

THE HARTREE-FOCK EQUATIONS IN MODULATION SPACES

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ABSTRACT. We establish both a local and a global well-posedness theories for the nonlinear Hartree-Fock equations and its reduced analog in the setting of the modulation spaces on $\mathbb{R}^d$. In addition, we prove similar results when a harmonic potential is added to the equations. In the process, we prove the boundedness of certain multilinear operators on products of the modulation spaces which may be of independent interest.

1. INTRODUCTION AND DESCRIPTION OF THE PROBLEM

1.1. Motivation. The Hartree equation, introduced by Hartree in the 1920s, arises as the mean-field limit of large systems of identical bosons, e.g., the Gross-Pitaevskii equation for Bose-Einstein condensates [25, 37], when taking into account the self-interactions of the bosons. A semirelativistic version of the Hartree equation was considered in [20, 33] for modeling boson stars. The Hartree-Fock equation, also developed by Fock [22] describes large systems of identical fermions by taking into account the self-interactions of charged fermions as well as an exchange term resulting from Pauli's principle. A semirelativistic version of the Hartree-Fock equation was developed in [23] for modeling white dwarfs. The Hartree equation is also used for fermions as an approximation of the Hartree-Fock equation neglecting the impact of their fermionic nature. Hartree and Hartree-Fock equations are used for several applications in many-particle physics [36, Section 2.2].

In [10, 32] fractional Laplacians have been applied to model physical phenomena. It was formulated by Laskin [32] as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. The harmonic oscillator (Hermite operator) $-\Delta + |x|^2$ is a fundamental operator in quantum physics and in analysis [40]. Hartree-Fock equations with harmonic potential model Bose-Einstein condensates with attractive inter-particle interactions under a magnetic trap. The isotropic harmonic potential $|x|^2$ describes a magnetic field whose role is to confine the movement of particles. A class of nonlinear Schrödinger equations with a “nonlocal” nonlinearity that we call “Hartree type” also occurs in the modeling of quantum semiconductor devices (see [11] and the references therein).

1.2. Hartree-Fock equations. Before giving the exact form of the Hartree-Fock equations, we set some notations that will be used through the paper. For two functions ϕ and h defined

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on $\mathbb{R}$ and $\mathbb{R}^d$ respectively, we set

$$\phi(h(D))f = \mathcal{F}^{-1}e^{it\phi\circ h(\cdot)}\mathcal{F}f$$

where $\mathcal{F}$ denotes the Fourier transform.

The Hartree-Fock equation of N particles is given by

$$(1.1) \quad \begin{cases} i\partial_t \psi_k = \phi(h(D))\psi_k - \sum_{l=1}^N \left(\frac{\kappa}{|x|^\gamma} * |\psi_l|^2 \right) \psi_k + \sum_{l=1}^N \psi_l \left(\frac{\kappa}{|x|^\gamma} * \{\overline{\psi_l} \psi_k\} \right), \\ \psi_k|_{t=0} = \psi_{0k}, \end{cases}$$

where $t \in \mathbb{R}$, $\psi_k : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$, $k = 1, 2, \dots, N$, $0 < \gamma < d$, κ is constant, and $*$ denotes the convolution in $\mathbb{R}^d$.

The Hartree factor

$$H = \sum_{l=1}^N \left(\frac{\kappa}{|x|^\gamma} * |\psi_l|^2 \right)$$

describes the self-interaction between charged particles as a repulsive force if $\kappa > 0$, and an attractive force if $\kappa < 0$. The last term on the right side of (1.1) is the so-called “exchange term (Fock term)”

$$F(\psi_k) = \sum_{l=1}^N \psi_l \left(\frac{\kappa}{|x|^\gamma} * \{\overline{\psi_l} \psi_k\} \right)$$

which is a consequence of the Pauli principle and thus applies to fermions. In the mean-field limit ($N \rightarrow \infty$), this term is negligible compared to the Hartree factor. In this case, (1.1) is replaced by the N coupled equations, the so-called **reduced Hartree-Fock equations**:

$$(1.2) \quad \begin{cases} i\partial_t \psi_k = \phi(h(D))\psi_k - \sum_{l=1}^N \left(\frac{\kappa}{|x|^\gamma} * |\psi_l|^2 \right) \psi_k, \\ \psi_k|_{t=0} = \psi_{0k}. \end{cases}$$

The rigorous time-dependent Hartree-Fock theory has been developed first by Chadam-Glassey [15] for (1.2) with $\phi(h(D)) = -\Delta$ in dimension $d = 3$. In this setting, (1.2) is equivalent to the von Neumann equation

$$(1.3) \quad iK'(t) = [G(t), K(t)]$$

for $K(t) = \sum_1^N |\psi_k(t)\rangle\langle\psi_k(t)|$ and $G(t) = \phi(h(D)) + H(x, t)$, see, e.g., [31, 34, 35]. In the above equation, we use Dirac’s notation $|u\rangle\langle v|$ for the operator $f \mapsto \langle v, f \rangle u$. The von Neumann equation (1.3) can also be considered for more general class of density matrices $K(t)$. For example, one can consider the class of nonnegative self-adjoint trace class operators, for which $K(t)$ satisfies the following conditions:

$$K^*(t) = K(t), K(t) \leq 1, \text{ tr} K = N$$

where the condition $K(t) \leq 1$ corresponds to the Pauli exclusion principle, and N is the “number of particles”.

The well-posedness for (1.3) was proved by Bove-Da Pardo-Fano [8, 9] for a short-range pairwise interaction potential $w(x - y)$ instead of Coulomb potential $\frac{1}{|x-y|}$ in the Hartree factor. The case of Coulomb potential was resolved by Chadam [14]. Lewin-Sabin [35]

have established the well-posedness for (1.3) with density matrices of infinite trace for pairwise interaction potentials $w \in L^1(\mathbb{R}^3)$. However, their investigation did not include the Coulomb potential case. Moreover, Lewin-Sabin [34] prove the asymptotic stability for the ground state in dimension $d = 2$. Fröhlich-Lenzmann [23] and Carles-Lucha-Moulay [13] studied the local and global well-posedness for (1.1) and (1.2) in L^2 -based Sobolev spaces, when $d = 3$. The existence of a global solution in $H^s(\mathbb{R}^d)$ ($s \geq 1/2$) to (1.1) and (1.2) was established in [23, Theorem 2.2] with smallness condition on the initial data in L^2 -norm. We note that this smallness condition is necessary. In fact, in [23, Theorem 2.3] and in [27], it is proved that radially symmetric data with negative energy lead to blow-up solutions for (1.1) and (1.2) in finite time in $H^{1/2}$ -norm. These results naturally raise the following question. Can we expect similar results in other functions spaces—which are not just based on L^2 -integrability?

We investigate this question in the setting of the modulation spaces $M^{p,q}(\mathbb{R}^d)$ (to be defined below), which have recently been considered as spaces of Cauchy data for certain nonlinear dispersive equations, see [1, 2, 3, 5, 4, 39, 42, 43, 44]. Generally modulation spaces are considered as low regularity spaces because they contain rougher functions than functions in any given fractional Bessel potential space (see Proposition 2.1 below). We refer to excellent survey [38] and the reference therein for details.

Taking these considerations into account, we initiate the study of (1.1) and (1.2) in modulation spaces. In particular, the two our main results can be stated as follows.

Theorem 1.1 (Local well-posedness). *Let $N, d \in \mathbb{N}$, and $\gamma \in (0, d)$ be given. Let X be given by*

$$X = \begin{cases} M^{p,q}(\mathbb{R}^d) & \text{if } 1 \leq p \leq 2, 1 \leq q \leq \frac{2d}{d+\gamma} \\ M_s^{p,1}(\mathbb{R}^d) & \text{if } 1 < p < \infty, \frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon} \end{cases}$$

for some $\epsilon > 0$ and $s \geq 0$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be such that there exist $m_1, m_2 > 0$ with

$$\begin{cases} |\phi^{(\mu)}(r)| \lesssim r^{m_1-\mu} & \text{if } r \geq 1 \\ |\phi^{(\mu)}(r)| \lesssim r^{m_2-\mu} & \text{if } 0 < r < 1 \end{cases}$$

for all $\mu \in \mathbb{N}_0$. Furthermore, assume that $h \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is either

- (a) a positive function of homogeneous type of degree $\lambda > 0$ with $m_1\lambda \leq 2$, or
- (b) $\phi \circ h(\xi) := P(\xi) = \sum_{|\beta| \leq m} c_\beta \xi^\beta$ is a polynomial with order $m \leq 2$.

Given initial data $(\psi_{01}, \dots, \psi_{0N}) \in X^N$, the following statements hold.

- (i) There exists $T > 0$ depending only on $\|\psi_{01}\|_X, \dots, \|\psi_{0N}\|_X$, d and γ such that (1.1) has a unique local solution

$$(\psi_1, \dots, \psi_N) \in (C([0, T], X))^N.$$

- (ii) There exists $T > 0$ depending only on $\|\psi_{01}\|_X, \dots, \|\psi_{0N}\|_X$, d and γ such that (1.2) has a unique local solution

$$(\psi_1, \dots, \psi_N) \in (C([0, T], X))^N.$$

Our second main result deal with the global well-posedness of these equations. In the statement, we denote by X_{rad} , space of radial functions in the Banach space X .

Theorem 1.2 (Global well-posedness). *Suppose that ϕ and h are defined on $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that $\phi \circ h(\xi) = |\xi|^\alpha$ for $\xi \in \mathbb{R}^d$, and where $\alpha > 0$. Assume that $0 < \gamma < \min\{\alpha, d/2\}$, and that one of the following two statements holds:*

(a) *For $\alpha = 2$ and $d \in \mathbb{N}$, let*

$$X = \begin{cases} M^{p,q}(\mathbb{R}^d) & \text{if } 1 \leq p \leq 2, 1 \leq q \leq \frac{2d}{d+\gamma} \\ M_s^{p,1}(\mathbb{R}^d) & \text{if } 1 < p < \infty, \frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon} \end{cases}$$

for some $\epsilon > 0$ and $s \geq 0$.

(b) *For $\alpha \in (\frac{2d}{2d-1}, 2)$ and $d \geq 2$, let*

$$X = \begin{cases} M_{rad}^{p,q}(\mathbb{R}^d) & \text{if } 1 \leq p \leq 2, 1 \leq q \leq \frac{2d}{d+\gamma} \\ M_s^{p,1}(\mathbb{R}^d) \cap L_{rad}^2(\mathbb{R}^d) & \text{if } 2 < p < \infty, \frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon} \end{cases}$$

for some $\epsilon > 0$ and $s \geq 0$.

Given initial data $(\psi_{01}, \dots, \psi_{0N}) \in X^N$, the following statements hold.

(i) *There exists a unique global solution of (1.1) such that*

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, X) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

(ii) *There exists a unique global solution of (1.2) such that*

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, X) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In the case $N = 1$, first author in [5, Theorem 1.1] established the global well-posedness of (1.2) in $M^{p,q}(\mathbb{R}^d)$ when $1 \leq p \leq 2$, and $1 \leq q < \frac{2d}{d+\gamma}$. Part (ii) of Theorem 1.2 proves this result for the end point case for any $N \geq 1$. We note that $M^{p,q}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ ($q \leq \min\{p, p'\}$) is sharp embedding and up to now we cannot get the global well-posedness of (1.1) in $L^p(\mathbb{R}^d)$ ($1 \leq p < 2$) but in $M^{p,q}(\mathbb{R}^d)$ (Theorem 1.2). Noticing for $s > \gamma/2$, we have sharp embedding $H^s(\mathbb{R}^d) \subset M^{2, \frac{2d}{d+\gamma}}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ (see Proposition 2.1 below), Theorem 1.2 reveals that we can solve (1.1) and (1.2) with Cauchy data beyond in $H^s(\mathbb{R}^d)$ ($s > \gamma/2$).

