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Some random paths with angle constraints

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Abstract. We propose a simple, geometrically-motivated construction of smooth random paths in the plane. The construction is such that, with probability one, the paths have finite curvature everywhere. Our construction is Markovian of order 2. We show that a simpler construction which is Markovian of order 1 fails to exhibit the desired finite curvature property.

Résumé. Nous étudions une manière élémentaire de construire des marches aléatoires du plan à l'aide d'angles aléatoires. Cette construction, issue de considérations géométriques, est telle que le processus limite possède presque sûrement des trajectoires dont la courbure est partout finie. Les marches aléatoires que nous exhibons sont markoviennes d'ordre 2, et nous montrons qu'une approche plus simple, avec des processus d'ordre 1, ne permet pas d'obtenir, à la limite, les propriétés désirées de courbure finie.

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1. Introduction

A random walk with independent increments having finite variance converges, when linearly interpolated, to a Brownian motion. This is the essence of the celebrated Donsker's theorem [5], and applies in any (finite) dimension. In fact, historically, Robert Brown's observations were of pollen particles moving in a solution, therefore in dimension two or three.

As is well-known, a Brownian motion is differentiable nowhere with probability one, and may be therefore inappropriate to model motion that is smoother. In the present paper, we are concerned with constructing a stochastic process in the plane that yields curves which have finite curvature almost surely. There are various relatively obvious constructions of such processes that fit the bill, such as integrating a Brownian motion twice (Figure 1), or interpolating a random sample of points using some splines such as cubic ones or GAMs (Generalized Additive Models) (Figure 2). In the first case, the realizations are less than pleasant in that they do not seem to curve much at all. In the later case, the construction is not particularly geometric in nature.

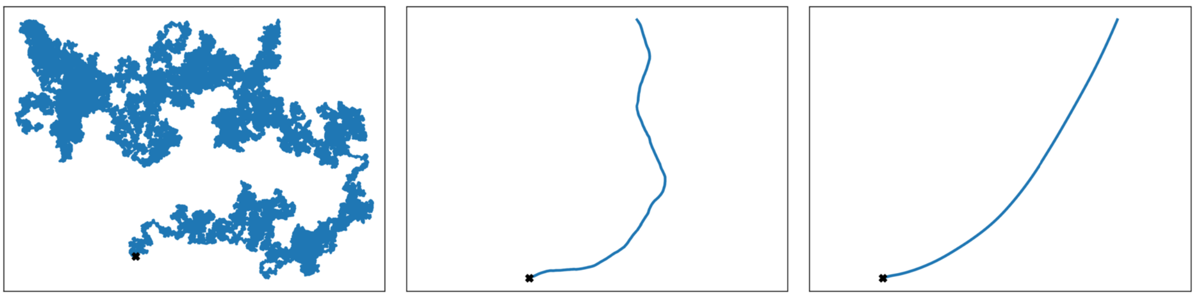

Fig. 1. A realization of a of 2-dimensional Brownian motion (left), which is then integrated once (center) and twice (right).

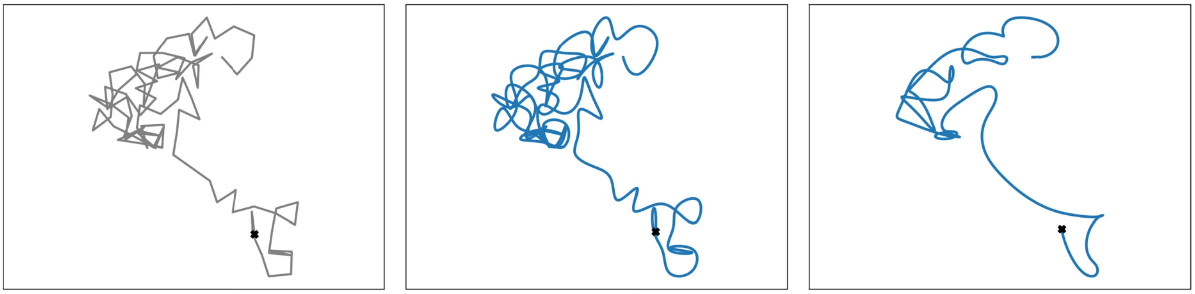

Fig. 2. Two realizations of smooth random processes using cubic splines interpolation (middle) and GAM regression (right) applied to a discrete random walk (left).

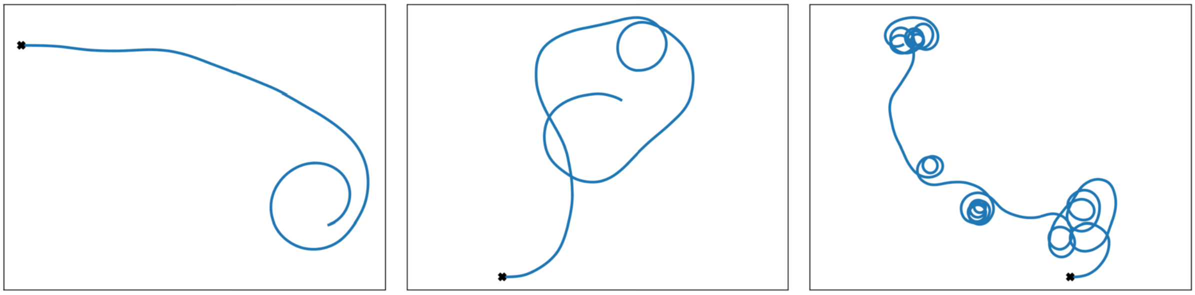

Fig. 3. A realization of the process defined in Section 4 for different values of the parameter defining it. Specifically, with the notation to be defined shortly, $n^{3/2}\alpha_n$ was taken to be 4 (left), 16 (middle), and 128 (right).

We propose a construction based on a random walk with nontrivial memory. Indeed, a random walk with no memory would again converge to a Brownian motion.

Our first attempt leads us to constraint the angle between two successive line segments in the polygonal line resulting from interpolating the random walk. In our construction, the line segments are all of unit length and the angles are drawn independently and uniformly at random in some interval – see (1) and (2) for a formal definition. It turns out that this construction fails in producing a smooth curve in the limit: when the angle interval remains constant, the process converges again to a Brownian motion (Theorem 1); when the angle interval has length tending to zero asymptotically, the smoothest limiting process we are able to obtain is only once differentiable (Theorem 3). Our second attempt is based on endowing the sequence of random angles in the construction with some memory. It turns out that a minimum amount of memory suffices for the construction to be successful (Theorem 4). A realization of this process is given in Figure 3.

Related work. There does not seem to be much literature on geometric constructions of random walks leading to processes with smooth paths. Tangentially related, [10] model polymer configurations as polygonal lines with random angles, but no continuous process limit is derived. We provide other pointers to the literature later on.

Content. The remainder of the article is organized as follows. In Section 2, we define and study a random walk where the successive angles are drawn iid from the uniform distribution on a fixed interval. We show that this construction results to a Brownian motion when taken to the limit (Theorem 1). In Section 3, we consider the same construction except that the interval from which the angles are sampled shrinks in size in the limit. We show that this construction results to either trivial limits (Proposition 3), to a Brownian motion (Theorem 2), or to a process whose realizations have infinite pointwise curvature everywhere with probability one (Theorem 3). In Section 4, we consider again the same basic construction, except that the angles are generated by a Markov process, and show that the limit is a process whose realizations have finite curvature everywhere with probability one (Theorem 4). We finish with a short discussion in Section 5.

2. Construction based on an iid sequence of angles

We consider a sequence of iid random variables $\{\Theta_i\}_{i\geq 2}$ with values in $\mathbb{R}$, which we use to define the following process: Starting with U_1 drawn uniformly at random from $\mathbb{S}^1$, recursively define

$$U_j = e^{i\Theta_j} U_{j-1}, \quad \text{for } j \geq 2. \tag{1}$$

Note that $U_1, U_2, \ldots$ are uniformly distributed on the unit circle, but not independent in general. Denote $\mathcal{G}_j$ the σ-field generated by $\{\Theta_k\}_{2\leq k\leq j}$ and U_1, so that U_j is $\mathcal{G}_j$-measurable for all j. We investigate the behavior of the piecewise-linear interpolation of this walk, namely

$$X_t^n = \sum_{j=1}^{\lfloor nt\rfloor} U_j + \big(nt - \lfloor nt\rfloor\big) U_{\lfloor nt\rfloor+1}, \quad \text{for } t \in [0,1]. \tag{2}$$

See Figure 4 for an illustration of this definition. We see X^n as a random variable taking its value in $\mathcal{C}_2 = C([0,1],\mathbb{R}^2)$, the set of continuous functions from $[0,1]$ to $\mathbb{R}^2$, endowed with the σ-field associated with the uniform topology.

Theorem 1. *If the random variables $\{\Theta_i\}_{i\geq 1}$ are uniformly distributed in $[-\alpha,\alpha]$, where $\alpha \in (0,\pi]$, then, as $n\to\infty$,*

$$\frac{1}{\sqrt{n}} X^n \Rightarrow \sigma_\alpha B^{(2)}, \quad \text{with } \sigma_\alpha^2 = \frac{1}{2}\frac{1+\operatorname{sinc}\alpha}{1-\operatorname{sinc}\alpha},$$

where $\Rightarrow$ stands for the weak convergence of probability measures, $B^{(2)}$ denotes the standard 2-dimensional Brownian motion, and $\operatorname{sinc}\alpha = \sin(\alpha)/\alpha$.

