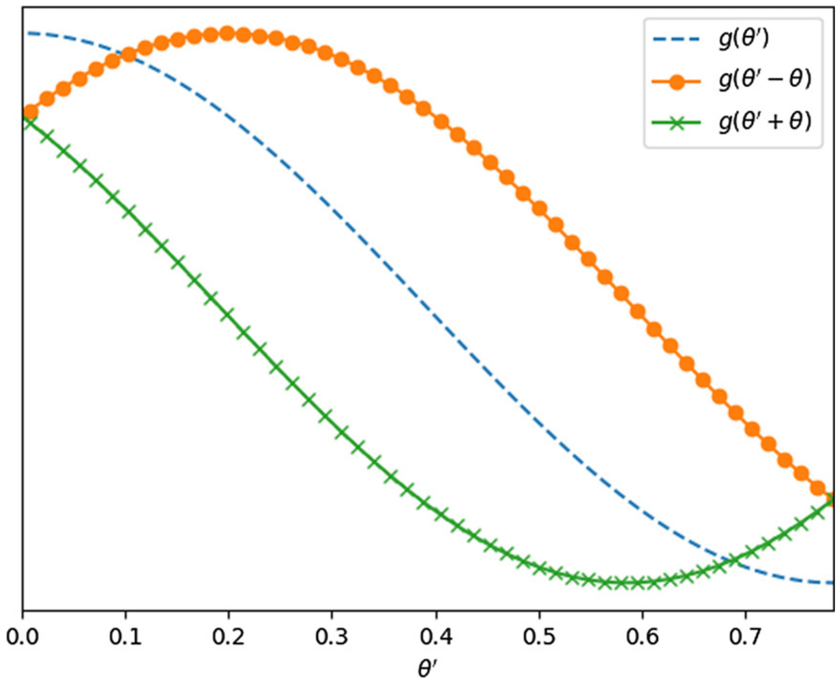

Figure 3. Fixing θ, the graph of $g(\theta'), g(\theta'-\theta), g(\theta'+\theta)$ for nonnegative θ'.

as long as it exists. So that $\gamma(t,\alpha)$ is increasing and for $\alpha > 0, \gamma(t,\alpha) \geqslant 0$ since $\gamma(t,0) \equiv 0$. Thus the claim's result is equivalent to for $\alpha \geqslant 0, u(t,\alpha) \geqslant 0$.

Recall that by (29),

$$u = \partial_\theta(4 + \partial_\theta^2)^{-1} g, \tag{50}$$

then

$$u(t,\theta) = C(\partial_\theta K * g(t,\cdot))(\theta), \tag{51}$$

where $K(\theta)$ is the kernel of operator $(4 + \partial_\theta^2)^{-1}$ and C is a universal constant. The kernel K is given by

$$K = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{2}{4 - 16k^2} \cos(4k\theta). \tag{52}$$

The above summation can be calculated explicitly (see [11]). The idea is to first calculate a 2π periodic kernel K_0 with its Fourier transform satisfying:

$$\widehat{K}_0(k) = \begin{cases} \dfrac{1}{4 - k^2}, & |k| \neq 2, \\ 0, & |k| = 2. \end{cases}$$

Then we extend K_0 as a four-fold rotationally symmetric function giving our desired kernel defined on $[-\frac{\pi}{4}, \frac{\pi}{4}]$:

$$K(\theta) = \frac{1}{4}\sum_{n=0}^{3} K_0\left(\theta + \frac{n\pi}{2}\right).$$

Taking advantages of the fact that the Fourier transform of the function $\pi\frac{\theta}{|\theta|} - \theta$ is $\frac{1}{ik}$ and using shift property give that for $|k| \neq 2$, the Fourier transform of

$$K_1(\theta) = \frac{\pi}{2}\,\sin(2\theta)\frac{\theta}{|\theta|} - \frac{1}{2}\,\sin(2\theta)\theta$$

is $\frac{1}{4-k^2}$ and $\widehat{K}_1(\pm 2) = \frac{1}{16}$. Thus we can assume that $K_0(\theta) = K_1(\theta) + C\cos(2\theta)$. The constant C is calculated through $\widehat{K}(\pm 2) = 0$ and finally we have

$$K_0(\theta) = \frac{\pi}{2}\,\sin(2\theta)\frac{\theta}{|\theta|} - \frac{1}{2}\,\sin(2\theta)\theta - \frac{1}{8}\,\cos(2\theta). \tag{53}$$

In conclusion,

$$K(\theta) = \frac{\pi}{8}\frac{\theta\,\sin(2\theta)}{|\theta|}. \tag{54}$$

Now we can prove the claim for u. For $\frac{\pi}{4} \geqslant \theta \geqslant 0$,

$$K_\theta = \frac{\pi}{4}\,\cos(2\theta) \geqslant 0,$$

which, plugging into (51) yields

$$u(t,\theta) = \int_{-\pi/4}^{\pi/4} K_\theta(\theta')g(t,\theta-\theta')\mathrm{d}\theta'. \tag{55}$$

Then by the oddness of K_θ and the evenness of g, (55) becomes

$$\begin{aligned} u(t,\theta) &= \int_0^{\frac{\pi}{4}} K_\theta(\theta')(g(t,\theta-\theta') - g(t,\theta+\theta'))\mathrm{d}\theta' \\ &= \int_0^{\frac{\pi}{4}} K_\theta(\theta')(g(t,\theta'-\theta) - g(t,\theta'+\theta))\mathrm{d}\theta'. \end{aligned}$$

Now fixing $\theta > 0$, $g(t,\theta'-\theta) - g(t,\theta'+\theta) \geqslant 0$ for all $\theta' \in [0, \frac{\pi}{4}]$ because of the evenness, periodicity and monotonicity of g (see figure 3). Then the claim is proved.

□

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ORCID iDs

Slim Ibrahim https://orcid.org/0000-0002-5072-8062

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