Remark 1.1. *The sign of κ in Hartree and Fock terms determines the defocusing and focusing character of the nonlinearity, but, as we shall see, this character will play no role in our analysis on modulation spaces, as we do not use the conservation of energy of (1.1) and (1.2) to achieve global existence.*

Remark 1.2. *We have several comments about Theorems 1.1 and 1.2.*

- (1) *Based on the trilinear estimate to be established in Section 3, we can prove some local well-posedness results for (1.1) in some modulation spaces (see Theorem 1.1) without any radial assumption of the initial data. In particular, in Theorem 1.1, we can take $\phi(r) = r^\alpha$ and $h(\xi) = |\xi|$ and so $\phi \circ h(\xi) = |\xi|^\alpha$, ($0 < \alpha \leq 2$).*

- (2) To extend this local existence globally in time, that is, to prove Theorem 1.2, in modulation spaces, we proceed as follows. First, we prove that (1.1) is globally well-posed (see Proposition 4.2 below) in $L^2_{rad}(\mathbb{R}^d)^N$ ($d \geq 2$) if $\frac{2d}{2d-1} < \alpha < 2$ and in $L^2(\mathbb{R}^d)^N$ ($d \geq 1$) if $\alpha = 2$. To prove this, we employ Strichartz estimates for fractional Schrödinger equation (see Proposition 4.1 below) –where we need the initial data to be a radial and dimension $d \geq 2$ ([26, p.26-27]) if $\alpha \neq 2$. Finally, we invoke Proposition 4.2 to prove that the modulation space norm of the growth term cannot blow-up in finite time (see Lemma 4.2 below) leading to the global existence in modulation spaces.
- (3) Note that to take advantage of Proposition 5.2, the hypotheses that the initial data is radial and $0 < \gamma < \min\{\alpha, d/2\}$ in Theorem 1.2 is necessary. Thus, the global time behavior of (1.1) with non-radial initial data in modulation spaces remains an interesting open question.

Remark 1.3. Kato-Naumkin [28, Theorems 1.1 and 1.2] proved some space-time estimates for the solution of the Dirac equation (free Schrödinger equation associated to Dirac operator) in modulation spaces. Combining these results with a trilinear estimate of Section 3, it is natural to expect some local well-posedness results for the Dirac equation with Hartree type non linearity. We plan to address the global well-posedness theory for the Dirac equation with Hartree type non linearity in modulation spaces in a future work.

1.3. Hartree-Fock equation with harmonic potential. The Hartree-Fock equation with the harmonic potential of N particles is given by

$$(1.4) \quad \begin{cases} i\partial_t \psi_k - (-\Delta + |x|^2) \psi_k = \sum_{l=1}^N \left(\frac{\kappa}{|x|^\gamma} * |\psi_l|^2 \right) \psi_k + \sum_{l=1}^N \psi_l \left(\frac{\kappa}{|x|^\gamma} * \{\overline{\psi_l} \psi_k\} \right), \\ \psi_k|_{t=0} = \psi_{0k} \end{cases}$$

and the corresponding reduced Hartree-Fock equation with the harmonic potential:

$$(1.5) \quad \begin{cases} i\partial_t \psi_k - (-\Delta + |x|^2) \psi_k = \sum_{l=1}^N \left(\frac{\kappa}{|x|^\gamma} * |\psi_l|^2 \right) \psi_k, \\ \psi_k|_{t=0} = \psi_{0k}, \end{cases}$$

where $t \in \mathbb{R}$, $\psi_k : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$, $k = 1, 2, \dots, N$, $0 < \gamma < d$, κ is constant. In this context we establish the following result.

Theorem 1.3. Let $0 < \gamma < \min\{2, d/2\}$, $d \in \mathbb{N}$ and $1 \leq p \leq \frac{2d}{d+\gamma}$. Given initial data $(\psi_{01}, \dots, \psi_{0N}) \in (M^{p,p}(\mathbb{R}^d))^N$, the following statements hold.

- (i) There exists a unique global solution of (1.4) such that

$$(\psi_1, \dots, \psi_N) \in \left(C([0, \infty), M^{p,p}(\mathbb{R}^d)) \cap L^{8/\gamma}_{loc}([0, \infty), L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

- (ii) There exists a unique global solution of (1.5) such that

$$(\psi_1, \dots, \psi_N) \in \left(C([0, \infty), M^{p,p}(\mathbb{R}^d)) \cap L^{8/\gamma}_{loc}([0, \infty), L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In the case $N = 1$, first author in [6, Theorem 1.1] proved that (1.5) is globally well-posed in $M^{p,p}(\mathbb{R}^d)$ for $1 \leq p < \frac{2d}{d+\gamma}$. Part (ii) of Theorem 1.3 establishes this result for the end point case for any $N \geq 1$.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminary results which will be used in the sequel. In Section 3, we prove the boundedness for Hartree nonlinearity on modulation spaces. In Section 4 we establish two of our main results, namely Theorems 1.1, and 1.2. Finally, in Section 5 we prove Theorem 1.3.

2. PRELIMINARIES

2.1. Notations. The notation $A \lesssim B$ means $A \leq cB$ for some constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$ for some $c \geq 1$. Given $a, b \in \mathbb{R}$ we let $a \wedge b = \min\{a, b\}$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space A_1 into A_2 . Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ is a multi-index, we set

$$|\beta| = \sum_1^d \beta_j, \quad \beta! = \prod_1^d \beta_j!, \quad \partial^\beta = D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\beta_d},$$

and if $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$x^\beta = \prod_1^d x_j^{\beta_j}.$$

The $L^p(\mathbb{R}^d)$ norm is denoted by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty),$$

the $L^\infty(\mathbb{R}^d)$ norm is $\|f\|_{L^\infty} = \text{ess. sup}_{x \in \mathbb{R}^d} |f(x)|$. For $1 \leq p \leq \infty$, p' denotes the Hölder conjugate of p , that is, $1/p + 1/p' = 1$. We use $L_t^r(I, X)$ to denote the space-time norm

$$\|u\|_{L_t^r(I, X)} = \left(\int_I \|u\|_X^r dt \right)^{1/r},$$

where $I \subset \mathbb{R}$ is an interval and X is a Banach space. The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$, and, its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^d x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(w) = \hat{f}(w) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot w} dt, \quad w \in \mathbb{R}^d.$$

Then $\mathcal{F}$ is an isomorphism on $\mathcal{S}(\mathbb{R}^d)$ which uniquely extends to an isomorphism on $\mathcal{S}'(\mathbb{R}^d)$.

The Fourier-Lebesgue spaces $\mathcal{FL}^p(\mathbb{R}^d)$ is defined by

$$\mathcal{FL}^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{FL}^p} := \|\hat{f}\|_{L^p} < \infty \right\}.$$

For $p \in (1, \infty)$ and $s \geq 0$, $W^{s,p}(\mathbb{R}^d)$ will denote standard Sobolev space. In particular, if s is an integer, then $W^{s,p}$ consists of L^p -functions with derivatives in L^p up to order s , hence

coincides with the L_s^p -Sobolev space, also known as Bessel potential space, defined for $s \in \mathbb{R}$ by

$$L_s^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{L_s^p} := \|\mathcal{F}^{-1}[\langle \cdot \rangle^s \mathcal{F}(f)]\|_{L^p} < \infty\},$$

where $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ ($\xi \in \mathbb{R}^d$). Note that $L_{s_1}^p(\mathbb{R}^d) \hookrightarrow L_{s_2}^p(\mathbb{R}^d)$ if $s_2 \leq s_1$.

2.2. Modulation spaces. Feichtinger [21] introduced the modulation spaces by imposing integrability conditions on the *short-time Fourier transform* (STFT) of functions or distributions defined on $\mathbb{R}^d$. To be specific, the STFT of a function f with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i w \cdot t} dt, \quad (x, w) \in \mathbb{R}^{2d}$$

whenever the integral exists. For $x, w \in \mathbb{R}^d$ the translation operator T_x and the modulation operator M_w are defined by $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w \cdot t} f(t)$. In terms of these operators the STFT may be expressed as

$$V_g f(x, w) = \langle f, M_w T_x g \rangle$$

where $\langle f, g \rangle$ denotes the inner product for L^2 functions, or the action of the tempered distribution f on the Schwartz class function g . Thus $V : (f, g) \rightarrow V_g(f)$ extends to a bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

Definition 2.1 (Modulation spaces). *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$. The weighted modulation space $M_s^{p,q}(\mathbb{R}^d)$ is defined to be the space of all tempered distributions f for which the following norm is finite:*

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, w)|^p dx \right)^{q/p} (1 + |w|^2)^{sq/2} dw \right)^{1/q},$$

for $1 \leq p, q < \infty$. If p or q is infinite, $\|f\|_{M_s^{p,q}}$ is defined by replacing the corresponding integral by the essential supremum. For $s = 0$, we write $M_0^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$.

It is standard to show that this definition is independent of the choice of the particular window function, e.g., see, [24, Proposition 11.3.2(c)].

Using a uniform partition of the frequency domain, one can obtain an equivalent definition of the modulation spaces [43] as follows. Let Q_k be the unit cube with the center at k , so $\{Q_k\}_{k \in \mathbb{Z}^d}$ constitutes a decomposition of $\mathbb{R}^d$, that is, $\mathbb{R}^d = \cup_{k \in \mathbb{Z}^d} Q_k$. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$, $\rho : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function satisfying $\rho(\xi) = 1$ if $|\xi|_\infty \leq \frac{1}{2}$ and $\rho(\xi) = 0$ if $|\xi|_\infty \geq 1$, where $|\xi|_\infty = \max_{k=1, \dots, d} |\xi_k|$. Let ρ_k be a translate of ρ , that is,

$$\rho_k(\xi) = \rho(\xi - k) \quad (k \in \mathbb{Z}^d).$$

For each $0 \neq k \in \mathbb{Z}$ let

$$\sigma_k(\xi) = \frac{\rho_k(\xi)}{\sum_{l \in \mathbb{Z}^d} \rho_l(\xi)}$$

and when $k = 0$, we simply write $\sigma_0 = \sigma$. Then $\{\sigma_k(\xi)\}_{k \in \mathbb{Z}^d}$ satisfies the following properties

$$\begin{cases} |\sigma_k(\xi)| \geq c, \forall \xi \in Q_k, \\ \text{supp } \sigma_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, \\ \sum_{k \in \mathbb{Z}^d} \sigma_k(\xi) \equiv 1, \forall \xi \in \mathbb{R}^d, \\ |D^\alpha \sigma_k(\xi)| \leq C_{|\alpha|}, \forall \xi \in \mathbb{R}^d, \alpha \in (\mathbb{N} \cup \{0\})^d \end{cases}$$

for some positive constant c .

The frequency-uniform decomposition operators can be defined by

$$(2.1) \quad \square_k = \mathcal{F}^{-1} \sigma_k \mathcal{F}.$$

For $1 \leq p, q \leq \infty, s \in \mathbb{R}$, it is known [21] that

$$(2.2) \quad \|f\|_{M_s^{p,q}} \asymp \left(\sum_{k \in \mathbb{Z}^d} \|\square_k(f)\|_{L^p}^q (1 + |k|)^{sq} \right)^{1/q},$$

with natural modifications for $p, q = \infty$. As observed in [44], the frequency-uniform decomposition operators obey an almost orthogonality relation: for each $k \in \mathbb{Z}$

$$(2.3) \quad \square_k = \sum_{\|\ell\|_\infty \leq 1} \square_{k+\ell} \square_k$$

where $\|\ell\|_\infty = \max\{|\ell_i| : \ell_i \in \mathbb{Z}, i = 1, \dots, d\}$.