In particular, we recover Donsker's theorem (in dimension 2) when $\alpha = \pi$, the situation in which $\{U_i\}_{i\geq 1}$ are de facto independent (and therefore iid, since they are uniformly distributed on the circle). In general, however, the limit process is a scaled Brownian motion. See Figure 5 and Figure 6 for an illustration of Theorem 1.

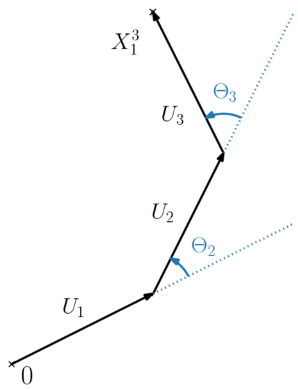

Fig. 4. The first steps of the random walk defined by (1)-(2) and its linear interpolation.

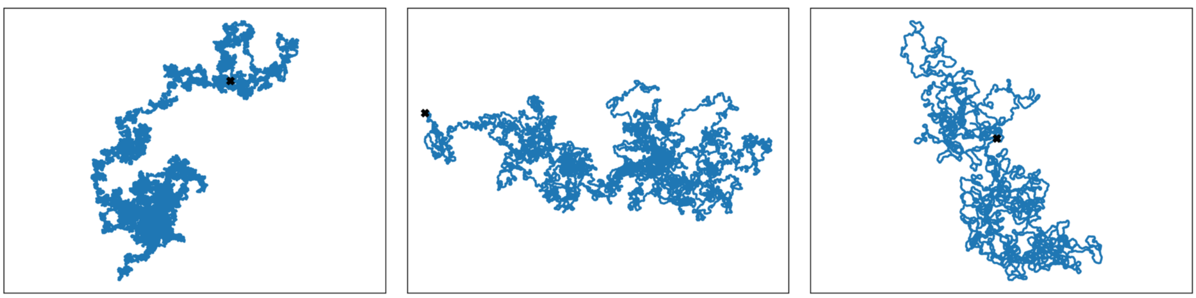

Fig. 5. A realization of the process defined in Theorem 1 for α being equal to $\pi/2$ (left), $\pi/8$ (center) or $\pi/16$ (right).

Fig. 6. A realization of the process defined in Theorem 1 for $\alpha = \pi/4$ observed at different scales.

When clear from the context, we will use the abbreviation B instead of $B^{(2)}$.

This first result shows that we cannot create smoothness from independent angles, no matter how small we constraint them to be. To prove Theorem 1, we will first show that the finite-dimensional laws of $\frac{1}{\sqrt{n}} X^n$ converge toward the ones of B, that is to say, as $n \to \infty$,

$$\frac{1}{\sqrt{n}}\left(X^n_{t_1}, \ldots, X^n_{t_k}\right) \Rightarrow \sigma_\alpha(B_{t_1}, \ldots, B_{t_k}),$$

for all $0 \le t_1 \le \cdots \le t_k \le 1$. (Here $\Rightarrow$ denotes the weak convergence of random vectors in the appropriate dimension, which is $2k$.) Once this is done, it will remain to show that the sequence of laws of $\frac{1}{\sqrt{n}} X^n$ is tight.[1]

Because the steps, $\{U_i\}_{i\ge 1}$, are not independent (at least when $\alpha < \pi$, which is the situation *not* covered by Donsker's theorem), we need a generalization of the central limit theorem for dependent random variables. (Unless otherwise specified, the convergence is as $n \to \infty$.)

Proposition 1. (Dependent CLT, [2]) *Let $\xi_{i,n}$ be centered with finite second moment random variables in $\mathbb{R}^d$. Let $k_n \to \infty$ be a sequence of integers. Suppose that the following conditions hold*:

$$\textit{There exists } 0 < \delta \le 1 \textit{ such that } A_n(\delta) = \sum_{i=1}^{k_n} \mathbb{E}\left[\|\xi_{i,n}\|^{2+\delta}\right] \to 0;$$

$$\textit{There exists a matrix } \Gamma \textit{ such that } \Gamma_n = \sum_{i=1}^{k_n} \mathrm{Cov}[\xi_{i,n}] \to \Gamma;$$

$$\textit{For any } t \in \mathbb{R}^d,\ T_n(t) = \sum_{i=1}^{k_n} \left|\mathrm{Cov}\left[f_t(\xi_{1,n} + \cdots + \xi_{i-1,n}),\ f_t(\xi_{i,n})\right]\right| \to 0,\ \textit{where } f_t : x \mapsto e^{i\langle x,t\rangle}.$$

Then,

$$S_n = \sum_{i=1}^{k_n} \xi_{i,n} \Rightarrow \mathcal{N}(0, \Gamma), \textit{ the centered normal law with covariance matrix } \Gamma.$$

We will apply Proposition 1, not to the steps U_j themselves, but instead to slices of the random walk, defined in our context as

$$\xi_{j,n} = \frac{1}{\sqrt{n}} \sum_{i=(j-1)(p_n+q_n)+1}^{(j-1)(p_n+q_n)+p_n} U_i. \tag{3}$$

Each slice contains p_n terms, and are q_n terms apart. We will need to have p_n large enough so that the sum $\sum_j \xi_{j,n}$ is close to $\frac{1}{\sqrt{n}} \sum_i U_i$, but also q_n large enough so that the $\xi_{j,n}$'s are all independent enough from each other.

We start with a covariance inequality.

[1] Because $\mathcal{C}_2$ is a polish space, tightness and relative compactness are classically equivalent notions, according to Prohorov's theorem. We will use these terms interchangeably.

Proposition 2. *Let $s_1 \leq \dots \leq s_u$ and $t_1 \leq \dots \leq t_v$ be positive integers. Suppose that $s_u \leq t_1$. Then, for any bounded measurable functions $f : \{\mathbb{R}^2\}^u \to \mathbb{R}$ and $g : \{\mathbb{R}^2\}^v \to \mathbb{R}$, we have*

$$\left|\operatorname{Cov}\left[f(U_{s_1}, \dots, U_{s_u}), g(U_{t_1}, \dots, U_{t_v})\right]\right| \leq \|f\|_\infty \|g\|_\infty \operatorname{TV}(\nu_{t_1 - s_u}, \nu),$$

where ν is the uniform law over $[0, 2\pi]$, ν_r is the law of $\sum_{i=1}^r \Theta_i \mod 2\pi$, and TV is the total variation distance.[2]

Remark 1. If f and g are complex-valued, this results remains true up to a numeric constant. Indeed, for any random variables $X, Y \in \mathbb{C}$, noting $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$, we have

$$\begin{aligned}\left|\operatorname{Cov}[X, Y]\right|^2 &= \left(\operatorname{Cov}[X_1, Y_1] - \operatorname{Cov}[X_2, Y_2]\right)^2 + \left(\operatorname{Cov}[X_1, Y_2] + \operatorname{Cov}[X_2, Y_1]\right)^2 \\ &\leq 8 \max_{i,j \in \{1,2\}} \left|\operatorname{Cov}[X_i, Y_j]\right|^2.\end{aligned}$$

Proof. We set $\Delta = |\operatorname{Cov}[f(U_{s_1}, \dots, U_{s_u}), g(U_{t_1}, \dots, U_{t_v})]|$. Recall that for any $j \in \mathbb{N}^*$, $\mathcal{G}_j$ is the σ-field generated by $\{\Theta_k\}_{2 \leq k \leq j}$ and U_1. We have

$$\begin{aligned}\Delta &= \left|\mathbb{E}\left[f(U_{s_1}, \dots, U_{s_u}) g(U_{t_1}, \dots, U_{t_v})\right] - \mathbb{E}\left[f(U_{s_1}, \dots, U_{s_u})\right]\mathbb{E}\left[g(U_{t_1}, \dots, U_{t_v})\right]\right| \\ &= \left|\mathbb{E}\left[f(U_{s_1}, \dots, U_{s_u})\left(\mathbb{E}\left[g(U_{t_1}, \dots, U_{t_v}) \mid \mathcal{F}_{s_u}\right] - \mathbb{E}\left[g(U_{t_1}, \dots, U_{t_v})\right]\right)\right]\right| \\ &\leq \|f\|_\infty \mathbb{E}\left[\left|\mathbb{E}\left[g(U_{t_1}, \dots, U_{t_v}) | \mathcal{G}_{s_u}\right] - \mathbb{E}\left[g(U_{t_1}, \dots, U_{t_v})\right]\right|\right].\end{aligned}$$

