A NOTE ON LIE ALGEBRA COHOMOLOGY

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Abstract. Given a finite dimensional Lie algebra L let I be the augmentation ideal in the universal enveloping algebra $U(L)$. We study the conditions on L under which the Ext-groups $\mathrm{Ext}(k,k)$ for the trivial L-module k are the same when computed in the category of all $U(L)$-modules or in the category of I-torsion $U(L)$-modules. An application to cohomology of equivariant sheaves is given.

1. Introduction

Let L be a finite dimensional Lie algebra over a field k. Consider the universal enveloping algebra $U(L)$ with the augmentation ideal $I \subset U(L)$. Denote by $U(L)$-mod the category of finitely generated left $U(L)$-modules and by $(U(L)\text{-mod})_I \subset U(L)$-mod the Serre subcategory of I-torsion modules. We have the obvious functor

$$\Phi_L : D^b((U(L)\text{-mod})_I) \to D^b_I(U(L)\text{-mod}) \tag{1.1}$$

where $D^b_I(U(L)\text{-mod}) \subset D^b(U(L)\text{-mod})$ is the full triangulated subcategory consisting of complexes with I-torsion cohomology. In this paper we study the question:

Question. When is Φ_L an equivalence?

The functor Φ_L being an equivalence means that the Ext-groups $\mathrm{Ext}^i(k,k)$ for the trivial L-module k are the same in the categories $U(L)$-mod and $(U(L)\text{-mod})_I$.

We answer this question in Theorem 1.1 below.

Define inductively the decreasing sequence of ideals in L:

$$L_1 = L, \quad L_n = [L, L_{n-1}]$$

and put $L_\infty = \bigcap_n L_n$. This is an ideal in L such that the quotient Lie algebra $L_{\mathrm{nil}} := L/L_\infty$ is nilpotent. We have $L_\infty = 0$ if and only if L is nilpotent.

For each i the cohomology $H^i(L_\infty, k)$ is naturally an L_{nil}-module. Denote by $H^{>0}(L_\infty, k)$ the positive degree cohomology.

Theorem 1.1. *The functor Φ_L is an equivalence if and only if $H^{>0}(L_\infty, k)^{L_{\mathrm{nil}}} = 0$. For example, Φ_L is an equivalence if L is nilpotent.*

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We find it natural to approach Theorem 1.1 by studying the graded Rees algebra

$$U(L)^* := \bigoplus_{n\geq 0} I^n = U(L) \oplus I \oplus I^2 \oplus \cdots$$

It is easy to prove the following result.

Proposition 1.2. *If the algebra $U(L)^*$ is graded left Noetherian, then the functor Φ_L is an equivalence.*

It is, however, not necessary for $U(L)^*$ to be graded left Noetherian in order for Φ_L to be an equivalence. The relevant result is the following theorem (see [SW], where the interesting "if" direction is proved).

Theorem 1.3. *The algebra $U(L)^*$ is graded left Noetherian if and only if L is nilpotent.*

In the last section of the paper we mention an application of Theorem 1.1 to the cohomology of quasi-coherent sheaves which are equivariant with respect to a unipotent group.

In this paper we consider only left modules, but all the results are also valid (with the same proofs) for right modules.

We fix a field k. All Lie algebras are finite dimensional over k. All associative rings are unital.

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2. A criterion for equivalence of categories

Let R be an associative left Noetherian ring with a 2-sided ideal $I \subset R$. Let M be a left R-module. An element $m \in M$ is called *I-torsion*, if $I^n m = 0$ for some $n > 0$. The collection of I-torsion elements in M is an R-submodule, which we denote by M_I. We say that M is torsion if $M_I = M$.

Let R-mod denote the abelian category of finitely generated left R-modules and let $(R\text{-mod})_I \subset R$-mod be its full Serre subcategory of I-torsion modules. Let $C^b(R\text{-mod})$ (resp. $C^b((R\text{-mod})_I)$) be the category of bounded complexes over R-mod (resp. over $(R\text{-mod})_I$) and let $C^b_I(R\text{-mod}) \subset C^b(R\text{-mod})$ be the full subcategory of complexes whose cohomology groups are torsion.