We now list some basic properties of the modulation spaces.

Lemma 2.1. *Let $p, q, p_i, q_i \in [1, \infty]$ ($i = 1, 2$), $s, s_1, s_2 \in \mathbb{R}$. Then*

- (1) $M_{s_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2}^{p_2, q_2}(\mathbb{R}^d)$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$ and $s_2 \leq s_1$.
- (2) $M^{p, q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p, q_2}(\mathbb{R}^d)$ holds for $q_1 \leq \min\{p, p'\}$ and $q_2 \geq \max\{p, p'\}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.
- (3) $M^{\min\{p', 2\}, p}(\mathbb{R}^d) \hookrightarrow \mathcal{FL}^p(\mathbb{R}^d) \hookrightarrow M^{\max\{p', 2\}, p}(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{p'} = 1$.
- (4) $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p, q}(\mathbb{R}^d)$ if p and $q < \infty$.
- (5) The Fourier transform $\mathcal{F} : M_s^{p, p}(\mathbb{R}^d) \rightarrow M_s^{p, p}(\mathbb{R}^d)$ is an isomorphism.
- (6) The space $M_s^{p, q}(\mathbb{R}^d)$ is a Banach space.
- (7) The space $M_s^{p, q}(\mathbb{R}^d)$ is invariant under complex conjugation.

Proof. For the proof of parts (1), (2), (3), and (4) see [24, Theorem 12.2.2], [41, Proposition 1.7], [18, Corollary 1.1] and [24, Proposition 11.3.4] respectively. The proof of statement (5) can be derived from the fundamental identity of time-frequency analysis:

$$V_g f(x, w) = e^{-i2\pi x \cdot w} V_{\hat{g}} \hat{f}(w, -x),$$

which is easy to obtain. The proof of statement (6) is trivial, indeed, we have $\|f\|_{M^{p, q}} = \|\bar{f}\|_{M^{p, q}}$. $\square$

We can obtain examples of functions in the modulation spaces via embedding relations with certain classical functions spaces. For example the following result can be proved.

Proposition 2.1 (Examples). *The following statements hold.*

(i) ([30], [38, Theorem 3.8]) Let $1 \leq p, q \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and

$$\tau(p, q) = \max \left\{ 0, d \left(\frac{1}{q} - \frac{1}{p} \right), d \left(\frac{1}{q} + \frac{1}{p} - 1 \right) \right\}.$$

Then $L_{s_1}^p(\mathbb{R}^d) \subset M_{s_2}^{p,q}(\mathbb{R}^d)$ if and only if one of the following conditions is satisfied:

$$\begin{cases} q \geq p > 1, s_1 \geq s_2 + \tau(p, q), \text{ or} \\ p > q, s_1 > s_2 + \tau(p, q), \text{ or} \\ p = 1, q = \infty, s_1 \geq s_2 + \tau(1, \infty), \text{ or} \\ p = 1, q \neq \infty, s_1 > s_2 + \tau(1, q). \end{cases}$$

(ii) ([30], [38, Theorem 3.8]) Let $1 \leq p, q \leq \infty$, $s_1, s_2 \in \mathbb{R}$, and

$$\sigma(p, q) = \max \left\{ 0, d \left(\frac{1}{p} - \frac{1}{q} \right), d \left(1 - \frac{1}{q} - \frac{1}{p} \right) \right\}.$$

Then $M_{s_1}^{p,q}(\mathbb{R}^d) \subset L_{s_2}^p(\mathbb{R}^d)$ if and only if one of the following conditions is satisfied:

$$\begin{cases} q \leq p < \infty, s_1 \geq s_2 + \sigma(p, q), \text{ or} \\ p < q, s_1 > s_2 + \sigma(p, q), \text{ or} \\ p = \infty, q = 1, s_1 \geq s_2 + \sigma(\infty, 1), \text{ or} \\ p = \infty, q \neq 1, s_1 > s_2 + \sigma(\infty, q). \end{cases}$$

- (iii) For $1 \leq p < 2$, $M^{p,p}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ and there exists $f \in L^p(\mathbb{R}^d)$ such that $f \notin M^{p,p}(\mathbb{R}^d)$.
- (iv) For $s > \frac{\gamma}{2} > 0$, $H^s(\mathbb{R}^d) \subset M^{2, \frac{2d}{d+\gamma}}(\mathbb{R}^d)$ and there exists $f \in M^{2, \frac{2d}{d+\gamma}}(\mathbb{R}^d)$ such that $f \notin H^s(\mathbb{R}^d)$.
- (v) For $2 < p < \infty$ and $s > d \left(1 - \frac{1}{p} \right)$, $L_s^p(\mathbb{R}^d) \subset M^{p,1}(\mathbb{R}^d)$ and there exists $f \in M^{p,1}(\mathbb{R}^d)$ such that $f \notin L_s^p(\mathbb{R}^d)$.

Proof. We only give proofs of the last three parts.

- (iii) For $1 \leq p < 2$, by part (2) of Lemma 2.1, we have $M^{p,p}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. We claim that $M^{p,p}(\mathbb{R}^d) \subsetneq L^p(\mathbb{R}^d)$. If possible, suppose that claim is not true, that is, for all $f \in L^p(\mathbb{R}^d)$, we have $f \in M^{p,p}(\mathbb{R}^d)$. It follows that $L^p(\mathbb{R}^d) = M^{p,p}(\mathbb{R}^d)$. But then by part (5) of Lemma 2.1, it follows that $L^p(\mathbb{R}^d)$ invariant under the Fourier transform, which is a contradiction. Hence, the claim. Similarly, for $2 < p \leq \infty$, we have $M^{p,p}(\mathbb{R}^d) \subsetneq L^p(\mathbb{R}^d)$.
- (iv) Noticing $\tau \left(2, \frac{2d}{d+\gamma} \right) = \frac{\gamma}{2}$, by part (i), we have $H^s(\mathbb{R}^d) \subset M^{2,1}(\mathbb{R}^d)$ for $s > \gamma/2$. We claim that $H^s(\mathbb{R}^d) \subsetneq M^{2, \frac{2d}{d+\gamma}}(\mathbb{R}^d)$. If possible, suppose that claim is not true. Then we have $H^s(\mathbb{R}^d) = M^{2, \frac{2d}{d+\gamma}}(\mathbb{R}^d)$. But then, noticing $\sigma \left(2, \frac{2d}{d+\gamma} \right) = -\frac{\gamma}{2}$, part (ii) gives contradiction. Hence, the claim.
- (v) Noticing $\tau(p, 1) = d \left(1 - \frac{1}{p} \right)$ and $\sigma(p, 1) = -\frac{d}{p}$, parts (i) and (ii) give $L_s^p(\mathbb{R}^d) \subsetneq M^{p,1}(\mathbb{R}^d)$.

□

Proposition 2.2. (Algebra property, [41, Theorem 2.4]) Let $s \geq 0$, and $p, q, p_i, q_i \in [1, \infty]$, where $i = 0, 1, 2$. If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0}$, then

$$M_s^{p_1, q_1}(\mathbb{R}^d) \cdot M_s^{p_2, q_2}(\mathbb{R}^d) \hookrightarrow M_s^{p_0, q_0}(\mathbb{R}^d)$$

with norm inequality $\|fg\|_{M_s^{p_0, q_0}} \lesssim \|f\|_{M_s^{p_1, q_1}} \|g\|_{M_s^{p_2, q_2}}$. In particular, the space $M^{p, q}(\mathbb{R}^d)$ is a pointwise $\mathcal{FL}^1(\mathbb{R}^d)$ -module, that is, it satisfies

$$\|fg\|_{M^{p, q}} \lesssim \|f\|_{\mathcal{FL}^1} \|g\|_{M^{p, q}}.$$

2.3. Modulation space estimates for unimodular Fourier multipliers. In this section, we consider the boundedness properties of a class of unimodular Fourier multipliers defined by

$$U(t)f(x) = e^{it\phi(h(D))}f(x) = \int_{\mathbb{R}^d} e^{i\pi t\phi \circ h(\xi)} \hat{f}(\xi) e^{2\pi i\xi \cdot x} d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, where $\phi \circ h : \mathbb{R}^d \rightarrow \mathbb{R}$ is the composition function of $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Proposition 2.3. Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$.

(i) ([19, Theorem 1.1]) Assume that there exist $m_1, m_2 > 0$ such that ϕ satisfies

$$\begin{cases} |\phi^{(\mu)}(r)| \lesssim r^{m_1 - \mu} & \text{if } r \geq 1 \\ |\phi^{(\mu)}(r)| \lesssim r^{m_2 - \mu} & \text{if } 0 < r < 1 \end{cases}$$

for all $\mu \in \mathbb{N}_0$ and $h \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is positive homogeneous function with degree $\lambda > 0$. Then we have

$$\|e^{it\phi(h(D))}f\|_{M_s^{p, q}} \lesssim \|f\|_{M_s^{p, q}} + |t|^{d|\frac{1}{2} - \frac{1}{p}|} \|f\|_{M_{s+\gamma(m_1, \lambda)}^{p, q}}$$

where $\gamma(m_1, \lambda) = d(m_1\lambda - 2)|1/2 - 1/p|$.

(ii) ([16, Theorems 1 and 2]) Let $h(\xi) = |\xi|$ and $\phi(r) = r^\alpha$, with $1/2 < \alpha \leq 2$. Then

$$\|U(t)f\|_{M^{p, q}} \leq (1 + |t|)^{d|\frac{1}{p} - \frac{1}{2}|} \|f\|_{M^{p, q}}$$

(iii) ([43, Proposition 4.1]) Let $2 \leq p \leq \infty, 1 \leq q \leq \infty$, $h(\xi) = |\xi|$ and $\phi(r) = r^\alpha$ ($\alpha \geq 2$). Then

$$\|U(t)f\|_{M^{p, q}} \leq (1 + |t|)^{-\frac{2d}{\alpha}(\frac{1}{2} - \frac{1}{p})} \|f\|_{M^{p', q}}$$

Another important class of unimodular Fourier multipliers that is not covered by Proposition 2.3, are the so-called Fourier multiplier with polynomial symbol. Specifically, for $f \in \mathcal{S}(\mathbb{R}^d)$ let

$$U(t)f(x) = e^{itP(D)}f(x) = \int_{\mathbb{R}^d} e^{i\pi tP(\xi)} \hat{f}(\xi) e^{2\pi i\xi \cdot x} d\xi,$$

where $P(\xi) = \sum_{|\beta| \leq m} c_\beta \xi^\beta$ is a polynomial with order $m \geq 1$. In this setting the following result was proved in [19].

Proposition 2.4. ([19, Theorem 4.3]) *Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $m \geq 2$. Then*

$$\|e^{itP(D)}f\|_{M^{p,q}} \lesssim \|f\|_{M_s^{p,q}} + |t|^{d|\frac{1}{2}-\frac{1}{p}|} \|f\|_{M_{s+\gamma(m)}^{p,q}}$$

where $\gamma(m) = d(m-2)|1/2 - 1/p|$.

