

PAPER

Bulk entanglement entropy and matrices

To cite this article: Sumit R Das *et al* 2020 *J. Phys. A: Math. Theor.* **53** 444002

View the [article online](#) for updates and enhancements.

Recent citations

- [Gauge invariant target space entanglement in D-brane holography](#)
Sumit R. Das *et al*
- [Partial deconfinement at strong coupling on the lattice](#)
Hiromasa Watanabe *et al*
- [A passion for theoretical physics: a special issue in memory of Peter G O Freund](#)
Jeffrey A Harvey *et al*



IOP | ebooks™

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

Bulk entanglement entropy and matrices

Sumit R Das^{1,*} , Anurag Kaushal², Gautam Mandal²  and Sandip P Trivedi²

¹ Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, United States of America

² Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai 400005, India

E-mail: sumit.das@uky.edu, das@pa.uky.edu, anurag.kaushal@theory.tifr.res.in, mandal@theory.tifr.res.in and sandip@theory.tifr.res.in

Received 11 March 2020, revised 23 June 2020

Accepted for publication 17 August 2020

Published 14 October 2020



CrossMark

Abstract

Motivated by the Bekenstein–Hawking formula and the area law behaviour of entanglement entropy, we propose that in any UV finite theory of quantum gravity with a smooth spacetime, the total entropy for a pure state in a co-dimension one spatial region, to leading order, is given by $S = \frac{A}{4G_N}$, where A is the area of the co-dimension two boundary. In the context of Dp brane holography we show that for some specially chosen regions bulk entanglement can be mapped to ‘target space’ entanglement in the boundary theory. Our conjecture then leads to a precise proposal for target space entanglement in the boundary theory at strong coupling and large N . In particular, it leads to the conclusion that the target space entanglement would scale like $O(N^2)$ which is quite plausible in a system with $O(N^2)$ degrees of freedom. Recent numerical advances in studying the $D0$ brane system hold out the hope that this proposal can be tested in a precise way in the future.

Keywords: entanglement entropy, holography, matrix model

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum entanglement plays a key role in gauge-gravity duality. In AdS/CFT correspondence, the Ryu–Takayanagi formula [1] and its covariant version [2], together with its extensions [3, 4], provide a strikingly simple geometric understanding of the entanglement entropy of a subregion in the boundary theory in terms of extremal surfaces in the bulk.

In this note we consider entanglement entropy in the bulk itself and its interpretation in the boundary theory. Consider some spatial subregion of the bulk and the entanglement of this

*Author to whom any correspondence should be addressed.

subregion with its complement. We may consider the associated entanglement entropy. Our aim is to ask: what is the value for this ‘bulk entanglement entropy’ and what is the meaning of this quantity in the dual quantum field theory? This bulk entanglement entropy is itself, of course, a tricky concept since the bulk theory is a theory of gravity. Nevertheless one would expect that in some smooth spacetime background with approximately local physics, this is the entanglement of quantum fields, including gravitons, across the co-dimension two boundary of the subregion. Our aim is to seek a definition of this quantity which agrees with this notion in the semiclassical regime, but can be extended beyond this. We will propose such a definition in terms of the holographically dual field theory.

The leading term in this entropy is expected to be proportional to the area of the boundary in units of an appropriate power of a UV cutoff. In the context of entanglement across a black hole horizon, it has been argued that this UV divergent term renormalizes the Newton constant [5–7]. In an UV complete theory of gravity the answer should be finite, and it is natural to then ask what provides this cutoff.

In this note we conjecture that given a consistent theory of quantum gravity, in any smooth spacetime the total entropy in a spatial co-dimension one region is given by the area formula,

$$S = \frac{A_E}{4G_N}, \quad (1)$$

with A_E denoting the area in the Einstein frame.

