Infinitely long isotropic Kirchhoff rods with helical centerlines cannot be stable

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It has long been known that every configuration of a planar elastic rod with clamped ends satisfies the property that if its centerline has constant nonzero curvature, then it is in stable equilibrium regardless of its length. In this paper, we show that for a certain class of nonplanar elastic rods, no configuration satisfies this property. In particular, using results from optimal control theory, we show that every configuration of an inextensible, unshearable, isotropic, and uniform Kirchhoff rod with clamped ends that has a helical centerline with constant nonzero curvature becomes unstable at a finite length. We also derive coordinates for computing this critical length that are independent of the rod's bending and torsional stiffness. Finally, we derive a scaling relationship between the length at which a helical rod becomes unstable and the rod's curvature, torsion, and twist. In a companion paper, these results are used to compute the set of all stable rods with helical centerlines.

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I. INTRODUCTION

If the centerline of a planar elastic rod with clamped ends has constant nonzero curvature—in other words, if the centerline is a circle—then the rod is in stable equilibrium regardless of its length. This property follows from a result proved by Born in his 1906 thesis, which states that any configuration of the rod with a noninflectional centerline is stable [1]. It is natural to ask if the same property holds for a nonplanar elastic rod. In particular, suppose that the centerline of an inextensible, unshearable, isotropic, and uniform Kirchhoff elastic rod with clamped ends has constant nonzero curvature. The centerline of such a rod must also have constant torsion and is therefore a circular helix. Is this helical rod in stable equilibrium, in the sense of locally minimizing elastic potential energy, regardless of its length?

Even prior to the work of Born, it had been established that the answer to this question is "no" in at least the degenerate case of a helical centerline with zero torsion (i.e., a circle). Indeed, Michell showed in 1889 that an inextensible, unshearable, isotropic, and uniform Kirchhoff elastic rod with a closed circular centerline—so, the same shape to which Born's result applies but of a rod in three-dimensional rather than two-dimensional space—becomes unstable when sufficiently twisted [2,3]. However, one might still ask if there exists *any* helical rod that is stable regardless of its length. In this paper, we use Jacobi's conjugate point condition from the calculus of variations and results from linear-quadratic optimal control theory to show that no such helical rod exists—in other words, that *every* configuration of an inextensible, unshearable, isotropic, and uniform Kirchhoff elastic rod with clamped ends and a helical centerline with constant nonzero curvature becomes unstable at a finite length.

This result builds upon a long history of applying Jacobi's conjugate point condition to determine if an equilibrium configuration of an elastic rod is a local minimum of elastic potential energy. Born was perhaps the first to apply the conjugate point condition to planar elastic rods in his 1906 thesis [1]. Following Born's seminal work, Maddocks was most likely the first to analyze stability in three dimensions using conjugate points [4]. In subsequent work, the conjugate point condition has been used to analyze the stability of three-dimensional elastic rods in a variety of configurations [5–9]. Although this paper focuses on applying Jacobi's conjugate point condition, we note that other approaches for determining stability exist, including dynamical methods, bifurcation methods, and direct analysis of the second variation of elastic energy. (See the description in the companion paper [10] and the references therein).

Of particular relevance to this paper is prior work on elastic rods with helical centerlines. The classification of inextensible, unshearable, isotropic, and uniform elastic rods in equilibrium with helical centerlines was completed by Kirchhoff [11,12]. This classification was later generalized to extensible and shearable rods by Antman [13] and Whitman and DeSilva [14], and to anisotropic rods by Chouaieb, Goriely, and Maddocks [9,15,16]. Conditions for stability of these helical equilibria were also derived by Chouaieb and Maddocks using the conjugate point condition [9]. Their formulation of the conjugate point condition relied on a Lagrangian framework, which is the typical approach in the calculus of variations [17], and these conditions for stability have been evaluated numerically for a few representative helical rods [6,9].

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In contrast to prior formulations of the conjugate point condition for helical rods, this paper uses a Hamiltonian formulation based on geometric optimal control theory [18]. Similar methods from optimal control theory have previously been used to generalize many of Born's original results and provide a nearly complete description of stability for the planar elastic rod [19,20]. Using this approach, we are able to draw an analogy between conjugate points in helical rods and conjugate points in linear-quadratic optimal control problems [21]. This analogy is the key insight that allows us to show that every helical rod becomes unstable at a finite length. In the process of establishing this result, we also derive coordinates for computing the length at which a helical rod loses stability that are independent of the rod's bending and torsional stiffness. Finally, after showing that every helical rod becomes unstable at a finite length, we derive a scaling relationship that compares this critical length along different helical rods. In a companion paper [10], these results are used to compute and visualize the set of all inextensible, unshearable, isotropic, and uniform elastic rods with helical centerlines that are stable.

II. EQUILIBRIUM AND STABILITY OF RODS WITH HELICAL CENTERLINES

In this section, we recall the conditions for a configuration of an inextensible, unshearable, isotropic, and uniform Kirchhoff elastic rod with a helical centerline to be in stable equilibrium, which are also described in the companion paper [10]. Equilibrium configurations of a rod with length L and clamped ends correspond to solutions of the differential equations

$$r' = Rv, \qquad R' = R\widehat{u}, \tag{1}$$

$$m' = m \times u + n \times v, \quad n' = n \times u, \tag{2}$$

$$u_1 = c^{-1}m_1, \quad u_2 = m_2, \quad u_3 = m_3, \tag{3}$$

subject to boundary conditions on $r(0)$, $r(L)$, $R(0)$, and $R(L)$. In the differential equations (1)–(3), primes denote differentiation with respect to arc length $s \in [0, L]$, $v = [1\ 0\ 0]^T$, and the map $\widehat{\ } : \mathbb{R}^3 \to \mathfrak{so}(3)$ satisfies $a \times b = \widehat{a}b$ for all $a, b \in \mathbb{R}^3$. The variable $r : [0, L] \to \mathbb{R}^3$ describes the rod's centerline, $R : [0, L] \to SO(3)$ describes the orientation of a triad of orthonormal vectors attached to the rod's centerline, and the vector u denotes the rod's strains, with u_1 being the twisting strain, and u_2 and u_3 the bending strains [22]. The parameter $c > 0$ is the ratio of torsional to bending stiffness and is constant since the rod is assumed to be uniform. The functions $m, n : [0, L] \to \mathbb{R}^3$ can be interpreted mechanically as the internal moments and forces, respectively, acting on the rod [22]. In the Hamiltonian formulation used in this paper, the functions m and n are adjoint variables, similar to Lagrange multipliers, associated with the constraints (1) [18,23].