In the bounded derived category $D^b(R\text{-mod})$ consider the full subcategory $D^b_I(R\text{-mod})$ of complexes with torsion cohomology groups. We have the obvious functor

$$\Phi = \Phi_R : D^b((R\text{-mod})_I) \to D^b_I(R\text{-mod})$$

Proposition 2.1. *Assume that for every finitely generated left R-module M there exists a submodule $N \subset M$ such that $N_I = 0$ and M/N is I-torsion. Then the functor Φ is an equivalence.*

Proof. Let $A^\bullet$ be an object of $C^b_I(R\text{-mod})$. We claim there exists an object $B^\bullet$ of $C^b((R\text{-mod})_I)$ and a morphism of complexes $f : A^\bullet \to B^\bullet$ which is a quasi-isomorphism. Indeed, let

$$A^\bullet = 0 \to A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots \xrightarrow{d^{n-1}} A^n \to 0$$

and let t be the lowest index such that $A^t_I \neq A^t$. By assumption there exists a submodule $P \subset A^t$ such that $P_I = 0$ and $(A^t/P)_I = A^t/P$. We claim that $P \cap \ker d^t = 0$. Indeed, since $H^t(A^\bullet)$ and A^{t-1} are torsion, it follows that $\ker d^t$ is torsion, so $P \cap \ker d^t = 0$. Therefore, the complex $A^\bullet$ contains an acyclic subcomplex $\tilde{P} := P \xrightarrow{\sim} d^t(P)$ and the components with index $\leq t$ of the quotient complex $A^\bullet/\tilde{P}$ are torsion. Iterating this process we find the required quasi-isomorphism $f : A^\bullet \to B^\bullet$. This shows that the functor Φ is essentially surjective.

For complexes $C^\bullet, D^\bullet$ representing objects in $D^b((R\text{-mod})_I)$, a morphism $\Phi(D^\bullet) \to \Phi(C^\bullet)$ is represented by a diagram of complexes $D^\bullet \to A^\bullet \xleftarrow{s} C^\bullet$, where $A^\bullet \in C^b_I(R\text{-mod})$ and s is a quasi-isomorphism. The fact that the functor Φ is full and faithful now follows, since (as shown above) there exists a complex $B^\bullet \in C^b((R\text{-mod})_I)$ and a morphism $f : A^\bullet \to B^\bullet$ of complexes that is a quasi-isomorphism. $\square$

Consider now the graded Rees algebra

$$R^* := \bigoplus_{n\geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

Lemma 2.2. *Assume that the algebra R^* is graded left Noetherian (i.e. every graded left ideal is finitely generated). Then the assumption of Proposition 2.1 holds: for any finitely generated left R-module M there exists a submodule $N \subset M$ such that $N_I = 0$ and $M/N = (M/N)_I$*

Proof. Let M be a finitely generated R-module. Consider the graded finitely generated R^*-module

$$\tilde{M} = M \oplus IM \oplus I^2M \oplus \cdots$$

and its graded submodule

$$P = M_I \oplus (IM \cap M_I) \oplus (I^2M \cap M_I) \oplus \cdots$$

By our assumption, P is finitely generated; hence, there exists $n > 0$ such that for all $m > 0$

$$I^m(I^nM \cap M_I) = I^{n+m}M \cap M_I$$

As $I^nM \cap M_I$ is finitely generated and I-torsion, it is annihilated by I^m for some m. Thus,

$$(I^{m+n}M)_I = I^{m+n}M \cap M_I = I^m(I^nM \cap M_I) = 0.$$

Putting $N = I^{m+n}M$, we have $N_I = 0$ and $(M/N)_I = M/N$. $\square$

Corollary 2.3. *Assume that the algebra R^* is graded left Noetherian. Then the functor Φ is an equivalence.*

3. When is the Rees algebra of a universal enveloping algebra Noetherian

Let L be a finite dimensional Lie algebra, $U(L)$ its universal enveloping algebra and $I \subset U(L)$ the augmentation ideal. As above, we consider the graded Rees algebra

$$U(L)^* = \bigoplus_{n\geq 0} I^n = U(L) \oplus I \oplus I^2 \oplus \cdots$$

Theorem 3.1. *The algebra $U(L)^*$ is graded left Noetherian if and only if the Lie algebra L is nilpotent.*

We get the following immediate consequence of Theorem 3.1 and Corollary 2.3

Corollary 3.2. *Let L be a nilpotent Lie algebra. Then the functor*

$$\Phi_{U(L)} : D^b((U(L)\text{-}mod)_I) \to D^b_I(U(L)\text{-}mod)$$

is an equivalence.