To make the paper self content, we outline the proof of Proposition 2.4 in the particular case when $P(\xi, \eta) = |\xi|^2 - |\eta|^2$, and note that the general case can be proved similarly. But first, we state a result that provides a criteria for the Fourier multiplier to be bounded on modulation spaces. In particular, it provides an application of the uniform decomposition operators given in (2.1).

Proposition 2.5. *Let $\square_k$ be defined as in (2.1) and $t \in \mathbb{R}$. Suppose that there is an integer $M > 0$ such that*

$$\|\square_k e^{itP(D)}f\|_{L^1} \lesssim \begin{cases} |t|^{b_1} \|f\|_{L^1} & \text{if } |k| < M \\ |t|^{b_2} \|f\|_{L^1} & \text{if } |k| \geq M \end{cases}$$

where $b_1 \geq b_2 \geq 0$ for all $f \in L^1(\mathbb{R}^d)$. Then we have

$$\|e^{itP(D)}f\|_{M_s^{p,q}} \lesssim \left(|t|^{2b_1|\frac{1}{p}-\frac{1}{2}|} + |t|^{2b_2|\frac{1}{p}-\frac{1}{2}|} \right) \|f\|_{M_s^{p,q}}$$

whenever $f \in M_s^{p,q}(\mathbb{R}^d)$.

Proof. By (2.1) and Plancherel theorem, we obtain

$$\|\square_k e^{itP(D)}f\|_{L^2} = \|\sigma_k e^{itP} \hat{f}\|_{L^2} \lesssim \|f\|_{L^2}$$

for all $k \in \mathbb{Z}^d$. By the Riesz-Thorin interpolation theorem, and for any $1 \leq p \leq 2$, we have

$$\|\square_k e^{itD}f\|_{L^p} \lesssim \begin{cases} |t|^{2b_1|\frac{1}{p}-\frac{1}{2}|} \|f\|_{L^p} & \text{if } |k| < M \\ |t|^{2b_2|\frac{1}{p}-\frac{1}{2}|} \|f\|_{L^p} & \text{if } |k| \geq M. \end{cases}$$

Using a duality argument, we obtain the above two inequality for all $1 \leq p \leq \infty$. Using (2.3), for $f \in \mathcal{S}(\mathbb{R}^d)$ and $|k| < M$, we obtain

$$\|\square_k e^{itP(D)}f\|_{L^p} \leq \sum_{\|\ell\|_\infty \leq 1} \|\pi_{k+\ell} e^{itP(D)} \square_k f\|_{L^p} \lesssim |t|^{2b_1|\frac{1}{p}-\frac{1}{2}|} \|\square_k f\|_{L^p}.$$

Similarly, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $|k| \geq M$, we obtain

$$\|\square_k e^{itP(D)}f\|_{L^p} \lesssim |t|^{2b_2|\frac{1}{p}-\frac{1}{2}|} \|\square_k f\|_{L^p}.$$

In view of (2.2), and the above two inequalities, we obtain

$$\begin{aligned}
\|e^{itP(D)}f\|_{M_s^{p,q}} &= \left\{ \sum_{|k| \leq M+1} (1+|k|)^{sq} \|\square_k(e^{itP(D)}f)\|_{L^p}^q \right\}^{1/q} \\
&\quad + \left\{ \sum_{|k| > M+1} (1+|k|)^{sq} \|\square_k(e^{itP(D)}f)\|_{L^p}^q \right\}^{1/q} \\
&\lesssim |t|^{2b_1|\frac{1}{p}-\frac{1}{2}|} \left\{ \sum_{|k| \leq M+1} (1+|k|)^{sq} \|\square_k f\|_{L^p}^q \right\}^{1/q} \\
&\quad + |t|^{2b_2|\frac{1}{p}-\frac{1}{2}|} \left\{ \sum_{|k| > M+1} (1+|k|)^{sq} \|\square_k f\|_{L^p}^q \right\}^{1/q} \\
&\lesssim \left(|t|^{2b_1|\frac{1}{p}-\frac{1}{2}|} + |t|^{2b_2|\frac{1}{p}-\frac{1}{2}|} \right) \|f\|_{M_s^{p,q}}.
\end{aligned}$$

This completes the proof. $\square$

Now to apply Proposition 2.5, we must have control on the L^1 -norm of the projection operator $\|\square_k(e^{itP(D)}f)\|_{L^1(\mathbb{R}^d)}$. Since $\square_k(e^{itP(D)}f) = \mathcal{F}^{-1}(\sigma_k e^{itP}) * f$, in view of Young's inequality, it suffices to control the norm $\|\mathcal{F}^{-1}(\sigma_k e^{itP})\|_{L^1}$, which we shall do in next two lemmas.

Lemma 2.2. ([19, Lemmas 4.1 and 4.2]) *Let $t \in \mathbb{R}$, $P(z) = P(\xi, \eta) = |\xi|^2 - |\eta|^2$, $z = (\xi, \eta) \in \mathbb{R}^{2d}$, $k \in \mathbb{Z}^{2d}$, and $M > 0$. Then we have*

$$\|\mathcal{F}^{-1}(\sigma_k e^{itP(D)})\|_{L^1(\mathbb{R}^{2d})} \lesssim \max\{|t|^d, 1\}.$$

Proof. Assume that $|k| > M + 1$. We introduce an auxiliary function defined by

$$\Lambda_k(z) = P(z+k) - P(k) - \nabla P(k) \cdot z$$

for all $k \in \mathbb{Z}^{2d}$. Since L^1 -norm is invariant under translation and modulation, we have

$$\begin{aligned}
\|\mathcal{F}^{-1}(\sigma_k(z)e^{itP(z)})\|_{L^1(\mathbb{R}^{2d})} &= \|\mathcal{F}^{-1}(\sigma(z)e^{itP(z+k)})\|_{L^1(\mathbb{R}^{2d})} \\
&= \|\mathcal{F}^{-1}(\sigma(z)e^{it(\Lambda_k(z)+P(k)+\nabla P(k) \cdot z)})\|_{L^1(\mathbb{R}^{2d})} \\
&= \|g^\vee(x + \nabla P(k))\|_{L_x^1(\mathbb{R}^{2d})} \\
&= \|g^\vee\|_{L^1(\mathbb{R}^{2d})},
\end{aligned}$$

where $g(z) = \sigma(z)e^{it\Lambda_k(z)}$. Thus, to prove Lemma 2.2, it suffices to prove

$$\|g^\vee\|_{L^1} \lesssim \max\{t^d, 1\}$$

for $t > 0$. We consider

$$\begin{aligned}
\|g^\vee\|_{L^1(\mathbb{R}^{2d})} &= \int_{|x| \leq t} \left| \int_{\mathbb{R}^{2d}} \sigma(z) e^{it\Lambda_k(z)} e^{ixz} dz \right| dx + \int_{|x| > t} \left| \int_{\mathbb{R}^{2d}} \sigma(z) e^{it\Lambda_k(z)} e^{ixz} dz \right| dx \\
(2.4) \quad &:= I_1 + I_2.
\end{aligned}$$

By Cauchy-Schwarz inequality and Plancherel's Theorem, we have

$$\begin{aligned} I_1 &\lesssim \left(\int_{|x| \leq t} 1 dx \right)^{1/2} \left(\int_{|x| \leq t} \left| \int_{\mathbb{R}^{2d}} \sigma(z) e^{it\Lambda_k(z)} e^{ixz} dz \right|^2 dx \right)^{1/2} \\ (2.5) \quad &\lesssim t^d \|g^\vee\|_{L^2(\mathbb{R}^{2d})} \lesssim t^d. \end{aligned}$$

Now we concentrate on I_2 . For $j \in \{1, 2, \dots, 2d\}$, let

$$\begin{aligned} E_t &= \{x \in \mathbb{R}^{2d} : |x| > t\}, \\ E_{j,t} &= \{x \in E_t : |x_j| \geq |x_l| \text{ for all } l \neq j\}. \end{aligned}$$

We note that

$$I_2 \lesssim \sum_{j=1}^{2d} \int_{E_{j,t}} \left| \int_{\mathbb{R}^{2d}} \sigma(z) e^{it\Lambda_k(z)} e^{ixz} dz \right| dx := \sum_{j=1}^{2d} I_{2j}.$$

Since σ is compactly supported and Λ_k is a smooth function, performing integration by parts and using Plancherel's theorem, we obtain that for each $j \in \{1, 2, \dots, 2d\}$

$$\begin{aligned} I_{2j} &\lesssim \int_{E_{j,t}} \frac{1}{|x_j|^L} \left| \int_{\mathbb{R}^{2d}} e^{ixz} D_j^L(\sigma(z) e^{it\Lambda_k(z)}) dz \right| dx \\ &\lesssim \left(\int_{E_{j,t}} \frac{1}{|x_j|^{2L}} dx \right)^{1/2} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})} \\ &\lesssim \left(\int_{|x| > t} \frac{1}{|x|^{2L}} dx \right)^{1/2} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})} \\ &= t^{-d} t^{-L} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})} \end{aligned}$$

where we choose $L > d$ as an integer. Where we have used the fact that since $|x|^2 = \sum_{j=1}^{2d} x_j^2 \leq 2d|x_j|^2$ for $x \in E_{j,t}$, we have $|x_j|^{-2L} \lesssim |x|^{-2L}$. Consequently, we have

$$(2.6) \quad I_2 \lesssim t^d t^{-L} \sum_{j=1}^{2d} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})}.$$

Next, we claim that

$$\sum_{j=1}^{2d} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})} \lesssim t^L.$$

Once this claim is established, the proof of the lemma will follow from (2.4), (2.5), and (2.6).

We now give a proof of this claim. To this end, we note that by Taylor's and Leibniz formula, we have

$$(2.7) \quad \Lambda_k(z) = 2 \sum_{|\beta|=2} \frac{z^\beta}{\beta!} \cdot \int_0^1 (1-s) D^\beta P(k+sz) ds,$$

and

$$(2.8) \quad D^\gamma \Lambda_k(z) = \sum_{\gamma_1 + \gamma_2 = \gamma} \sum_{|\beta|=2} C_{\beta, \gamma_1, \gamma_2} D^{\gamma_1} z^\beta \cdot \int_0^1 (1-s) D^{\beta + \gamma_2} P(k+sz) ds.$$

Since $P(z)$ is a polynomial of order 2, there exists C_γ such that

$$(2.9) \quad |D^\gamma P(z)| \leq C_\gamma |z|^{2-|\gamma|}$$

for all $\gamma \in \mathbb{N}_0^{2d}$. We note that for $z \in \text{supp } \sigma$, and $s \in [0, 1]$, we have $|k + sz| \lesssim |k|$, and in view of (2.7)-(2.9), we have that for all $|k| > M + 1$

$$|D^\gamma \Lambda_k(z)| \lesssim C_{\beta, \gamma} \sum_{\gamma_1 + \gamma_2 = \gamma} \sum_{|\beta|=2} |D^{\gamma_1} z^\beta| \int_0^1 |k|^{-|\gamma_2|} ds \lesssim 1, \quad \gamma \in \mathbb{N}_0^{2d},$$

which implies that

$$(2.10) \quad \begin{aligned} |D^\gamma e^{it\Lambda_k(z)}| &= \left| \sum_{l=1}^{|\gamma|} \sum_{|v_l|=|\gamma|} C_v t^l \Lambda_1^{(v_1)} \cdots \Lambda_k^{(v_l)} \right| \\ &\lesssim \sum_{l=1}^{|\gamma|} t^l \lesssim t^{|\gamma|} \end{aligned}$$

for all $\gamma \in \mathbb{N}^{2d}$, where for each $l \in \{1, \dots, |\gamma|\}$, $v_l = (v_1, \dots, v_l) \in \mathbb{N}^l$. For fixed j , by Leibniz formula, we have

$$D_j^L(\sigma e^{it\Lambda_k(z)}) = \sum_{n=0}^L D_j^n(e^{it\Lambda_k(z)}) D_j^{L-n}(\sigma(z)).$$