A solution of the differential equations (1)–(3) is an extremum of the elastic energy functional

$$\frac{1}{2}\int_0^L \left(cu_1^2 + u_2^2 + u_3^2\right) ds, \tag{4}$$

subject to the boundary conditions on r and R. For a given solution of (1)–(3), Jacobi's conjugate point condition can be used to determine if this extremum is a local minimum of the elastic energy functional (4) [5–9]. As described in the companion paper [10], the conjugate point condition is applied by linearizing the differential equations (1)–(3), resulting in a linear system of the form

$$J' = HJ + GM, \quad M' = FM, \tag{5}$$

where

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{c}m_3 & 0 & \bar{c}m_1 & 0 & 0 & 1 \\ -\bar{c}m_2 & -\bar{c}m_1 & 0 & 0 & -1 & 0 \\ 0 & -n_3 & n_2 & 0 & m_3 & -m_2 \\ c^{-1}n_3 & 0 & -n_1 & -m_3 & 0 & c^{-1}m_1 \\ -c^{-1}n_2 & n_1 & 0 & m_2 & -c^{-1}m_1 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & m_3 & -m_2 & 0 & 0 & 0 \\ -m_3 & 0 & c^{-1}m_1 & 0 & 0 & 0 \\ m_2 & -c^{-1}m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_3 & -m_2 \\ 0 & 0 & 1 & -m_3 & 0 & c^{-1}m_1 \\ 0 & -1 & 0 & m_2 & -c^{-1}m_1 & 0 \end{bmatrix},$$

$$G = \mathrm{diag}(c^{-1}, 1, 1, 0, 0, 0), \tag{6}$$

where $\bar{c} = (c^{-1} - 1)$. The system (5) is solved with the initial conditions $J(0) = 0_{6\times6}$ and $M(0) = I_{6\times6}$, where $0_{6\times6}$ is the 6×6 matrix containing all zeros, and $I_{6\times6}$ is the 6×6 identity matrix. If $\det(J(s)) = 0$ for some $s \in (0, L)$, then s corresponds to a conjugate point, and the equilibrium configuration is not stable. If $\det(J(s)) \neq 0$ for all $s \in (0, L]$, then there are no conjugate points and the equilibrium configuration is stable [18].

In this paper, we are interested in solutions of (1)–(3) that correspond to rod configurations whose centerlines have constant nonzero curvature. As stated in the Introduction, the centerline of such a rod also has constant torsion and is therefore a circular helix. To see that this is true, we first note that in terms of the functions m and n, the curvature $\kappa \geqslant 0$ and torsion τ of the rod's centerline are given by [23]

$$\kappa^2 = m_2^2 + m_3^3, \quad \tau = m_1 - \frac{m_2n_2 + m_3n_3}{\kappa^2}.$$

Differentiating these expressions with respect to arc length s and simplifying gives

$$2\kappa\kappa' = -n_1', \quad \tau' = n_1'\frac{2\tau - m_1}{\kappa^2}.$$

It is well known that the rod's centerline is a circular helix if and only if the axial force n_1 is constant [9,15,16]. However, from the above expressions, we can further conclude that if the centerline's curvature is constant, then $n_1' = 0$ and its torsion must also be constant. Thus all rod configurations whose centerlines have constant curvature are helical, and there are no configurations having constant curvature and nonconstant torsion.

Solutions of the system (2) and (3) corresponding to rod configurations with helical centerlines having curvature $\kappa \geqslant 0$ and torsion τ have the form

$$m(s) = \begin{bmatrix} \omega \\ \kappa\cos(\gamma s + \phi) \\ \kappa\sin(\gamma s + \phi) \end{bmatrix}, \quad n(s) = (\omega - \tau)\begin{bmatrix} \tau \\ m_2(s) \\ m_3(s) \end{bmatrix}, \tag{7}$$

where ω is the twisting moment, ϕ is a phase parameter, and $\gamma = \tau - \omega/c$ [cf., Eqs. (86) and (94) of Ref. [15], and Eqs. (4.7)– (4.9) of Ref. [24]]. We call such configurations helical rods.

The conjugate point condition described above can be used to determine which rod configurations with helical centerlines locally minimize the elastic potential energy functional (4). To apply the conjugate point condition, the expressions (7) can be substituted into the coefficient matrices F and H in (6), and the matrix differential equations (5) can be solved to determine the existence of conjugate points. Finally, we note that the conjugate point condition can only be applied in the case when $\kappa > 0$, and it cannot be used when $\kappa = 0$, i.e., in the degenerate case of a straight and twisted rod. Due to the clamped boundary conditions and the assumption of inextensibility, the straight, twisted rod is an isolated configuration [i.e., there are no nearby configurations with nonstraight centerlines that satisfy the constraints in (1) and the boundary conditions on r and R], and is therefore an abnormal extremal of the elastic energy functional (4). Since this paper is focused on configurations with centerlines having constant nonzero curvature, we will not consider these abnormal extremals.

III. A CLOSED-FORM SOLUTION OF THE STABILITY CONDITIONS

In this section, we show that for the functions m and n given in (7), the linear system of differential equations (5) can be solved in closed form. In the following section, this closed-form solution is used to derive coordinates for determining the stability of helical rods that are independent of the stiffness parameter c. We begin by applying a coordinate transformation to the system (5) of the form

$$\widetilde{J}(s) = K(s)J(s), \quad \widetilde{M}(s) = K(s)M(s), \tag{8}$$

where

$$K(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\gamma s+\phi) & \sin(\gamma s+\phi) & 0 & 0 & 0 \\ 0 & -\sin(\gamma s+\phi) & \cos(\gamma s+\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\gamma s+\phi) & \sin(\gamma s+\phi) \\ 0 & 0 & 0 & 0 & -\sin(\gamma s+\phi) & \cos(\gamma s+\phi) \end{bmatrix}.$$