Proof. The more interesting "if" direction is contained in Theorem 2.1 in [SW] and we prove the "only if" direction which is the easy one.

For a Lie algebra L we consider the lower central series $L_1 = L$, $L_n = [L, L_{n-1}]$. Thus

$$L = L_1 \supset L_2 \supset L_3 \supset \cdots$$

is a nonincreasing sequence of ideals in L. We put

$$L_\infty := \bigcap_n L_n, \quad L_{\mathrm{nil}} = L/L_\infty$$

The Lie algebra L_{nil} is nilpotent, and L is nilpotent if and only if $L_\infty = 0$.

We have the short exact sequence of Lie algebras

$$0 \to L_\infty \to L \to L_{\mathrm{nil}} \to 0$$

This induces the surjection $\theta : U(L) \to U(L_{\mathrm{nil}})$ and $\ker\theta$ is the ideal $U(L)L_\infty U(L)$. As before, let $I \subset U(L)$ be the augmentation ideal. We have by construction $L_n \subset I^n$ for all n, hence $L_\infty \subset \bigcap_n I^n$.

Assume that the Lie algebra L is not nilpotent, i.e. $L_\infty \neq 0$. Fix $0 \neq x \in L_\infty$ and consider the graded left ideal

$$J = \bigoplus_n U(L)x_n \subset U(L)^*$$

where x_n denotes the copy of x in I^n. We claim that J is not finitely generated. Assume, on the contrary, that

$$J = \sum_i U(L)^* f_i$$

for a finite number of homogeneous elements $f_i \in U(L)x_{n_i}$. Choose $m > n_i$ for all i. We claim that

$$x_m \notin \sum_i U(L)^* f_i$$

Indeed, it suffices to notice that $x \notin Ix$: if $x = fx$, then $f = 1$, (since $U(L)$ is a domain) and hence $f \notin I$. This completes the proof of Theorem 3.1. □

4. Main theorem

Let L be a finite dimensional Lie algebra, $U(L)$ its universal enveloping algebra, $I \subset U(L)$ the augmentation ideal. As in section 3, consider the ideal

$$L_\infty = \bigcap_n L_n \subset L$$

and the quotient nilpotent Lie algebra $L_{\mathrm{nil}} = L/L_\infty$.

Each cohomology space $H^i(L_\infty, k)$ is naturally a L_{nil}-module, so we have the Hochschild-Serre spectral sequence [HS]:

$$E_2^{pq} = H^p(L_{\mathrm{nil}}, H^q(L_\infty, k)) \Rightarrow H^{p+q}(L, k). \tag{4.1}$$

Theorem 4.1. *Let L be a finite dimensional Lie algebra over a field k. The following conditions are equivalent:*

(1) The natural functor

$$\Phi_L : D^b((U(L)\text{-}mod)_I) \to D^b_I(U(L)\text{-}mod)$$

is an equivalence.

(2) The natural map $H^\bullet(L_{\mathrm{nil}}, k) \to H^\bullet(L, k)$ is an isomorphism.

(3) The positive degree cohomology $H^{>0}(L_\infty, k)$ considered as an L_{nil}-module satisfies $H^{>0}(L_\infty, k)^{L_{\mathrm{nil}}} = 0$.

Proof. We first notice that the 3 conditions in the theorem hold in case L is nilpotent. Indeed, then $L_\infty = 0$, so (2) and (3) hold trivially. Also (1) holds by Corollary 3.2.