Using this and (2.10), we obtain

$$\begin{aligned} \sum_{j=1}^{2d} \|D_j^L(\sigma e^{it\Lambda_k(z)})\|_{L^2(\mathbb{R}^{2d})} &\lesssim \sum_{j=1}^{2d} \sum_{n=0}^L t^n \|D_j^{L-n} \sigma\|_{L^2(\mathbb{R}^{2d})} \\ &\lesssim \sum_{n=0}^L t^n \left(\sum_{j=1}^{2d} \|D_j^{L-n} \sigma\|_{L^2(\mathbb{R}^{2d})} \right) \lesssim t^L. \end{aligned}$$

This proves the claim when $|k| > M + 1$. The case $|k| \leq M + 1$ can be consider similarly (see e.g. [19, Lemma 4.2]). $\square$

Sketch Proof of Proposition 2.4. Taking Proposition 2.5 and Lemma 2.2 into account, the proof follows when $P(\xi, \eta) = |\xi|^2 - |\eta|^2$, $(\xi, \eta) \in \mathbb{R}^{2d}$. The general case can be done similarly. $\square$

3. TRILINEAR $M^{p,q}$ ESTIMATES

One of the main technical results needed to prove our main result is establishing a trilinear estimate for the following Hartree type trilinear operator. For $0 < \gamma < d$, let

$$H_\gamma(f, g, h) := (|\cdot|^{-\gamma} * (f\bar{g})) h$$

where $f, g, h \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 3.1. *Let $0 < \gamma < d$, $1 \leq p \leq 2$, and $1 \leq q \leq \frac{2d}{d+\gamma}$. Given $f, g, h \in M^{p,q}(\mathbb{R}^d)$, then $H_\gamma(f, g, h) \in M^{p,q}(\mathbb{R}^d)$, and the following estimate holds*

$$\|H_\gamma(f, g, h)\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}} \|g\|_{M^{p,q}} \|h\|_{M^{p,q}}.$$

Proof. By Proposition 2.2, we have

$$\begin{aligned} \|H_\gamma(f, g, h)\|_{M^{p,q}} &\lesssim \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{M^{\infty,1}} \|h\|_{M^{p,q}} \\ &\lesssim \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{\mathcal{FL}^1} \|h\|_{M^{p,q}}. \end{aligned}$$

We note that

$$\begin{aligned} \left| |\xi|^{-(d-\gamma)} \widehat{f\bar{g}}(\xi) \right| &= \frac{1}{|\xi|^{d-\gamma}} \left| \int_{\mathbb{R}^d} \hat{f}(\xi - \eta) \widehat{\bar{g}}(\eta) d\eta \right| \\ &\leq \frac{1}{|\xi|^{d-\gamma}} \int_{\mathbb{R}^d} |\hat{f}(\xi - \eta)| |\widehat{\bar{g}}(\eta)| d\eta \end{aligned}$$

and integrating with respect to ξ , we get

$$\| |\cdot|^{-\gamma} * (f\bar{g}) \|_{\mathcal{FL}^1} \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi_1)| |\widehat{\bar{g}}(\xi_2)|}{|\xi_1 - \xi_2|^{d-\gamma}} d\xi_1 d\xi_2 = \left\langle |I^\gamma \hat{f}|, |\widehat{\bar{g}}| \right\rangle_{L^2(\mathbb{R}^d)}$$

where I^γ denotes the Riesz potential of order γ :

$$I^\gamma \hat{f}(x) = C_\gamma \int_{\mathbb{R}^d} \frac{\hat{f}(y)}{|x - y|^{d-\gamma}} dy.$$

By Hölder and Hardy-Littlewood Sobolev inequalities and Lemma 2.1, we have

$$\begin{aligned} \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{\mathcal{FL}^1} &= \| I^\gamma \hat{f} \|_{L^{\frac{2d}{d-\gamma}}} \| \widehat{\bar{g}} \|_{L^{\frac{2d}{d+\gamma}}} \\ &\lesssim \| \hat{f} \|_{L^{\frac{2d}{d+\gamma}}} \| \widehat{\bar{g}} \|_{L^{\frac{2d}{d+\gamma}}} \\ &\lesssim \| f \|_{M^{\min\{\frac{2d}{d-\gamma}, 2\}, \frac{2d}{d+\gamma}}} \| g \|_{M^{\min\{\frac{2d}{d-\gamma}, 2\}, \frac{2d}{d+\gamma}}} \\ &= \| f \|_{M^{2, \frac{2d}{d+\gamma}}} \| g \|_{M^{2, \frac{2d}{d+\gamma}}} \\ &\lesssim \| f \|_{M^{p,q}} \| g \|_{M^{p,q}}. \end{aligned}$$

This completes the proof. $\square$

We next prove a related result for weighted modulation spaces $M_s^{p,q}$.

Proposition 3.2. *Assume that $0 < \gamma < d$. The following statements hold*

- (i) *If $1 < p_1 < p_2 < \infty$ with $\frac{1}{p_1} + \frac{\gamma}{d} - 1 = \frac{1}{p_2}$ and $1 \leq q \leq \infty, s \geq 0$. For any $f \in M_s^{p_1,q}(\mathbb{R}^d)$, we have $\| |\cdot|^{-\gamma} * f \|_{M_s^{p_2,q}} \lesssim \| f \|_{M_s^{p_1,q}}$.*
- (ii) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}$ for some $\epsilon > 0$. For any $f, g, h \in M_s^{p,1}(\mathbb{R}^d)$, we have*

$$\|H_\gamma(f, g, h)\|_{M_s^{p,1}} \lesssim \|f\|_{M_s^{p,1}} \|g\|_{M_s^{p,1}} \|h\|_{M_s^{p,1}}.$$

Proof. We may rewrite the STFT as $V_g(x, w) = e^{-2\pi i x \cdot w} (f * M_w g^*)(x)$ where $g^*(y) = \overline{g(-y)}$.

(i) Using Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned} \| |\cdot|^{-\gamma} * f \|_{M_s^{p_2,q}} &= \| \| |\cdot|^{-\gamma} * (f * M_w g^*) \|_{L^{p_2}} \langle w \rangle^s \|_{L_w^q} \\ &\lesssim \| \| f * M_w g^* \|_{L^{p_1}} \langle w \rangle^s \|_{L_w^q} \\ &\lesssim \| f \|_{M_s^{p_1,q}}. \end{aligned}$$

This completes the proof of part (i).

(ii) By Proposition 2.2 and part (1) of Lemma 2.1, we have

$$\begin{aligned} \|H_\gamma(f, g, h)\|_{M_s^{p,1}} &\lesssim \| |\cdot|^{-\gamma} * (fg) \|_{M_s^{\infty,1}} \|h\|_{M_s^{p,1}} \\ &\lesssim \| |\cdot|^{-\gamma} * (fg) \|_{M_s^{p+\epsilon,1}} \|h\|_{M_s^{p,1}}, \end{aligned}$$

for some $\epsilon > 0$. By part (i) and Proposition 2.2, we have $\|T_\gamma(fg)\|_{M_s^{p+\epsilon,1}} \lesssim \|fg\|_{M_s^{p,1}} \lesssim \|f\|_{M_s^{p,1}} \|g\|_{M_s^{p,1}}$.

□

The following result immediately follows.

Proposition 3.3. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}$ for some $\epsilon > 0$. For any $f, g, h \in M^{p,1}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have*

$$\|(|\cdot|^{-\gamma} * (f\bar{g}))h\|_{M^{p,1} \cap L^2} \lesssim \|f\|_{M^{p,1} \cap L^2} \|g\|_{M^{p,1} \cap L^2} \|h\|_{M^{p,1} \cap L^2}.$$

Proof. By part (2) of Lemma 2.1, we have

$$\begin{aligned} \|(|\cdot|^{-\gamma} * f\bar{g})h\|_{M^{p,1} \cap L^2} &:= \|(|\cdot|^{-\gamma} * (f\bar{g}))h\|_{M^{p,1}} + \|(|\cdot|^{-\gamma} * (f\bar{g}))h\|_{L^2} \\ &\lesssim \|f\|_{M^{p,1}} \|g\|_{M^{p,1}} \|h\|_{M^{p,1}} + \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{L^\infty} \|h\|_{L^2} \\ &\lesssim \|f\|_{M^{p,1}} \|g\|_{M^{p,1}} \|h\|_{M^{p,1}} + \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{M^{\infty,1}} \|h\|_{L^2} \\ &\lesssim \|f\|_{M^{p,1}} \|g\|_{M^{p,1}} \|h\|_{M^{p,1}} + \| |\cdot|^{-\gamma} * (f\bar{g}) \|_{M^{p+\epsilon,1}} \|h\|_{L^2} \\ &\lesssim \|f\|_{M^{p,1}} \|g\|_{M^{p,1}} \|h\|_{M^{p,1}} + \|f\|_{M^{p,1}} \|g\|_{M^{p,1}} \|h\|_{L^2} \\ &\lesssim \|f\|_{M^{p,1} \cap L^2} \|g\|_{M^{p,1} \cap L^2} \|h\|_{M^{p,1} \cap L^2}. \end{aligned}$$

This completes the proof. □

We will also need the following result.

Lemma 3.1. *Let $0 < \gamma < d$.*

(i) *Let $1 \leq p \leq 2, 1 \leq q \leq \frac{2d}{d+\gamma}$. For any $f, g \in M^{p,q}(\mathbb{R}^d)$, we have*

$$\|(|\cdot|^{-\gamma} * |f|^2)f - (|\cdot|^{-\gamma} * |g|^2)g\|_{M^{p,q}} \lesssim (\|f\|_{M^{p,q}}^2 + \|f\|_{M^{p,q}} \|g\|_{M^{p,q}} + \|g\|_{M^{p,q}}^2) \|f - g\|_{M^{p,q}}.$$

(ii) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}$ for some $\epsilon > 0$. For any $f, g \in M^{p,1}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have*

$$\begin{aligned} \|(|\cdot|^{-\gamma} * |f|^2)f - (|\cdot|^{-\gamma} * |g|^2)g\|_{M^{p,1} \cap L^2} &\lesssim (\|f\|_{M^{p,1} \cap L^2}^2 + \|f\|_{M^{p,1} \cap L^2} \|g\|_{M^{p,1} \cap L^2} \\ &\quad + \|g\|_{M^{p,1} \cap L^2}^2) \|f - g\|_{M^{p,1} \cap L^2}. \end{aligned}$$

Proof. Notice that

$$\|(|\cdot|^{-\gamma} * |f|^2)(f - g)\|_{M^{p,1} \cap L^2} \lesssim \|f\|_{M^{p,1} \cap L^2}^2 \|f - g\|_{M^{p,1} \cap L^2},$$

and

$$\begin{aligned} \|(|\cdot|^{-\gamma} * (|f|^2 - |g|^2))g\|_{M^{p,1} \cap L^2} &\lesssim (\|f\|_{M_s^{p,1}} \|g\|_{M^{p,1}} + \|g\|_{M^{p,1}}^2) \|f - g\|_{M^{p,1}} \\ &\quad + \| |\cdot|^{-\gamma} * (|f|^2 - |g|^2) \|_{L^2} \|g\|_{L^2} \\ &\lesssim (\|f\|_{M_s^{p,1} \cap L^2} \|g\|_{M^{p,1}} + \|g\|_{M^{p,1} \cap L^2}^2) \|f - g\|_{M^{p,1} \cap L^2}. \end{aligned}$$