After using this coordinate transformation and substituting the expressions in (7) for m and n into the resulting coefficient matrices, the differential equations (5) become

$$\widetilde{J}' = \widetilde{H}\widetilde{J} + \widetilde{G}\widetilde{M}, \quad \widetilde{M}' = \widetilde{F}\widetilde{M}, \tag{9}$$

where

$$\widetilde{F} = K'K^T + KFK^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau-\omega & 0 & 0 & 1 \\ -\bar{c}\kappa & -(\tau-\omega) & 0 & 0 & -1 & 0 \\ 0 & 0 & -\kappa(\tau-\omega) & 0 & 0 & -\kappa \\ 0 & 0 & \tau(\tau-\omega) & 0 & 0 & \tau \\ c^{-1}\kappa(\tau-\omega) & -\tau(\tau-\omega) & 0 & \kappa & -\tau & 0 \end{bmatrix},$$

$$\widetilde{H} = K'K^T + KHK^T = \begin{bmatrix} 0 & 0 & -\kappa & 0 & 0 & 0 \\ 0 & 0 & \tau & 0 & 0 & 0 \\ \kappa & -\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa \\ 0 & 0 & 1 & 0 & 0 & \tau \\ 0 & -1 & 0 & \kappa & -\tau & 0 \end{bmatrix},$$

and $\widetilde{G} = KGK^T = G$. Note that since $\det(K(s)) = 1$ for all $s \in [0, L]$, conjugate points are invariant under the coordinate transformation (8), i.e., $\det(J(s)) = 0$ if and only if $\det(\widetilde{J}(s)) = 0$.

Since the coefficient matrices $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{H}$ in the system (9) are independent of ϕ, we conclude that conjugate points are independent of the parameter ϕ as well. This independence results from the assumption that the rod is isotropic. Without loss of generality, we assume that $\phi = 0$ in the remainder of this paper. With this choice of ϕ, the initial conditions for the system (9) are $\widetilde{J}(0) = 0_{6\times6}$ and $\widetilde{M}(0) = I_{6\times6}$, just as they were for the original system (5).

In addition to removing the dependence upon ϕ, the coordinate transformation (8) removes the dependence upon arc length s in the coefficient matrices $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{H}$ so that the differential equation (9) becomes a constant coefficient linear system. We now show that this linear system can be integrated in closed form. We begin by considering the ith column of the matrices $\widetilde{J}$ and

$\widetilde{M}$, which we denote by $[\widetilde{J}_{1i} \ \cdots \ \widetilde{J}_{6i}]^T$ and $[\widetilde{M}_{1i} \ \cdots \ \widetilde{M}_{6i}]^T$, respectively. A short calculation shows that

$$\widetilde{M}_{2i}''' = \frac{\sigma^2}{\kappa}\widetilde{M}_{4i}',$$
$$\widetilde{M}_{3i}'' = \frac{2\tau - \omega}{\kappa}\widetilde{M}_{4i}',$$
$$\widetilde{M}_{4i}''' = -\sigma^2\widetilde{M}_{4i}',$$

where $\sigma = \sqrt{\kappa^2 + (2\tau - \omega)^2}$. The solution for M_{4i} is given by

$$\widetilde{M}_{4i}(s) = A_{1i}\cos(\sigma s) + A_{2i}\sin(\sigma s) + A_{3i}, \tag{10}$$

where the constants A_{1i}, A_{2i}, and A_{3i} are determined by the ith column of the initial condition $\widetilde{M}(0)$. The solutions for $\widetilde{M}_{1i}$, $\widetilde{M}_{2i}$, and $\widetilde{M}_{3i}$ are then found to be

$$\begin{aligned}
\widetilde{M}_{1i}(s) &= \widetilde{M}_{1i}(0),\\
\widetilde{M}_{2i}(s) &= -\frac{1}{\kappa}[A_{1i}\cos(\sigma s) + A_{2i}\sin(\sigma s)] + B_{1i}s^2 + B_{2i}s + B_{3i},\\
\widetilde{M}_{3i}(s) &= \frac{\omega - 2\tau}{\sigma\kappa}[A_{1i}\sin(\sigma s) - A_{2i}\cos(\sigma s)] + C_{1i}s + C_{2i},
\end{aligned} \tag{11}$$

where the constants B_{1i}, B_{2i}, B_{3i}, C_{1i}, and C_{2i} are determined by the ith column of the initial condition $\widetilde{M}(0)$. Expressions for the constants A_{1i}, A_{2i}, A_{3i}, B_{1i}, B_{2i}, B_{3i}, C_{1i}, and C_{2i} in terms of $\widetilde{M}(0)$ can be found by evaluating Eqs. (10) and (11) and their derivatives at $s = 0$, and these expressions are given in the Appendix.

Now consider the differential equation for the ith column of $\widetilde{J}'$. These equations can be rewritten as

$$\begin{bmatrix}\widetilde{J}_{1i}'\\ \widetilde{J}_{2i}'\\ \widetilde{J}_{3i}'\end{bmatrix} = \begin{bmatrix}0 & 0 & -\kappa\\ 0 & 0 & \tau\\ \kappa & -\tau & 0\end{bmatrix}\begin{bmatrix}\widetilde{J}_{1i}\\ \widetilde{J}_{2i}\\ \widetilde{J}_{3i}\end{bmatrix} + \begin{bmatrix}c^{-1}\widetilde{M}_{1i}\\ \widetilde{M}_{2i}\\ \widetilde{M}_{3i}\end{bmatrix} \tag{12}$$

and

$$\begin{bmatrix}\widetilde{J}_{4i}'\\ \widetilde{J}_{5i}'\\ \widetilde{J}_{6i}'\end{bmatrix} = \begin{bmatrix}0 & 0 & -\kappa\\ 0 & 0 & \tau\\ \kappa & -\tau & 0\end{bmatrix}\begin{bmatrix}\widetilde{J}_{4i}\\ \widetilde{J}_{5i}\\ \widetilde{J}_{6i}\end{bmatrix} + \begin{bmatrix}0\\ \widetilde{J}_{3i}\\ -\widetilde{J}_{2i}\end{bmatrix}. \tag{13}$$

Let

$$D = \begin{bmatrix}0 & 0 & -\kappa\\ 0 & 0 & \tau\\ \kappa & -\tau & 0\end{bmatrix}.$$