Let now L be general. As in section 3 we consider the short exact sequence of Lie algebras

$$0 \to L_\infty \to L \to L_{\mathrm{nil}} \to 0$$

and the induced surjection $\theta : U(L) \to U(L_{\mathrm{nil}})$ with the kernel $\ker\theta = U(L)L_\infty U(L)$. Denote by $\overline{I}$ the augmentation ideal in $U(L_{\mathrm{nil}})$. Since $L_\infty \subset \cap_n I^n$, any $U(L)$-module M such that $M = M_I$ is actually a $U(L_{\mathrm{nil}})$-module (and $M = M_{\overline{I}}$). Hence the functor of restriction of scalars

$$\theta_* : U(L_{\mathrm{nil}})\text{-mod} \to U(L)\text{-mod}$$

induces the equivalence of categories

$$(U(L_{\mathrm{nil}})\text{-mod})_{\overline{I}} \xrightarrow{\sim} (U(L)\text{-mod})_I$$

and therefore the equivalence of categories

$$D^b((U(L_{\mathrm{nil}})\text{-mod})_{\overline{I}}) \xrightarrow{\sim} D^b((U(L)\text{-mod})_I) \tag{4.2}$$

We have the commutative diagram of functors

$$\begin{tikzcd} D^b((U(L)\text{-mod})_I) \arrow[r, "\Phi_L"] & D^b_I(U(L)\text{-mod}) \\ D^b((U(L_{\mathrm{nil}})\text{-mod})_{\bar I}) \arrow[u] \arrow[r, "\Phi_{L_{\mathrm{nil}}}"] & D^b_{\bar I}(U(L_{\mathrm{nil}})\text{-mod}) \arrow[u] \end{tikzcd} \tag{4.3}$$

As explained above the left vertical arrow is an equivalence. Also $\Phi_{L_{\mathrm{nil}}}$ is an equivalence (Corollary 3.2). Hence Φ_L is an equivalence if and only if the functor

$$\theta_* : D^b_{\bar I}(U(L_{\mathrm{nil}})\text{-mod}) \to D^b_I(U(L)\text{-mod}) \tag{4.4}$$

is an equivalence. Every finitely generated I-torsion $U(L)$-module (resp. $\bar I$-torsion $U(L_{\mathrm{nil}})$-module) is a finite dimensional k-vector space on which L (resp. L_{nil}) acts nilpotently, so by Engel's theorem, it admits a stable flag with trivial 1-dimensional quotients. As triangulated categories, therefore, both sides of (4.4) are generated by the trivial module k, and so the functor in (4.4) is an equivalence if and only if the natural map

$$\theta_* : \mathrm{Ext}^\bullet_{U(L_{\mathrm{nil}})}(k,k) \to \mathrm{Ext}^\bullet_{U(L)}(k,k) \tag{4.5}$$

is an isomorphism. This proves the equivalence of conditions (1) and (2) in the theorem. It remains to prove the equivalence of (2) and (3).

The Hochschild-Serre spectral sequence (4.1) has E_2 page

$$\begin{tikzcd} E_2^{01} \arrow[drr] & E_2^{11} \arrow[drr] & \cdots & \cdots \\ E_2^{00} & E_2^{10} & E_2^{20} & \cdots \end{tikzcd} \tag{4.6}$$

We have $H^0(L_\infty, k) = k$ – the trivial L_{nil}-module and the graded space $\mathrm{Ext}^\bullet_{U(L_{\mathrm{nil}})}(k,k)$ identifies naturally with the bottom row of this spectral sequence. The map $\theta^* : \mathrm{Ext}^\bullet_{U(L_{\mathrm{nil}})}(k,k) \to \mathrm{Ext}^\bullet_{U(L)}(k,k)$ then coincides with the projection

$$\mathrm{Ext}^\bullet_{U(L_{\mathrm{nil}})}(k,k) = H^\bullet(L_{\mathrm{nil}}, H^0(L_\infty, k)) \to H^\bullet(L,k)$$

Assume that the condition (3) holds, i.e. $H^{>0}(L_\infty,k)^{L_{\mathrm{nil}}} = 0$. Then by [Ba, Lemma 3], we have

$$H^\bullet(L_{\mathrm{nil}}, H^{>0}(L_\infty, k)) = 0,$$

and hence only the bottom row of the spectral sequence (4.6) is nonzero. Therefore the natural map $H^\bullet(L_{\mathrm{nil}}, k) \to H^\bullet(L,k)$ is an isomorphism, i.e., condition (2) of the theorem holds.