This together with the following identity

$$(| \cdot |^{-\gamma} * |f|^2)f - (| \cdot |^{-\gamma} * |g|^2)g = (| \cdot |^{-\gamma} * |f|^2)(f - g) + (| \cdot |^{-\gamma} * (|f|^2 - |g|^2))g,$$

gives the desired inequality. $\square$

4. PROOFS OF MAIN RESULTS

4.1. Local well-posedness for Hartree-Fock equations. We can now prove our main results, beginning with Theorem 1.1.

Proof of Theorem 1.1. By Duhamel's principle, we rewrite the Cauchy problem (1.1) in an integral form: for $k = 1, \dots, N$,

$$\Psi_k(\psi_1, \dots, \psi_N) := \psi_k(t) = U(t)\psi_{0k} - i \int_0^t U(t-s)(H\psi_k)(s)ds + i \int_0^t U(t-s)(F_k(\psi_k))(s)ds.$$

We shall show that Ψ has a unique fixed point in an appropriate function space, for small t . For this, we consider Banach space $(C([0, T], X))^N$, with the norm

$$\|u\|_{(C([0, T], X))^N} = \max_{1 \leq k \leq N} \sup_{t \in [0, T]} \|u_k(t)\|_X$$

where $u = (u_1, \dots, u_N) \in (C([0, T], X))^N$. By Propositions 2.3 and 2.4, we have

$$\|U(t)\psi_{0k}\|_X \lesssim C_T \|\psi_{0k}\|_X$$

where $C_T = C(1 + |t|)^{d|\frac{1}{p} - \frac{1}{2}|}$. By Minkowski's inequality for integrals, Propositions 2.3 and 2.4 and Propositions 3.1, and 3.2, we obtain

$$\begin{aligned} \left\| \int_0^t U(t-s)(H\psi_k)(s)ds \right\|_X &\lesssim TC_T \sum_{\ell=1}^N \left\| \left(\frac{\kappa}{|x|^\gamma} * |\psi_\ell|^2 \right) \psi_k \right\|_X \\ &\lesssim TC_T \sum_{\ell=1}^N \|\psi_k\|_X \|\psi_\ell\|_X^2. \end{aligned}$$

Similarly,

$$\left\| \int_0^t U(t-s)(F\psi_k)(s)ds \right\|_X \lesssim TC_T \sum_{\ell=1}^N \|\psi_k\|_X \|\psi_\ell\|_X^2.$$

Thus, we have

$$\|\Psi_k\|_{L^\infty([0, T], X)} \lesssim C_T \left(\|\psi_{0k}\|_X + cT \sum_{\ell=1}^N \|\psi_k\|_X \|\psi_\ell\|_X^2 \right)$$

for some universal constant c .

For $M > 0$, put

$$B_{T, M} = \{(\psi_1, \dots, \psi_N) \in (C([0, T], X))^N : \|\psi_k\|_{L^\infty([0, T], X)} \leq M \text{ for } k = 1, \dots, N\}$$

which is the closed ball of radius M and centered at the origin in $(C([0, T], X))^N$. Next, we show that the mapping Ψ_k takes $B_{T,M}$ into itself for suitable choice of M and small $T > 0$. Indeed, if we take $M = 2C_T \max\{\|\psi_{0k}\|_X : k = 1, \dots, N\}$ and $\bar{\psi} = (\psi_1, \dots, \psi_N) \in B_{T,M}$, we obtain

$$\|\Psi_k(\bar{\psi})\|_{C([0,T],X)} \leq \frac{M}{2} + cC_TM^3$$

for all $k = 1, \dots, N$. We choose a T such that $cC_TM^2 \leq 1/2$, that is, $T \leq \tilde{T}(\|\psi_{01}\|_X, \dots, \|\psi_{0N}\|_X, d, \gamma)$ and as a consequence we have

$$\|\Psi_k(\bar{\psi})\|_{C([0,T],X)} \leq M \text{ for all } k = 1, \dots, N.$$

So $B_{T,M}$ is invariant under the action of Ψ provided that $T > 0$ is sufficiently small. Up to diminishing T , contraction follows readily, since H_γ is a trilinear operator. So there exist a unique (in $B_{T,M}$) fixed point for Ψ , that is, a solution to (1.1). This completes the proof of Theorem 1.1 part (i). Similarly, we can produce the proof of Theorem 1.1 part (ii) of which we omit the details. $\square$

4.2. Global Well-posedness for Hartree-Fock Equations. In this section we prove Theorem 1.2.

Definition 4.1. A pair (q, r) is α -fractional admissible if $q \geq 2, r \geq 2$ and

$$\frac{\alpha}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right).$$

We recall the following results. For details, see [29, 26].

Proposition 4.1 (Strichartz estimates). Denote

$$DF(t, x) := e^{-it(-\Delta)^{\alpha/2}} \phi(x) + \int_0^t U(t - \tau) F(\tau, x) d\tau.$$

(i) Let $\phi \in L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$ and $\alpha = 2$. Then for any time slab I and admissible pairs (p_i, q_i) , $i = 1, 2$, there exists a constant $C = C(|I|, q_1)$ such that for all intervals $I \ni 0$,

$$\|D(F)\|_{L_{t,x}^{p_1, q_1}} \leq C\|\phi\|_{L^2} + C\|F\|_{L_{t,x}^{p'_2, q'_2}}, \quad \forall F \in L^{p'_2}(I, L^{q'_2})$$

where p'_i and q'_i are Hölder conjugates of p_i and q_i respectively [29].

(ii) Let $d \geq 2$ and $\frac{2d}{2d-1} < \alpha < 2$. Assume that ϕ and F are radial. Then for any time slab I and admissible pairs (p_i, q_i) , $i = 1, 2$, there exists a constant $C = C(|I|, q_1)$ such that for all intervals $I \ni 0$,

$$\|D(F)\|_{L_{t,x}^{p_1, q_1}} \leq C\|\phi\|_{L^2} + C\|F\|_{L_{t,x}^{p'_2, q'_2}}, \quad \forall F \in L^{p'_2}(I, L^{q'_2})$$

where p'_i and q'_i are Hölder conjugates of p_i and q_i [26, Corollary 3.4].

We first establish the following preliminary results.

Proposition 4.2. Let $\phi \circ h(\xi) = |\xi|^\alpha$ where $\xi \in \mathbb{R}^d, \alpha > 0$ and $0 < \gamma < \min\{\alpha, d\}$.

(i) Let $d \in \mathbb{N}$ and $\alpha = 2$. If $(\psi_{01}, \dots, \psi_{0N}) \in (L^2(\mathbb{R}^d))^N$ then (1.1) has a unique global solution

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all α -fractional admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

(ii) Let $d \geq 2$ and $\frac{2d}{2d-1} < \alpha < 2$. If $(\psi_{01}, \dots, \psi_{0N}) \in (L_{rad}^2(\mathbb{R}^d))^N$ then (1.1) has a unique global solution

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, L_{rad}^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all α -fractional admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

Proof. We first establish part (ii). By Duhamel's formula, we write (1.1) as

$$\Phi(\psi_1, \dots, \psi_N) := \psi_k(t) = U(t)\psi_{0k} - i \int_0^t U(t-s)(H\psi_k)(s)ds + i \int_0^t U(t-s)(F_k(\psi_k))(s)ds$$

where Hartree factor $H = \sum_{l=1}^N \left(\frac{1}{|x|^\gamma} * |\psi_l|^2 \right)$ and Fock term $F(\psi_k) = \sum_{l=1}^N \psi_l \left(\frac{\kappa}{|x|^\gamma} * \{\bar{\psi}_l \psi_k\} \right)$. Put $s = \frac{\alpha}{2}$. We introduce the space

$$\begin{aligned} Y(T) &= \{(\psi_1, \dots, \psi_N) \in (C([0, T], L_{rad}^2(\mathbb{R}^d)))^N : \|\psi_k\|_{L^\infty([0, T], L^2)} \leq 2\|\psi_{0k}\|_{L^2}, \\ &\quad \|\psi_k\|_{L^{\frac{8s}{\gamma}}([0, T], L^{\frac{4d}{2d-\gamma}})} \lesssim \|\psi_{0k}\|_{L^2}\} \end{aligned}$$

and the distance

$$d(\phi_1, \phi_2) = \max \left\{ \|f_i - g_i\|_{L^{\frac{8s}{\gamma}}([0, T], L^{\frac{4d}{2d-\gamma}})} : i = 1, \dots, N \right\},$$

where $\phi_1 = (f_1, \dots, f_N)$ and $\phi_2 = (g_1, \dots, g_N)$. Then (Y, d) is a complete metric space. Now we show that Φ takes $Y(T)$ to $Y(T)$ for some $T > 0$. We put

$$q = \frac{8s}{\gamma}, \quad r = \frac{4d}{2d-\gamma}.$$

Note that (q, r) is α -fractional admissible and

$$\frac{1}{q'} = \frac{4s-\gamma}{4s} + \frac{1}{q}, \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r}.$$

Let $(\bar{q}, \bar{r}) \in \{(q, r), (\infty, 2)\}$. By part (ii) of Proposition 4.1 and Hölder's inequality, we have

$$\begin{aligned}
I &:= \|\Phi(\psi_1, \dots, \psi_N)\|_{L_{t,x}^{\bar{q}, \bar{r}}} \\
&\lesssim \|\psi_{0k}\|_{L^2} + \|H\psi_k\|_{L_{t,x}^{q', r'}} + \|F\psi_k\|_{L_{t,x}^{q', r'}} \\
&\lesssim \|\psi_{0k}\|_{L^2} + \sum_{l=1}^N \|(|\cdot|^{-\gamma} * |\psi_l|^2) \psi_k\|_{L_{t,x}^{q', r'}} + \|(|\cdot|^{-\gamma} * (\bar{\psi}_l \psi_k)) \psi_l\|_{L_{t,x}^{q', r'}} \\
&\lesssim \|\psi_{0k}\|_{L^2} + \sum_{l=1}^N \| |\cdot|^{-\gamma} * |\psi_l|^2 \|_{L_{t,x}^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}} \|\psi_k\|_{L_{t,x}^{q, r}} + \| |\cdot|^{-\gamma} * (\bar{\psi}_l \psi_k) \|_{L_{t,x}^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}} \|\psi_l\|_{L_{t,x}^{q, r}}.
\end{aligned}$$

Since $0 < \gamma < \min\{\alpha, d\}$, by the Hardy-Littlewood-Sobolev lemma, we have

$$\begin{aligned}
\| |\cdot|^{-\gamma} * (\bar{\psi}_l \psi_k) \|_{L_{t,x}^{\frac{4s}{4s-\gamma}, \frac{2d}{\gamma}}} &= \left\| \| |\cdot|^{-\gamma} * (\bar{\psi}_l \psi_k) \|_{L_x^{\frac{2d}{\gamma}}} \right\|_{L_t^{\frac{4s}{4s-\gamma}}} \\
&\lesssim \left\| \| \bar{\psi}_l \psi_k \|_{L_x^{\frac{2d}{2d-\gamma}}} \right\|_{L_t^{\frac{4s}{4s-\gamma}}} \\
&\lesssim \|\psi_l\|_{L_{t,x}^{\frac{8s}{4s-\gamma}, r}} \|\psi_k\|_{L_{t,x}^{\frac{8s}{4s-\gamma}, r}} \\
&\lesssim T^{1-\frac{\gamma}{2s}} \|\psi_l\|_{L_{t,x}^{q, r}} \|\psi_k\|_{L_{t,x}^{q, r}}.
\end{aligned}$$