After recalling that the initial condition for $\widetilde{J}$ is $\widetilde{J}(0) = 0_{6\times 6}$, the solution of (12) is found to be

$$\begin{bmatrix}\widetilde{J}_{1i}(s)\\ \widetilde{J}_{2i}(s)\\ \widetilde{J}_{3i}(s)\end{bmatrix} = e^{Ds}\int_0^s e^{-Dt}\begin{bmatrix}c^{-1}\widetilde{M}_{1i}(t)\\ \widetilde{M}_{2i}(t)\\ \widetilde{M}_{3i}(t)\end{bmatrix}dt, \tag{14}$$

and the solution of (13) is given by

$$\begin{bmatrix}\widetilde{J}_{4i}(s)\\ \widetilde{J}_{5i}(s)\\ \widetilde{J}_{6i}(s)\end{bmatrix} = e^{Ds}\int_0^s e^{-Dt}\begin{bmatrix}0\\ \widetilde{J}_{3i}(t)\\ -\widetilde{J}_{2i}(t)\end{bmatrix}dt. \tag{15}$$

The exponential terms in (14) and (15) can be computed using the Rodrigues rotation formula, given by

$$e^{Ds} = I + \frac{\sin(\sqrt{\kappa^2+\tau^2}s)}{\sqrt{\kappa^2+\tau^2}}D + \frac{1-\cos(\sqrt{\kappa^2+\tau^2}s)}{\kappa^2+\tau^2}D^2.$$

The expressions (11) can be used in (14) to obtained closed-form solutions for $\widetilde{J}_{1i}$, $\widetilde{J}_{2i}$, and $\widetilde{J}_{3i}$. These expressions can then be used in (15) to obtained closed-form solutions for $\widetilde{J}_{4i}$, $\widetilde{J}_{5i}$, and $\widetilde{J}_{6i}$. These solutions can be computed for each of the six columns in $\widetilde{J}$.

TABLE I. Expressions for the constants defined in the Appendix when $\widetilde{M}(0)$ is given by (16).

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
A_{1i}	0	$-\frac{\kappa(\tau-\omega)(2\tau-\omega)}{\sigma^2}$	0	$\frac{\kappa^2}{\sigma^2}$	$-\frac{\kappa(2\tau-\omega)}{\sigma^2}$	0
A_{2i}	0	0	$-\frac{\kappa(\tau-\omega)}{\sigma}$	0	0	$-\frac{\kappa}{\sigma}$
A_{3i}	τ	$\frac{\kappa(\tau-\omega)(2\tau-\omega)}{\sigma^2}$	0	$1-\frac{\kappa^2}{\sigma^2}$	$\frac{\kappa(2\tau-\omega)}{\sigma^2}$	0
B_{1i}	0	0	0	0	0	0
B_{2i}	0	0	0	0	0	0
B_{3i}	0	$1-\frac{(\tau-\omega)(2\tau-\omega)}{\sigma^2}$	0	$\frac{\kappa}{\sigma^2}$	$-\frac{2\tau-\omega}{\sigma^2}$	0
C_{1i}	$-c^{-1}\kappa$	$-\frac{(\tau-\omega)(\sigma^2+(2\tau-\omega)^2)}{\sigma^2}$	0	$\frac{\kappa(2\tau-\omega)}{\sigma^2}$	$-1-\frac{(2\tau-\omega)^2}{\sigma^2}$	0
C_{2i}	0	0	$1+\frac{(\tau-\omega)(2\tau-\omega)}{\sigma^2}$	0	0	$\frac{2\tau-\omega}{\sigma^2}$

IV. STIFFNESS-INVARIANT COORDINATES FOR DETERMINING STABILITY

In this section, we show that the values of the parameters κ, τ, and ω appearing in the expressions for m and n given in (7) determine whether or not the corresponding helical rod is a local minimum of the elastic energy functional (4). In other words, if the parameters κ, τ, and ω are fixed, then varying the stiffness parameter c does not change the arc lengths s at which $\det(\widetilde{J}(s)) = 0$ and therefore does not affect the rod's stability. The key observation that allows us to establish this property is that by changing the initial condition $\widetilde{M}(0)$ in the system (9), we can make the resulting matrix $\widetilde{J}$ depend on the stiffness parameter c in a simple way.

We first consider a solution $\widetilde{J}$ and $\widetilde{M}$ of the system (9) with the initial conditions $\widetilde{J}(0) = 0_{6\times6}$ and $\widetilde{M}(0) = I_{6\times6}$. If the initial condition $\widetilde{M}(0)$ is changed from $I_{6\times6}$ to any 6×6 matrix R, then the resulting solution of the system (9) is given by $\widetilde{J}R$ and $\widetilde{M}R$. If R is invertible, then $\widetilde{J}$ and $\widetilde{J}R$ are singular at the same values of arc length s, and this change in initial condition $\widetilde{M}(0)$ does not affect the locations of the conjugate points. Since we are now free to choose the initial condition $\widetilde{M}(0)$, we will consider the particular case when $\widetilde{M}(0)$ is the invertible matrix

$$\widetilde{M}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \tau & 0 & 0 & 1 & 0 & 0 \\ \kappa & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

With this initial condition for $\widetilde{M}(0)$, the expressions given in the Appendix for the constants A_{1i}, A_{2i}, A_{3i}, B_{1i}, B_{2i}, B_{3i}, C_{1i}, and C_{2i} appearing in (10) and (11) can be computed, and they are given in Table I.

We will first consider columns 2–6 of the matrices $\widetilde{M}$ and $\widetilde{J}$. Note that the parameter c does not explicitly appear in the expressions (11) for $\widetilde{M}_{1i}$, $\widetilde{M}_{2i}$, and $\widetilde{M}_{3i}$. Furthermore, for $i \in \{2, \ldots, 6\}$, the parameter c does not appear in the expressions for A_{1i}, A_{2i}, A_{3i}, B_{1i}, B_{2i}, B_{3i}, C_{1i}, and C_{2i} given in Table I. We therefore conclude that $\widetilde{M}_{1i}$, $\widetilde{M}_{2i}$, and $\widetilde{M}_{3i}$ are independent of c when $i \in \{2, \ldots, 6\}$. Next, note that for $i \in \{2, \ldots, 6\}$, we have $\widetilde{M}_{1i}(s) = 0$. In the expressions (14) and (15) for the columns of $\widetilde{J}$, the parameter c only appears as a coefficient of $\widetilde{M}_{1i}$. Therefore, columns 2–6 of the matrix $\widetilde{J}$ are independent of c.