Assume, conversely, that condition (2) holds. Let d be the maximal integer such that $H^d(L_{\mathrm{nil}}, k) \neq 0$. If N is any indecomposable finite-dimensional L_{nil}-module, then by [Cu, Theorem 1], all irreducible subquotients of N are equivalent. Then using Lemmas 3 and 4 in [Ba], we conclude that

$H^\bullet(L_{\rm nil}, N) = 0$ unless every subquotient of N is isomorphic to the trivial module k. In this case, k is a quotient representation of N, and by induction on $\dim N$, $H^{>d}(L_{\rm nil}, N) = 0$. For indecomposable representations, and therefore for all representations, it follows that if $N^{L_{\rm nil}} \neq 0$, then $H^d(L_{\rm nil}, N) \neq 0$.

Assume for the sake of contradiction that for some $i > 0$, $H^i(L_\infty, k)^{L_{\rm nil}} \neq 0$. Let i be the maximal such. Then $E_2^{d,i} = H^d(L_{\rm nil}, H^i(L_\infty, k))$ is nonzero and it survives in $H^{i+d}(L, k)$. This is a contradiction and finishes the proof of the theorem. □

4.1. **Some examples.** (A) Consider the 2-dimensional Lie algebra with basis x, y and the relation $[x, y] = y$. This Lie algebra is solvable but not nilpotent, $L_\infty = ky$. The standard complex, which computes the cohomology $H^\bullet(L_\infty, k)$, has terms in degrees 0 and 1 and zero differential:

$$k \xrightarrow{0} \mathrm{Hom}_k(ky, k).$$

The element $x \in L_{\rm nil}$ acts on the space $\mathrm{Hom}_k(ky, k) = H^1(L_\infty, k)$ as minus the identity, so the condition (3) of Theorem 4.1 is satisfied.

(B) This is a generalization of example (A) above: assume that

$$(\bigwedge\nolimits^{>0} L_\infty)^{L_{\rm nil}} = 0.$$

Then condition (3) of Theorem 4.1 holds. For example this is the case when L is the Lie algebra of upper-triangular matrices. Then L_∞ is the ideal of strictly triangular matrices and $L_{\rm nil}$ is the abelian quotient.

(C) However, there exist solvable algebras L for which condition (3) does not hold. Let $L = kt \oplus kx_1 \oplus kx_{-1}$ be the 3-dimensional solvable algebra with $[t, x_i] = ix_i$ and $[x_{-1}, x_1] = 0$. Then $L_\infty = kx_{-1} \oplus kx_1$, and $L_{\rm nil} = kt$. By construction, $L_{\rm nil}$ acts trivially on $H^2(L_\infty, k) = \bigwedge^2 L_\infty^*$.

(D) It may happen that condition (3) holds even though $\bigwedge^{>0} L_\infty$ admits k as an $L_{\rm nil}$-subquotient. See Proposition 4.4 below.

(E) If k is not of characteristic 2 and $L_\infty \neq 0$ has a non-degenerate Killing form, then $H^3(L_\infty, k) \neq 0$ by [Se, p. 103]. By [Za, Satz 16], every derivation of L_∞ is inner, so the action of $L_{\rm nil}$ on the cohomology of L_∞ is trivial.

(F) We formulate this as a proposition:

Proposition 4.2. *Assume that the equivalent conditions of Theorem 4.1 are satisfied and the characteristic of k is zero. Then the algebra L is solvable.*

Proof. As L/L^∞ is solvable, it suffices to prove that L^∞ is solvable. Let $L_\infty^{\rm rad}$ denote the radical of L^∞, so that

$$L_\infty^{\rm ss} := L_\infty / L_\infty^{\rm rad}$$

is semi-simple. Setting $\mathfrak{g} := L_\infty^{\rm ss}$, we need to prove that $\mathfrak{g} = 0$.