Note that, there exists random variables Φ, Ψ and Z' such that the vector $Z = (U_{t_1}, \dots, U_{t_v})$, which takes values in $\{\mathbb{R}^2\}^v$, can be written in the form $Z = \exp\{i(\Phi + \Psi)\}Z'$ as follows

$$\begin{aligned}Z &= (U_{t_1}, \dots, U_{t_v}) \\ &= \left(U_1 \exp\left(i \sum_{j=2}^{t_1} \Theta_j\right), \dots, U_1 \exp\left(i \sum_{j=2}^{t_v} \Theta_j\right)\right) \\ &= \exp\left(i \sum_{j=2}^{s_u} \Theta_j\right) \exp\left(i \sum_{j=s_u+1}^{t_1} \Theta_j\right)\left(U_1, U_1 \exp\left(\sum_{j=t_1+1}^{t_2} \Theta_j\right), \dots, U_1 \exp\left(\sum_{j=t_1+1}^{t_v} \Theta_j\right)\right) \\ &=: \exp(i\Phi)\exp(i\Psi)Z'.\end{aligned}$$

The random variable Z' has same law as $(U_1, U_{t_2 - t_1 + 1}, \dots, U_{t_v - t_1 + 1})$, which is the same as Z by strong stationarity of $(U_1, U_2, \dots)$. Furthermore, Φ is $\mathcal{G}_{s_u}$-measurable, and Ψ and Z' are independent of $\mathcal{G}_{s_u}$. Using the fact that the law of Z is rotationally-invariant and letting Θ be a random variable with law ν and independent from Z', and denoting by ζ the law of Z', we get

$$\begin{aligned}\Delta &\leq \|f\|_\infty \mathbb{E}\left[\left|\mathbb{E}\left[g\left(\exp\left\{i(\Psi + \Phi)\right\}Z'\right) | \mathcal{G}_{s_u}\right] - \mathbb{E}\left[g\left(\exp\{i\Theta\}Z'\right)\right]\right|\right] \\ &\leq \|f\|_\infty \sup_{\phi \in [0, 2\pi]} \left|\mathbb{E}\left[g\left(\exp\left\{i(\Psi + \phi)\right\}Z'\right)\right] - \mathbb{E}\left[g\left(\exp\{i\Theta\}Z'\right)\right]\right| \\ &\leq \|f\|_\infty \|g\|_\infty \operatorname{TV}(\nu_{t_1 - s_u} \otimes \zeta, \nu \otimes \zeta) \\ &\leq \|f\|_\infty \|g\|_\infty \operatorname{TV}(\nu_{t_1 - s_u}, \nu). \end{aligned} \tag{4}$$

In (4), we used the fact that the function $g_\phi : (\psi, z) \in [0, 2\pi] \times \{\mathbb{R}^2\}^v \mapsto g(e^{i(\psi + \phi)}z)$ is bounded by $\|g\|_\infty$, the definition of the total variation distance, and in the last inequality we used the subadditivity of the latter. □

We turn now to bounding $\operatorname{TV}(\nu_r, \nu)$, which again is the total variation between ν_r, the law of $\sum_{i=1}^r \Theta_i$ (modulo 2π), and ν, the uniform distribution on $[0, 2\pi]$.

[2] Recall that for any probability laws $\mathbb{P}$ and $\mathbb{Q}$ on some measurable space $(\mathcal{X}, \mathcal{B})$, $\operatorname{TV}(\mathbb{P}, \mathbb{Q}) = \sup_f \{|\mathbb{P}(f) - \mathbb{Q}(f)|\}$ where the supremum is over $f : \mathcal{X} \to \mathbb{R}$ measurable such that $\|f\|_\infty \leq 1$.

Lemma 1. *Let μ be a symmetric and absolutely continuous distribution over $\mathbb{R}$. Letting ν_r denote the distribution $\mu_r = \mu^{*r}$, but modulo 2π, we have*

$$\mathrm{TV}(\nu_r, \nu) \le \sum_{k\ge 1} \left|\phi_\mu(k)\right|^r, \tag{5}$$

where ϕ_μ is the characteristic function of μ. In the special case where μ is the uniform distribution on $[-\alpha, \alpha]$, where $\alpha \in (0, \pi]$, there exists a positive numeric constant A such that, for $r \ge 2$,

$$\mathrm{TV}(\nu_r, \nu) \le \frac{A}{\alpha}\big(\mathrm{sinc}(\alpha) \vee 2/\pi\big)^r$$

and so the total variation distance between ν_r and ν decreases exponentially fast as $r \to \infty$.

Proof. Since μ is absolutely continuous with respect to the Lebesgue measure, so is μ_r, and for any Borel set A of $[0, 2\pi]$ we have

$$\nu_r(A) = \int_{[0,2\pi]} 1_A \, d\nu_r = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} 1_{A+2k\pi} \, d\mu_r = \sum_{k\in\mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} 1_{A+2k\pi}(x) \frac{d\mu_r}{dx}(x)\, dx$$
$$= \int_0^{2\pi} 1_A(x) \sum_{k\in\mathbb{Z}} \frac{d\mu_r}{dx}(x + 2k\pi)\, dx.$$

The law of ν_r is thus absolutely continuous with respect to ν, and $\frac{d\nu_r}{d\nu}(x) = 2\pi \sum_{k\in\mathbb{Z}} \frac{d\mu_r}{dx}(x + 2k\pi)$. The RHS can be computed with the Poisson summation formula

$$2\pi \sum_{k\in\mathbb{Z}} \frac{d\mu_r}{dx}(x + 2k\pi) = \sum_{k\in\mathbb{Z}} \mathcal{F}\left[\frac{d\mu_r}{dx}\right](k) e^{ikx},$$

where $\mathcal{F}$ is the Fourier transform. With the classical property of the convolution product, we can get

$$\mathcal{F}\left[\frac{d\mu_r}{dx}\right](k) = \mathcal{F}\left[\frac{d\mu}{dx}\right]^r(k) = \phi_\mu\big(|k|\big)^r,$$

since μ is symmetric, so that

$$\mathrm{TV}(\nu_r, \nu) = \frac{1}{2} \int \left|\frac{d\nu_r}{d\nu} - 1\right| d\nu \le \sum_{k\ge 1} \left|\phi_\mu(k)\right|^r.$$

This proves the stated bound (5).

When μ is uniform over $[-\alpha, \alpha]$, we have $\phi_\mu(k) = \mathrm{sinc}(k\alpha)$. We use this in order to bound the sum on the RHS of (5). We distinguish two cases according to the value of α. If $\alpha > \pi/2$, we immediately get that

$$\sum_{k\ge 1} \left|\phi_\mu(k)\right|^r \le \sum_{k\ge 1} 1/(k\alpha)^r \le \big(\zeta(r) - 1\big)(\pi/2)^{-r} \le \big(\zeta(2) - 1\big)(\pi/\alpha)(\pi/2)^{-r},$$

where ζ is the Riemann zeta function. If $\alpha \le \pi/2$, we split the sum at $n_\alpha = \lfloor \pi/\alpha \rfloor$. For the first part of the sum, we simply have

$$\sum_{k=1}^{n_\alpha} \left|\phi_\mu(k)\right|^r \le n_\alpha (\mathrm{sinc}\,\alpha)^r \le (\pi/\alpha)(\mathrm{sinc}\,\alpha)^r,$$

which is justified because sinc is decreasing on the segment $[0, \pi]$ and $k\alpha \le \pi$ for all $k \le n_\alpha$. For the second part of the sum,

$$\sum_{k>n_\alpha} \frac{1}{(k\alpha)^r} \le \frac{1}{\alpha^r} \int_{n_\alpha}^{\infty} \frac{dx}{x^r} = \frac{1}{(r-1)\alpha^r n_\alpha^{r-1}} \le \frac{n_\alpha}{(r-1)(\pi - \alpha)^r} \le \frac{\pi}{(r-1)\alpha} (\pi/2)^{-r}.$$

Summing these two parts, all in all, we indeed get a bound of the desired form. □

A very simple and straightforward computation of the covariance gives the following

$$\mathrm{Cov}[U_j, U_{j+k}] = \frac{1}{2}(\operatorname{sinc}\alpha)^k \,\mathrm{Id}_2, \quad \text{for all } j, k \in \mathbb{N}^*. \tag{6}$$

Recall the definition (3). We have the following.

Lemma 2. *If p_n and q_n are two sequences of integers diverging to ∞ such that $p_n + q_n \le n$ and $q_n \ll p_n \ll n$, then $\mathbb{E}[\|S_n - S_n^*\|^2] \to 0$, where $S_n = \frac{1}{\sqrt{n}}\sum_{j=1}^n U_j$, $S_n^* = \sum_{k=1}^{k_n} \xi_{k,n}$ and $k_n = \lfloor n/(p_n + q_n)\rfloor$.*

Here, we write $a_n \ll b_n$ for any two real-valued sequences a_n and b_n for $a_n = o(b_n)$ as $n \to \infty$. This result appears in [6, Section 4.3.1] in the context of real-valued time series. Although this is not difficult, we extend the result to bivariate time series for the sake of completeness.