Now consider the first column of the matrices $\widetilde{M}$ and $\widetilde{J}$. Using the initial condition (16), along with the expressions in Table I for $i = 1$ and the expressions in (11), we have

$$\widetilde{M}_{11}(s) = 1, \quad \widetilde{M}_{21}(s) = 0, \quad \widetilde{M}_{31}(s) = -c^{-1}\kappa s.$$

After substituting these expressions for $\widetilde{M}_{11}$, $\widetilde{M}_{21}$, and $\widetilde{M}_{31}$ into the differential equations (12) and (13), it is straightforward to verify that the resulting solution for the first column of $\widetilde{J}$ is given by

$$\widetilde{J}_{11}(s) = c^{-1}s, \quad \widetilde{J}_{k1}(s) = 0, \; k \in \{2, \ldots, 6\}.$$

We can now decompose the matrix $\widetilde{J}$ into four blocks as

$$\widetilde{J}(s) = \begin{bmatrix} \widetilde{J}^{(11)}(s) & \widetilde{J}^{(12)}(s) \\ \widetilde{J}^{(21)}(s) & \widetilde{J}^{(22)}(s) \end{bmatrix},$$

where

$$\widetilde{J}^{(11)}(s) = c^{-1}s, \quad \widetilde{J}^{(21)}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

and where $\widetilde{J}^{(22)}(s)$ is a 5×5 matrix that is independent of c. The determinant of $\widetilde{J}$ can now be written as

$$\det(\widetilde{J}(s)) = c^{-1}s \det(\widetilde{J}^{(22)}(s)).$$

So, varying the value of c does not affect the arc lengths at which $\det(\widetilde{J}(s)) = 0$, and we conclude that if the parameters κ, τ, and ω are fixed, then the conjugate points are invariant under changes in the stiffness parameter c. Therefore, it is sufficient to know the values of κ, τ, and ω in order to determine if a helical rod locally minimizes the elastic potential energy functional (4).

When solving the linear system (9), we are now free to choose any positive value for c, and our results will be applicable to all other positive values of c. For simplicity, we will choose $c = 1$ in the remainder of this paper. However, we note that in the case when both the torsion and twisting moment are nonzero and have the same sign, one can choose $c = \omega/\tau$, which gives $\gamma = 0$ in the expressions (7). If we further choose $\phi = \pi/2$, then $u = [\tau \;\; 0 \;\; \kappa]^T$, and the frame R corresponds to the Frenet-Serret frame of the curve r.

V. ALL HELICAL RODS BECOME UNSTABLE AT A FINITE LENGTH

In this section, we show that for any given values of $\kappa > 0$, τ, and ω, there exists a finite arc length at which the corresponding helical rod becomes unstable, i.e., there exists $0 < s^* < \infty$ at which the solution of (9) satisfies $\det(\widetilde{J}(s^*)) =$

0. To establish this result, we draw an analogy between the system (9) used to compute conjugate points along helical rods and the differential equations used to compute conjugate points in linear-quadratic optimal control problems. Before making this connection, we briefly review the main results from linear-quadratic optimal control theory that are needed for our analysis.

We consider linear-quadratic optimal control problems with fixed boundary conditions of the form

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & \frac{1}{2}\int_0^T \left(z^T z - x^T Q x\right) dt \\ \text{subject to} \quad & x' = Ax + Bz, \\ & x(0) = x_0,\ x(T) = x_T, \end{aligned} \tag{17}$$

where $x : [0, T] \to \mathbb{R}^d$ and $z : [0, T] \to \mathbb{R}^k$ for some $d, k > 0$. In the problem (17), the terminal time $T > 0$; the matrices $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times k}$, $Q \in \mathbb{R}^{d\times d}$, and the boundary conditions $x_0, x_T \in \mathbb{R}^d$ are fixed; and Q is symmetric. In this optimal control problem, x is called the state and z is called the control. Although we consider the above problem with fixed boundary conditions, we note that it is more common in the optimal control literature to allow the terminal boundary condition $x(T)$ to be free, which is often referred to as the linear-quadratic regulator [25]. We will assume that the system is controllable, i.e.,

$$\text{rank}([B\ AB\ A^2B\ \cdots\ A^{d-1}B]) = d, \tag{18}$$

which ensures that for any choice of T, x_0, and x_T, there exists a control z such that the corresponding state x satisfies the given boundary conditions [26].

Necessary conditions for a control z and corresponding state x to be a local minimum of the problem (17) are given by the Pontryagin maximum principle [27]. To apply the maximum principle, we define the control Hamiltonian function h for the problem (17), given by

$$h(x, z, p) = p^T(Ax + Bz) - \tfrac{1}{2}\left(z^T z - x^T Q x\right),$$

where p is called the costate. Then if (x, z) is a local minimum of (17), there exists a costate trajectory $p : [0, T] \to \mathbb{R}^d$ such that x and p satisfy Hamilton's canonical equations

$$\begin{aligned} x' &= \partial_p h = Ax + Bz, \\ p' &= -\partial_x h = -A^T p - Qx, \end{aligned} \tag{19}$$

and the control z satisfies $\partial_z h = 0$, which gives $z = B^T p$ [25]. State and control trajectories (x, z) that satisfy these conditions are local extrema of the problem (17).

If (x, z) satisfies the above necessary conditions, then the conjugate point condition can be used to determine if (x, z) is indeed a local minimum of (17). To apply the conjugate point condition, we solve the system

$$\begin{bmatrix} P' \\ X' \end{bmatrix} = \begin{bmatrix} -A^T & -Q \\ BB^T & A \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix} = Z \begin{bmatrix} P \\ X \end{bmatrix} \tag{20}$$

on the interval $t \in [0, T]$ with the initial conditions $P(0) = I_{d\times d}$ and $X(0) = 0_{d\times d}$, where we have used Z to denote the $2d \times 2d$ coefficient matrix in the linear system. The above system results from substituting the expression $z = B^T p$ for the control into the system (19). In a similar fashion to the conjugate point condition for the elastic rod described in Sec. II, (x, z) is a local minimum of (17) if $\det(X(t)) \neq 0$ for all $t \in (0, T]$. If $\det(X(t)) = 0$ for some $t \in (0, T)$, then t is called a conjugate time and (x, z) is not a local minimum.

In the work of Agrachev, Rizzi, and Silveira [21], linear systems of the form (20), with A and B satisfying the controllability assumption (18), were classified into the following two categories:

(i) If the Jordan normal form of the matrix Z in (20) has at least one odd-dimensional Jordan block corresponding to a purely imaginary eigenvalue, then the number of times at which $\det(X(t)) = 0$ in the interval $(0, T)$ grows to infinity as $T \to \infty$.