Lemma 4.3. *In the above notation the natural map*

$$H^\bullet(\mathfrak{g}, k) = \mathrm{Ext}^\bullet_{U(\mathfrak{g})}(k, k) \to \mathrm{Ext}^\bullet_{U(L_\infty)}(k, k) = H^\bullet(L_\infty, k)$$

is injective.

Proof. By the Levi theorem we know that the surjection of Lie algebras $p : L_\infty \to \mathfrak{g}$ has a splitting $s : \mathfrak{g} \to L_\infty$. These induce chain maps $p^* : C^\bullet(\mathfrak{g}, k) \to C^\bullet(L_\infty, k)$ and $s^* : C^\bullet(L_\infty, k) \to C^\bullet(\mathfrak{g}, k)$ such that $s^* \cdot p^* = id$. Hence the map

$$H^\bullet(p^*) : H^\bullet(\mathfrak{g}, k) \to H^\bullet(L_\infty, k)$$

is injective.

□

In characteristic 0, the Killing form of a semi-simple algebra is non-degenerate. Therefore, if $\mathfrak{g} \neq 0$, then $H^3(\mathfrak{g}, k) \neq 0$ [Se, p 103]. The L_{nil} action on L_∞ stabilizes L_∞^{rad} and therefore induces an action on $\mathfrak{g}$. For any $x \in L_{\mathrm{nil}}$ the operator $[x, -]$ on $\mathfrak{g}$ is a derivation, so is inner. Therefore the action of L_{nil} on the cohomology is trivial, and so the condition (3) of Theorem 4.1 fails. □

Proposition 4.4. *The conditions of Theorem 4.1 are strictly weaker than the condition that* $(\bigwedge^{>0} L_\infty^*)$ *has a non-trivial* L_{nil}*-invariant subquotient.*

Proof. As $H^{>0}(L_\infty, k)$ is a subquotient of $\bigwedge^{>0} L_\infty^*$, if the former has a non-trivial L_{nil}-invariant subquotient, the latter does as well.

We show that converse does not hold by exhibiting a case in which L_{nil} acts semisimply on $\bigwedge^\bullet L_\infty^*$ and therefore on every L_{nil}-stable subquotient and for which

$$\dim(\bigwedge\nolimits^\bullet L_\infty^*)^{L_{\mathrm{nil}}} > 1 = \dim H^\bullet(L_\infty, k)^{L_{\mathrm{nil}}}.$$

The free Lie algebra on two generators x and y admits a unique bigrading for which x and y have bidegree $(1, 0)$ and $(0, 1)$ respectively. Let M be the quotient of this algebra by the graded ideal generated by all elements of total degree ≥ 4 and also $[[x, y], y]$. Then M has basis: x, y, z, w of bidegree $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(2, 1)$ respectively, satisfying the following relations:

$$[x, y] = z,\ [x, z] = w,\ [x, w] = [y, z] = [y, w] = [z, w] = 0$$

(see [Bo, II, §2, no. 11, Théorème 1] and the computation of the Hall set for 2 generators given at the end of no. 10.)

We define t to be the derivation which acts on the bidegree (a, b) part of M by $2a - 3b$. Let $L := M \oplus kt$ denote the semi-direct sum, so

$$[t, x] = 2x,\ [t, y] = -3y,\ [t, z] = -z,\ [t, w] = w.$$

We confirm that $[L, L] = [L, M] = M$, so $L_\infty = M$, and L_{nil} is the 1-dimensional algebra spanned by the class of t.

