A GENERIC SLICE OF THE MODULI SPACE OF LINE ARRANGEMENTS

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Abstract. We study the compactification of the locus parametrizing lines having a fixed intersection with a given line, inside the moduli space of line arrangements in the projective plane constructed for weight one by Hacking-Keel-Tevelev and Alexeev for general weights. We show that this space is smooth, with normal crossing boundary, and that it has a morphism to the moduli space of marked rational curves which can be understood as a natural continuation of the blow up construction of Kapranov. In addition, we prove that our space is isomorphic to a closed subvariety inside a non-reductive Chow quotient.

1. Introduction

The compact moduli space of weighted hyperplane arrangements in $\mathbb{P}^2$ is a higher dimensional generalization of $\overline{M}_{0,n}$, and has a main component parameterizing equivalence classes of n weighted lines in $\mathbb{P}^2$ and their log canonical degenerations. The moduli space $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n)$ was constructed for lines of weight one by Hacking-Keel-Tevelev [HKT06], and for more general weights $\vec{\beta}$ as a generalization of the weighted Hassett spaces, by Alexeev [Ale13]. The space is expected to satisfy Murphy's law– it can be arbitrarily singular, and can contain many irreducible components. The goal of this paper is to describe a naturally appearing locus inside this moduli space which has perhaps unexpected properties – it is smooth with normal crossings boundary.

Given an arrangement of $(n+1)$ labeled lines in $\mathbb{P}^2$, there is a natural restriction morphism: label the line l_{n+1} as l_A, and obtain an arrangement of n labeled points on $l_A \cong \mathbb{P}^1$, by intersecting the other n lines with l_A. The restriction morphism induces a morphism $M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}}$ that has rational fibers of dimension $n-3$ (see Lemma 3.3). Given a generic point $q \in M_{0,\vec{w}}$, we study the closure, which we denote by $R_{\vec{w}}(q)$, in $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ of the fiber of $M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}}$ over q (see Definition 3.1).

In other words, $R_{\vec{w}}(q)$ compactifies the locus parametrizing equivalence classes of $n+1$ labeled lines having a fixed intersection with the line l_A. Our first theorem characterizes $R_{\vec{w}}(q)$.

Theorem 1.1 (see Theorem 5.14 and Theorem 5.16). *For weights $\vec{w}$ in the set of admissible weights $\mathcal{D}_n^R$ (see Definition 4.1) and generic choice of $q \in M_{0,\vec{w}}$, the locus $R_{\vec{w}}(q)$ is smooth with normal crossings boundary and there are birational morphisms*

$$R_{\vec{w}}(q) \xrightarrow{\pi_2} \overline{M}_{0,\vec{w}} \xrightarrow{\pi_1} \mathbb{P}^{n-3}.$$

By results of Kapranov [Kap93b] the morphism $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ factors into the sequence of following morphisms: The blow up of $(n-1)$ points $q_i \in \mathbb{P}^{n-3}$ which are in general position;

the blow up of the strict transforms of the $\mathbb{P}^1$'s spanned by pairs of the points q_i, and so forth. For $\vec{w} = (1, \dots, 1)$, the morphism π_2 factors in a similar fashion.

Corollary 1.2. *(see Corollary 5.18) The morphism $R_{1^n}(q) \to \overline{M}_{0,n}$ factors into the sequence of following morphisms: The blow up of a point q_n in the interior of $\overline{M}_{0,n}$; the blow up of the strict transforms of the $\mathbb{P}^1$s spanned by pairs $\{q_i, q_n\}$; the blow up of the strict transforms of the $\mathbb{P}^2$s spanned by triples $\{q_i, q_j, q_n\}$, and so forth.*

In contrast to $\overline{M}_{0,n}$, the centers used to construct $R_{1^n}(q)$ are not projectively equivalent to each other. As a result, $R_{1^n}(q)$ depends on the choice of q_n, and in general different q_n yields non-isomorphic spaces. Moreover we show the following.

Theorem 1.3. *For a generic choice of q and $n \geq 5$, there do not exist weights $\vec{w}$ such that $R_{\vec{w}}(q) \cong \overline{M}_{0,n}$.*

The objects parametrized by $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ are called stable hyperplane arrangements, or *shas* (see [Ale13, Def 5.3.1]), and they are stable pairs in the sense of the Minimal Model Program (see [Ale13, Thm 5.3.2]). The shas parametrized by $R_{\vec{w}}(q)$ are described in Section 2. In particular, our setting restricts the possible singularities that appear in our shas (see Remark 3.4 and Proposition 4.8) (see Figure 1).

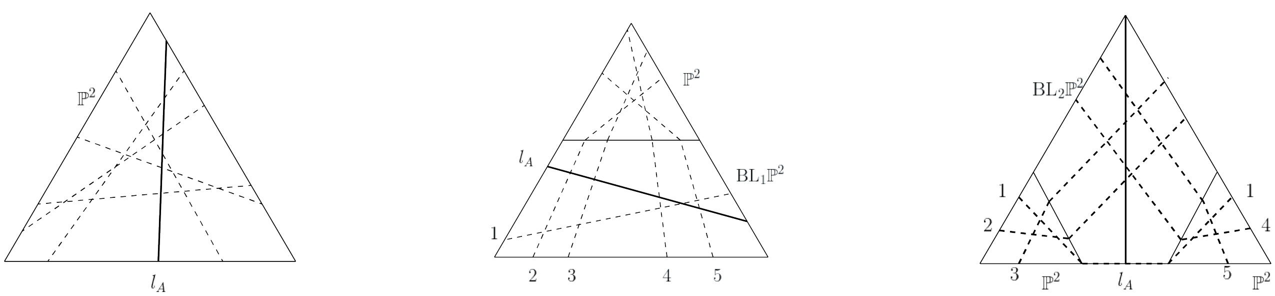

FIGURE 1. Examples of generic and non-generic shas parametrized by $R_{1^5}(q)$

Our next main result is that the locus $R_{1^n}(q)$ is the normalization of a non-reductive Chow quotient. In particular, our result fits into a library of examples (see [GG], [Tha99], [H+05], [Gia13] and [KSZ91]) where Chow quotients are used to study the geometry of moduli spaces. The following outline generalizes the construction of Kapranov [Kap93a] in the setting of $R_{1^n}(q)$ (see Remark 6.1): Given the collection of n points p_i in the dual projective space $\hat{\mathbb{P}}^2$ such that the point p_i is dual to the line l_i, we consider the locus, in an appropriate Chow variety, that parametrizes the cycles associated to the orbits $\overline{G \cdot (p_1, \dots, p_n)}$ where $G \subset SL(3, \mathbb{C})$ is the group that *fixes the intersection* of the associated lines l_i with l_A. By normalizing the closure of this locus in the Chow variety, we recover $R_{1^n}(q)$ (see Section 6).

Theorem 1.4 (see Theorem 6.12)**.** *For a generic choice of q, the space $R_{1^n}(q)$ is isomorphic to the normalization of a closed subvariety of the Chow quotient $(\hat{\mathbb{P}}^2)^n /\!/_{Ch} G$ where $G \subset SL(3, \mathbb{C})$ is the group fixing the line l_A pointwise.*

1.5. **Method of proof of Theorem 1.1.** We give an outline of our proof that $R_{\vec{w}}(q)$ is smooth with normal crossings boundary. The overall strategy is to prove that $R_{\vec{w}}(q)$ is isomorphic to a wonderful compactification, which is smooth with normal crossings boundary by definition (see Theorem 5.6).

We first construct our space with smallest admissible weights $\vec{w}_0$, show that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ (see Lemma 4.4), and construct a family over $R_{\vec{w}_0}$ (see Lemma 4.6). In Section 5.3 we construct the wonderful compactification $Bl_{\vec{w}} R_{\vec{w}_0}$, and in Lemma 5.10 we construct a family of shas over the wonderful compactification. Using this family, we obtain a finite birational (i.e. normalization) morphism from the wonderful compactification to our space: $Bl_{\vec{w}} R_{\vec{w}_0} \to R_{\vec{w}}$. We prove normality of $R_{\vec{w}}$ in Theorem 5.14, which implies that $R_{\vec{w}} \cong Bl_{\vec{w}} R_{\vec{w}_0}$ by Zariski's main theorem. Finally, we note that the key lemma required to prove normality of $R_{\vec{w}}$ is Lemma 4.9.

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2. Definition and basic properties

We work only over $\mathbb{C}$ for convenience. We begin with the necessary background on the moduli space $\overline{M}_{\vec{w}}(\mathbb{P}^2, n+1)$, see [HKT06] and [Ale13] for a full exposition.

Configurations of $(n+1)$ labeled lines $(l_1, ..., l_{n+1})$ in $\mathbb{P}^2$ up to projective equivalence are parametrized by the open moduli space $M(\mathbb{P}^2, n+1)$, which has a family of geometric compactifications $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ depending on a weight vector $\vec{\beta} := (\beta_1, \dots, \beta_{n+1})$ (see [Ale13, Theorem 5.4.2]).

The weight domain of possible weights $\vec{\beta}$ is

$$\mathcal{D}(3, n+1) = \left\{ \vec{\beta} \in \mathbb{Q}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} \beta_i > 3,\ 0 < \beta_i \leq 1 \right\} \tag{2.0.1}$$

In general these compactifications are *not* irreducible. However, they do contain a main irreducible component parameterizing stable pairs in the sense of MMP $\left(X, \sum_{k=1}^{n+1} \beta_k l_k\right)$ appearing as degenerations of the $(n+1)$ lines in $\mathbb{P}^2$.

Definition 2.1. *The stable pairs* $(X, D) := \left(X, \sum_{i=1}^{n+1} \beta_k l_k\right)$ *parametrized by* $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ *are called* ***shas*** *of weight* $\vec{\beta}$ *or just shas if the weight* $\vec{\beta}$ *is clear from the context.*

Notation 2.2. Let $I \subset \{1,2,...,n\}$ be an index set. A sha (X,D) has a multiple point $p(I)$ if there exists a component X_i of X and divisors $\{l_i = D|_{X_i} \mid i \in I\}$ such that the divisors l_i are concurrent at a point $p(I) \in X_i$.

Remark 2.3. The admissible singularities of the divisors D in the sha (X,D) depend completely on the weights $\vec{\beta}$. Indeed, we cannot have coincident lines $\{l_i \mid i \in I\}$ with weight $\sum_{i\in I}\beta_i > 1$ or multiple points $p(I)$ defined by the concurrent lines $\{l_i \mid i \in I\}$ with total weight $\sum_{i\in I}\beta_i > 2$.

Definition 2.4. *Let $\vec{\beta}$ and $\vec{\alpha}$ be two weights vector in $\mathcal{D}(3,n+1)$. We say that $\vec{\beta} \geq \vec{\alpha}$ if $\beta_i \geq \alpha_i$ for all i.*

As in the Hassett spaces $\overline{M}_{0,\vec{w}}$, the shas parametrized by $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ depend solely on the weights $\vec{w}$, and the weight domain admits a wall and chamber decomposition.

Theorem 2.5. *(see* [Ale13, Thm 5.5.2]*) The domain $\mathcal{D}(3,n+1)$ is divided into finitely many walls and chambers. There are two types of walls:*

$$W(I) := \left(\sum_{i\in I}\beta_i - 2 = 0\right), \qquad \widetilde{W}(I) := \left(\sum_{i\in I}\beta_i - 1 = 0\right). \tag{2.5.1}$$

for all $I \subset \{1,\dots,n+1\}$, $2 \leq |I| \leq (n-1)$. Moreover,

(1) *if $\vec{\beta}$ and $\vec{\alpha}$ lie in the same chamber, then the weighted moduli spaces and their families of shas are the same.*
(2) *If $\vec{\beta}$ is in the closure of the chamber containing $\vec{\alpha}$, then there exists a contraction*
$$\overline{M}_{\vec{\alpha}}(\mathbb{P}^2, n+1) \to \overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$$
(3) *Further, if $\vec{\beta}$ is in the closure of the chamber containing $\vec{\alpha}$ and $\alpha \leq \vec{\beta}$ then*
$$\overline{M}_{\vec{\alpha}}(\mathbb{P}^2, n+1) = \overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1).$$

Remark 2.6. Recall from Remark 2.3 that there are two types of singularities appearing in shas. In this setting, the walls $W(I)$ correspond to multiple points $p(I)$, and the walls $\widetilde{W}(I)$ correspond to coincident lines.

3. Definition of $R_{\vec{w}}(q)$

To construct $R_{\vec{w}}(q)$, we consider arrangements of $n+1$ labeled lines in $\mathbb{P}^2$, and we label the $(n+1)^{\text{st}}$-line as l_A to distinguish it. We will always assume l_A has weight 1, and thus will denote our weight set $\beta \in \mathcal{D}(3,n+1)$ as $(\vec{w},1)$. In this section, there is no need to restrict the set of weights $\vec{w}$. However in the following sections, we will consider an additional restriction on the weights (see Definition 4.1).