(ii) If the Jordan normal form of the matrix Z in (20) has no odd-dimensional Jordan blocks corresponding to a purely imaginary eigenvalue, then $\det(X(t)) \neq 0$ for all $t \in (0, \infty)$.

In both cases (i) and (ii), a state and control trajectory (x, z) that satisfies the necessary conditions provided by Pontryagin's maximum principle is a local minimum of (17) for sufficiently small values of $T > 0$. In case (i), there exists $0 < T^* < \infty$ such that if $T > T^*$, then (x, z) is no longer a local minimum of (17). In case (ii), (x, z) remains a local minimum of (17) for arbitrarily large values of T.

We now use the above classification of linear systems having the form (20) to show that every helical rod becomes unstable at a finite length. To establish this result, we will construct a linear system of the form (20) whose conjugate times [i.e., times when $\det(X(t)) = 0$] correspond to the arc lengths in the system (9) at which $\det(\widetilde{J}(s)) = 0$, and we will show that this linear system falls into the first of the two categories described above. We can rewrite the system (9) as

$$\begin{bmatrix} \widetilde{M}' \\ \widetilde{J}' \end{bmatrix} = \begin{bmatrix} \widetilde{F} & 0 \\ \widetilde{G} & \widetilde{H} \end{bmatrix} \begin{bmatrix} \widetilde{M} \\ \widetilde{J} \end{bmatrix}.$$

Since the coefficient matrix in this system does not have the same form as in (20), we cannot immediately apply the classification from [21]. We therefore consider a coordinate transformation of the above system of the form

$$P = \widetilde{M} + S\widetilde{J}, \quad X = \widetilde{J},$$

where S is some 6×6 constant matrix. Under this transformation, the arc lengths at which $\widetilde{J}$ and X are singular trivially coincide. Differentiating the above expressions and simplifying leads to the linear system

$$\begin{bmatrix} P' \\ X' \end{bmatrix} = \begin{bmatrix} \widetilde{F} + S\widetilde{G} & S\widetilde{H} - \widetilde{F}S - S\widetilde{G}S \\ \widetilde{G} & \widetilde{H} - \widetilde{G}S \end{bmatrix} \begin{bmatrix} P \\ X \end{bmatrix}. \tag{21}$$

For the above system to be written in the form of (20), we must have $-(\widetilde{F} + S\widetilde{G})^T = \widetilde{H} - \widetilde{G}S$, or equivalently $\widetilde{G}(S - S^T) = \widetilde{F}^T + \widetilde{H}$. Many choices of S can be found that satisfy this expression. One such matrix is

$$S = \begin{bmatrix} 0 & 0 & -\kappa/2 & 0 & 0 & 0 \\ 0 & 0 & \omega/2 & 0 & 0 & 0 \\ \kappa/2 & -\omega/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa(\tau - \omega) & 0 & 0 & 0 \\ 0 & 0 & -\tau(\tau - \omega) & 0 & 0 & 0 \\ -\kappa(\tau - \omega) & \tau(\tau - \omega) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With this choice of S, the system (21) can be written in the form of (20), with

$$A = \begin{bmatrix} 0 & 0 & -\kappa/2 & 0 & 0 & 0 \\ 0 & 0 & \tau-\omega/2 & 0 & 0 & 0 \\ \kappa/2 & \omega/2-\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa \\ 0 & 0 & 1 & 0 & 0 & \tau \\ 0 & -1 & 0 & \kappa & -\tau & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$Q = \frac{1}{4}\begin{bmatrix} \kappa^2 & -\kappa(2\tau-\omega) & 0 & 0 & 0 & 0 \\ -\kappa(2\tau-\omega) & (2\tau-\omega)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa^2+(2\tau-\omega)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

Note that with this choice of S, we have that $Q = -(S\widetilde{H} - \widetilde{F}S - S\widetilde{G}S)$ is symmetric, as is required. Finally, we note that a long but straightforward computation shows that the above matrices A and B satisfy the controllability assumption (18), which can be checked, e.g., using symbolic computation software.

For given values of curvature $\kappa > 0$, torsion τ, and twist moment ω, we have constructed a linear system of the form (20) whose first conjugate time corresponds to the arc length at which the helical rod becomes unstable. We can now determine in which of the two categories this linear system falls. Computing the Jordan normal form of the coefficient matrix Z in the system (20) using the matrices A, B, and Q in (22) produces a matrix with seven Jordan blocks $Z_1, \dots, Z_7$, given by

$$Z_1 = Z_2 = Z_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$Z_4 = \begin{bmatrix} \sqrt{-\kappa^2-\tau^2} & 1 \\ 0 & \sqrt{-\kappa^2-\tau^2} \end{bmatrix}, \quad Z_5 = -Z_4,$$
$$Z_6 = \sqrt{-\kappa^2-(2\tau-\omega)^2}, \quad Z_7 = -Z_6.$$

Both Z_6 and Z_7 are odd-dimensional Jordan blocks corresponding to purely imaginary eigenvalues. Therefore, the first conjugate time in the linear system is finite.

For the corresponding helical rod, we conclude that there exists a finite arc length $0 < s^* < \infty$ at which the solution of (9) satisfies $\det(\widetilde{J}(s^*)) = 0$ and the helical rod becomes unstable. This property is in direct contrast to Born's result for planar rods, which showed that every configuration having constant nonzero curvature is stable—regardless of the rod's length [1]. The results in this section show that within the class of inextensible, unshearable, isotropic, and uniform rods having clamped ends and centerlines with constant nonzero curvature, there exists no configuration that is stable for arbitrary length.

VI. A STABILITY SCALING RELATIONSHIP

Thus far, we have shown that the stability of a helical rod is determined by the parameters $\kappa > 0$, τ, and ω, and we have shown that for any choice of these three parameters, the corresponding helical rod becomes unstable at a finite length.

We now derive a scaling relationship between the length at which the helical rod becomes unstable and the parameters $\kappa > 0$, τ, and ω.

Based on the results in the previous sections, we were able to set $\phi = 0$ and $c = 1$ without loss of generality. Therefore, the solutions $\widetilde{J}$ and $\widetilde{M}$ of the linear system (9) only depend upon the choice of κ, τ, and ω. Let us denote the solutions of (9) by $\widetilde{J}_{(\kappa,\tau,\omega)}$ and $\widetilde{M}_{(\kappa,\tau,\omega)}$. Let us also explicitly show the dependence of the coefficient matrices in (9) on κ, τ, and ω by denoting them $\widetilde{F}_{(\kappa,\tau,\omega)}$ and $\widetilde{H}_{(\kappa,\tau,\omega)}$. (Note that $\widetilde{G}$ is independent of κ, τ, and ω.)