Observe that in the last inequality we use the inclusion relation for the finite measure space $L^p([0, T])$. Thus, we have

$$\|\Phi(\psi_1, \dots, \psi_N)\|_{L_{t,x}^{\bar{q}, \bar{r}}} \lesssim \|\psi_{0k}\|_{L^2} + T^{1-\frac{\gamma}{2s}} \sum_{l=1}^N \|\psi_l\|_{L_{t,x}^{q, r}}^2 \|\psi_k\|_{L_{t,x}^{q, r}}.$$

This shows that Φ maps $Y(T)$ to $Y(T)$. Next, we show Φ is a contraction. To this end, we notice the following identity: for fixed $j \in \{1, \dots, N\}$ and $K(x) = |x|^{-\gamma}$, we have

$$(4.1) \quad \sum_{i=1}^N (K * |u_i|^2) u_j - (K * |v_i|^2) v_j = \sum_{i=1}^N (K * |u_i|^2) (u_j - v_j) + (K * (|u_i|^2 - |v_i|^2)) v_j$$

and

$$(4.2) \quad \sum_{i=1}^N (K * (\bar{u}_i u_j)) u_i - (K * (\bar{v}_i v_j)) v_i = \sum_{i=1}^N (K * (\bar{u}_i u_j)) (u_i - v_i) + (K * (\bar{u}_i u_j - \bar{v}_i v_j)) v_i.$$

It follows that

$$(4.3) \quad \|(K * (\bar{u}_i u_j)) (u_i - v_i)\|_{L_{t,x}^{q', r'}} \lesssim T^{1-\frac{\gamma}{2s}} \|u_i\|_{L_{t,x}^{q, r}} \|u_j\|_{L_{t,x}^{q, r}} \|u_i - v_i\|_{L_{t,x}^{q, r}}.$$

Put $\delta = \frac{8s}{4s-\gamma}$. Notice that $\frac{1}{q'} = \frac{1}{2} + \frac{1}{\delta}$, $\frac{1}{2} = \frac{1}{\delta} + \frac{1}{q}$, and thus by Hölder's inequality, we obtain

$$\begin{aligned}
\|(K * (|u_i|^2 - |v_i|^2))v_j\|_{L_{t,x}^{q',r'}} &\lesssim \|K * (|u_i|^2 - |v_i|^2)\|_{L_{t,x}^{2,\frac{2d}{\gamma}}} \|v_j\|_{L_{t,x}^{\delta,r}} \\
&\lesssim (\|K * (u_i(\bar{u}_i - \bar{v}_i))\|_{L_{t,x}^{2,\frac{2d}{\gamma}}} \\
&\quad + \|K * \bar{v}_i(u_i - v_i)\|_{L_{t,x}^{2,\frac{2d}{\gamma}}}) \|v_j\|_{L_{t,x}^{\delta,r}} \\
&\lesssim \left(\|u_i\|_{L_{t,x}^{\delta,r}} \|v_j\|_{L_{t,x}^{\delta,r}} + \|v_i\|_{L_{t,x}^{\delta,r}} \|v_j\|_{L_{t,x}^{\delta,r}} \right) \|u_i - v_i\|_{L_{t,x}^{q,r}} \\
(4.4) \quad &\lesssim T^{1-\frac{\gamma}{2s}} \left(\|u_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}} + \|v_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}} \right) \|u_i - v_i\|_{L_{t,x}^{q,r}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|(K * (\bar{u}_i u_j - (\bar{v}_i v_j)))v_i\|_{L_{t,x}^{q',r'}} &\lesssim T^{1-\frac{\gamma}{2s}} \|u_i\|_{L_{t,x}^{q,r}} \|v_i\|_{L_{t,x}^{q,r}} \|u_j - v_j\|_{L_{t,x}^{q,r}} \\
&\quad + T^{1-\frac{\gamma}{2s}} \|v_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}} \|u_j - v_j\|_{L_{t,x}^{q,r}}.
\end{aligned}$$

Let $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$. Now in view of (4.1), (4.3), and (4.4), we have

$$\begin{aligned}
d(\Phi(u), \Phi(v)) &\lesssim \sum_{i=1}^N \|(K * |u_i|^2)(u_j - v_j)\|_{L_{t,x}^{q',r'}} + \|(K * (|u_i|^2 - |v_i|^2))v_j\|_{L_{t,x}^{q',r'}} \\
&\quad + \|(K * (\bar{u}_i u_j))(u_i - v_i)\|_{L_{t,x}^{q',r'}} + \|(K * (\bar{u}_i u_j - (\bar{v}_i v_j)))v_i\|_{L_{t,x}^{q',r'}} \\
&\lesssim T^{1-\frac{\gamma}{2s}} \sum_{i=1}^N [\|u_i\|_{L_{t,x}^{q,r}}^2 + \|u_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}} + \|v_i\|_{L_{t,x}^{q,r}}^2 \|v_j\|_{L_{t,x}^{q,r}} \\
&\quad + \|v_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}} + \|v_i\|_{L_{t,x}^{q,r}} \|v_j\|_{L_{t,x}^{q,r}}] d(u, v).
\end{aligned}$$

Thus Φ is a contraction from $Y(T)$ to $Y(T)$ provided that T is sufficiently small. Then there exists a unique $(\psi_1, \dots, \psi_N) \in Y(T)$ solving (1.1). The global existence of the solution (1.1) follows from the conservation of the L^2 -norm of ψ_k . The last property of the proposition then follows from the Strichartz estimates applied with an arbitrary α -fractional admissible pair on the left hand side and the same pairs as above on the right hand side. This completes the proof of part (ii).

The proof of part (i) follows by setting $\alpha = 2$ and using Proposition 4.1 part (i). $\square$

Proposition 4.3. *Let $\phi \circ h(\xi) = |\xi|^\alpha$ for $\xi \in \mathbb{R}^d$, $\alpha > 0$ and $0 < \gamma < \min\{\alpha, d\}$.*

(i) *Let $d \in \mathbb{N}$ and $\alpha = 2$. If $(\psi_{01}, \dots, \psi_{0N}) \in (L^2(\mathbb{R}^d))^N$ then (1.2) has a unique global solution*

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all α -fractional admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

(ii) Let $d \geq 2$ and $\frac{2d}{2d-1} < \alpha < 2$. If $(\psi_{01}, \dots, \psi_{0N}) \in (L_{rad}^2(\mathbb{R}^d))^N$ then (1.2) has a unique global solution

$$(\psi_1, \dots, \psi_N) \in \left(C(\mathbb{R}, L_{rad}^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all α -fractional admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

Proof. Since the proof is similar to that of Proposition 4.2, we omit its details. $\square$

Let $\Psi = (\psi_1, \dots, \psi_N) : (\mathbb{R} \times \mathbb{R}^d)^N \rightarrow \mathbb{C}$ be a global L^2 -solution given by Proposition 4.2. Let T_+ denotes the maximal time of existence:

$$T_+(\Psi) = \sup \left\{ T > 0 : \Psi(t) \mid_{([0, T] \times \mathbb{R}^d)^N} \in (C([0, T], X))^N \right\}.$$

Theorem 1.1 tells us that $T_+(\Psi) > 0$ if initial data $(\psi_{01}, \dots, \psi_{0N}) \in (C([0, T], X \cap L^2(\mathbb{R}^d)))^N$.

Lemma 4.1. Assume that $0 < T_+ < \infty$. Then

$$\lim_{t \rightarrow T_+} \sum_{k=1}^N \|\psi_k(t)\|_X = \infty.$$

Proof. We proceed by contradiction and assume that there exist $M > 0$ and $\{t_n\}_{n=1}^\infty$ such that

$$t_n \rightarrow T_+ \text{ as } n \rightarrow \infty \text{ and } \sum_{k=1}^N \|\psi_k(t_n)\|_X \leq M.$$

Recall that the life span of the local solution in Theorem 1.1 depends on the norm of the initial data. Therefore, there is $T = T(M) > 0$ such that for each $n \in \mathbb{N}$, the solution $\Psi(t) = (\psi_1(t), \dots, \psi_N(t))$ of (1.1) can be established on the time interval $[t_n, t_n + T(M)]$. By uniqueness, $\psi_k(t)$ coincides with standard global L^2 -solution on this interval, which implies

$$\psi_k(t) \mid_{[0, T_+ + \epsilon] \times \mathbb{R}^d} \in C([0, T_+ + \epsilon], X)$$

for some $\epsilon \in (0, T(M))$ and for $k = 1, 2, \dots, N$ but this is a contradiction. $\square$

Now we shall see that the solution constructed before is global in time. In fact, in view of Proposition 4.2, to prove Theorem 1.2, it suffices to prove that the modulation space norm of ψ_k , that is, $\|\psi_k\|_{M^{p,q}}$ cannot become unbounded in finite time for all $k = 1, \dots, N$. To this end, let $T_0 > 0$ and $\psi_k : [0, T_0] \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a local solution to (1.1) such that

$$\psi_k(t) \in C([0, T], X \cap L^2(\mathbb{R}^d))$$

for any $T \in (0, T_0)$ and for $k = 1, \dots, N$.

Lemma 4.2. Assume that $0 < \gamma < \min\{\alpha, d/2\}$. Then

$$\sup_{t \in [0, T_0]} \sum_{k=1}^N \|\psi_k(t)\|_X < \infty.$$

Proof. There exists $C = C(d, \gamma)$ such that the Fourier transform of $K(x) = \kappa|x|^{-\gamma}$ is

$$\widehat{K}(\xi) = \frac{\kappa C}{|\xi|^{d-\gamma}}.$$

We can decompose the Fourier transform of Hartree potential into Lebesgue spaces: indeed, we have

$$(4.5) \quad \widehat{K} = k_1 + k_2 \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d),$$

where $k_1 := \chi_{\{|\xi| \leq 1\}} \widehat{K} \in L^p(\mathbb{R}^d)$ for all $p \in [1, \frac{d}{d-\gamma})$ and $k_2 := \chi_{\{|\xi| > 1\}} \widehat{K} \in L^q(\mathbb{R}^d)$ for all $q \in (\frac{d}{d-\gamma}, \infty]$.