More precisely, we conjecture that the formula above is true for a pure state where the entropy is fully accounted for by the entanglement entropy and given by

$$S_{EE} = \frac{A_E}{4G_N}. \quad (2)$$

Note that equation (1) gives the leading order behaviour of the entropy, there can be subleading terms, e.g., arising in string theory from g_s and α' corrections, which could give corrections to equations (1), (2), which may not be universal.

For mixed states, the formula equation (1) can still be sometimes true if the density matrix for the mixed state can be obtained starting from a pure state in a bigger system which also admits a gravity description. In fact in such cases equation (1) follows from the conjecture for pure states, equation (2). Such a situation arises for a thermal state in AdS/CFT which is described by a black hole in the gravity dual. The thermal state’s density matrix can be obtained from that of a pure state in the thermo-field double system and the corresponding dual geometry is the double sided black hole. More generally for mixed states we suspect that some version of equation (1) is still valid, once it is stated precisely in terms of the fine-grained entropy of the degrees of freedom of a light sheet associated with a boundary of area A [10], but we have not understood this well yet and leave a detailed discussion for general mixed states for the future³.

We can be most definite about our conjecture equation (2) in the context of AdS/CFT like situations where the gravity theory has a dual which lives on a time-like boundary. In such situations, which arise for example in the near-horizon geometries of Dp brane theories, the state in the bulk can be mapped to a state in the boundary dual. The bulk entanglement then maps to the von Neumann entropy of a suitably defined density matrix in the boundary theory, which we will see shortly is associated with target space entanglement. Even in such situations

³ We are grateful to R Bousso for drawing our attention to this issue and in particular for bring reference [10] to our notice.

though, once a black hole is present, one can consider a bulk region inside the horizon and the precise map of the bulk entanglement to the boundary theory in such cases is not well understood by us as yet and left for further investigation.

Our conjecture stems from the following, admittedly intuitive, reasoning. One expects in any quantum mechanically complete theory to get a finite result for the bulk entanglement, for example, in a closed string theory describing the bulk. For a black hole horizon one also expects the result to agree with the Bekenstein–Hawking formula,

$$S_{\text{BH}} = \frac{A_{\text{H}}}{4G_N}. \quad (3)$$

This suggests that the cut-off rendering the entanglement finite is provided by G_N in general, leading to the relations equations (1), (2). In particular, for a black hole that forms from the collapse of a pure state it is quite plausible that S_{BH} is accounted for completely by entanglement, leading to equation (2).

Let us note that in the context of Einstein gravity with some matter, the entanglement entropy of matter and gravitons across the black hole horizon appears as a quantum correction to the Bekenstein–Hawking entropy which may be regarded as a ‘classical’ contribution [7]. However if Einstein gravity itself is an effective theory obtained by integrating out massive closed string modes, such a classical contribution itself can be considered as an entanglement entropy of the fundamental degrees of freedom. This viewpoint is consistent with what happens in models of induced gravity [8].

We also note that our conjecture equation (1) is equivalent to saying that the Bekenstein bound is saturated by our notion of the bulk entropy, to leading order, in any smooth background. For mixed states, the total entropy is a sum of ‘quantum’ entanglements and a ‘disorder’ part inherited from the mixed state. Recall that for a quantum system, an initial mixed state $\rho = \sum_i w_i |\alpha_i\rangle\langle\alpha_i|$, the form of the reduced density matrix (RDM) is $\rho_A = \sum_i w_i \rho_A^i$ whose von Neumann entropy combines the quantum entanglement from the $\rho_A^i = \text{tr}_{A^c} |\alpha_i\rangle\langle\alpha_i|$ (inherent in the pure states $|\alpha_i\rangle$) with the classical or ‘disorder’ contribution $-\sum_i w_i \log w_i$ which would be present even if the states $|\alpha_i\rangle$ are factorizable states. For target space entropies, which will be of relevance for us, these concepts are suitably generalized. We envisage the saturation of the bound, if (1) holds for mixed states, as a trade off between these two parts of the entropy such that the sum equals the bound.