Proof. We start with the fact that

$$S_n - S_n^* = \sum_{i=1}^{k_n+1} \xi^*_{i,n},$$

where

$$\xi^*_{k,n} = \frac{1}{\sqrt{n}} \sum_{i=(k-1)(p_n+q_n)+p_n+1}^{k(p_n+q_n)} U_i, \quad \text{for } k \le k_n; \quad \text{and} \quad \xi^*_{k_n+1,n} = \frac{1}{\sqrt{n}} \sum_{i=k_n(p_n+q_n)}^{n} U_i.$$

Simple calculations give

$$\begin{aligned}
\mathbb{E}\big[\|S_n - S_n^*\|^2\big] &= \mathbb{E}\left[\left\|\sum_{i=1}^{k_n+1} \xi^*_{i,n}\right\|^2\right] \le 2\mathbb{E}\left[\left\|\sum_{i=1}^{k_n} \xi^*_{i,n}\right\|^2\right] + 2\mathbb{E}\big[\|\xi^*_{k_n+1,n}\|^2\big] \\
&\le 2\sum_{1\le i,j\le k_n} \operatorname{tr}\big(\mathrm{Cov}\big[\xi^*_{i,n}, \xi^*_{j,n}\big]\big) + 2\operatorname{tr}\big(\mathrm{Cov}\big[\xi^*_{k_n+1,n}\big]\big).
\end{aligned}$$

When $i = j$, we have, since U_j is strongly stationary and using formula (6),

$$\begin{aligned}
\operatorname{tr}\big(\mathrm{Cov}\big[\xi^*_{i,n}, \xi^*_{i,n}\big]\big) &= \frac{q_n}{n}\mathbb{E}\big[\|U_0\|^2\big] + \frac{2}{n}\sum_{1\le j<k\le q_n} \operatorname{tr}\big(\mathrm{Cov}[U_j, U_k]\big) \\
&= \frac{q_n}{n} + \frac{2}{n}\sum_{k=1}^{q_n-1}(q_n - k)(\operatorname{sinc}\alpha)^k = O\left(\frac{q_n}{n}\right).
\end{aligned}$$

We have, likewise, $\operatorname{tr}(\mathrm{Cov}[\xi^*_{k_n+1,n}]) = O(p_n/n)$.

When $i \ne j$, the steps of $\xi_{i,n}$ and $\xi_{j,n}$ are at least $|i-j|p_n$ apart, so that, using again (6),

$$\operatorname{tr}\big(\mathrm{Cov}\big[\xi^*_{i,n}, \xi^*_{j,n}\big]\big) \le \frac{q_n^2}{n} \sup_{k\ge|i-j|p_n} \operatorname{tr}\big(\mathrm{Cov}[U_1, U_{k+1}]\big) \le \frac{q_n^2}{n}(\operatorname{sinc}\alpha)^{|i-j|p_n}.$$

Combining these bounds, we get that

$$\begin{aligned}
\mathbb{E}\big[\|S_n - S_n^*\|^2\big] &= O\left(\frac{k_n q_n}{n} + \frac{p_n}{n} + \frac{q_n^2}{n}\sum_{k=1}^{k_n}(k_n - k)(\operatorname{sinc}\alpha)^{kp_n}\right) \\
&= O\left(\frac{q_n}{p_n} + \frac{p_n}{n} + \frac{q_n^2 k_n}{n}(\operatorname{sinc}\alpha)^{p_n}\right) \\
&= O\left(\frac{q_n}{p_n} + \frac{p_n}{n}\right) \longrightarrow 0,
\end{aligned}$$

which ends the proof. □

In view of Lemma 2, it is thus sufficient to establish the convergence in law for S_n^* to deduce the same for S_n. This is exactly what we do next.

Lemma 3. *The finite-dimensional laws of* $\frac{1}{\sqrt{n}}X^n$ *converge towards the ones of* $\sigma_\alpha B$.

Proof. We use the notation introduced in Lemma 2. We apply Proposition 1 to $S_n^* = \sum_{k=1}^{k_n} \xi_{k,n}$, and we also use the notation introduced there.

For the first condition, by stationarity, for any $\delta > 0$ we have

$$A_n(\delta) = \sum_{k=1}^{k_n} \mathbb{E}\big[\|\xi_{k,n}\|^{2+\delta}\big] = k_n \mathbb{E}\big[\|\xi_{1,n}\|^{2+\delta}\big] \le k_n \frac{p_n^{2+\delta}}{n^{1+\delta/2}} \le \frac{p_n^{1+\delta}}{n^{\delta/2}},$$

where the first inequality comes from the fact that $\|\xi_{k,n}\| \le p_n/\sqrt{n}$ (due to the triangle inequality and the fact that $U_j \in \mathbb{S}^1$ for all j), and the second inequality comes from the definition of k_n. It thus suffices to show that $p_n \ll n^{\delta/(2\delta+2)}$ to have that $A_n(\delta)$ converges towards 0.

For the third condition, we control $T_n(t)$ with a straightforward application of Proposition 2 and Lemma 1, as follows

$$\begin{aligned} T_n(t) &= \sum_{j=1}^{k_n} \big|\mathrm{Cov}\big[f_t(\xi_{1,n} + \cdots + \xi_{j-1,n}), f_t(\xi_{j,n})\big]\big| \\ &\le \sum_{j=1}^{k_n} 4\,\mathrm{TV}(\nu_{q_n}, \nu) \\ &\le 4k_n \frac{A}{\alpha}(\operatorname{sinc}\alpha \vee 2/\pi)^{q_n} = O\big(n\theta^{q_n}\big), \quad \text{where } \theta = \operatorname{sinc}\alpha \vee 2/\pi, \end{aligned} \tag{7}$$

which yields that $T_n(t) \to 0$ as soon as $q_n \gg \log n$. In (7) we used the fact that $f_t(\xi_{1,n} + \cdots + \xi_{j-1,n})$ and $f_t(\xi_{j,n})$ are bounded functions of $U_1, U_2, \ldots, U_{(j-2)(p_n+q_n)+p_n}$ and $U_{(j-1)(p_n+q_n)+1}, \ldots, U_{(j-1)(p_n+q_n)+p_n}$, respectively.

Finally, for the second condition, using again stationarity and using (6), we get that

$$\begin{aligned} \Gamma_n &= k_n \,\mathrm{Cov}[\xi_{1,n}] = \frac{k_n}{n} \sum_{1 \le i,j \le p_n} \mathrm{Cov}[U_i, U_j] \\ &= \frac{k_n}{2n}\left(p_n + \sum_{p=1}^{p_n}(p_n - p)(\operatorname{sinc}\alpha)^p\right)\mathrm{Id}_2 \\ &= \frac{k_n}{2n}\left(p_n + 2(\operatorname{sinc}\alpha)\frac{p_n(1-\operatorname{sinc}\alpha) + (\operatorname{sinc}\alpha)^{p_n} - 1}{(1-\operatorname{sinc}\alpha)^2}\right)\mathrm{Id}_2 \to \frac{1}{2}\frac{1+\operatorname{sinc}\alpha}{1-\operatorname{sinc}\alpha}\mathrm{Id}_2, \end{aligned} \tag{8}$$

where, in the convergence, we used the fact that $p_n k_n \sim n$ and $p_n \to \infty$.

Thus, for the conditions of Proposition 1 to be fulfilled, it suffices to choose sequences p_n and q_n such that $\log n \ll q_n \ll p_n \ll n^{\delta/(2\delta+2)}$, which we do. We may then apply Proposition 1, to get that S_n^* converges weakly to $\mathcal{N}(0, \sigma_\alpha^2 \mathrm{Id}_2)$ or, equivalently, to $\sigma_\alpha B_1$. And in light of Lemma 2, we may conclude that the same is true for $S_n = \frac{1}{\sqrt{n}}X_1^n$.

The same argumentation leads as easily to establishing that $\frac{1}{\sqrt{n}}X_t^n$ converges weakly to $\sigma_\alpha B_t$, and even that $\frac{1}{\sqrt{n}}(X_t^n - X_s^n)$ converges weakly to $\sigma_\alpha(B_t - B_s)$ for any $0 \le s \le t \le 1$.

Now, let $0 \le t_1 \le \cdots \le t_k \le 1$ be a sequence of real numbers and $t_0 = 0$. We set

$$Z^n = \frac{1}{\sqrt{n}}\big(X_{t_1}^n, X_{t_2}^n - X_{t_1}^n, \ldots, X_{t_k}^n - X_{t_{k-1}}^n\big),$$

with values in $\{\mathbb{R}^2\}^k$, and we write $Z^n = Y^n + \epsilon^n$, where

$$Y_j^n = \frac{1}{\sqrt{n}} \sum_{q=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j\rfloor - q_n} U_q \quad \text{and} \quad \epsilon_j^n = Z_j^n - Y_j^n.$$

Similar arguments lead to the fact that $\mathbb{E}[\|\epsilon_j^n\|^2] = O(q_n/n) \to 0$ as soon as $q_n \ll n$, thus $\mathbb{E}[\|Z^n - Y^n\|^2] \to 0$, implying that Z^n and Y^n have thereby the same limit law, provided that one of them has a limit law. In particular, we know that Y_j^n converges weakly towards $\sigma_\alpha(B_{t_j} - B_{t_{j-1}})$ for all j. Let $u = (u_1, \dots, u_k) \in \{\mathbb{R}^2\}^k$. By recurrence on k, it is easy to show the following formula

$$\left|\mathbb{E}\big[e^{i\langle u, Y^n\rangle}\big] - \prod_{j=1}^k \mathbb{E}\big[e^{i\langle u_j, Y_j^n\rangle}\big]\right| \le \sum_{j=2}^k \big|\mathrm{Cov}\big[e^{i(\langle u_1, Y_1^n\rangle + \dots + \langle u_{j-1}, Y_{j-1}^n\rangle)}, e^{i\langle u_j, Y_j^n\rangle}\big]\big|.$$

With Proposition 2 and Lemma 1, the RHS is bounded from above by $\sum_j 4\alpha^{-1} A\theta^{q_n} = O(\theta^{q_n}) \to 0$ as soon as $q_n \to \infty$. Since we already know that Y_j^n converges weakly towards $\sigma_\alpha(B_{t_j} - B_{t_{j-1}})$, we can conclude using the Levy continuity theorem. □

We conclude the proof of Theorem 1 with the following result.