In view of (4.5) and to use the Hausdorff-Young inequality we let $1 < \frac{d}{d-\gamma} < q \leq 2$, and we obtain

$$\begin{aligned} I_k &:= \|\psi_k(t)\|_X \\ &\lesssim C_T \left(\|\psi_{0k}\|_X + \sum_{l=1}^N \int_0^t \|(K * |\psi_l(\tau)|^2) \psi_k(\tau)\|_X + \|(K * (\bar{\psi}_l \psi_k)) \psi_l(\tau)\|_X d\tau \right) \\ &\lesssim C_T \|\psi_{0k}\|_X + C_T \sum_{l=1}^N \int_0^t \|K * |\psi_l(\tau)|^2\|_{\mathcal{FL}^1} \|\psi_k(\tau)\|_X + \|K * (\bar{\psi}_l \psi_k)\|_{\mathcal{FL}^1} \|\psi_l(\tau)\|_X d\tau \\ &\lesssim C_T \|\psi_{0k}\|_X + C_T \sum_{l=1}^N \int_0^t \left(\|k_1\|_{L^1} \|\psi_l(\tau)\|_{L^2}^2 + \|k_2\|_{L^q} \|\widehat{|\psi_l(\tau)|^2}\|_{L^{q'}} \right) \|\psi_k(\tau)\|_X d\tau \\ &\quad + C_T \sum_{l=1}^N \int_0^t \left(\|k_1\|_{L^1} \|\psi_l(\tau) \psi_k(\tau)\|_{L^1} + \|k_2\|_{L^q} \|\widehat{\bar{\psi}_l(\tau) \psi_k(\tau)}\|_{L^{q'}} \right) \|\psi_l(\tau)\|_X d\tau \\ &\lesssim C_T \|\psi_{0k}\|_X + C_T \sum_{l=1}^N \int_0^t \left(\|k_1\|_{L^1} \|\psi_{0l}\|_{L^2}^2 + \|k_2\|_{L^q} \|\psi_l(\tau)\|_{L^q}^2 \right) \|\psi_k(\tau)\|_X d\tau \\ &\quad + C_T \sum_{l=1}^N \int_0^t \left(\|k_1\|_{L^1} \|\psi_{0l}\|_{L^2} \|\psi_{0k}\|_{L^2} + \|k_2\|_{L^q} \|\bar{\psi}_l(\tau) \psi_k(\tau)\|_{L^q} \right) \|\psi_l(\tau)\|_X d\tau \\ &\lesssim C_T \|\psi_{0k}\|_X + C_T(N) \int_0^t \|\psi_k(\tau)\|_X d\tau + C_T \sum_{l=1}^N \int_0^t \|\psi_l(\tau)\|_{L^{2q}}^2 \|\psi_k(\tau)\|_X d\tau \\ &\quad + C_T \sum_{l=1}^N \int_0^t \|\psi_l(\tau)\|_X d\tau + C_T \sum_{l=1}^N \int_0^t \|\psi_l(\tau)\|_{L^{2q}} \|\psi_k(\tau)\|_{L^{2q}} \|\psi_l(\tau)\|_X d\tau \end{aligned}$$

where we have used Proposition 2.2, Hölder's inequality, and the conservation of the L^2 -norm of ψ_k ($k = 1, \dots, N$) and C_T is defined as in the proof of Theorem 1.1. We note that the requirement on q can be fulfilled if and only if $0 < \gamma < d/2$. To apply Proposition 4.3, we let $\beta > 1$ and $(2\beta, 2q)$ is α -fractional admissible, that is, $\frac{\alpha}{2\beta} = d \left(\frac{1}{2} - \frac{1}{2q} \right)$ such that $\frac{1}{\beta} = \frac{d}{\alpha} \left(1 - \frac{1}{q} \right) < 1$. This is possible provided $\frac{q-1}{q} < \frac{\alpha}{d}$: this condition is compatible with the requirement $q > \frac{d}{d-\gamma}$ if and only if $\gamma < \alpha$. Using Hölder's inequality for the last integral,

we obtain

$$\begin{aligned} I_k &\lesssim C_T \|\psi_{0k}\|_X + C_T(N) \int_0^t \|\psi_k(\tau)\|_X d\tau + C_T \sum_{l=1}^N \|\psi_l\|_{L^{2\beta}([0,T],L^{2q})}^2 \|\psi_k\|_{L^{\beta'}([0,T],X)} \\ &\quad + C_T \sum_{l=1}^N \int_0^t \|\psi_l(\tau)\|_X d\tau + C_T \sum_{l=1}^N \|\psi_l\|_{L^{2\beta}([0,T],L^{2q})} \|\psi_k\|_{L^{2\beta}([0,T],L^{2q})} \|\psi_l\|_{L^{\beta'}([0,T],X)} \end{aligned}$$

where β' is the Hölder conjugate exponent of β . Let

$$h(t) = \sup_{s \in [0,t]} \sum_{k=1}^N \|\psi_k(s)\|_X.$$

For a given $T > 0$, h satisfies an estimate of the form,

$$h(t) \lesssim C_T \sum_{k=1}^N \|\psi_{0k}\|_X + C_T(N) \int_0^t h(\tau) d\tau + C_T C_0(T, N) \left(\int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}},$$

provided that $0 \leq t \leq T$, and where we have used the fact that β' is finite. Using Hölder's inequality we infer that,

$$h(t) \lesssim C_T \sum_{k=1}^N \|\psi_{0k}\|_X + C_1(T, N) \left(\int_0^t h(\tau)^{\beta'} d\tau \right)^{\frac{1}{\beta'}}.$$

Raising the above estimate to the power β' , we find that

$$h(t)^{\beta'} \lesssim C_2(T, N) \left(1 + \int_0^t h(\tau)^{\beta'} d\tau \right).$$

In view of Gronwall inequality, one may conclude that $h \in L^\infty([0, T])$. Since $T > 0$ is arbitrary, $h \in L_{loc}^\infty(\mathbb{R})$. This completes the proof. $\square$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Taking Theorem 1.1 into account and combining Lemmas 4.2 and 4.1, the proof of Theorem 1.2 part (i) follows. Similarly, we can produce the proof of Theorem 1.2 part (ii), we shall omit the details. $\square$

5. WELL-POSEDNESS FOR HARTREE-FOCK EQUATIONS WITH HARMONIC POTENTIAL

In this final section we consider the Hartree-Fock and reduced Hartree-Fock equations with a harmonic potential as given by (1.4) and (1.5).

5.1. Schrödinger propagator associated to harmonic oscillator. We start by recalling the spectral decomposition of $H = -\Delta + |x|^2$ by the Hermite expansion. Let $\Phi_\alpha(x)$, $\alpha \in \mathbb{N}^d$ be the normalized Hermite functions which are products of one dimensional Hermite functions. More precisely, $\Phi_\alpha(x) = \prod_{j=1}^d h_{\alpha_j}(x_j)$ where

$$h_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} (-1)^k e^{\frac{1}{2}x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

The Hermite functions Φ_α are eigenfunctions of H with eigenvalues $(2|\alpha| + d)$ where $|\alpha| = \alpha_1 + \dots + \alpha_d$. Moreover, they form an orthonormal basis for $L^2(\mathbb{R}^d)$. The spectral decomposition of H is then written as

$$H = \sum_{k=0}^{\infty} (2k + d) P_k \quad \text{with} \quad P_k f(x) = \sum_{|\alpha|=k} \langle f, \Phi_\alpha \rangle \Phi_\alpha$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^d)$. Given a function m defined and bounded on the set of all natural numbers we can use the spectral theorem to define $m(H)$. The action of $m(H)$ on a function f is given by

$$m(H)f = \sum_{\alpha \in \mathbb{N}^d} m(2|\alpha| + d) \langle f, \Phi_\alpha \rangle \Phi_\alpha = \sum_{k=0}^{\infty} m(2k + d) P_k f.$$

This operator $m(H)$ is bounded on $L^2(\mathbb{R}^d)$. This follows immediately from the Plancherel theorem for the Hermite expansions as m is bounded. On the other hand, the mere boundedness of m is not sufficient to imply the L^p boundedness of $m(H)$ for $p \neq 2$ (see [40]). We define **Schrödinger propagator associated to harmonic oscillator**

$$m(H) = e^{it(-\Delta + |x|^2)} f = \sum_{k=0}^{\infty} e^{it(2k+d)} P_k f$$

with $m(n) = e^{itn}$ for $n \in \mathbb{N}, t \in \mathbb{R}$. The next result proves that $e^{it(-\Delta + |x|^2)}$ is uniformly bounded on $M^{p,p}(\mathbb{R}^d)$. More specifically, we have.

Theorem 5.1. ([7, Theorem 5], cf. [17]) *The Schrödinger propagator associated to harmonic oscillator $e^{it(-\Delta + |x|^2)}$ is bounded on $M^{p,p}(\mathbb{R}^d)$ for each $t \in \mathbb{R}$, and all $1 \leq p < \infty$. Moreover, we have*

$$\|e^{it(-\Delta + |x|^2)} f\|_{M^{p,p}} = \|f\|_{M^{p,p}}.$$

5.2. Proof of Theorem 1.3. In this section we give a proof of Theorem 1.3. But first, we state the following definition and some preliminary results.

Definition 5.1. *A pair (q, r) is admissible if $2 \leq r < \frac{2d}{d-2}$ with $2 \leq r \leq \infty$ if $d = 1$, and $2 \leq r < \infty$ if $d = 2$, whenever*

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right).$$

Proposition 5.1. ([12, Proposition 2.2]) *Let $\phi \in L^2(\mathbb{R}^d)$ and*

$$DF(t, x) := U(t)\phi(x) + \int_0^t U(t - \tau)F(\tau, x)d\tau.$$

Then for any time slab I and admissible pairs (p_i, q_i) , $i = 1, 2$, there exists a constant $C = C(|I|, q_1)$ such that for all intervals $I \ni 0$,

$$\|D(F)\|_{L_{t,x}^{p_1, q_1}} \leq C\|\phi\|_{L^2} + C\|F\|_{L_{t,x}^{p'_2, q'_2}}, \quad \forall F \in L^{p'_2}(I, L^{q'_2})$$

where p'_i and q'_i are Hölder conjugates of p_i and q_i respectively.

Proposition 5.2. *Let $0 < \gamma < \min\{2, d\}$, $d \in \mathbb{N}$. Assume that $(\psi_{01}, \dots, \psi_{0N}) \in (L^2(\mathbb{R}^d))^N$. Then*

(i) *There exists a unique global solution of (1.4) such that*

$$(\psi_1, \dots, \psi_N) \in \left(C([0, \infty), L^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}([0, \infty), L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

(ii) *There exists a unique global solution of (1.5) such that*

$$(\psi_1, \dots, \psi_N) \in \left(C([0, \infty), L^2(\mathbb{R}^d)) \cap L_{loc}^{4\alpha/\gamma}([0, \infty), L^{4d/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.$$

In addition, its L^2 -norm is conserved,

$$\|\psi_k(t)\|_{L^2} = \|\psi_{k0}\|_{L^2}, \quad \forall t \in \mathbb{R}, k = 1, 2, \dots, N$$

and for all admissible pairs (p, q) , and $(\psi_1, \dots, \psi_N) \in (L_{loc}^p(\mathbb{R}, L^q(\mathbb{R}^d)))^N$.

Proof. The proof follows from Proposition 5.1 and using ideas similar to the proof of Proposition 4.2. $\square$

We can now establish local well-posedness results for (1.4) and (1.5).

Theorem 5.2 (Local well-posedness). *Let $1 \leq p \leq \frac{2d}{d+\gamma}$ and $0 < \gamma < d$. Assume that $(\psi_{01}, \dots, \psi_{0N}) \in (M^{p,p}(\mathbb{R}^d))^N$. Then*

(i) *There exists $T > 0$ depending only on $\|\psi_{01}\|_{M^{p,p}}, \dots, \|\psi_{0N}\|_{M^{p,p}}$, d and γ such that (1.4) has a unique local solution*

$$(\psi_1, \dots, \psi_N) \in (C([0, T], M^{p,p}(\mathbb{R}^d)))^N.$$

(ii) *There exists $T > 0$ depending only on $\|\psi_{01}\|_{M^{p,p}}, \dots, \|\psi_{0N}\|_{M^{p,p}}$, d and γ such that (1.5) has a unique local solution*

$$(\psi_1, \dots, \psi_N) \in (C([0, T], M^{p,p}(\mathbb{R}^d)))^N.$$

Proof. The results are established by applying a standard contraction mapping argument and using Theorem 5.1 and Proposition 3.1. $\square$

Sketch proof of Theorem 1.3. The proof is similar to that of Theorem 1.2 using Proposition 5.1 and Theorem 5.2. $\square$

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