We first show that for any values of $\kappa > 0$, τ, and ω, and for any positive number λ, we have the relationship

$$\det(\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s)) = \lambda^{-12}\det(\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s)). \tag{23}$$

To establish this result, define the matrices

$$V_J = \mathrm{diag}\,(1, 1, 1, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}),$$
$$V_M = \mathrm{diag}\,(\lambda, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2).$$

We claim that the solutions of the differential equations (9) for the parameter values (κ, τ, ω) and $(\lambda\kappa, \lambda\tau, \lambda\omega)$ are related by

$$\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) = V_J(\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1},$$
$$\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) = V_M(\widetilde{M}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1}. \tag{24}$$

To verify this claim, first note that a short calculation leads to

$$\widetilde{F}_{(\lambda\kappa,\lambda\tau,\lambda\omega)} = \lambda V_M \widetilde{F}_{(\kappa,\tau,\omega)} V_M^{-1},$$
$$\widetilde{H}_{(\lambda\kappa,\lambda\tau,\lambda\omega)} = \lambda V_J \widetilde{H}_{(\kappa,\tau,\omega)} V_J^{-1},$$
$$\widetilde{G} = \lambda V_J \widetilde{G} V_M^{-1}. \tag{25}$$

Next, differentiating the first expression in (24) gives

$$\begin{aligned}\frac{d}{ds}\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) &= \frac{d}{ds}V_J(\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1} \\ &= \lambda V_J(\widetilde{H}_{(\kappa,\tau,\omega)}\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s) + \widetilde{G}\widetilde{M}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1} \\ &= \left(\lambda V_J \widetilde{H}_{(\kappa,\tau,\omega)} V_J^{-1}\right)\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) \\ &\quad + \left(\lambda V_J \widetilde{G} V_M^{-1}\right)\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) \\ &= \widetilde{H}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) + \widetilde{G}\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s),\end{aligned}$$

and differentiating the second expression in (24) gives

$$\begin{aligned}\frac{d}{ds}\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s) &= \frac{d}{ds}V_M(\widetilde{M}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1}\\ &= \lambda V_M(\widetilde{F}_{(\kappa,\tau,\omega)}\widetilde{M}_{(\kappa,\tau,\omega)}(\lambda s))V_M^{-1}\\ &= (\lambda V_M\widetilde{F}_{(\kappa,\tau,\omega)}V_M^{-1})\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s)\\ &= \widetilde{F}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s),\end{aligned}$$

where the last equalities in the above expressions follow from (25). The above computations show that $\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}$ and $\widetilde{M}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}$, as they are defined in (24), satisfy the differential equations (9) for the parameters $\lambda\kappa$, $\lambda\tau$, and $\lambda\omega$. The result (23) follows from (24), since $\det(V_J) = L^{-3}$ and $\det(V_M^{-1}) = L^{-9}$.

Based upon the expression (23), we can now derive a scaling relationship between the length at which the helical rod becomes unstable and the values of κ, τ, and ω. Let $s_c(\kappa, \tau, \omega)$ denote the location of the first conjugate point along a helical rod with curvature $\kappa > 0$, torsion τ, and twist moment ω, i.e., the first positive value of s at which $\det(\widetilde{J}_{(\kappa,\tau,\omega)}(s)) = 0$. From the previous section, we know that $s_c(\kappa, \tau, \omega)$ is finite. Next, for any values of $\kappa > 0$, τ, and ω, and for any positive number λ, we have

$$\det(\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s_c(\lambda\kappa, \lambda\tau, \lambda\omega))) = 0$$

and

$$\det(\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s)) \neq 0 \text{ for all } s \in (0, s_c(\lambda\kappa, \lambda\tau, \lambda\omega)).$$

Using (23), we then have

$$\begin{aligned}&\lambda^{-12}\det(\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s_c(\lambda\kappa, \lambda\tau, \lambda\omega)))\\ &\quad = \det(\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s_c(\lambda\kappa, \lambda\tau, \lambda\omega)))\\ &\quad = 0\end{aligned}$$

and

$$\begin{aligned}&\lambda^{-12}\det(\widetilde{J}_{(\kappa,\tau,\omega)}(\lambda s))\\ &\quad = \det(\widetilde{J}_{(\lambda\kappa,\lambda\tau,\lambda\omega)}(s))\\ &\quad \neq 0 \text{ for all } s \in (0, s_c(\lambda\kappa, \lambda\tau, \lambda\omega)).\end{aligned}$$

We can therefore conclude that $\lambda s_c(\lambda\kappa, \lambda\tau, \lambda\omega)$ is the first conjugate point along the helical rod corresponding to the parameters κ, τ, and ω, i.e., $s_c(\kappa, \tau, \omega) = \lambda s_c(\lambda\kappa, \lambda\tau, \lambda\omega)$. In the companion paper [10], we rearrange this expression into the more useful form

$$s_c(\lambda\kappa, \lambda\tau, \lambda\omega) = \lambda^{-1}s_c(\kappa, \tau, \omega). \tag{26}$$

This scaling relationship allows us to compare the locations of conjugate points along different helical rods. Furthermore, in the companion paper [10], we use this relationship to prove that the closure of the set of points within the κ-τ-ω parameter space that correspond to stable helical rods is star convex. This geometric property is an essential component of the process used in [10] to compute and visualize the set of all stable rods with helical centerlines.

VII. DISCUSSION

We have shown that every configuration of an inextensible, unshearable, isotropic, and uniform elastic rod having fixed boundary conditions and a centerline with constant nonzero curvature becomes unstable at a finite length. In the process of establishing this result, we also showed that if the rod's curvature, torsion, and twist moment are fixed, then changes in the rod's material parameters do not affect stability. Finally, we derived a scaling relationship for comparing conjugate points along different rods with helical centerlines. These three properties are used in the companion paper [10] to compute and visualize the boundary between sets of stable and unstable helical elastic rods.