Lemma 4. *The sequence of laws of* $\frac{1}{\sqrt{n}} X^n$ *is relatively compact.*

Proof. For $n \in \mathbb{N}$, we now note $S_n = \sum_{k=1}^n U_k$. We have

$$\mathbb{E}\big[\|S_n\|^4\big] = \mathbb{E}\Bigg[\bigg(\sum_{1\le i,j\le n} \langle U_i, U_j\rangle\bigg)^2\Bigg] = \sum_{1\le i,j,k,l\le n} \mathbb{E}\big[\langle U_i, U_j\rangle\langle U_k, U_l\rangle\big].$$

Using that $ab \le a \wedge b$ for any $a, b \in [0, 1]$, and using formula (6), we find that

$$\begin{aligned}\mathbb{E}\big[\|S_n\|^4\big] &\le \sum_{1\le i,j,k,l\le n} \mathbb{E}\big[\langle U_i, U_j\rangle\big] \wedge \mathbb{E}\big[\langle U_k, U_l\rangle\big] = \sum_{1\le a,b\le n} (n-a)(n-b)(\operatorname{sinc}\alpha)^{a\vee b} \\ &= 2\sum_{k=1}^n (n-k)\bigg(nk - \frac{k(k+1)}{2}\bigg)(\operatorname{sinc}\alpha)^k \le 2n^2 \sum_{k=1}^n k(\operatorname{sinc}\alpha)^k \\ &\le \frac{2n^2}{(1-\operatorname{sinc}\alpha)^2}.\end{aligned}$$

Using [4, Thm. 10.2], which we may since the process $\{U_k\}$ is stationary, we get that there exists a numeric constant $K > 0$ such that, for any $\lambda > 0$,

$$\mathbb{P}\Big(\max_{k\le n} \|S_k\| \ge \lambda\Big) \le \frac{Kn^2}{(1-\operatorname{sinc}\alpha)^2\lambda^4}. \tag{9}$$

Then, Lemma 5 below yields tightness, and hence relative compactness, of the sequence of law of $\frac{1}{\sqrt{n}} X^n$. □

Lemma 5 ([4, Lem. on p. 88]). *Let* ξ_i *be stationary, real-valued and square integrable random variables with variance* σ^2. *Let* $W_t^n = \frac{1}{\sigma\sqrt{n}}(S_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor)\xi_{\lfloor nt\rfloor+1})$, *where* $S_k = \sum_{j=1}^k \xi_j$. *If*

$$\lim_{\lambda\to+\infty} \limsup_{n\to\infty} \lambda^2 \mathbb{P}\Big(\max_{1\le k\le n} |S_k| \ge \lambda\sigma\sqrt{n}\Big) = 0,$$

then, the sequence of the laws of W^n *is tight.*

3. Construction based on a triangular array of angles

We now place ourselves in the setting where the laws of the angles Θ_j can vary with n. Let $\{\Theta_{j,n}\}_{j\ge 2, n\ge 1}$ be a collection of real valued random variables. As in Section 2, define the following process: Starting with $U_{1,n}$ drawn uniformly at random from $\mathbb{S}^1$, recursively define

$$U_{j,n} = e^{i\Theta_{j,n}} U_{j-1,n}, \quad \text{for } j \ge 2,$$

and then

$$X_t^n = \sum_{j=1}^{\lfloor nt \rfloor} U_{j,n} + \big(nt - \lfloor nt \rfloor\big) U_{\lfloor nt \rfloor+1,n} \quad \text{for } t \in [0,1].$$

Contrary to what has been done in the previous section, we will this time normalize X^n with n instead of $\sqrt{n}$. Note that, if one wants to obtain a smooth – and thus rectifiable – curve at the limit, this is the only reasonable normalization.

Lemma 6. *For any $n \geq 1$, as a function on $[0,1]$ with values in $\mathbb{R}^2$, $\frac{1}{n}X^n$ is 1-Lipschitz.*

Proof. For $0 \leq s \leq t \leq 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\left\| \frac{1}{n}X_t^n - \frac{1}{n}X_s^n \right\| &= \frac{1}{n} \left\| \sum_{k=\lfloor ns \rfloor+2}^{\lfloor nt \rfloor} U_{k,n} + \big(nt - \lfloor nt \rfloor\big) U_{\lfloor nt \rfloor+1,n} + \big(1 - ns + \lfloor ns \rfloor\big) U_{\lfloor ns \rfloor+1,n} \right\| \\ &\leq \frac{1}{n}\Big(\lfloor nt \rfloor - \lfloor ns \rfloor - 1 + \big(nt - \lfloor nt \rfloor\big) + \big(1 - ns + \lfloor ns \rfloor\big)\Big) = t - s,\end{aligned}$$

by a simple application of the triangle inequality and the fact that $U_{k,n} \in \mathbb{S}^1$ for all k. □

Corollary 1. *As a sequence of laws on $\mathcal{C}_2$, $\{\frac{1}{n}X^n\}$ is relatively compact.*

Proof. This is an immediate consequence of Lemma 6 and the fact that the set of 1-Lipschitz functions from $[0,1]$ to $\mathbb{R}^2$ taking value $(0,0) \in \mathbb{R}^2$ at 0 is relatively compact by the Arzelà–Ascoli theorem; see [4, Thm. 7.2 p. 81]. □

We first investigate the case where

$$\begin{aligned}&\Theta_{j,n},\ j \geq 1 \text{ are iid from the uniform distribution on } [-\alpha_n, \alpha_n], \\ &\alpha_n \in (0, \pi] \text{ is a sequence of angles converging to } 0.\end{aligned} \tag{10}$$

We observe two degenerate regimes when α_n converges either too fast or too slow towards 0.

Proposition 3. *Consider a sequence of angles as in (10). If $n\alpha_n^2 \to \infty$, then $\frac{1}{n}X^n \Rightarrow 0$ in $\mathcal{C}_2$. If $n\alpha_n^2 \to 0$, then $\frac{1}{n}X_t^n \Rightarrow tU$ in $\mathcal{C}_2$, where U denotes a random vector with the uniform distribution on $\mathbb{S}^1$.*

Proof. We first suppose that $n\alpha_n^2 \to \infty$. In this case, we have that for any t, developing the square like we did in formula (8),

$$\mathbb{E}\left[\left\| \frac{1}{n}X_t^n \right\|^2\right] = 2(\operatorname{sinc}\alpha_n)\frac{\lfloor nt \rfloor(1 - \operatorname{sinc}\alpha_n) + (\operatorname{sinc}\alpha_n)^{\lfloor nt \rfloor} - 1}{n^2(1 - \operatorname{sinc}\alpha_n)^2} + O\left(\frac{1}{n}\right), \tag{11}$$

where the $O(1/n)$ term corresponds to the one coming from $U_{\lfloor nt \rfloor+1,n}$ in the definition of X_t^n. Since

$$(\operatorname{sinc}\alpha_n)^{\lfloor nt \rfloor} = \exp\big\{\lfloor nt \rfloor \log\big(1 - \alpha_n^2/6 + o\big(\alpha_n^2\big)\big)\big\} = \exp\big\{-\lfloor nt \rfloor \alpha_n^2 + o\big(n\alpha_n^2\big)\big\} \to 0,$$

and $n(1 - \operatorname{sinc}\alpha_n) \sim n\alpha_n^2/6$, we find that

$$\mathbb{E}\left[\left\| \frac{1}{n}X_t^n \right\|^2\right] = O\left(\frac{1}{n\alpha_n^2}\right) + O\left(\frac{1}{n}\right) \longrightarrow 0.$$

Finite-dimensional laws of $\frac{1}{n}X^n$ all converge to 0 and thus $\frac{1}{n}X^n \Rightarrow 0$ in $\mathcal{C}_2$ by relative compactness (Corollary 1).