We have a naturally induced *restriction* morphism

$$\varphi_A : M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}},$$

induced by considering the intersection of l_A with the lines l_i where $i \in \{1,\dots,n\}$. Next, we take the fiber of this restriction over a generic point $q \in M_{0,\vec{w}}$, and then take closure of this fiber in the compact moduli space of weighted hyperplane arrangements.

Definition 3.1. *Let $q \in M_{0,\vec{w}} \subset \overline{M}_{0,\vec{w}}$ be a generic point. We define $R_{\vec{w}}(q)$ as the closure in $\overline{M}_{\vec{w},1}(\mathbb{P}^2, n+1)$ of the fiber product of the following diagram:*

$$\begin{array}{ccc} R_{\vec{w}}(q) & \longrightarrow & \overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \\ \downarrow & & \downarrow \varphi_A \\ q & \longrightarrow & \overline{M}_{0,\vec{w}} \end{array}$$

Remark 3.2. B. Hassett gave an example of families $\left(\mathcal{X}, \frac{1}{2}\mathcal{D}\right) \to \operatorname{Spec}(\mathbb{C}[[t]])$ where $\mathcal{D}|_{t=0}$ has embedded points. In general for pairs, the components of the boundary with fractional coefficients $\leq \frac{1}{2}$ need not be Cohen-Macaulay. By [Ale13, Lemma 1.5.1], the mentioned difficulty will *not* occur for very generic coefficients of the form $\vec{w}$ for which one entry satisfies $w_i = 1$.

Lemma 3.3. *The dimension* $\dim(R_{\vec{w}}(q)) = n - 3$.

Proof. By the fiber product construction we see that

$$\dim\left(R_{\vec{w}}(q)\right) = \dim\left(\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)\right) - \dim(\overline{M}_{0,n})$$

The result follows since $\dim\left(\overline{M}_w(\mathbb{P}^2, n+1)\right) = 2(n-3)$ (see [Ale13, pg 84]). □

Remark 3.4. We will show in Proposition 4.8 that

1. the only singularities in the shas parametrized by $R_{\vec{w}}(q)$ are multiple points (no overlapping lines), as each line l_i with $1 \leq i \leq n$ intersects the fixed line l_A in a *distinct* point.
2. The dual graph of X is a rooted tree (see Proposition 4.8 [II]). This allows us to fully describe the shas parametrized by $R_{1^n}(q)$ (see Figure 1).
3. Each *broken line* l_i can be seen as a chain of lines that starts in the rooted component. The l_i may have several branches, and can be contained in several components.

Definition 3.5. *We say that the weight $\vec{\beta}$ **destabilizes** the multiple point $p(K)$ if the sum $\sum_{k \in K} \beta_i > 2$. We also say $\vec{\beta}$ destabilizes the sha (X, D) if the pair has a singularity destabilized by $\vec{\beta}$.*

In what follows, we discuss the stable replacement of shas with multiple points which will be relevant for us (see [Ale13, Chapter 5] for a complete discussion).

3.6. **Stable replacement.** Let $I \subset \{1, 2, ..., n\}$ be an index set. We consider two chambers in $\mathcal{D}(3, n+1)$ separated by the wall $W(I)$ as defined in Theorem 2.5. Let $\vec{w} \leq \vec{v}$ be weights in those chambers such that $\sum_{i \in I} w_i < 2$ and $\sum_{i \in I} v_i > 2$. Let $\vec{u}$ be a weight in the wall that separates those chambers, so in particular $\sum_{i \in I} u_i = 2$.

Let (X, D) be a sha parametrized by $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$, and suppose that the sha has only a multiple point $p(I)$; notice that the point $p(I)$ will never be supported on l_A (Remark 3.4 (1)). By (3) in Theorem 2.5, changing the weights from $\vec{w}$ to $\vec{u}$ will not modify the moduli spaces, so

$$\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \cong \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1).$$

The singularity $p(I)$ is still log canonical with respect to the weights $(\vec{u}, 1)$. Therefore, (X, D) is in the universal family associated to weights $\vec{u}$.

Next, we change the weights from $\vec{u}$ to $\vec{v}$. By (2) in Theorem 2.5, there is a contraction

$$\pi_{\vec{v},\vec{u}} : \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1)$$

By moduli theory, we know that the center of this morphism is the locus parametrizing shas with singularities that are destabilized respect to the new weights $(\vec{v}, 1)$. In particular, the sha (X, D) is no longer parametrized by $\overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1)$ because $\sum_{i\in I} v_i > 2$.

Let $z \in \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1)$ be the point parametrizing the sha (X, D). Next, we describe the sha $(\tilde{X}, \tilde{D})$ parametrized by a generic point in $\pi^{-1}_{\vec{v},\vec{u}}(z)$. We first blow up X at $p(I)$, and we attach a $\mathbb{P}^2$ along the exceptional divisor $E_{p(I)}$ to obtain a new surface

$$\tilde{X} = Bl_{p(I)}X \cup_{E_{p(I)}} \mathbb{P}^2$$

with the lines $(l_i\ ,\ i \in I)$ crossing into the new $\mathbb{P}^2$ and defining a new divisor $\tilde{D}$ (see Figure 2). The multiple lines defining $p(I)$ are separated in $Bl_{p(I)}X$, and they are generically separated in the new component $\mathbb{P}^2$. They may acquire a multiple point, but they cannot overlap with each other, because they are already separated in the double locus.

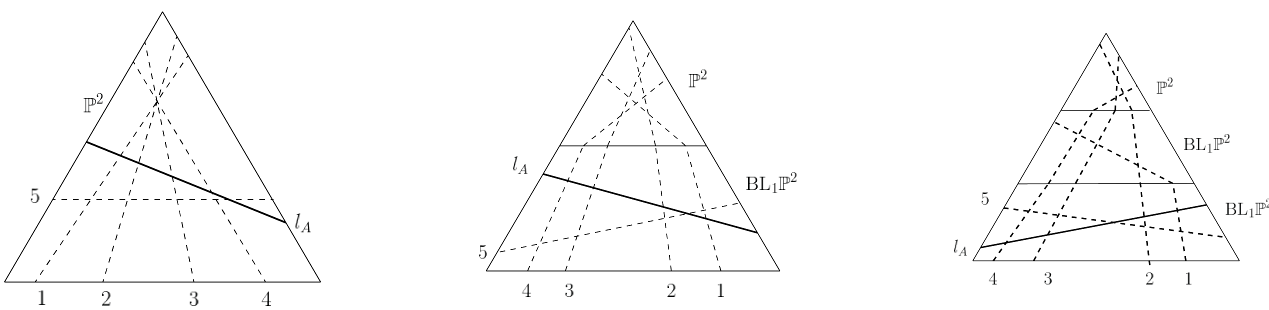

FIGURE 2. Quadruple point and its generic and non-generic stable replacement.

Example 3.7. Consider a quadruple point in an arrangement of 6 lines– then there are two possible stable replacements. The starting configuration is stable if the total weight of the intersection point of the four lines $l_1, .., l_4$ is ≤ 2. Increasing the weights of all the lines to one causes any singularity with multiplicity larger than two to become unstable. Generically, the stable replacement has a new component where the 4 lines are separated. The four lines plus the double locus in $\mathbb{P}^2$ have two dimensional moduli, so that we can further degenerate the configuration to a triple point. In this case, we must blow up the new component, obtaining a surface with three components. Here, the additional surface is a $\mathbb{P}^2$ with three lines. Since a configuration of three lines and the double locus in $\mathbb{P}^2$ has no moduli, we cannot degenerate the configuration any further. These two cases are all of the possible stable replacements.

4. $R_{\vec{w}_0}$ AS A GIT QUOTIENT AND SOME PROPERTIES OF $R_{\vec{w}}$

The starting point of this section is Lemma 4.4, where we show that there are weights $\vec{w}_0$ such that $R_{\vec{w}_0}(q) \cong \mathbb{P}^{n-3}$. Afterwards, we study some geometric properties of $R_{\vec{w}}$ in general, such as the surfaces parametrized and the singularities that appear (Proposition 4.8), as well as the outcome of wall-crossing on our moduli spaces (Lemma 4.9).

The results of this section do not depend on the q used in the definition of $R_{\vec{w}}(q)$, so we simplify our notation and we just write $R_{\vec{w}}$. First, we define our admissible weights.

Definition 4.1. *Let $\vec{w}_0 = (w_{0_1}, \dots, w_{0_n})$ be a set of rational numbers such that for every subset $I \subsetneq \{1, \dots, n\}$ the inequality $\sum_{i\in I} w_{0_i} \leq 2$ holds. The* ***set of admissible weights*** *is*

$$\mathcal{D}_n^R = \{(w_1, \dots, w_n) \in \mathbb{Q}^n \mid 1 \geq w_i > 0,\ \sum_{i=1}^n w_i \geq 2,\ w_i \geq w_{0_i}\}$$

The chamber decomposition of $\mathcal{D}(3, n+1)$ induces a chamber decomposition on $\mathcal{D}_n^R$ where the chambers are separated by the walls $W(I)$ (see Theorem 2.5).

Definition 4.2. *We say that two weights $\vec{v}$ and $\vec{u}$ are* ***adjacent*** *if each of them belongs to a chamber in $\mathcal{D}_n^R$ and those chambers are separated by a single wall $W(I)$. Sometimes, we say that the weights $\vec{u}$ and $\vec{v}$ are* ***separated*** *by $W(I)$.*

Moreover, by Remark 3.2, to avoid any subtle technicalities, we will assume all our weighs are very generic.

Before showing that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ (Lemma 4.4), we prove a key lemma.

Lemma 4.3. *The subgroup of $\mathrm{SL}(3, \mathbb{C})$ that fixes:*

- *three lines l_n, l_{n-1} and l_A in general position, and*
- *n distinct points $\{l_1 \cap l_A, \dots, l_n \cap l_A\}$ in l_A.*

is equal to $\mathbb{C}^*$.

Proof. We can suppose without lost of generality that the lines are

$$l_A := (x_0 = 0), \qquad l_{n-1} := (x_1 = 0), \qquad l_n := (x_2 = 0)$$

The subgroup that fixes those lines in $\mathbb{P}^2$ is $(\mathbb{C}^*)^2$, and it is given by matrices of the form $g = \mathrm{diag}((g_2 g_1)^{-1}, g_1, g_2)$ which acts on any point in the line l_A by $g\cdot[0 : q_1 : q_2] \to [0 : g_1 q_1 : g_2 q_2]$. By hypothesis, the points $\{l_1 \cap l_A, \dots, l_n \cap l_A\}$ on l_A are fixed, implying that $g_1 = g_2$. □

Lemma 4.4. *Let $\vec{w}_0$ be as in Definition 4.1. Then*

$$R_{\vec{w}_0} \cong \mathbb{P}^{n-3} \subset \overline{M}_{(\vec{w}_0, 1)}(\mathbb{P}^2, n+1),$$

and each fiber of the universal family over $R_{\vec{w}_0}$ is a pair $(\mathbb{P}^2, \sum_{k=1}^n w_{0_k} l_k + l_A)$ such that

1. *the n lines l_i cannot all meet at an n-tuple point,*
2. *any multiple point of multiplicity strictly smaller than n is allowed,*
3. *none of the lines l_i can overlap with l_A,*

Proof. Let l_A be the line with weight $w_A = 1$ that induces the restriction morphism

$$M_{(\vec{w}_0, 1)}(\mathbb{P}^2, n+1) \to M_{0, \vec{w}_0}.$$

To prove (1), recall that an n-tuple point is unstable if and only if the sum of the weights $\sum_{i=1}^n w_{0_i} > 2$, which is true by assumption.

Following the proof of (1), we note that (2) holds because of the assumption that for every subset $I \subsetneq \{1, ..., n\}$ the sum of the weights is ≤ 2.

To prove (3), we recall that a multiple line is unstable if the sum of the weights is greater than 1. Since the weight w_A of the line l_A is already 1, no other line can overlap with it.

Let (X, D) be any configuration parametrized by $R_{\vec{w}_0}$. By (1) and (3), we can suppose that the lines l_{n-1}, l_n and l_A are fixed and in general position. By definition, the points $\{l_1 \cap l_A, \ldots, l_n \cap l_A\} \subset l_A$ induce the equivalence class $q \in M_{0,n}$, and thus we can fix these points.

We can now demonstrate that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$. First note that the parameter space of each line l_i with $1 \leq i \leq n-2$ is $\mathbb{A}^1$, because the intersection $l_i \cap l_A$ is fixed. We can choose coordinates on each $\mathbb{A}^1$ so that the point $0 \in \mathbb{A}^1$ parametrizes whenever the line l_i coincides with the fixed intersection $l_n \cap l_{n-1}$. Then the parameter space of the $(n-2)$ lines $l_1, \ldots, l_{n-2}$ is $(\mathbb{A}^1)^{n-2} \setminus (0, \cdots, 0)$, since we cannot have an n-tuple point by (1). Therefore, by Lemma 4.3, we conclude that

$$R_{\vec{w}_0} \cong \mathbb{A}^{n-2} \setminus (0, \cdots, 0) /\!\!/ \mathbb{C}^* \cong \mathbb{P}^{n-3}.$$

$\square$

Next, we construct a family of shas over $R_{\vec{w}_0}$. Before doing that, we set up some notation.