Each of the three properties described above may be extended to more general models of thin elastic structures. First, the conjugate point condition described in Sec. II can be used to determine the stability of elastic rods that are extensible, shearable, and anisotropic [15]. An approach similar to that used in Sec. V might show that all helical configurations within this larger class of elastic rods become unstable at a finite length, or more interestingly, that only some helical rods become unstable at a finite length while others remain stable for arbitrary length. Second, including the effects of extension, shear, and anisotropy introduces additional material parameters in the elastic energy functional (4). If the differential equations used to compute conjugate points can be integrated in closed form as in Sec. III, it might be possible to show that some of these parameters have no effect on the stability of helical configurations.

Third, the scaling relationship that we found in Sec. VI can be generalized in multiple ways. A result similar to (26) can be derived for helical rods that are anisotropic, but the inclusion of extension or shear appears to break the scaling relationship. Furthermore, while our derivation of this scaling relationship is specific to helical equilibria, a similar result can be obtained for more general configurations of elastic rods, which has applications to the problem of manipulating an elastic rod with a robot [28]. Finally, we note that the relationship (26) can be viewed as a comparison theorem akin to those in Riemannian geometry, which are used to compare conjugate points along geodesics [29]. These methods have been extended to sub-Riemannian geometry [30], and it may be possible to further generalize them to include problems such as the elastic rod considered in this paper. Such a generalization would allow a comparison of conjugate points along configurations that are not necessarily helical.

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APPENDIX

Evaluating the expressions in (10) and (11) and their derivatives at $s = 0$ provides expressions for the constants A_{1i}, A_{2i}, A_{3i}, B_{1i}, B_{2i}, B_{3i}, C_{1i}, and C_{2i} in terms of $\widetilde{M}(0)$, given by

$$A_{1i} = \frac{\kappa}{\sigma^2}\bigg(\kappa(\tau-\omega)\widetilde{M}_{1i}(0) + \kappa\widetilde{M}_{4i}(0) - (2\tau-\omega)\big((\tau-\omega)\widetilde{M}_{2i}(0) + \widetilde{M}_{5i}(0)\big)\bigg),$$

$$A_{2i} = -\frac{\kappa}{\sigma}\big((\tau-\omega)\widetilde{M}_{3i}(0) + \widetilde{M}_{6i}(0)\big),$$

$$A_{3i} = \widetilde{M}_{4i}(0) - A_{1i},$$

$$B_{1i} = \frac{1}{2}\bigg(\kappa(\tau-\omega)\widetilde{M}_{1i}(0) + \kappa\widetilde{M}_{4i}(0) - (2\tau-\omega)\big((\tau-\omega)\widetilde{M}_{2i}(0) + \widetilde{M}_{5i}(0)\big) - \frac{\sigma^2}{\kappa}A_{1i}\bigg),$$

$$B_{2i} = (\tau-\omega)\widetilde{M}_{3i}(0) + \widetilde{M}_{6i}(0) + \frac{\sigma}{\kappa}A_{2i},$$

$$B_{3i} = \widetilde{M}_{2i}(0) + \frac{1}{\kappa}A_{1i},$$

$$C_{1i} = -\bar{c}\kappa\widetilde{M}_{1i}(0) - (\tau-\omega)\widetilde{M}_{2i}(0) - \widetilde{M}_{5i}(0) - \frac{\omega-2\tau}{\kappa}A_{1i},$$

$$C_{2i} = \widetilde{M}_{3i}(0) + \frac{\omega-2\tau}{\sigma\kappa}A_{2i}.$$

[1] M. Born, Ph.D. thesis, Dieterichsche Universitäts-Buchdruckerei Göttingen (1906).
[2] J. H. Michell, Messenger Math. **11**, 181 (1889–1890).
[3] A. Goriely, J. Elasticity **84**, 281 (2006).
[4] J. H. Maddocks, Arch. Ration. Mech. Anal. **85**, 311 (1984).
[5] K. A. Hoffman, Philos. Trans. R. Soc. London A **362**, 1301 (2004).
[6] M. Cicalese, M. Ruf, and F. Solombrino, Calc. Var. **56**, 115 (2017).
[7] R. S. Manning, K. A. Rogers, and J. H. Maddocks, Proc. R. Soc. London A **454**, 3047 (1998).
[8] K. A. Hoffman, R. S. Manning, and R. C. Paffenroth, SIAM J. Appl. Dyn. Syst. **1**, 115 (2002).
[9] N. Chouaieb, Ph.D. thesis, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland, EPFL Thesis No. 2717, 2013.
[10] A. Borum and T. Bretl, Phys. Rev. Lett. **125**, 088001 (2020).
[11] G. Kirchhoff, J. Reine Angew. Math. (Crelle) **56**, 285 (1859).
[12] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944).
[13] S. S. Antman, Quart. Appl. Math. **32**, 221 (1974).
[14] A. B. Whitman and C. N. DeSilva, J. Elasticity **4**, 265 (1974).
[15] N. Chouaieb and J. H. Maddocks, J. Elasticity **77**, 221 (2004).
[16] N. Chouaieb, A. Goriely, and J. H. Maddocks, Proc. Natl. Acad. Sci. (USA) **103**, 9398 (2006).
[17] I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Prentice-Hall, Englewood Cliffs, NJ, 1964).
[18] T. Bretl and Z. McCarthy, Int. J. Robotics Res. **33**, 48 (2014).
[19] Y. Savhkov, J. Dyn. Control Syst. **14**, 169 (2008).
[20] Y. Savhkov, J. Dyn. Control Syst. **14**, 409 (2008).
[21] A. Agrachev, L. Rizzi, and P. Silveira, J. Dyn. Control Syst. **21**, 625 (2015).
[22] S. S. Antman, *Nonlinear Problems of Elasticity*, 2nd ed. (Springer-Verlag, New York, 2005).
[23] J. Biggs, W. Holderbaum, and V. Jurjdevic, IEEE Trans. Autom. Control **52**, 1027 (2007).
[24] A. Goriely and M. Tabor, Proc. R. Soc. London A **453**, 2583 (1997).
[25] D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction* (Princeton University Press, Princeton, NJ, 2012).
[26] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed. (Springer, New York, 1998).
[27] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (Interscience, New York, 1962).
[28] A. Borum and T. Bretl, in *2015 IEEE International Conference on Robotics and Automation (ICRA)* (IEEE, Seattle, WA, USA, 2015), pp. 2958–2964.
[29] J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry* (North-Holland, Amsterdam, 1975).
[30] D. Barilari and L. Rizzi, ESAIM Contr. Optim. Ca. **22**, 439 (2016).