We now assume that $n\alpha_n^2 \to 0$. We then get

$$1 - (\operatorname{sinc}\alpha_n)^{\lfloor nt \rfloor} = 1 - \exp\left(-\frac{1}{6}\lfloor nt \rfloor \alpha_n^2 + o\big(n\alpha_n^2\big)\right) = \frac{1}{6}\lfloor nt \rfloor \alpha_n^2 + o\big(n\alpha_n^2\big),$$

so that

$$\frac{1}{n}\frac{1-(\operatorname{sinc}\alpha_n)^{\lfloor nt\rfloor}}{1-\operatorname{sinc}\alpha_n} \longrightarrow t, \quad \text{for any } t \in [0,1].$$

Developing (11) to the next order, we find

$$\lfloor nt\rfloor(1-\operatorname{sinc}\alpha_n)+(\operatorname{sinc}\alpha_n)^{\lfloor nt\rfloor}-1=\frac{1}{72}\lfloor nt\rfloor^2\alpha_n^4+o\big(n^2\alpha_n^4\big),$$

and this leads to $\mathbb{E}[\|\frac{1}{n}X_t^n\|^2]\to t^2$. We then conclude with

$$\begin{aligned}\mathbb{E}\left[\left\|\frac{1}{n}X_t^n-tU_{1,n}\right\|^2\right]&=\mathbb{E}\left[\left\|\frac{1}{n}X_t^n\right\|^2\right]+t^2-2\frac{t}{n}\mathbb{E}\big[\langle X_t^n,U_1^n\rangle\big]\\&=t^2+o(1)+t^2+2\frac{t}{n}\sum_{j=1}^{n}\mathbb{E}\big[\langle U_j,U_1^n\rangle\big]\\&=2t^2-2t\frac{1}{n}\frac{1-(\operatorname{sinc}\alpha_n)^{\lfloor nt\rfloor}}{1-\operatorname{sinc}\alpha_n}+o(1)\longrightarrow 0,\end{aligned} \tag{12}$$

where at (12) we used (6), together with the relative compactness of $\{\frac{1}{n}X^n\}$ as a sequence of laws (Corollary 1). □

When $n\alpha_n^2\to\infty$ sufficiently fast, with a different normalization, X^n in fact converges to a Brownian motion. The precise normalization that results in this is given below. (In a sense, Theorem 1 is a special case of the following.) See Figure 7 for an illustration of Proposition 3 and Theorem 2.

Theorem 2. *Consider a sequence of angles as in* (10). *If $n\alpha_n^2\gg n^\omega$ for some $\omega\in(0,1)$, then*

$$\frac{\alpha_n}{\sqrt{n}}X^n \Rightarrow \sqrt{3}B.$$

Proof. The arguments are similar to those given in the proof of Theorem 1 in Section 2, so we will omit some details. Let $q_n \ll p_n \ll n$ be two sequences of integers with $p_n, q_n\to\infty$ and such that $p_n+q_n<n$. Let $k_n=\lfloor n/(p_n+q_n)\rfloor$. We introduce the random variables

$$\xi_{k,n}=\frac{\alpha_n}{\sqrt{n}}\sum_{i=(k-1)(p_n+q_n)+1}^{(k-1)(p_n+q_n)+p_n}U_{i,n},$$

and $S_n^*=\sum_{i=1}^{k_n}\xi_{i,n}$. We set $S_n=\frac{\alpha_n}{\sqrt{n}}X_1^n$. Mimicking the proof of Lemma 2, and using again Proposition 2 and Lemma 1, we get

$$\begin{aligned}\mathbb{E}\big[\|S_n-S_n^*\|^2\big]&=O\left(\frac{k_nq_n}{n}+\frac{p_n}{n}+\frac{q_n^2}{n\alpha_n}\sum_{k=1}^{k_n}(k_n-k)(\operatorname{sinc}\alpha_n)^{kp_n}\right)\\&=O\left(\frac{q_n}{p_n}+\frac{p_n}{n}+\frac{q_n^2k_n}{n\alpha_n}\frac{(\operatorname{sinc}\alpha_n)^{p_n}}{1-(\operatorname{sinc}\alpha_n)^{p_n}}\right).\end{aligned}$$

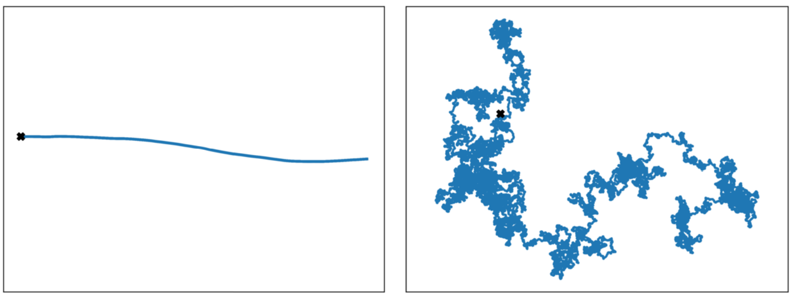

Fig. 7. A realization of the process defined in (10) for $\alpha_n=2\pi n^{-3/4}$ (left) and $\alpha_n=2\pi n^{-1/4}$ (right).

If $p_n\alpha_n^2 \gg \log n$, then $(\operatorname{sinc}\alpha_n)^{p_n} = \exp(-p_n\alpha_n^2/6 + o(p_n\alpha_n^2)) \to 0$, thus

$$\mathbb{E}\big[\|S_n - S_n^*\|^2\big] = O\left(\frac{q_n}{p_n} + \frac{p_n}{n} + n^{2-\omega}(\operatorname{sinc}\alpha_n)^{p_n}\right) \to 0.$$

We now investigate the control of the three quantities underlying the conditions necessary for Proposition 1 to apply. For the first condition, for any $\delta \in (0, 1]$, we have

$$\sum_{i=1}^{k_n} \|\xi_{i,n}\|^{2+\delta} \le k_n(p_n\alpha_n/\sqrt{n})^{2+\delta} \le \alpha_n^{2+\delta} p_n^{1+\delta}/n^{\delta/2},$$

using the triangle inequality and the fact that $U_{j,n} \in \mathbb{S}^1$. This implies that $A_n(\delta) \to 0$ as soon as the RHS converges to 0.

For the third condition, for $t \in \mathbb{R}^2$, we have, according to Proposition 2 and Lemma 1, for any n large enough so that $\operatorname{sinc}\alpha_n \ge 2/\pi$,

$$T_n(t) \le 4k_n\frac{A}{\alpha_n}(\operatorname{sinc}\alpha_n)^{q_n} = O\big(n^{2-\omega}(\operatorname{sinc}\alpha_n)^{q_n}\big).$$

Thereby, $T_n(t) \to 0$ as soon as $q_n\alpha_n^2 \gg \log n$.

For the second condition, using the same development as in the proof of Proposition 3, we find

$$\Gamma_n = \frac{\alpha_n^2 k_n}{2n}\left\{p_n + 2(\operatorname{sinc}\alpha_n)\frac{p_n(1-\operatorname{sinc}\alpha_n) + (\operatorname{sinc}\alpha_n)^{p_n} - 1}{(1-\operatorname{sinc}\alpha_n)^2}\right\}\mathrm{Id}_2,$$

and in particular, if $p_n\alpha_n^2 \to \infty$,

$$\Gamma_n = \left\{O\big(\alpha_n^2\big) + o(1) + \frac{3k_n p_n}{n}\right\}\mathrm{Id}_2 \to 3\,\mathrm{Id}_2\,.$$

Thus, if we can find two sequences, p_n and q_n, verifying all the conditions above, we can then apply Proposition 1 and, in the same fashion as in the proof of Lemma 3, we show that the finite-dimensional laws of $\frac{\alpha_n}{\sqrt{n}}X^n$ converge weakly to the appropriate limit.

It only remains to find two such sequences. The conditions are, in order of appearance: $q_n \ll p_n \ll n$; $\log n \ll p_n\alpha_n^2$; and $\alpha_n^{2+\delta}p_n^{1+\delta} \ll n^{\delta/2}$ for some $\delta \in (0, 1]$; and $\log n \ll q_n\alpha_n^2$. Denoting $u_n = n^{1-\omega/2}\alpha_n^2/\log n$, set $p_n = \alpha_n^{-2}(\log n)u_n^{\epsilon}$ and $q_n = \alpha_n^{-2}(\log n)u_n^{\eta}$ with $0 < \eta < \epsilon < 1$ fixed. The first, second and fourth conditions are immediate consequences of the fact that $u_n \to \infty$ (since $n^{1-\omega/2}\alpha_n^2 \gg n^{\omega/2} \gg \log n$) and $\alpha_n \to 0$. The third condition is equivalent to $u_n^{\varepsilon(1+\delta)-\delta/2} \ll n^{\omega\delta/4}(\log n)^{-1-\delta/2}$ which is true as soon as we pick ϵ smaller than $\frac{\delta}{2(1+\delta)}$.

It remains to show that the family of laws defined by $\{\frac{\alpha_n}{\sqrt{n}}X^n\}$ is tight. To do this, we do as in the proof Lemma 4, and reinstate the notation defined there. The inequality at (9) applies in the same way, but with α replaced here by α_n, and thus

$$\limsup_{n\in\mathbb{N}} \lambda^2\mathbb{P}\Big(\max_{k\le n}\|S_k\| \ge \lambda\sqrt{n}/\alpha_n\Big) \le \limsup_{n\in\mathbb{N}} \frac{\alpha_n^4 K}{(1-\operatorname{sinc}\alpha_n)^2\lambda^2} = \frac{6K}{\lambda^2} \xrightarrow[\lambda\to\infty]{} 0,$$

which, by Lemma 5, implies relative compactness of the sequence of laws. □

Remark 2. We conjecture that the conditions of Theorem 2 can be weakened to a mere divergence, $n\alpha_n^2 \to \infty$, although our proof technique does not seem capable to confirm this conjecture.