Notation 4.5. We choose a coordinate system $[t_0 : t_1 : t_2] \in \mathbb{P}^2$ such that:

$$l_A := (t_0 = 0), \qquad l_{n-2} \cap l_A := [0 : 0 : 1], \qquad l_{n-1} := (t_2 = 0), \qquad l_n := (t_1 - t_2 = 0).$$

and we select the point $q \in M_{0,\vec{w}_0}$ induced by the following configuration of points in l_A

$$\{[0 : a_1 : 1], \ldots, [0 : a_{n-3} : 1], [0 : 0 : 1], [0 : 1 : 0], [0 : 1 : 1]\},$$

Under this choice of coordinates, $[s_1 : \ldots : s_{n-2}] \in R_{\vec{w}_0}(q)$ parametrizes the following configuration of lines with $1 \leq i \leq (n-3)$

$$l_i := (t_1 - a_i t_2 + s_i t_0 = 0), \quad l_{n-2} := (s_{n-2} t_0 + t_1 = 0), \quad l_{n-1} := (t_2 = 0), \quad l_n := (t_1 - t_2 = 0).$$

In the following lemma, we consider $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ with coordinates $[s_1, \ldots, s_{n-2}]$ as above and the projective space $\mathbb{P}^{n-1}$ with coordinates $[z_1, \ldots, z_n]$. We exclude the $n = 4$ case for convenience of notation (see Remark 5.13).

Lemma 4.6. *For $n \geq 5$, let $\mathcal{U}_{\vec{w}_0}$ be the blow up of $\mathbb{P}^{n-1}$ at the line defined by*

$$Z := \{z_k - z_{k+2} = 0 \mid 1 \leq k \leq n-2\}.$$

and let σ_i be the strict transform of the following n hyperplanes in $\mathbb{P}^{n-1}$ with $1 \leq i \leq n-3$.

$$\begin{aligned}
H_i &:= (a_2 z_3 - a_1 z_4) - a_i(z_3 - z_4) + (a_2 - a_1)(z_i - z_{i+2}) = 0 \\
H_{n-2} &:= (a_2 - a_1)(z_{n-2} - z_n) + a_2 z_3 - a_1 z_4 = 0 \\
H_{n-1} &:= z_3 - z_4 = 0 \\
H_n &:= (a_2 - 1) z_3 - (a_1 - 1) z_4 = 0
\end{aligned}$$

Then there exists a flat, proper morphism $\phi_{\vec{w}_0} : \mathcal{U}_{\vec{w}_0} \to R_{\vec{w}_0}$ *such that for every* $\vec{s} \in R_{\vec{w}_0}$ *the fiber* $\phi_{\vec{w}_0}^{-1}(\vec{s})$ *is isomorphic to* $\mathbb{P}^2$*. Moreover, if* $E_{\vec{w}_0} \subset \mathcal{U}_{\vec{w}_0}$ *is the exceptional divisor, then the configuration of lines*

$$l_i := \phi_{\vec{w}_0}^{-1}(\vec{s}) \cap \hat{\sigma}_i \qquad\qquad l_A := \phi_{\vec{w}_0}^{-1}(\vec{s}) \cap E_{\vec{w}_0}$$

define the stable sha of weight $\vec{w}_0$ *parametrized by* $\vec{s}$*.*

Proof. Let $\pi_Z : \mathbb{P}^{n-1} \to R_{\vec{w}_0}$ be the projection defined by $\{s_k = z_k - z_{k+2} \mid 1 \leq k \leq n-2\}$. Note that Z is the indeterminacy loci of π_Z, and that given a point $\vec{s} \in R_{\vec{w}_0}$, we have $\pi_Z^{-1}(\vec{s}) \cong \mathbb{P}^2$. Therefore, the map $\mathcal{U}_{\vec{w}_0} \to R_{\vec{w}_0}$ is a $\mathbb{P}^2$-fibration obtained by the composition $\mathcal{U}_{\vec{w}_0} \to \mathbb{P}^{n-1} \to R_{\vec{w}_0}$.

The following functions with $2 \leq m \leq \frac{n}{2}$ if n is even, and $2 \leq m \leq \frac{(n+1)}{2}$ if n is odd.

$$\zeta_1 = t_1 - a_1 t_2 + s_1 t_0, \qquad\qquad \zeta_2 = t_1 - a_2 t_2 + s_2 t_0,$$

$$\zeta_{2m-1} = \zeta_1 - t_0 \sum_{k=0}^{m-2} s_{2k+1}, \qquad\qquad \zeta_{2m} = \zeta_2 - t_0 \sum_{k=1}^{m-1} s_{2k}$$

define, for a fixed $\pi_Z^{-1}(\vec{s})$, a map $\zeta_{\vec{s}} : \mathbb{P}^2 \to \pi_Z^{-1}(\vec{s})$ given by

$$\zeta_{\vec{s}} : [t_0, t_1, t_2] \to [\zeta_1, \zeta_2, \ldots, \zeta_n].$$

Indeed, we can verify the image of the map $\zeta_{\vec{s}}$ is $\pi_Z^{-1}(\vec{s})$ since

$$\pi_Z\left(\zeta_{\vec{s}}[t_0, t_1, t_2]\right) = [\zeta_1 - \zeta_3, \zeta_2 - \zeta_4, \ldots, \zeta_{n-2} - \zeta_n] = [s_1 t_0, s_2 t_0, \ldots, s_n t_0].$$

We also note that the map is not defined for $(t_0 = 0)$ because $\zeta_{\vec{s}}^{-1}(Z) = (t_0 = 0)$. Moreover, by the definition of the H_i above, and the equations of the lines given in Notation 4.5 it holds that

$$\zeta_{\vec{s}}(l_i) = \pi_Z^{-1}(\vec{X}) \cap H_i \qquad\qquad \zeta_{\vec{s}}(l_A) = Z$$

These equalities follow at once by observing that $\zeta_3 = \zeta_1 - t_0 s_1$, $\zeta_4 = \zeta_2 - t_0 s_2$ as well as

$$a_2\zeta_3 - a_1\zeta_4 = (a_2 - a_1)t_1 \qquad \zeta_3 - \zeta_4 = (a_2 - a_1)t_2 \qquad \zeta_i - \zeta_{i+2} = s_i t_0.$$

Finally, we assign the weights given by $\vec{w}_0$ to the n hyperplanes and weight 1 to the exceptional divisor, we get a family of shas with respect to the weights $\vec{w}_0$. □

4.7. **Generalities on $R_{\vec{w}}$.** We start with a explicit description of the surfaces parametrized by $R_{\vec{w}}$.

Proposition 4.8. *Let* (X, D) *be a sha parametrized by* $R_{\vec{w}}$*, then the following hold:*

- I *The only singularities in* (X, D) *are of the form* $p(J)$ *(see Notation 2.2). In particular, the shas never have overlapping lines.*
- II *The dual graph* $Graph(X)$ *of* X *is a rooted tree where the rooted vertex is the unique surface containing the line* l_A*.*
- III *All the components of* X *are a blow up of* $\mathbb{P}^2$ *at* $k \geq 0$ *points. In particular, the stable replacement of any sha parametrized by* $R_{\vec{w}}$ *is obtained by blowing up isolated points. That is, we never have to blow down a* (-1)*-curve.*

Proof. Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and consider a sequence of weights $\vec{\gamma}_1, \dots, \vec{\gamma}_m$ such that $\vec{\gamma}_1 := \vec{w}_0$, $\vec{\gamma}_m := \vec{w}$, the weights $\vec{\gamma}_i \leq \vec{\gamma}_{i+1}$ are adjacent to each other (see Definition 4.2), and m is the minimal length of such sequences. We prove our proposition by induction on m. The case $m = 1$ follows from Lemma 4.4. In that case, the dual graph for every pair is a point.

We suppose the statement holds for $m-1$. Let $\vec{\gamma}_m := \vec{w}$ and let $\vec{\gamma}_{m-1} := \vec{v}$ be two adjacent weights separated by the wall $W(I)$. We highlight that walls of type $\widetilde{W}(K)$ in $\mathcal{D}(3, n+1)$ do not modify neither $R_{\vec{v}}$ nor the shas parametrized by it because the space $R_{\vec{v}}$ only parametrizes shas with isolated multiple points by our inductive hypothesis. By case (2) in Theorem 2.5, there is a contraction

$$\pi_m : \overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1).$$

Let (X', D') be an arbitrary sha with at least one $p(I)$ singularity and parametrized by a point $z \in R_{\vec{v}}$. We will show that any shas (X', D') parametrized by $\pi_m^{-1}(z)$ have only multiple point singularities.

By Subsection 3.6, the fibers of π_m parametrize a new sha (X, D) containing a new $\mathbb{P}^2$ component with the lines $\{l_{i_1} \mid i_k \in I\}$. Therefore, the fiber of π_m over the point parametrizing (X', D') is the moduli associated to the pairs $(\mathbb{P}^2, l_{i_1} + \ldots + l_{i_k})$ that satisfy the following conditions:

(1) The lines cannot all overlap in an $|I|$-tuple point, because this is precisely the singularity we destabilized.
(2) The pair can have any singularity of the form $p(J) := \cap_{i_k \in J} l_{i_k}$ with J properly contained in I, because we are only destabilizing one type of singularitiy. We must cross more walls to destabilize $p(J)$.
(3) Let H_0 be the hyperplane obtained by intersecting the new $\mathbb{P}^2$ with the other components of $\tilde{X}$. Then the lines l_{i_s} cannot overlap with H_0.
(4) The equivalence class induced by the intersection of the lines l_{i_s} with the gluing locus is fixed because the sha (X', D') is fixed.

These are precisely the same conditions used in the proof of Lemma 4.4 with the gluing locus playing the role of l_A. Therefore, every positive dimensional fiber of π_m is isomorphic to $\mathbb{P}^{(|I|-3)}$. The new shas (X, D) have at worst mulitple point singularities, because the lines $\{l_{i_1} \mid i_k \in I\}$ cannot overlap in the new component $\mathbb{P}^2 \subset X$ by the fourth condition above. The singularities of (X, D) away from this $\mathbb{P}^2$ are also multiple points by our hypothesis on the singularities of (X', D').

Part (II) follows from the previous argument because the wall crossing between two adjacent weights $\vec{v}$ and $\vec{u}$ adds a new vertex to *Graph*(X') corresponding to the new $\mathbb{P}^2$. The multiple points never occur in l_A, and so l_A is always contained in a single surface which will be our root.

Finally, we prove Part (III). In the absence of overlapping lines, as in our case, [Ale13, Thm 5.7.2 (ii)] states that a $\mathbb{P}^1 \times \mathbb{P}^1$ component is only obtained from a configuration of points with the following characteristics:

(1) Given a $\mathbb{P}^2$-component with lines $\{l_i\}$, there are exactly two non-log-canonical points in the configuration of those lines.
(2) The line l_k between the two-non log canonical points have weight 1.
(3) There is not an additional line l_s or a component of the double locus intersecting l_k transversally.

Under the above conditions, one must blow up the two points and contract the strict transform of the line between them (see [Ale13, Figure 5.8]).

To clarify this last condition, the reader should compare the following shas from [Ale13, Fig 5.12]. In sha #3, line l_3 intersects l_4 and prevents a line from being contracted in the $\mathrm{Bl}_2\mathbb{P}^2$ component, so that we do *not* obtain a $\mathbb{P}^1 \times \mathbb{P}^1$. In contrast, in sha #8, there does not exist a similar line intersecting l_1, in which case the sha has a $\mathbb{P}^1 \times \mathbb{P}^1$ as the corresponding component.

In particular, condition (3) will never happen in our case, as we always have either the double locus or the line l_A intersecting the line l_k transversally. ☐

The following result will be important for proving that $R_{\vec{w}}$ is smooth.

Lemma 4.9. *Let $\vec{v} \geq \vec{u}$ be adjacent weights in $\mathcal{D}_n^R$ separated by the wall $W(I)$. Let*

$$\pi_{\vec{v},\vec{u}} : \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1)$$

be the associated wall crossing morphism. Then its restriction $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ has (scheme-theoretic) fibers equal to $\mathbb{P}^{(|I|-3)}$.

The morphism $\phi_{\vec{v},\vec{u}}$ has positive dimensional fibers over the loci parametrizing shas that become unstable with respect to the weights $\vec{v}$. In our case, those are the shas with a isolated multiple point $p(I)$ and its fibers are described in the proof of Proposition 4.8. We now prove this scheme-theoretically.

Proof of Lemma 4.9. Let $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ be the wall crossing morphism where $\vec{v} \geq \vec{u}$, let A be the spectrum of an Artinian ring, and let $\psi : A \to R_{\vec{v}}$ be a deformation of $R_{\vec{v}}$. Furthermore, suppose that the total space of the composition $\phi_{\vec{v},\vec{u}} \circ \psi : A \to R_{\vec{u}}$ is constant. We wish to show, by contradiction, that this forces the total space of $\psi : A \to R_{\vec{v}}$ to be the trivial deformation as well.