Next, we consider the Chevalley-Eilenberg complex of M. The underlying graded space is $\bigwedge^\bullet M^*$, which is spanned by wedge products of the dual basis x^*, y^*, z^*, w^* of M. The differential is given by

$$\delta(x^*) = 0,\ \delta(y^*) = 0,\ \delta(z^*) = y^* \wedge x^*,\ \delta(w^*) = z^* \wedge x^*.$$

The bigrading on M induces a bigrading on $\bigwedge^\bullet M^*$, and δ preserves bidegree. The degree $(3,2)$-part of $\bigwedge^\bullet M^*$ is spanned by $z^* \wedge w^*$ and $x^* \wedge y^* \wedge w^*$. As

$$\delta(z^* \wedge w^*) = -x^* \wedge y^* \wedge z^*,$$

the degree $(3,2)$-part of $H^*(L_\infty, k)$ is zero. On the other hand, the t-invariant part of $H^*(L_\infty, k)$ is the sum of the $(3n, 2n)$-part over all integers n.

Now $H^*(L_\infty, k)$ is a subquotient of $\bigwedge^\bullet M^*$, and the latter has non-trivial degree $(3n, 2n)$-part only for $n = 0, 1$. Thus, $\dim(\bigwedge^\bullet L_\infty)^{L_{\mathrm{nil}}} = 3$ but

$$\dim H^\bullet(L_\infty, k)^{L_{\mathrm{nil}}} = 1.$$

□

5. An application

Let k be an algebraically closed field of characteristic zero. Let V be a linear unipotent algebraic group over k, $L = \operatorname{Lie} V$ the corresponding nilpotent Lie algebra.

Denote by V-Mod the abelian category of rational representations of V. Recall that an object of V-mod is by definition a V-module M which is a union of finite dimensional submodules M_i, such that the V-action on M_i comes from a homomorphism of k-algebraic groups $V \to GL(M_i)$. In particular every element of V acts on M_i via a unipotent operator.

Notice that we have a natural equivalence of abelian categories

$$\log : V\text{-Mod} \to (U(L)\text{-Mod})_I$$

where $(U(L)\text{-Mod})_I$ is the abelian category of (all) $U(L)$-modules which are I-torsion.

This induces the equivalence of derived categories

$$\log : D^b(V\text{-Mod}) \to D^b((U(L)\text{-Mod})_I) \tag{5.1}$$

Recall that for $M \in V$-mod its cohomology is by definition

$$H_V^\bullet(M) := \operatorname{Ext}^\bullet_{V\text{-Mod}}(k, M)$$

where k is the trivial rational V-module.

Corollary 5.1. *For any $M \in V$-Mod we have the isomorphism*

$$H_V^\bullet(M) \simeq H^\bullet(L, \log(M)) \tag{5.2}$$

In particular, the cohomology $H_V^\bullet(M)$ can be computed using the standard complex for the Lie algebra L.

Proof. The equivalence (5.1) implies the isomorphism

$$\operatorname{Ext}^\bullet_{V\text{-Mod}}(k, M) = \operatorname{Ext}^\bullet_{(U(L)\text{-Mod})_I}(k, \log(M)) \tag{5.3}$$

The module M is a direct limit (union) of its finite dimensional submodules. The cohomology on both sides of (5.2) commutes with direct limits, hence we may assume that $\dim_k M < \infty$ and so the $\log(M) \in U(L)$-mod. Using the standard methods one can show that

$$\mathrm{Ext}^{\bullet}_{(U(L)\text{-Mod})_I}(k, \log(M)) = \mathrm{Ext}^{\bullet}_{(U(L)\text{-mod})_I}(k, \log(M))$$

Finally, Corollary 3.2 implies the isomorphism

$$\mathrm{Ext}^{\bullet}_{(U(L)\text{-mod})_I}(k, \log(M)) = \mathrm{Ext}^{\bullet}_{U(L)\text{-mod}}(k, \log(M))$$

which proves the corollary. □

Let X be a k-scheme with an action of the group V. For a V-equivariant quasi-coherent sheaf F, its cohomology can be computed as

$$H^{\bullet}_V(X, F) = \mathrm{Ext}^{\bullet}_{V\text{-Mod}}(k, \mathbb{R}\Gamma(X, F)),$$

and sometimes one wants to know that the Ext-space $\mathrm{Ext}^{\bullet}_{V\text{-Mod}}(k, -)$ can be computed using the standard complex for the Lie algebra L (by Corollary 5.1). This fact was used, for example, in the key computation on p. 8 of [Te].

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