So far, our constructions have only yielded a (scaled) Brownian motion, or trivial limits. However, in the critical regime where $n\alpha_n^2$ converges to a positive real, the limit process is different, and, in particular, it is strictly smoother than the Brownian motion itself. See Figure 8 for an illustration of Theorem 3.

Theorem 3. *Consider a sequence of angles as defined in* (10). *If* $n\alpha_n^2 \to \kappa > 0$, *then*

$$\frac{1}{n}X_t^n \Rightarrow U\int_0^t \exp\left\{i\frac{2}{3}\kappa B_s^{(1)}\right\}ds, \tag{13}$$

where U *and* $B^{(1)}$ *are independent, with* U *uniform over* $\mathbb{S}^1$ *and* $B^{(1)}$ *a standard* 1*-dimensional Brownian motion.*

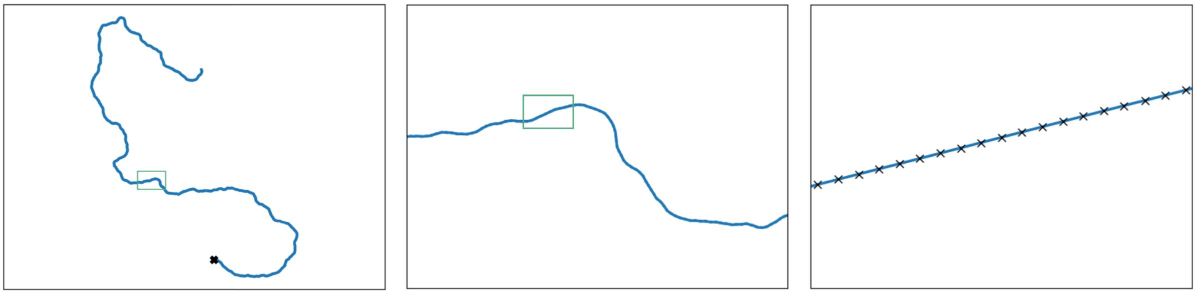

Fig. 8. A realization of the process defined in (10) for $\alpha_n = 2\pi n^{-1/2}$ observed at different scales.

Proof. We set $\mathcal{C}_1 = C([0,1],\mathbb{R})$, and we introduce the sequence of processes

$$\Phi_t^n = \sum_{i=2}^{\lfloor nt\rfloor} \Theta_{i,n} + \big(nt - \lfloor nt\rfloor\big)\Theta_{\lfloor nt\rfloor+1,n}.$$

Since the angle variables $\Theta_{i,n}$, $i \geq 1$, are iid, a simple application of the Lyapunov central limit theorem, in conjunction with the use of [4, Lem. on p. 88] and of the Etemadi inequality [3, Thm. 22.5 on p. 288], immediately show that $\Phi_t^n \Rightarrow \Phi_t = \frac{2}{3}\kappa B_t^{(1)}$ in the space $\mathcal{C}_1$.

Set

$$f_n : x \in \mathcal{C}_1 \mapsto \left(t \mapsto \frac{1}{n}\left\{\sum_{k=1}^{\lfloor nt\rfloor} e^{ix(k/n)} + \big(nt - \lfloor nt\rfloor\big)e^{ix((\lfloor nt\rfloor+1)/n)}\right\}\right) \in \mathcal{C}_2, \quad \text{and}$$

$$f : x \in \mathcal{C}_1 \mapsto \left(t \mapsto \int_0^t e^{ix(s)}\, ds\right) \in \mathcal{C}_2.$$

These two maps are continuous from $\mathcal{C}_1$ to $\mathcal{C}_2$ for the uniform topology – they are even 1-Lipschitz for the supnorm. Furthermore, we notice that $\frac{1}{n}X^n = U_{1,n} f_n \Phi^n$, with $U_{1,n}$ being independent from $f_n\Phi^n$. Since f is continuous, we immediately have that $f\Phi^n \Rightarrow f\Phi$ in the space $\mathcal{C}_2$.

Take a test function $g : \mathcal{C}_2 \to \mathbb{R}$ that is both bounded and Lipschitz,[3] and denote by $\operatorname{Lip} g$ its Lipschitz constant. We have

$$\begin{aligned}\big|\mathbb{E}\big[g\big(f_n\Phi^n\big)\big] - \mathbb{E}\big[g(f\Phi)\big]\big| &\leq \big|\mathbb{E}\big[g\big(f_n\Phi^n\big)\big] - \mathbb{E}\big[g\big(f\Phi^n\big)\big]\big| + \big|\mathbb{E}\big[g\big(f\Phi^n\big)\big] - \mathbb{E}\big[g(f\Phi)\big]\big| \\ &\leq \operatorname{Lip}(g)\mathbb{E}\big[\big\|f_n\Phi^n - f\Phi^n\big\|_\infty\big] + o(1).\end{aligned}$$

The second term is indeed $o(1)$ because $f\Phi^n$ converges weakly to $f\Phi$. With an analogous reasoning as the one underlying Lemma 6, we see that for any $s,t \in [0,1]$, $|\Phi_t^n - \Phi_s^n| \leq n\alpha_n|t-s|$, and thus, for any $t \in [0,1]$,

$$\big|f\Phi^n[t] - f_n\Phi^n[t]\big| \leq \sum_{k=1}^{\lfloor nt\rfloor} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \big|e^{i\Phi_s^n} - e^{i\Phi_{k/n}^n}\big|\, ds + \int_{\frac{\lfloor nt\rfloor}{n}}^{t} \big|e^{i\Phi_s^n} - e^{i\Phi^n_{(\lfloor nt\rfloor+1)/n}}\big|\, ds \tag{14}$$

$$\leq \sum_{k=1}^{\lfloor nt\rfloor} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \big|\Phi_s^n - \Phi_{k/n}^n\big|\, ds + \int_{\frac{\lfloor nt\rfloor}{n}}^{t} \big|\Phi_s^n - \Phi^n_{(\lfloor nt\rfloor+1)/n}\big|\, ds$$

$$\leq \sum_{k=1}^{\lfloor nt\rfloor} \frac{1}{n}(n\alpha_n)\frac{1}{n} + \frac{nt - \lfloor nt\rfloor}{n}(n\alpha_n)\frac{1}{n} \leq t\alpha_n. \tag{15}$$

Hence, $\|f_n\Phi^n - f\Phi^n\|_\infty \leq \alpha_n \to 0$. We may thus conclude that $\mathbb{E}[g(f_n\Phi^n)] \to \mathbb{E}[g(f\Phi)]$, and so for any g bounded-Lipschitz, thus implying that $f_n\Phi^n$ converges weakly to $f\Phi$ in $\mathcal{C}_2$. □

[3] Because $\mathcal{C}_2$ is a polish space, the bounded-Lipschitz distance metrizes the weak convergence of probability measures [7, Thm. 11.3.3].

Let $(X_t : t \in [0,1])$ denote the limit process in (13). Because a Brownian motion has continuous paths, X has continuously differentiable paths. Furthermore, as a parametrization of a curve, it is unit-speed, its velocity at time t being given by

$$\dot{X}_t = U \exp\left\{ i \frac{2}{3} \kappa B_t^{(1)} \right\}.$$

4. Construction based on a Markov sequence of angles

The limit process derived for the construction studied in Theorem 3 is not twice differentiable. Our goal in this section is to construct a random walk with limiting process having finite curvature, which from a geometric standpoint is appealing. Given our investigations in the previous two sections, such a construction appears to require some memory in the angle processes. It turns out that just a little memory is sufficient.

We consider a sequence of angles constructed as follows:

$$\begin{aligned} &\text{Given } \delta_{j,n} \text{ that are iid uniform on } [-\alpha_n, \alpha_n], \\ &\text{define } \Theta_{1,n} = \delta_{1,n}, \text{ and for } j \geq 2,\ \Theta_{j,n} = \Theta_{j-1,n} + \delta_{j,n}. \end{aligned} \tag{16}$$

See Figure 9 for an illustration of this definition.