We may assume that the total space of $\phi_{\vec{v},\vec{u}} \circ \psi : A \to R_{\vec{u}}$ is the trivial deformation of a pair (X, D) where (X, D) is stable with respect to the weights $\vec{u}$ but unstable with respect to $\vec{v}$. Indeed, if (X, D) was stable with respect to both weights, then the morphism $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ is an isomorphism on this locus, and there is nothing to prove.

In particular, there exists $D' \subset D$ such that $D' = \cup_{i \in I} L_i$ with $\sum_{i \in I} u_i \leq 2$ and $\sum_{i \in I} v_i > 2$. Then the definition of $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ implies that the preimage of the sha (X, D) is $(Y, D_Y + Z)$, where $Y = X' \cup \mathbb{P}^2$ with $X' = \mathrm{Bl}_{p(I)} X$. Recall that $p(I)$ denotes the point we are required to blowup, as there are too many weighted lines passing through that point with respect to $\vec{v}$.

If we denote the gluing locus by $Z_1 \subset X'$ and $Z_2 \subset \mathbb{P}^2$, then it suffices to show that the deformation restricted to the three pairs, $(X', Z_1), (\mathbb{P}^2, Z_2)$, and $Z = Z_1 \cong Z_2$ (the gluing locus $X' \cap \mathbb{P}^2$) is trivial. Indeed, we first note that $(\mathbb{P}^2, Z_2)$ is rigid. Furthermore, the deformation restricted to (X', Z_1) is trivial, as the pair (X', Z_1) is uniquely determined by (X, D), which is assumed to be fixed. In particular, (X', Z_1) is obtained as the blowup of a fixed variety at a fixed point. Therefore, it suffices to show that the deformation is trivial on the gluing locus, Z. To do so, we recall how our construction yields this line Z.

Recall that we are blowing up a point $p(I)$ inside a surface X living inside a total space $\bar{X} := X \times A$. In particular, there is an inclusion of normal bundles

$$N_X := N_{p(I)/X} \subset N_{A/\bar{X}} := N_{\bar{X}},$$

where N_X is also the restriction of $N_{\bar{X}}$ on X. Indeed, we obtain $N_{A/\bar{X}}$ as we are blowing up a $p(I)$ inside each fiber, and an entire family of them, thus blowing up a section $A \subset \bar{X}$. The exceptional divisor of the blowup of $p(I)$ inside $X \subset \bar{X}$, is defined by the projectivization of these normal bundles – indeed, the $\mathbb{P}^2$ arises from the projectivization of $N_{\bar{X}}$, and the gluing locus $Z \cong \mathbb{P}^1$ arises from the projectivization of N_X.

As $\phi_{\vec{v},\vec{u}} \circ \psi$ is assumed to be the trivial deformation, the normal bundles N_X and $N_{\bar{X}}$, as well as the inclusion $N_X \to N_{\bar{X}}$ never change. Now it suffices to note that any non-trivial deformation of Z, when composed with the wall crossing $\phi_{\vec{v},\vec{u}}$, would change the inclusion $N_X \to N_{\bar{X}}$, thus contradicting the fact that $\phi_{\vec{v},\vec{u}} \circ \psi$ is a trivial deformation.

Therefore, the moduli is determined by the moduli of the lines $\sum_{i \in I} L_i + Z$ inside $\mathbb{P}^2$, such that $\sum_{i \in I} L_i = 2 + \epsilon$ and $L_I \cap Z$ is a fixed point of $M_{0,n}$, which is $\mathbb{P}^{|I|-3}$ by Lemma 4.4. □

5. Construction of $R_{\vec{w}}$ via wonderful compactifications

As in the previous section, the results of this section do not depend on the q used in the definition of $R_{\vec{w}}(q)$,as long as it is a generic point of $M_{0,\vec{w}_0}$. We simplify our notation and just write $R_{\vec{w}}$.

Recall in Notation 4.5 we showed that the equivalence class of the n lines parametrized by $[s_1 : \ldots : s_{n-2}] \in R_{\vec{w}_0}$ is induced by the lines

$$l_i := (x_1 - a_i x_2 + s_i x_0 = 0), \qquad l_{n-2} := (s_{n-2} x_0 + x_1 = 0), \qquad l_{n-1} := (x_2 = 0),$$
$$l_A := (x_0 = 0), \qquad l_n := (x_1 - x_2 = 0).$$

Therefore, the point $[1 : 0 : \ldots : 0] \in R_{\vec{w}_0}$ parametrizes a pair with an $(n-1)$-tuple point at $[1 : 0 : 0] \in \mathbb{P}^2$ induced by the intersection of the lines $l_2, \ldots, l_n$. Similarly, the hyperplane $(s_1 = 0) \subset R_{\vec{w}_0}$ parametrizes a pair with a triple point at $[1 : 0 : 0]$.

We now show that this behavior holds in general.

Lemma 5.1. *For every $I \subset \{1, \dots, n\}$, there is a linear subspace $\mathbb{P}^{(n-|I|-1)} \cong H(I) \subset R_{\vec{w}_0}$ that generically parametrizes a configuration with an $|I|$-tuple point $p(I)$ given by the intersection of the lines $\{l_i \mid i \in I\}$.*

Proof. A set of lines $\{l_i \mid i \in I\}$ has an $|I|$-multiple point if and only if their dual points $\{y_i \mid i \in I\}$ are collinear. Taking any subset of three of these points, the associated matrix $[y_j, y_k, y_l]$ has determinant equal to zero. In particular, these equations are linear on the variables s_i and define $H(I)$. Finally, the dimension count is $(n-3)-(|I|-2) = n-|I|-1$. □

Example 5.2. We use the equation of the lines as given in Notation 4.5. For example associated to the points $y_1 = [s_1, 1, -a_1]$, $y_2 = [s_1, 1, -a_2]$, and $y_3 = [s_1, 1, -a_3]$, we have the equation

$$s_1(a_2 - a_3) - s_2(a_1 - a_3) + s_3(a_1 - a_2) = 0.$$

The sets $H(I)$ will generate the centers of the morphism $R_{\vec{1}^n} \to R_{\vec{w}_0}$. These morphisms are induced by changing the weights, and the description of these linear subspaces will be crucial for the next subsection.

5.3. **Wonderful compactifications.** In what follows, we review the pertinent definitions of *wonderful compactifications* following [Li09]. We note that the theory of wonderful compactifications originiated in [DCP95].

Definition 5.4. *An **arrangement** of subvarieties of a nonsingular variety Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties $S_i \subset Y$ closed under scheme-theoretic intersection. Given* $\dim(Y) = (n-3)$*, we say that a finite collection of k nonsingular subvarieties $S_1, ..., S_k$ intersect **transversely**, if either $k = 1$ or for any $y \in Y$ the following conditions holds (see* [Li09, Sec 5.1.2]*)*

(a) *there exist a system of local parameters $x_1, ..., x_{(n-3)}$ on Y at y that are regular on an affine neighborhood U of y such that y is defined by the maximal ideal $(x_1, ..., x_{(n-3)})$ as well as*
(b) *integers $0 = r_0 \le r_1 \le ... \le r_k \le (n-3)$ such that the subvariety S_i is defined by the ideal*
$$(x_{r_{i-1}+1}, x_{r_{i-1}+2}, ..., x_{r_i})$$
for all $1 \le i \le k$

If $r_{i-1} = r_i$ then the ideal is assumed to be the ideal containing units, which means geometrically that the restriction of S_i to U is empty.

Definition 5.5. *A subset $\mathcal{G} \subset \mathcal{S}$ is called a **building set** of $\mathcal{S}$ if for all $S_k \in \mathcal{S}$, the minimal elements of $\mathcal{G}$ containing S_k intersect transversally and their intersection is equal to S_k (by convention, the condition is satisfied if $S_k \in \mathcal{G}$). These minimal elements are called the **$\mathcal{G}$-factors** of S_k. Let $\mathcal{G}$ be a building set and set $Y^o := Y \setminus \bigcup_{S_k \in \mathcal{G}} S_k$. The closure of the image of the natural locally closed embedding (*[Li09, Def 1.1]*)*

$$Y^o \hookrightarrow \prod_{S_k \in \mathcal{G}} Bl_{S_k} Y$$

*is called the **wonderful compactification** of Y with respect to $\mathcal{G}$.*

Theorem 5.6. [Li09, Theorem 1.3] *Let $\mathcal{G}$ be a building set and let $Bl_{\mathcal{G}}Y$ be the wonderfuld compactification of Y with respect to $\mathcal{G}$. Then $Bl_{\mathcal{G}}Y$ is smooth with normal crossing boundary, and that for each $S_k \in \mathcal{G}$ there is a nonsingular divisor $D_{S_k} \subset Y_{\mathcal{G}}$. Moreover, the union of the divisors is $Y_{\mathcal{G}} \setminus Y^o$, and any set of these divisors, with nonempty interesction, meet transversally.*

Example 5.7. A building set $\mathcal{H}$ in $R_{\vec{w}_0}$ is given by 5 points $H(J)$ with $|J| = 4$ and 10 lines $H(I)$ with $|I| = 3$ parametrizing configurations with either a quadruple or a triple point respectively. The arrangement $\mathcal{S}$ is the set of all possible intersections among them. The 10 lines, which are not in general position, intersect along 20 points given by:

(1) The point $H(I) \cap H(J)$ with $|I \cap J| = 2$ parametrizes the quadruple point $p(I \cup J)$.
(2) The point $H(I) \cap H(J)$ with $|I \cap J| = 1$ parametrizes a configuration with two triple points associated to I and J. There are 15 of these points.

The above example illustrates the general behavior.

Lemma 5.8. *Let $\mathcal{S}_{\vec{w}}$ be the set of all possible intersections of collections of subvarieties from*

$$\mathcal{H}_{\vec{w}} = \{H(J) \mid \sum_{i \in J} w_i > 2,\ |J| \subset \{1, \dots, n\}\}.$$

Then, $\mathcal{S}_{\vec{w}}$ is an arrangement and $\mathcal{H}_{\vec{w}}$ is a building set.

Proof. $\mathcal{S}_{\vec{w}}$ is an arragement by Definition 5.4. For the last statement, let S_k be an arbitrary element of $\mathcal{S}_{\vec{w}}$. By definition, S_k is an arbitrary nonempty intersection $S_k := H(I_1) \cap \cdots \cap H(I_m)$. We need to prove two conditions: (I) that the minimal elements of $\mathcal{H}_{\vec{w}}$ that contain S_k intersect transversally, and (II) that their intersection is equal to S_k.

For (I), we first observe that any S_k can be written uniquely as an intersection of the form $H(J_1) \cap \cdots \cap H(J_s)$, where $|J_i \cap J_k| \leq 1$ and each of the J_i is a union of I_j. Indeed, if $|I_1 \cap I_2| \geq 2$ and $I_1 \cap I_2 \neq \{1, \dots, n\}$, then their intersection must parametrize an $(|I_1| + |I_2|)$-tuple point. This implies that $H(I_1) \cap H(I_2)$ is either the empty set or $H(I_1 \cup I_2) \in \mathcal{H}_{\vec{w}}$. In the latter case, we can dismiss $H(I_1)$ and $H(I_2)$ while keeping $H(I_1) \cap H(I_2)$. Iterating this process, we can find all the minimal elements $J_i \in \mathcal{H}_{\vec{w}}$ containing S_k.

Part (I) now reduces to showing that the intersection of the linear subspaces $\mathbb{P}^{(n-|J_i|-1)}$, $1 \leq i \leq s$, along S_k is transversal. By Definition 5.4, it is enough to exhibit numbers $0 = r_0 \leq r_1 \leq \ldots \leq r_s \leq (n-3)$ that satisfy the conditions of the aforementioned definition. We can take

$$r_0 := 0, \qquad r_m := \sum_{i=1}^{m} (|J_1| - 2) \qquad \text{with} \qquad 1 \leq m \leq s.$$

Indeed, $r_s \leq (n-3)$ because

$$0 \leq \dim(S_k) = (n-3) - \sum_{i=1}^{s} (|J_1| - 2)$$

since S_k is non-empty. We can take the linear subspace $H(J_m) = \mathbb{P}^{n-|J_m|-1}$ to be defined by the ideal

$$\left(x_{(r_{m-1}+1)}, \dots, x_{r_m} \right),$$

because counting its number of generators, we obtain

$$r_m - (r_{m-1}+1) + 1 = \left(\sum_{i=1}^{i=m}(|J_1|-2)\right) - \left(\sum_{i=1}^{i=m-1}(|J_1|-2)\right) = (|J_m|-2)$$

which is the codimension of $H(J_m)$.

Finally, as we are intersecting linear subspaces in projective space condition (II) follows by the definition of the $H(J_i)$. □

Definition 5.9. *Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and let $\mathcal{H}_{\vec{w}}$ be as in Lemma 5.8. Then the **wonderful compactification** of $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ with respect to $\mathcal{H}_{\vec{w}}$ is denoted by $Bl_{\vec{w}}R_{\vec{w}_0}$.*

Lemma 5.10. *Let $\vec{w}$ be an admissible weight vector in $\mathcal{D}_n^R$. There exists a smooth variety $\mathcal{U}_{\vec{w}}$, a flat proper morphism $\phi_{\vec{w}}$,*

$$\begin{array}{ccc} \mathcal{U}_{\vec{w}} & \xrightarrow{\tau} & \mathcal{U}_{\vec{w}_0} \\ {\scriptstyle \phi_{\vec{w}}}\downarrow & & \downarrow{\scriptstyle \phi_{\vec{w}_0}} \\ Bl_{\vec{w}}R_{\vec{w}_0} & \longrightarrow & R_{\vec{w}_0} \end{array}$$

and n hypersurfaces $\sigma_i(\vec{w}) \subset \mathcal{U}_{\vec{w}}$ such that for every $\vec{s} \in Bl_{\vec{w}}R_{\vec{w}_0}$ the fiber $\phi_{\vec{w}}^{-1}(\vec{s})$ and the divisors

$$\phi_{\vec{w}}^{-1}(\vec{s}) \cap \sigma_i(\vec{w}) \qquad\qquad l_A := \phi_{\vec{w}}^{-1}(\vec{s}) \cap \tau^{-1}(E_{\vec{w}_0})$$

define a stable sha of weight $\vec{w}$ ($E_{\vec{w}_0}$ is defined in Lemma 4.6).

Proof. Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and consider a sequence of weights $\vec{\gamma}_1, \dots, \vec{\gamma}_{m+1}$ such that $\vec{\gamma}_1 := \vec{w}_0$, $\vec{\gamma}_{m+1} := \vec{w}$, the weights $\vec{\gamma}_i \leq \vec{\gamma}_{i+1}$ are adjacent to each other (see Definition 4.2), and $m+1$ is the minimal length of such sequences. We prove our Lemma by induction. The base case is proven in Lemma 4.6.

Next, we describe the inductive step. We suppose that the statement holds for γ_m. In particular, there exists a smooth variety $\mathcal{U}_{\vec{\gamma_m}}$ with a flat proper morphism $\phi_{\vec{\gamma_m}} : \mathcal{U}_{\vec{\gamma_m}} \to Bl_{\vec{\gamma_m}}R_{\vec{w}_0}$, and n hypersurfaces $\sigma_i(\vec{\gamma_m}) \subset \mathcal{U}_{\vec{\gamma_m}}$ such that for every $\vec{s} \in Bl_{\vec{\gamma_m}}R_{\vec{w}_0}$ the fiber $\phi_{\vec{\gamma_m}}^{-1}(\vec{s})$ and the divisors $\phi_{\vec{\gamma_m}}^{-1}(\vec{s}) \cap \sigma_i(\vec{\gamma_m})$ and $l_A := \phi_{\vec{\gamma_m}}^{-1}(\vec{s}) \cap \tau^{-1}(E_{\vec{w}_0})$ define a stable sha of weight γ_m.

Let $W(I)$ be the wall separating $\vec{\gamma}_m$ and $\vec{\gamma}_{m+1} = \vec{w}$, we denote the singularity destabilized by this wall crossing by $p(I)$.

Let $\overline{H}(I)$ be the closure of the locus in $Bl_{\vec{\gamma}_m}R_{\vec{w}_0}$ parametrizing all shas (X, D) with a multiple point $p(I)$, and let $S(I) \subset \mathcal{U}_{\gamma_m}$ be the locus supporting $p(I)$. We will show that the following diagram

$$\begin{array}{ccccc} \mathcal{U}_{\vec{w}} := Bl_{\eta^{-1}(S(I))}\left(Bl_{\vec{w}}R_{\vec{w}_0} \times_{\left(Bl_{\vec{\gamma}_m}R_{\vec{w}_0}\right)} \mathcal{U}_{\vec{\gamma_m}}\right) & \longrightarrow & Bl_{\vec{w}}R_{\vec{w}_0} \times_{\left(Bl_{\vec{\gamma}_m}R_{\vec{w}_0}\right)} \mathcal{U}_{\vec{\gamma}_m} & \xrightarrow{\eta} & \mathcal{U}_{\vec{\gamma}_m} \\ & {\scriptstyle \phi_{\vec{w}}}\searrow & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \gamma_{\vec{\gamma}_m}} \\ & & Bl_{\vec{w}}R_{\vec{w}_0} & \xrightarrow{\rho} & Bl_{\vec{\gamma}_m}R_{\vec{w}_0} \end{array}$$

yields our family $\phi_{\vec{w}} : \mathcal{U}_{\vec{w}} \to Bl_{\vec{w}}R_{\vec{w}_0}$.

Notice that $S(I) \cong \overline{H}(I)$ because the projection $S(I) \to \overline{H}(I)$ is finite, generically one-to-one, and $\overline{H}(I)$ is the smooth strict transform of $H(I) \subset R_{\vec{w}_0}$ in $Bl_{\vec{\gamma}_m} R_{\vec{w}_0}$. Therefore, the isomorphism $S(I) \cong \overline{H}(I)$ follows by Zariski's main theorem.

By definition of the wonderful blow up, we have that

$$Bl_{\vec{w}} R_{\vec{w}_0} = Bl_{\overline{H}(I)} \left(Bl_{\vec{\gamma}_m} R_{\vec{w}_0}\right).$$

On another hand, by the inductive hypothesis, $\phi_{\vec{\gamma}_m} : \mathcal{U}_{\vec{\gamma}_m} \to Bl_{\vec{\gamma}_m} R_{\vec{w}_0}$ is flat. Since blowing up commutes with flat base change, we obtain

$$Bl_{\vec{w}} R_{\vec{w}_0} \times_{\left(Bl_{\vec{\gamma}_m} R_{\vec{w}_0}\right)} \mathcal{U}_{\vec{\gamma}_m} \cong Bl_{\phi^{-1}_{\vec{\gamma}_m}(\overline{H}(I))} \mathcal{U}_{\vec{\gamma}_m} \tag{5.10.1}$$

which implies

$$Bl_{\eta^{-1}(S(I))} \left(Bl_{\vec{w}} R_{\vec{w}_0} \times_{\left(Bl_{\vec{\gamma}_m} R_{\vec{w}_0}\right)} \mathcal{U}_{\vec{\gamma}_m}\right) = Bl_{\eta^{-1}(S(I))} \left(Bl_{\phi^{-1}_{\vec{\gamma}_m}(\overline{H}(I))} \mathcal{U}_{\vec{\gamma}_m}\right).$$

Let E_ρ and E_η be the exceptional divisors of ρ and η respectively. Next, we describe the fiber $\tilde{\pi}^{-1}(z)$ for $z \in E_\rho$. Given $y \in \overline{H}(I)$, the fiber $\phi^{-1}_{\vec{\gamma}_m}(y)$ is a surface X.

We find, by dimension counting, that $\rho^{-1}(y) \cong \mathbb{P}^{(|I|-3)}$, and $\eta^{-1}(\phi^{-1}_{\vec{\gamma}_m}(y))$ is a $\mathbb{P}^{(|I|-3)}$-fibration over X. Due to the fiber product construction, there is a morphism $\eta^{-1}(\phi^{-1}_{\vec{\gamma}_m}(z)) \to \rho^{-1}(z)$. So $\eta^{-1}(\phi^{-1}_{\vec{\gamma}_m}(z))$ is a fibration over $\mathbb{P}^{(|I|-3)}$ with fibers isomorphic to X.

Therefore, for all $z \in E_\rho$ it holds that $\tilde{\pi}^{-1}(z) \cong X$, and the strict transform

$$\{\eta_*^{-1}(\sigma_i(\vec{\gamma}_m)) \mid i \in I\}$$

of the sections $\{\sigma_i(\vec{\gamma}_m) \mid i \in I\}$ induces a divisor in $\tilde{\pi}^{-1}(z)$ with an $(n-1)$ multiple point. Blowing up $\eta^{-1}(S(I))$ generically separates those sections in $\mathcal{U}_{\vec{w}}$, because the intersection of the hypersurfaces $\{\eta_*^{-1}(\sigma_i(\vec{\gamma}_m)) \mid i \in I\}$ is locally an intersection of $|I|$ hyperplanes in affine space. Indeed, recall our sections are the strict transforms of $\sigma_i \subset R_{\vec{w}_0}$ and that $\mathcal{U}_{\vec{w}_0} \cong Bl_Z\mathbb{P}^{n-1}$ with $Z \cong \mathbb{P}^1$ and $Z \cap \sigma_i = \emptyset$.

Finally, we describe the fibers of $\phi_{\vec{w}}$. The locus $\eta^{-1}(q_I) \cong \mathbb{P}^{|I|-3}$ intersects $\tilde{\pi}(z) \cong X$ at the point x supporting the multiple point $q(I)$. The locus $S(I) \subset \mathcal{U}_{\vec{\gamma}_m}$ has dimension $(n-|I|-1)$. Therefore, $\dim(\eta^{-1}(S(I)) = (n-4)$ which implies the divisor of the blow up

$$\mathcal{U}_{\vec{w}} \to Bl_{\phi^{-1}_{\vec{\gamma}_m}(\overline{H}(I))} \left(\mathcal{U}_{\vec{\gamma}_m}\right)$$

is a $\mathbb{P}^2$-fibration over $\eta^{-1}(S(I))$. So, $\phi^{-1}_{\vec{w}}(z)$ is equal to

$$\mathbb{P}^2 \bigcup_{L=E} Bl_x\left(\tilde{\pi}^{-1}(z)\right) \cong \mathbb{P}^2 \bigcup_{L=E} Bl_x X. \tag{5.10.2}$$

where $E \subset Bl_x X$ is the exceptional divisor obtained by blowing up x and L is a line in $\mathbb{P}^2$.

The $\mathbb{P}^2$ component is a fiber of $\mathcal{U}_{\vec{w}} \to Bl_{\phi^{-1}_{\vec{\gamma}_m}(p_I)} \mathcal{U}_{\vec{\gamma}_m}$, so the strict transforms of the sections $\{\sigma_i(\vec{\gamma}_m) | \, i \in I\}$ define a configuration of lines on it. Those lines do not overlap in a $|I|$-tuple point, because that is the multiple point we just separated. Therefore, the resultant pair defined by the surface in 5.10.2, and its intersection with the strict transform $\sigma_i(\vec{w})$ of the hypersurfaces $\sigma_i(\vec{\gamma}_m)$ in $\mathcal{U}_{\vec{w}}$ defines a stable sha with respect to $\vec{w}$. $\square$

In the following Lemma, we recall that n points in $\mathbb{P}^{n-3}$ are in general position if there are no two of them supported in a point, no three of them contained on a line, no four of them contained in a plane, and so forth.

Lemma 5.11. *For $n \geq 5$, there are n points $q_1, \ldots, q_n$ in $R_{\vec{w}_0}$ in general position, a sequence of weights $\vec{w}_k$ with $1 \leq k \leq (n-3)$, and morphisms of smooth varieties*

$$Bl_{\vec{w}_{(n-3)}} R_{\vec{w}_0} \longrightarrow \ldots \longrightarrow Bl_{\vec{w}_k} R_{\vec{w}_0} \longrightarrow \ldots \longrightarrow R_{\vec{w}_0}$$

where

- *$Bl_{\vec{w}_1} R_{\vec{w}_0}$ is the blow up of $R_{\vec{w}_0}$ along $q_1, \ldots, q_n$ in any order.*
- *$Bl_{\vec{w}_2} R_{\vec{w}_0}$ is the blow up of $Bl_{\vec{w}_1} R_{\vec{w}_0}$ along the strict transform of lines spanned by all pairs of points $\{q_i, q_j\}$, in any order*

$$\vdots$$

- *$Bl_{\vec{w}_{(n-3)}} R_{\vec{w}_0}$ is the blow up of $Bl_{\vec{w}_{(n-4)}} R_{\vec{w}_0}$ along the strict transforms of the $(n-4)$-planes spanned by all $(n-3)$-tuples of the q_i, $i = 1, \ldots, n$ in any order.*

Proof. The wonderful blowup is by definition a sequence of iterative blow ups along the strict transforms of the elements in the building set $\mathcal{H}_{1^n}$. The points q_i correspond to $H(I)$ with $|I| = (n-1)$, the lines spanned by the points q_i correspond to $H(J)$ with $|J| = (n-2)$, and so on. The order of the blow-ups can be taken to be any order of increasing dimension by [Li09, Thm 1.3]. □

5.12. $R_{\vec{w}}$ **is isomorphic to a wonderful compactification.** Our aim is to show that $R_{\vec{w}}$ is isomorphic to the wonderful compactification $Bl_{\vec{w}}\mathbb{P}^{n-3}$. First we review $R_{\vec{w}}$ for small values of n.

Example 5.13.