Theorem 4. *Consider a sequence of angles as defined in* (16). *If* $n^3 \alpha_n^2 \to \kappa > 0$, *then*

$$\frac{1}{n} X_t^n \Rightarrow U \int_0^t \exp\left\{ i \frac{2}{3} \kappa \int_0^s B_u^{(1)}\, du \right\} ds. \tag{17}$$

Proof. The proof is similar to that of Theorem 3, and we reinstate the notation used there. We have $\Theta_{k,n} = \sum_{i=2}^k \delta_{i,n}$ (denoting $\delta_{2,n} = \Theta_{2,n}$). We then define

$$\Psi_t^n = n \left\{ \sum_{i=2}^{\lfloor nt \rfloor} \delta_{i,n} + \big(nt - \lfloor nt \rfloor\big) \delta_{\lfloor nt \rfloor + 1, n} \right\},$$

so that $\Theta_{k,n} = \frac{1}{n} \Psi_{k/n}^n$. As in the proof of Theorem 3, we have $\Psi_t^n \Rightarrow \Psi_t = \frac{2}{3} \kappa B_t^{(1)}$ in the space $\mathcal{C}_1$. We introduce the functions

$$h_n : x \in \mathcal{C}_1 \mapsto \left(t \mapsto \frac{1}{n} \left\{ \sum_{k=1}^{\lfloor nt \rfloor} x(k/n) + \big(nt - \lfloor nt \rfloor\big) x\left(\frac{\lfloor nt \rfloor + 1}{n} \right) \right\} \right) \in \mathcal{C}_1, \quad \text{and}$$

$$h : x \in \mathcal{C}_1 \mapsto \left(t \mapsto \int_0^t x(s)\, ds \right) \in \mathcal{C}_1,$$

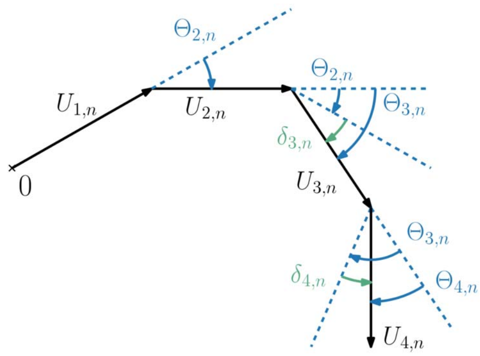

Fig. 9. The first steps of the random walk with a Markov sequence of angles. Because the angles keep track of their former values, we can expect a smoother process at the limit.

which are 1-Lipschitz for the supnorm. Furthermore, we have

$$\frac{1}{n}X^n = U_{1,n} f_n \Phi^n = U_{1,n} f_n h_n \Psi^n.$$

As before, $U_{1,n}$ are independent from $f_n h_n \Phi^n$. Take a test function $g \in \mathrm{BL}(\mathcal{C}_2)$. We have

$$\begin{aligned}\left|\mathbb{E}\left[g\left(f_n h_n \Psi^n\right)\right] - \mathbb{E}\left[g(fh\Psi)\right]\right| \le \left|\mathbb{E}\left[g\left(fh\Psi^n\right)\right] - \mathbb{E}\left[g(fh\Psi)\right]\right| + \left|\mathbb{E}\left[g\left(f_n h\Psi^n\right)\right] - \mathbb{E}\left[g\left(fh\Psi^n\right)\right]\right| \\ + \left|\mathbb{E}\left[g\left(f_n h_n \Psi^n\right)\right] - \mathbb{E}\left[g\left(f_n h\Psi^n\right)\right]\right|. \qquad (18)\end{aligned}$$

The first term on the RHS of (18) converges to 0 because $\Phi^n \Rightarrow \Phi$. The second term on the RHS of (18) can be bounded as follows

$$\begin{aligned}\left|\mathbb{E}\left[g\left(f_n h\Psi^n\right)\right] - \mathbb{E}\left[g\left(fh\Psi^n\right)\right]\right| &\le \mathrm{Lip}(g)\mathbb{E}\left[\left\| f_n h\Psi^n - fh\Psi^n\right\|_\infty\right] \\ &\le \mathrm{Lip}(g)\frac{1}{n}\mathbb{E}\left[\mathrm{Lip}\left(h\Psi^n\right)\right] \\ &\le \mathrm{Lip}(g)\frac{1}{n}\mathbb{E}\left[\left\|\Psi^n\right\|_\infty\right] \le \mathrm{Lip}(g) n\alpha_n \longrightarrow 0, \qquad (19)\end{aligned}$$

where the inequality $\|f_n x - fx\|_\infty \le \frac{1}{n}\mathrm{Lip}(x)$ comes from a computation similar to the one done in the proof of Theorem 3 (see formulas (14) and (15)). The inequality $\mathrm{Lip}(h\Psi^n) \le \|\Psi^n\|_\infty$ that we use at (19) comes from the definition of h: for any $x \in \mathcal{C}_1$, we have $|hx(t) - hx(s)| \le \int_s^t |x| \le \|x\|_\infty |t-s|$ for any $0 \le s \le t \le 1$. The convergence to 0 holds because $n = O(\alpha_n^{-2/3})$. The last term on the RHS of (18) is bounded as follows

$$\begin{aligned}\left|\mathbb{E}\left[g\left(f_n h_n \Psi^n\right)\right] - \mathbb{E}\left[g\left(f_n h\Psi^n\right)\right]\right| &\le \mathrm{Lip}(g)\mathbb{E}\left[\left\| f_n h_n \Psi^n - f_n h\Psi^n\right\|_\infty\right] \\ &\le \mathrm{Lip}(g)\mathbb{E}\left[\left\| h_n \Psi^n - h\Psi^n\right\|_\infty\right] \\ &\le \mathrm{Lip}(g)\frac{1}{n}\mathbb{E}\left[\mathrm{Lip}\left(\Psi^n\right)\right] \le \mathrm{Lip}(g) n\alpha_n \to 0,\end{aligned}$$

where we used the fact that f_n is 1-Lipschitz, and a few inequalities that we already used in the previous bounds.

We conclude that $\frac{1}{n}X^n = U_{1,n} f_n h_n \Psi^n$ converges weakly in $\mathcal{C}_2$ to $Ufh\Psi$, which is exactly the convergence stated in the theorem. □

Let X denote the limit process in (17). It is clearly twice differentiable, with velocity given by

$$\dot{X}_t = U \exp\left\{i\frac{2}{3}\kappa \int_0^t B_s^{(1)}\, ds\right\},$$

and acceleration is given by

$$\ddot{X}_t = i\frac{2}{3}\kappa B_t^{(1)} U \exp\left\{i\frac{2}{3}\kappa \int_0^t B_s^{(1)}\, ds\right\}.$$

In particular, as a parameterization, it is unit-speed (since $\|\dot{X}_t\| = 1$ for all t), and its (unsigned) curvature at time t is given by $\|\ddot{X}_t\| = \frac{2}{3}\kappa|B_t^{(1)}|$. See Figure 10 for a realization of such a process.

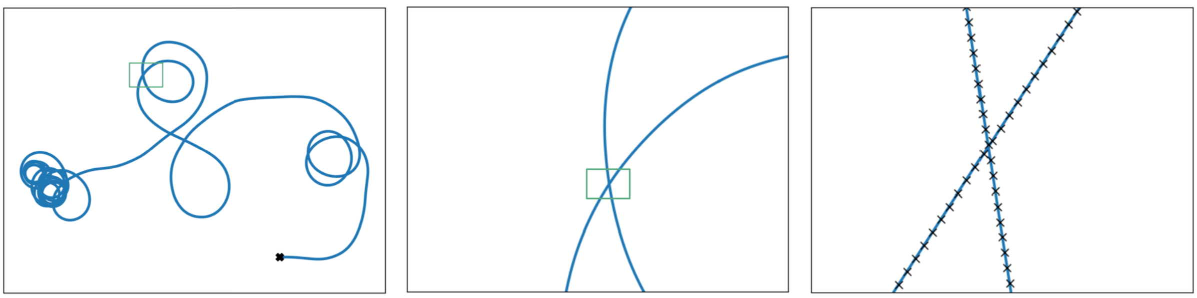

Fig. 10. A realization of the process defined in Section 4 for $\alpha_n = 64\pi n^{-3/2}$, observed at different scales.

5. Discussion

Retrospectively, our construction in Section 2 appears naive. Yet, the fact that the construction failed to produce a process with curves with finite curvature was initially surprising to us due to the fact that the polygonal lines resulting from the construction do have bounded curvature (independent of n) in the sense of [1]. In that paper, the curvature of a polygonal line at a vertex is defined as the inverse of the circumradius of the triangle that this vertex forms with the two adjacent vertices on the polygonal line – a rather natural definition that is shown there to enjoy good properties. However, as we have shown, such a construction can only yield a Brownian motion in the limit, or at best, a process with once differentiable realizations if we let the angle interval shrink at a very specific rate.

Otherwise, we believe that the limits established here have the sort of universality expected for random walk constructions, in that the edges defining polygonal line do not need to have the exact same length, and that the angles or their increments do not need to be selected uniformly at random.

We also anticipate that similar constructions, with similar limits, are possible in arbitrary dimension. The most interesting case, besides the planar case presented here, may well be that of random walks and curves in dimension three, where an analogous goal would be to construct random walks with limits that exhibit finite curvature and torsion (almost surely).

Finally, we mention that processes that look like (13) naturally appear in many applications ranging from mathematical finance to quantum optics [9] and have been thoroughly considered in the literature, in particular by [11], who studies exponential functionals of the Brownian motion of the form

$$X_t = \int_0^t \exp\{aB_s^{(1)} + bs\}\, ds,$$

in the situation where a and b are real. [8] study similar processes with a allowed to be complex, but with $b < 0$. To the authors' knowledge, the case when both $a \in i\mathbb{R}$ and $b = 0$ remains to be studied.

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