(1) If $\vec{w} \in \mathcal{D}_3^R$, then $R_{\vec{w}}$ is a point.
(2) If $\vec{w} \in \mathcal{D}_4^R$, then $R_{\vec{w}} \cong \mathbb{P}^1$, as $\overline{M}_{1^6}(\mathbb{P}^2, 5) \cong \overline{M}_{0,5}$.
(3) If $\vec{w} \in \mathcal{D}_5^R$, then $R_{\vec{w}} \cong Bl_{\vec{w}}\mathbb{P}^2$. In particular, the morphism $R_{1^5} \to R_{\vec{w}_0} \cong \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at five points and the morphisms induced by wall crossings inside $\mathcal{D}_5^R$ are either smooth blow ups or isomorphisms. Indeed, it is known that $\overline{M}_{1^6}(\mathbb{P}^2, 6)$ has isolated singularities (see [Lux08, Thm 4.2.4]). Therefore, by the construction of $R_{\vec{w}}$ as in Definition 3.1 it follows that R_{1^5} is smooth. We note that the building set $\mathcal{H}_{1^5}$ is described in Example 5.7, and that the smoothness of $R_{\vec{w}}$ follows from the smoothness of R_{1^5} and Theorem 5.16.

Theorem 5.14. *For any choice of n and $\vec{w} \in \mathcal{D}_n^R$, it holds that $R_{\vec{w}} \cong Bl_{\vec{w}} R_{\vec{w}_0}$, and thus $R_{\vec{w}}$ is smooth with normal crossings boundary.*

Proof. Our proof is by induction on the weight vector. The base case is $R_{\vec{w}_0}$ which is discussed in Lemmas 4.4 and 4.6. Let $\vec{v} \geq \vec{u}$ be two adjacent weights separated by the wall $W(I)$ which destabilizes the multiple point $p(I)$. Now consider the following diagram

$$\begin{array}{ccc} Bl_{\vec{v}}R_{\vec{w}_0} & \xrightarrow{f_{\vec{v}}} & R_{\vec{v}} \\ \downarrow{\scriptstyle \psi_{\vec{v},\vec{u}}} & & \downarrow{\scriptstyle \phi_{\vec{v},\vec{u}}} \\ Bl_{\vec{u}}R_{\vec{w}_0} & \xrightarrow{\cong} & R_{\vec{u}} \end{array}$$

where the morphism $\psi_{\vec{v},\vec{u}}$ is the blowup

$$Bl_{\vec{v}}R_{\vec{w}_0} \cong Bl_{\overline{H}(I)}\left(Bl_{\vec{u}}R_{\vec{w}_0}\right) \to Bl_{\vec{u}}R_{\vec{w}_0}$$

induced by the wonderful compactification, and $\phi_{\vec{v},\vec{u}}$ is the wall crossing morphism induced by changing the weights. By induction, we assume that $Bl_{\vec{u}}R_{\vec{w}_0} \cong R_{\vec{u}}$ and thus $R_{\vec{u}}$ is smooth. We must now show that $R_{\vec{v}}$ is also smooth.

By Lemma 5.10, there is a flat family $\mathcal{U}_{\vec{v}} \to Bl_{\vec{v}}R_{\vec{w}_0}$ whose fibers are stable shas with respect to $\vec{v}$. On the other hand, $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ is a fine moduli space by [Ale08, Lemma 7.7]. Therefore, there is a map $f_{\vec{v}} : Bl_{\vec{v}}R_{\vec{w}_0} \to R_{\vec{v}}$. Let $E_{\vec{v}} \subset Bl_{\vec{v}R_{\vec{w}_0}}$ and $F_{\vec{v}} \subset R_{\vec{v}}$ be the exceptional divisors of $\phi_{\vec{v},\vec{u}}$ and $\varphi_{\vec{v},\vec{u}}$ respectively. By construction $f_{\vec{v}}$ is an isomorphism when restricted to the open sets

$$(Bl_{\vec{v}}R_{\vec{w}_0}) \setminus E_{\vec{v}} \to R_{\vec{v}} \setminus F_{\vec{v}}$$

and the restriction $f_{\vec{v}} : E_{\vec{v}} \to F_{\vec{v}}$ is a finite morphism because both exceptional divisors are $\mathbb{P}^{|I|-3}$ fibrations over $\overline{H}(I)$.

In particular, the above argument implies the morphism $f_{\vec{v}}$ is the normalization. Therefore, since $Bl_{\vec{v}}R_{\vec{w}_0}$ is smooth, by Zariski's main theorem, it suffices to show that $R_{\vec{v}}$ is normal. To do so, we consider the exact sequence arising from normalization:

$$0 \to \mathcal{O}_{R_{\vec{v}}} \to f_*\mathcal{O}_{Bl_{\vec{v}}R_{\vec{w}_0}} \to \delta \to 0.$$

Our goal is to prove that $\delta = 0$. If $p \in R_{\vec{u}}$ is a point parametrizing a configuration which is stable with respect to both weights $\vec{v}$ and $\vec{u}$, then the morphisms $\psi_{\vec{v},\vec{u}}$ and $\phi_{\vec{v},\vec{u}}$ are both isomorphisms, and there is nothing to prove. Therefore, we may assume that p is a point which induces a blowup.

To look at the fiber over the point p we tensor by $\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}$ to obtain:

$$\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}} \to (f_*\mathcal{O}_{Bl_{\vec{v}}\mathbb{P}^{n-3}}) \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) \to \delta \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) \to 0.$$

The wonderful compactification is a sequence of iterative smooth blows, so by dimension counting the fiber of $\psi_{\vec{v},\vec{u}}$ over p is a $\mathbb{P}^{|I|-3}$. Furthermore, by Lemma 4.9, the fiber of $\phi_{\vec{v},\vec{u}}$ over p is also a $\mathbb{P}^{|I|-3}$. As $f_{\vec{v}}$ is the normalization, and both $\phi^{-1}_{\vec{v},\vec{u}}(p)$ and $\psi^{-1}_{\vec{v},\vec{u}}(p)$ are scheme theoretically $\mathbb{P}^{|I|-3}$, the projective spaces must be isomorphic. As the first arrow above is an isomorphism, we see that

$$\delta \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) = 0.$$

As this is true for all $p \in R_{\vec{v}}$, we see that $\delta = 0$, and thus $R_{\vec{v}}$ is normal. □

5.15. **Consequences of the blow up construction.**

Theorem 5.16. *There is a birational projective morphism $R_{\vec{w}} \to R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ that can be understood as a sequence of smooth blowups. In particular, the morphism $R_{1^n} \to \mathbb{P}^{n-3}$ can be understood as completing the steps descried in Lemma 5.11.*

Proof. The theorem follows from Theorem 5.14. □

Lemma 5.17. [Has03] *Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a set of weights where*

$$\alpha_1 = 1, \qquad \alpha_2 = 1 - \frac{(n-2)}{n-1} + \frac{1}{2(n-1)}, \qquad \alpha_3 = \ldots = \alpha_n = \frac{1}{n-1}.$$

Then $\overline{M}_{0,\vec{\alpha}} = \mathbb{P}^{n-3}$. Let $\delta_I \subset \mathbb{P}^{n-3}$ be the locus parametrizing configurations of n points in $\mathbb{P}^1$ such that $\{p_{i_1} = \ldots = p_{i_s} \mid i_k \in I\}$. Then, for every $\vec{w} > \vec{\alpha}$, it follows that $\overline{M}_{0,\vec{w}}$ is the wonderful compactification of $\mathbb{P}^{n-3}$ with respect to the building set

$$\mathcal{S}_{\vec{w}} := \{\mathbb{P}^{(n-|I|)-2} \cong \delta_I \subset \mathbb{P}^{n-3} \mid \sum_{i\in I} w_i > 1,\ I \subset \{2, \dots, n\},\ 2 \leq |I| \leq (n-2)\}.$$

Proof. The existence of a set of weights $\vec{\alpha}$ such that $\overline{M}_{0,\vec{\alpha}} \cong \mathbb{P}^{n-3}$ is well-known (see [Has03, Sec 6.2]). The condition $\vec{w} > \vec{\alpha}$ guarantees the existence of a morphism $\overline{M}_{0,\vec{w}} \to \overline{M}_{0,\vec{\alpha}}$ (see [Has03, Thm 4.1]). The set $\mathcal{S}_{\vec{w}}$ is the locus in $\mathbb{P}^{n-3}$ that becomes unstable with respect to the weights $\vec{w}$. In particular, the condition $1 \notin I$ is necessary for $\delta_I \subset \mathbb{P}^{n-3}$, otherwise δ_I is unstable with respect to $\vec{\alpha}$. □

Corollary 5.18. *Given a set of weights $\vec{w} = (1, w_2, \dots, w_n)$, there is a morphism $R_{\vec{w}} \to \overline{M}_{0,\vec{w}}$ which can be interpreted as a continuation of a blow up construction $\overline{M}_{0,\vec{w}} \to \mathbb{P}^{n-3}$.*

Proof. The weights of l_A and l_1 are one by construction, then we can define the morphism $\psi : R_{\vec{w}} \to \overline{M}_{0,\vec{w}}$ by intersecting the broken lines $\{l_A, l_2, \dots, l_n\}$ with l_1. That is

$$\left(X, l_A + \sum_{k=1}^{n} w_k l_k\right) \longrightarrow \left(l_1, (l_A + \sum_{k=2}^{n} w_k l_k)\Big|_{l_1}\right).$$

The morphism is well defined by adjunction. Notice that the set $\{H(I) \in \mathcal{H}_{\vec{w}} \mid 1 \in I\}$ is isomorphic to $\mathcal{S}_{\vec{w}}$ as in Lemma 5.17 above. Indeed, for an index set $I \subset \{1, \dots n\}$ such that $1 \in I$, it holds that

$$\sum_{i\in I} w_i > 2 \Longleftrightarrow \sum_{i\in I\setminus 1} w_i > 1.$$

Moreover, $\mathbb{P}^{(n-|I-1|)-2} \cong \delta_{I\setminus 1} \cong H(I) \cong \mathbb{P}^{(n-|I|)-1}$ by Lemma 5.1, and if I and K are indices containing 1, then $\delta_{I\setminus 1} \cap \delta_{K\setminus 1} \neq \emptyset$ if and only if $H(I) \cap H(K) \neq \emptyset$. Finally, we use that $\mathbb{P}^{n-3} \cong R_{\vec{\alpha}} \cong \overline{M}_{0,\vec{\alpha}}$ to identify these sets.

By [Li09, Thm 1.3.ii], the wonderful blowup does not change if we rearrange the elements of $\mathcal{H}_{\vec{w}}$ so that the first k terms form a building set for any $1 \leq k \leq n$. Therefore, by Theorem 5.14, we have

$$R_{\vec{w}} = Bl_{\mathcal{H}_{\vec{w}}}\left(\mathbb{P}^{n-3}\right) = Bl_{\mathcal{H}_{\vec{w}}\setminus\mathcal{S}_{\vec{w}}}\left(Bl_{\mathcal{S}_{\vec{w}}}\mathbb{P}^{n-3}\right) = Bl_{\mathcal{H}_{\vec{w}}\setminus\mathcal{S}_{\vec{w}}}\left(\overline{M}_{0,\vec{w}}\right)$$

where $Bl_{\mathcal{H}_\beta \setminus S_\beta}$ denotes the blow up along the strict transform of the elements in the set $\mathcal{H}_\beta \setminus S_\beta$.

The description in the statement of our result follows by comparing Lemma 5.11 with the blow up construction of $\overline{M}_{0,n}$ outlined in the introduction. □

We now show that there do not exist weights $\vec{w}$ so that $R_{\vec{w}} \cong \overline{M}_{0,n}$.

Proof of Theorem 1.3. Let $\vec{w} \in \mathcal{D}_n^R$, we will show that $\mathcal{H}_{\vec{w}}$ cannot be equal to the locus $S_{\vec{w}}$ required to construct $\overline{M}_{0,n}$ as described in [Has03, Sec 6.2]. If we suppose otherwise, then $\vec{w}$ destabilizes $(n-1)$ points and all the linear subspaces spanned by them while the nth point is stable with respect to $\vec{w}$. In other words, let $H(I_k) \in \mathcal{H}_{\vec{w}}$, where $|I_k| = (n-1)$ for $k = 1, ..., n-1$ and $H(I_n) \notin \mathcal{H}_w$ where $|I_n| = (n-1)$. The existence of $\vec{w}$ is equivalent to the existence of a solution for the following system of inequalities.

$$w_{i_1} + w_{i_2} + w_{i_3} > 2 \qquad \forall \{i_1, i_2, i_3\} \subset I_k \tag{5.18.1}$$

$$\sum_{i \in I_n} w_i \leq 2, \qquad 0 < w_i \leq 1. \tag{5.18.2}$$

The inequality (5.18.1) is associated to destabilizing the $(n-2)$-planes generated by $H(I_k)$ with $1 \leq i \leq n$. The inequality (5.18.2) follows because $H(I_n)$ is stable with respect to $\vec{w}$. Without loss of generality, we set $I_n = \{2, \ldots, n\}$. Since $|I_k| = (n-1)$, there is at least one I_k such that $I_k \cap I_n$ has at least three distinct elements i_1, i_2, i_3 and so the inequality (5.18.1) for these three elements contradicts (5.18.2). □

6. R_{1^n} as a Non-reductive Chow Quotient

In this section, we discuss the proof of Theorem 1.4. An important step of the proof is based on the fact that the dual graphs of the pairs parametrized by $R_{\vec{w}}$ are always rooted trees, with the root vertex corresponding to the component containing the line l_A. To keep track of the lines l_i, we mark the vertices corresponding to the last component containing the broken line l_i.

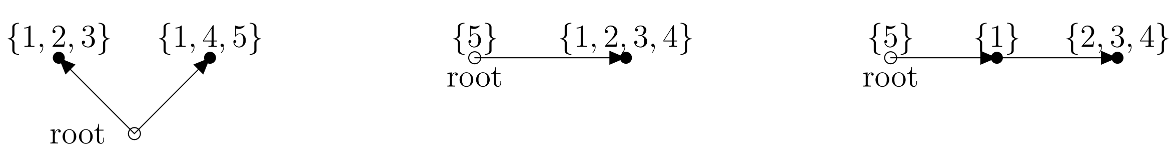

Figure 3. Left to right: Dual graphs associated to the last sha of Fig. 1 and last 2 of Fig. 2 resp.

We highlight that there is a configuration space known as $T_{d,n}$ which generalizes $\overline{M}_{0,n}$ (see [CGK09]), and is a non reductive Chow quotient under the same group [GG]. The objects parametrized by $T_{d,n}$ are known as stable rooted trees, and are the union of surfaces $X \cong Bl_m\mathbb{P}^2$, as in our space, but with markings given by points rather than lines.

Remark 6.1. We recall Kapranov's construction of $\overline{M}_{0,n}$ as a Chow quotient (see [Kap93a]). Given a collection of n generic points p_i in $\mathbb{P}^1$, we consider the cycle associated to the closure of the orbit: $\overline{\mathrm{SL}_3 \cdot (p_1, \ldots, p_n)} \subset (\mathbb{P}^1)^n$. Varying the points, we obtain cycles parametrized by an open locus in the appropriate Chow variety. Taking the closure of this open set, we obtain the Chow quotient $(\mathbb{P}^1)^n /\!/_{Ch} \mathrm{SL}_2$ which is isomorphic to $\overline{M}_{0,n}$.

We fix our line l_A once and for all, and denote by $\hat{\mathbb{P}}^2$ the dual projective space. The lines $\{l_1, \ldots l_n\}$ are parametrized by points $p_1, ..., p_n \in (\hat{\mathbb{P}}^2)^n$. Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be the group acting on $\mathbb{P}^2$ that fixes the line $l_A \subset \mathbb{P}^2$ pointwise. Then $G \cong \mathbb{G}_m \rtimes \mathbb{G}_a^2$, $\dim(G) = 3$, and if $l_A := (x_0 = 0)$, the group consists of elements of the form:

$$G = \begin{pmatrix} t^{-2} & 0 & 0 \\ s_0 & t & 0 \\ s_1 & 0 & t \end{pmatrix}$$

Given a point $p_i = [a_0 : a_1 : a_2] \in \hat{\mathbb{P}}^2$, the line associated to it by projective duality can be written as $l(\vec{x}) := (p_i \cdot x = 0)$. Then we have $l(g \cdot x) = (p_i \cdot g)(x)$ from which we obtain the following action of G on $\hat{\mathbb{P}}^2$.

Definition 6.2. *Let $g \in G$ be as above, then we define the action on $\hat{\mathbb{P}}^2$ as*

$$g \cdot [a_0 : a_1 : a_2] := [t^{-3} a_0 + \frac{s_0}{t} a_1 + \frac{s_1}{t} a_2 : a_2 : a_3]$$

After acting with the group, the line $l(x) = (a_0 x_0 + a_1 x_1 + a_2 x_2 = 0)$ becomes

$$\left(t^{-3} a_0 + \frac{s_0}{t} a_1 + \frac{s_1}{t} a_2 \right) x_0 + a_1 x_1 + a_2 x_2$$

In particular, the intersection point $l(x) \cap (x_0 = 0)$ is invariant under the action of G.

Inside $(\hat{\mathbb{P}}^2)^n$, we define the loci

$$U(q) := \{(p_1, \ldots, p_n) \in (\hat{\mathbb{P}}^2)^n \mid l_i \cap l_A \text{ are fixed with equivalence class } q \in M_{0,n}.\}$$

Notice that $\dim(U_n(q)) = n$. We select once and for all a connected component of the closure of $U(q_n)$ and we denote it, by abuse of notation, as $\overline{U}(q_n)$. In particular, we fix an intersection $\{l_i \cap l_A\}$ once and for all for the rest of this chapter, so we omit it after here and just write $\overline{U}$.

Proposition 6.3. *The Chow quotient $\overline{U} /\!/_{Ch} G$ is birational to R_{1^n}.*

Proof. By shrinking if necessary, we can find an open subet $U' \subset U$ contained in a G-invariant open locus in $(\hat{\mathbb{P}}^2)^n$, so that there is a natural map $\psi : U' \to R_{1^n}$. Furthermore, the G-action fixes the line l_A pointwise, and thus fixes $l_i \cap l_A$. As a result, all configurations in the orbit $G \cdot l_i$ are isomorphic as line arrangements in $\mathbb{P}^2$, and thus are equivalent in R_{1^n}. Therefore, ψ is G-invariant and induces a morphism $\overline{\psi} : U'/G \to R_{1^n}$. This morphism is injective on an open set in R_{1^n}, because if generic $p, p' \in U'$ satisfy $\overline{\psi}(p) = \overline{\psi}(p')$, then there is a $g \in SL(3, \mathbb{C})$ such that $g \cdot p = p'$. This last equality implies g fixes the line l_A as well as all of the intersections $l_i \cap l_A$, and so $g \in G$ and p and p' are in the same G-orbit. The map $\overline{\psi}$ is dominant, because for a generic isomorphic class of lines parametrized by R_{1^n}, we can choose a representative where l_A and $l_i \cap l_A$ are as in the beginning of this section, and that representative is parametrized by U'. □

Next, we show that the birational map $\rho : R_{1^n} \dashrightarrow \overline{U}/\!\!/_{Ch}G$ is a regular morphism. This is done by associating a cycle to each sha X parametrized by $\overline{R}_{1^n}$. We recall that each component X_v of X is either $\mathbb{P}^2$, the blow up of $\mathbb{P}^2$ at finite number of points, or $\mathbb{P}^1 \times \mathbb{P}^1$ (see Proposition 4.8), and that there is a *contraction morphism* $\varphi_v : X \to \mathbb{P}^2$ that contracts X_v to $\mathbb{P}^2$ while also contracting all other components. For each $v \in I$, the contraction morphism induces a line arrangement $\varphi_v(X)$ defined up to choice of coordinates. We always select a representative which, by an abuse of notation, we denote by $\varphi_v(X)$, so $l_A := (x_0 = 0)$ and the points $l_A \cap l_i$ are the same as the ones used to define U.

Definition 6.4. *Fix a closed point of R_{1^n} parametrizing the sha $X = \cup_{v\in I} X_v$. The* ***configuration cycle*** *$Z(X)$ is:*

$$Z(X) := \sum_{v\in I} \overline{G \cdot \varphi_v(X)} \subsetneq (\hat{\mathbb{P}}^2)^n.$$

We must show that these configuration cycles all have the same dimension and homology class. Let $\vec{m} := \{m_1, \ldots, m_n\}$ be a set of integers such that $\sum_{i=1}^n m_i = 3$ and $0 \leq m_i \leq 2$. By the Künneth formula, a basis for the homology in $(\hat{\mathbb{P}}^2)^n$ is $[\hat{\mathbb{P}}^{m_1}] \otimes \cdots \otimes [\hat{\mathbb{P}}^{m_n}]$. Let $\mathbb{L}_{\vec{m}} := L_1 \times \cdots \times L_n$ be a collection of generic linear subspaces $L_i \subseteq \hat{\mathbb{P}}^2$ of codimension m_i. The homology class of the orbit $\overline{G \cdot p}$ is

$$[\overline{G} \cdot p] = \sum_{\vec{m}} c_{\vec{m}} \left([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]\right)$$

where $\left(\overline{G \cdot p}\right) \cdot \mathbb{L}_{\vec{m}}$ is the intersection of the orbit $\overline{G \cdot p}$ with the generic linear subspaces $\mathbb{L}_{\vec{m}}$.

Proposition 6.5. *Let $\vec{m}$ be as above and $X = \cup_{v\in I} X_v$, then the homology class $[Z(X)]$ of the cycle $Z(X)$ is*

$$[Z(X)] := \sum_{\vec{m}=m_1,\ldots,m_n} \left(\sum_{v\in I} \overline{G \cdot \varphi_v(X)} \cdot \mathbb{L}_{\vec{m}} \right) \left([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]\right) \tag{6.5.1}$$

In particular, if X is a generic point of R_{1^n} (i.e. X is supported on a single $\mathbb{P}^2$). Then

$$[Z(X)] = \sum_{\vec{m}} c_{\vec{m}} \left([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]\right)$$

where $c_{\vec{m}}$ is either 0 or 1.

Proof. The result follows verbatim from the analogous [Kap93a, Proposition 2.1.7]. The main idea is as follows: let $p_i \in \hat{\mathbb{P}}^2$ be the points parametrizing the lines l_i in $\varphi_v(X)$. Then, $c_{\vec{m}} = 1$ if and only if there is a unique $g \in G \subset SL(3, \mathbb{C})$ such that $g \cdot p_i \in L_i$ for all $1 \leq i \leq n$; and $c_{\vec{m}}$ is zero if there is no such as $g \in G$. For generic X those are the only cases, so we only have those coefficients. □

It will turn out that we only need to calculate the homology of the cycles associated to the maximal degenerations parametrized by R_{1^n}.

Lemma 6.6. *A closed point $X = \cup_{v\in I} X_v$ in R_{1^n} is* maximally degenerate*, that is it lies on a minimal (i.e., deepest) stratum of the boundary stratification, if and only if the configuration* *of lines $\varphi_v(X_v)$ has exactly three lines l_i with $1 \leq i \leq n$ in general position for every $v \in I$, not including l_A or its image.*

Proof. Recall that the group G is three dimensional. If $\varphi_v(X_v)$ has more than three lines, not including l_A or its image, in general position, then X_v has moduli larger than zero, and it can be degenerated further. $\square$

Proposition 6.7. *If the sha $X \in R_{1^n}$ is maximally degenerated, then the homology class of $Z(X)$ has all coefficients $c_{\vec{m}}$ equal to 1 if and only if for all $m_i \in \vec{m}$ we have that $m_i \neq 2$.*

Proof. First we show the ($\Rightarrow$) direction by proving the contrapositive. Suppose that there is an $m_i \in \vec{m}$ such that $m_i = 2$. Then we claim that for each component X_v of X, we have that $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}'} = 0$. Indeed, $m_i = 2$ implies that there is a generic linear subspace $L_i \in \mathbb{L}_{\vec{m}'}$ such that $L_i \cong \mathbb{P}^0_i \subset \hat{\mathbb{P}}^2$ is a point. By projective duality, we obtain a line $\mathbb{P}^1_i$ in $\mathbb{P}^2$ that has generic intersection with l_A. However, there does not exist a $g \in G$ such that $g \cdot l_i = \mathbb{P}^1_i$, because this would imply that both l_i and the $\mathbb{P}^1_i$ would intersect l_A at the same point. This is impossible given our action of G, because G restricts to the identity in l_A.

Next, we show the ($\Leftarrow$) direction. We divide the set of lines in $\varphi_v(X) \cong \mathbb{P}^2$ into sets $I_i(v)$ and $I_A(v)$, where $I_i(v)$ denotes the set of lines associated to the the multiple points $p(I_i(v)) \in \varphi_v(X)$ (i.e. points of multiplicity ≥ 3), and the set $I_A(v)$ denotes the lines overlapping with l_A. By construction, $I_i(v) \cap I_A(v) = \emptyset$. However, the sets $I_i(v)$ are not necessarily disjoint, as lines can support more than one multiple point. Of course, if the configuration only has double points, then $I_i(v) = \emptyset$. We define the numbers $m_i(v) := \sum_{k \in I_i(v)} m_k$ and $m_A(v) := \sum_{k \in I_A(v)} m_k$. If $I_i(v) = \emptyset$, then we take $m_i(v) := 0$, and similarly for $I_A(v)$. We make the following claim.

Claim 6.8. $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} > 0 \iff m_A(v) = 0$*,* $m_i(v) \leq 2$*, and* $m_k \leq 1$ *for all* i *and* $m_k \in \vec{m}$*.*

Proof of Claim 6.8. We start with the ($\Rightarrow$) direction. If $m_A(v) > 0$, then we have a generic line $L_i \subset \hat{\mathbb{P}}^2$ with $i \in I_A(v)$, and thus a generic point $\mathbb{P}^0_i \subset \mathbb{P}^2$ in the dual space. We must find a $g \in G$ such that $\mathbb{P}^0 \in g(l_i)$ for a line l_i that overlaps with l_A. This is impossible, because G does not move l_A, and so $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

Next, suppose that $m_i(v) = 3$. By the previous argument, we know that if $m_i = 2$, then $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$. Then up to relabelling, we can assume that $m_1 = m_2 = m_3 = 1$ and that $\{1, 2, 3\} \subset I_1(v)$. The generic lines L_1, L_2, L_3 in $\hat{\mathbb{P}}^2$ induce three generic points $\mathbb{P}^0_s$ in $\mathbb{P}^2$. We need to find a $g \in G$ such that the points $\mathbb{P}^0_s \in g \cdot l_s$ for $s \in 1, 2, 3$. Again, this is impossible by the geometry of the problem. Indeed, recall that the intersection points of the lines $l_s \cap l_A$ are fixed. We can find two lines passing through $\mathbb{P}^0_1$ and $\mathbb{P}^0_2$, but those two lines will intersect at $p(I_1)$, and thus determine the position of all the other lines in $I_1(v)$. Therefore, a generic $\mathbb{P}^0_3$ will not be contained in $g \cdot l_3$, and therefore $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

We continue with the ($\Leftarrow$) direction of the claim. There are three L_s of codimension one, and we can suppose that $s \in \{1, 2, 3\}$. By duality, they induce three points in general position in $\mathbb{P}^2$. The statement follows because we can find three lines that pass through these three points as along as the lines are in general position. This holds, because $m_i(v) \leq 2$ implies that $\{1, 2, 3\}$ is not a subset of $I_i(v)$ for any i. $\square$

By Expression 6.5.1 in Proposition 6.5, our statement follows if we prove that for a given $\vec{m}$, and any sha $X = \cup_v X_v$ parametrized by R_{1^n}, there exists a unique component X_v satisfying the criteria of Claim 6.8. The following argument uses the description of the dual graph of the X, which is a rooted tree by Lemma 4.8. We start with the root component X_0. There is no line coinciding with l_A in $\varphi_0(X)$, and so $m_A(0) = 0$. Thus there are two options:

(1) Either $m_i(0) \leq 2$ for all i, or
(2) there exists an i such that $m_i(0) = 3$.

Case (1): If $m_i(0) \leq 2$ for all i, then $\varphi_0(X) \cdot \mathbb{L} > 0$. To show uniqueness, recall that $\sum_{i=1,2,3} m_i = 3$, and that $m_i(v) \leq 2$ for all i. Therefore, the root has at least two branches, and each of those branches has at least one index i_0 such that $m_{i_0} = 1$. Then $I_A(v)$ contains at least one of these indices for every other component $v \neq 0$, because at least one those branches is contracted with its line i_0 that overlaps with l_A. Therefore, $m_A(v) > 0$, and thus $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

Case (2): If there exists an i such that $m_i(0) = 3$, then $\varphi_0(X) \cdot \mathbb{L}_{\vec{m}} = 0$. Thus we may suppose after relabeling, that $m_1(0) = 3$, and that $m_1 = m_2 = m_3 = 1$ with $\{1, 2, 3\} \subset I_1(0)$. This means that there is a branch starting from the root which contains the lines $\{1, 2, 3\}$. Let $X_{v'}$ be the component in that branch that intersects with the rooted component. We claim that $m_A(v') = 0$, because $I_A(v')$ denotes the set of lines in the other branches which are not in I_1. Those indices do not include $\{1, 2, 3\}$, and these indices are the only ones of weight one. Thus, we have two options:

(1) If $m_i(v') \leq 2$ for every i, then we have that $\varphi_{v'}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. Uniqueness follows by same argument used above. There are at least two branches starting from v' with an index j such that $m_j = 1$. Any other $\varphi_v(X)$ will contain that index in $I_A(v)$, and so $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.
(2) If $m_i(v') = 3$ for some i, then there is a branch starting from the vertex v' that contains the lines $\{1, 2, 3\}$.

In the last case, we repeat the above argument with the surface $X_{v''}$ that intersects v' and belongs to the branch containing the lines with indices $\{1, 2, 3\}$. Since for any sha the tree is finite, one of the next two things must happen.

(1) We find a component $\hat{v}$ such that $\varphi_{\hat{v}}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. It is unique by above arguments, or
(2) we arrive to the last vertex of a branch that we call v_f.

In the last case, we have at most three lines in general position on X_{v_f}, because by assupmtion X is maximally degenerated; and there are no multiple points. Following our labeling, those lines are precisely $\{1, 2, 3\}$, and so $m_A(v_f) = m_i(v_f) = 0$, and $\varphi_{v_f}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. □

Next, we extend the birational map $\rho : R_{1^n} \dashrightarrow \overline{U} /\!/_{Ch} G$ to a regular morphism. Note that there exists at most one extension, since the image is dense and the Chow variety is separated. Furthermore, the image of an extension as above is contained in $\overline{U} /\!/_{Ch} G$, since this Chow quotient is closed in the Chow variety. We begin with a crucial lemma.

Definition 6.9. [GG14, Definition 7.2] *Let $(A, \mathfrak{m})$ be a DVR with residue field k and fraction field K, and let Y be a proper scheme. By the valuative criterion, any map $g : \operatorname{Spec} K \to Y$ extends to a map $g : \operatorname{Spec} A \to Y$. We write $\lim g$ for the point $g(\mathfrak{m}) \in Y$.*

Lemma 6.10. [GG14, Theorem 7.3] *Suppose X_1, X_2 are proper schemes over a noetherian scheme S with X_1 normal. Let $U \subset X_1$ be an open dense set and $f : U \to X_2$ an S-morphism. Then f extends to an S-morphism $\hat{f} : X_1 \to X_2$ if and only if for any DVR $(K, \mathfrak{m})$ and any morphism $g : Spec(K) \to U$, the point $\lim fg$ of X_2 is uniquely determined by the point $\lim g$ of X_1.*

Our argument for the following result follows the same structure as the one used for the proof of $\overline{M}_{0,n}$ (see [Gia13, Thm 1.1]), and $T_{d,n}$ (see [GG, Sec. 4.3]).

Proposition 6.11. *There is a morphism $\rho : R_{1^n} \to (\hat{\mathbb{P}}^2)^n /\!/_{Ch} G$ that associates to each closed point $X = \cup_{v \in I} X_v$ of R_{1^n} a cycle with homology class*

$$\sum_{\vec{m}:=(m_1,\dots,m_n)} ([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]) \qquad 0 \leq m_i \leq 1, \qquad \sum_{i=1}^{n} m_i = 3.$$

Proof. Consider a flat proper 1-parameter family $X_\Delta \to \Delta$ where the generic fiber X_t is a sha parametrized by the interior $R^\circ_{1^n}$. Then X_t is supported in $\mathbb{P}^2$ without any multiple point of multiplicity larger than two, and the central fiber $X_\mathbb{C} \to \operatorname{Spec} \mathbb{C}$ is an arbitrary closed point of R_{1^n}. The cycle $[Z(X_t)]$ associated to a generic fiber in X_t is three dimensional, and its homology class is δ (see Proposition 6.5). Therefore, we have a 1-parameter family of cycles whose limit in the Chow variety we denote as $\lim_{t\to 0}[Z(X_t)]$. By Proposition 6.3 and Lemma 6.10, the existence of the morphism then follows if we show that $\lim_{t\to 0}[Z(X_t)]$ is uniquely determined by $X_\mathbb{C}$. It suffices to show that:

$$\lim_{t\to 0}[Z(X_t)] = [Z(X_\mathbb{C})] \tag{6.11.1}$$

where $[Z(X_\mathbb{C})]$ is equal to the cycle defined in Proposition 6.5.

First we show that $Z(X_\mathbb{C}) \subseteq \lim_{t\to 0} Z(X_t)$ as subvarieties of $(\hat{\mathbb{P}}^2)^n$. Since $X_\mathbb{C} = \cup_v X_v$, by definition of $Z(X_\mathbb{C})$, our claim follows if for every component X_v of $X_\mathbb{C}$, we have that:

$$\varphi_v(X_\mathbb{C}) \subset \lim_{t\to 0} Z(X_t) \subset (\hat{\mathbb{P}}^2)^n$$

By construction $\lim_{t\to 0} Z(X_t)$ is closed and G-invariant. Therefore, our claim follows if φ_v maps the points $(p_{1_0}, \dots p_{n_0}) \in (\hat{\mathbb{P}}^2)^n$ associated to the lines in $\varphi_v(X_\mathbb{C})$, into

$$\lim_{t\to 0} Z(X_t) \subset (\hat{\mathbb{P}}^2)^n.$$

We recall that in general for shas, the contraction morphism $\varphi_v : X_\mathbb{C} \to \mathbb{P}^2$ is induced by a line bundle L_v that satisfies $h^i(X, L_v) = 0$ for all $1 \geq i$, since φ_v is degree 1 on the X_v component and degree 0 elsewhere. Then, by Grauert's Theorem (see Corollary III.12.9 of [Har77]), the morphism φ_v lifts to a morphism from the central fiber to our 1-parameter family X_Δ. Let $\varphi_v : X_\Delta \to (\hat{\mathbb{P}}^2)^n$ be that lift. For $t \neq 0$, the map φ_v sends the points $p_{i_t} \in (\hat{\mathbb{P}}^2)^n$ associated to the lines in $\varphi_v(X_t)$ to $Z(X_t)$, and the morphism φ_v is continuous. Then, $\varphi_v(X_\mathbb{C}) \subset \lim_{t\to 0} Z(X_t)$; and we have

$$[Z(X_\mathbb{C})] \leq \lim_{t\to 0}[Z(X_t)]. \tag{6.11.2}$$

Next, we show the equality. By Proposition 6.5, we know that the homology class of the generic orbit has coeficients equal to either 0 or 1. By the argument in the proof of Proposition 6.7, we conclude that the homology class of the generic orbit has coefficient $c_{\vec{m}} = 0$ if there is an $m_i \in \vec{m}$ such that $m_i = 2$. Indeed, it will induce a generic line $\mathbb{P}^1_i \subset \mathbb{P}^2$; and we cannot move any lines l_i to such a line because the intersections $l_i \cap l_A$ are fixed. On the other hand, for $t_0 \neq 0$ we see that:

$$\lim_{t \to 0} [Z(X_t)] = [Z(X_{t_0})] \tag{6.11.3}$$

because we are taking the limit inside a Chow variety. Consequently, the homology class of the limit is the same as the homology class of the generic fiber

Expressions 6.11.2 and 6.11.3 imply that the coefficients $c^{gen}_{\vec{m}}$ in the homology class of the generic element $Z(X_{t_0})$ are necessarily larger than or equal to the coefficients $c^0_{\vec{m}}$ associated to the central fiber $Z(X_{\mathbb{C}})$. Therefore we have the following inequality

$$1 \leq c^0_{\vec{m}} \leq c^{gen}_{\vec{m}} \leq 1, \tag{6.11.4}$$

The left inequality follows by Proposition 6.7 and because the homology class only decreases whenever degenerating, as seen in (6.11.2). The right inequality follows from Proposition 6.5. We conclude that there is a morphism $\rho : R_{1^n} \to (\hat{\mathbb{P}}^2)^n /\!/_{Ch} G$. □

Finally, we prove that R_{1^n} is isomorphic to the normalization of our Chow quotient.

Theorem 6.12. *Let $\overline{U}^n /\!/_{Ch} G$ be the normalization of the Chow quotient, and let ρ^n be the morphism obtained from the Stein factorization of ρ. Then the morphism*

$$\rho^n : R_{1^n} \to \overline{U}^n /\!/_{Ch} G$$

is an isomorphism.

Proof. We use the Zariski's Main Theorem which asserts that a quasi-finite birational morphism to a normal, Noetherian scheme is an open immersion. R_{1^n} is normal, and our morphism ρ factors through the normalization of the Chow quotient. Then, ρ^n is surjective and birational; and the crux of the result is to prove that ρ is quasi-finite. By Proposition 6.3, we already know the map ρ is injective on the interior $R^\circ_{1^n}$; and we observe that no point of the boundary divisor in R_{1^n} can be sent to the same cycle as a point of the open stratum, since the image of the latter is an irreducible cycle whereas the image of the former is not. Therefore, we only need to show that the restriction of ρ to the boundary in R_{1^n} is quasi-finite. The boundary is the union of a finite number of divisors, and so it will be enough to show our claim for a single component D_I of the boundary. The general point of the divisor D_I parametrizes a sha $X = \mathbb{P}^2 \cup \mathrm{Bl}_x(\mathbb{P}^2)$, where $\mathrm{Bl}_x(\mathbb{P}^2)$ contains the line l_A. For example, the second sha in Figure 1 is parametrized by D_{2345}. The morphism ρ sends X to the union of the two cycles:

$$\overline{G \cdot \varphi_0(X)} \cup \overline{G \cdot \varphi_1(X)}$$

If another sha $\tilde{X}$ parametrized by the interior of D_I has the same image as X, that is $\rho(X) = \rho(\tilde{X})$, then their cycles coincide. This means that the image of their reduction morphisms satisfy $\varphi_i(\tilde{X}) \in G \cdot \varphi_i(X)$. However, $G \subset SL(3, \mathbb{C})$, which implies that $X \cong \tilde{X}$. Therefore, ρ is injective on the interior of D_I. A straightforward iteration of this argument, using the fact that our dual graphs are always trees, applies to the deeper strata, and shows that ρ is injective on D_I itself. □

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