What makes the conjecture above interesting is that improvements in numerical techniques now hold out the hope that we can test it precisely in the future. With this motivation in mind here we consider this problem in the context of Dp brane holography for some special spatial co-dimension one regions. We show that the bulk entanglement entropy can be mapped to the boundary theory in a fairly precise manner. The boundary theory, for example for $D0$ branes, and more generally for Dp branes with $p < 3$, has no dimensionless parameter other than N —the number of branes. This constrains the form of the result, and one finds that the expression in equation (2) agrees with what could arise in the boundary theory. In fact the result equation (2), when expressed in terms of the appropriate dimensionless variables, scales like N^2 which is quite plausible in a system with $O(N^2)$ degrees of freedom.

We find that for the special co-dimension one regions we consider the bulk entanglement maps in a fairly precise way to a quantity sometimes referred to as ‘target space entanglement’ in the boundary theory. It is worth pausing to briefly explain this idea here. Consider a quantum mechanical system where the degrees of freedom live in time alone. Some of these degrees of freedom include target space directions along which the system can move, such a system arises for example in the case of the field theory limit of the $D0$ brane theory. There is no spatial extent in the quantum mechanical case so we cannot consider a spatial sub-region and define

an entanglement in that manner, as is often done in a field theory. However we can consider some restriction in the target space and associate an entanglement with this restriction—this is referred to as the target space entanglement. The simplest example is a single particle, say an harmonic oscillator, in one dimension x ; we may want to restrict ourselves to some region $a < x < b$ and only concern ourselves with measurements which can be made when we restrict ourselves to this region. Even if the state of the system is pure, this restriction on the set of all observables we have access to gives rise to a density metric whose von Neumann entropy is then the target space entanglement entropy. If $\Psi(x)$ is the wave function, the density matrix is given in the position basis by

$$\rho(x, x') = \Psi^*(x)\Psi(x') \quad (4)$$

with $x \in [a, b]$. The sub algebra of observables one is correspondingly restricted to is given by operators of the form

$$\hat{O} = \int_a^b \int_a^b dx dx' C(x, x') |x\rangle \langle x'| \quad (5)$$

where C is Hermitian, satisfying, $C(x, x') = C^*(x', x)$. One can think of the entanglement entropy as being associated with this sub algebra. While target space entanglement agrees with the usual notion of entanglement of a bulk region in the regime of couplings where the bulk is semi-classical, it clearly remains a well-defined quantity for any regime of couplings. The target space entanglement, therefore, provides the general notion of ‘bulk entanglement’ that we are seeking.

Target space entanglement has been implicitly used to define notions of entanglement entropy in several situations. One example involves worldsheet formulations of string theory [6, 9]. It is also the basis for discussion of entanglement entropy in the $c = 1$ matrix model [11] dual to two dimensional non-critical string theory [12, 13]. This also appears in a slightly different context in [14]. The formalism of general target space entanglement has been recently developed in [15, 16].

In general due to the non-commuting nature of the target space spatial coordinates in the $D0$ brane theory (and similarly the higher Dp systems) it is not possible to precisely map a region in the bulk to an appropriate restriction in the target space of the boundary theory. However for some carefully chosen co dimension one regions in the bulk we show that this is possible. This then allows us to map the bulk entanglement entropy quite precisely to target space entropy in the boundary theory.

Our mapping is not totally precise though, and we find that there are two natural possibilities which arise in the boundary theory. Distinguishing between them and checking whether the target space entanglement in either case agrees with (2) would require numerical work. In fact there have been great strides recently in studying some of the field theories which arise in the context of AdS/CFT numerically. For example the free energy at strong coupling for the $D0$ brane matrix theory has been studied by [19] and shown to agree quite precisely with the bulk result coming from a black hole. While calculations of target space entanglement will be much more challenging, these advances allow us to hope that such a calculation can be carried out in the not so distant future, allowing a test of whether either of the two possibilities for the target space entanglement agrees with equation (2). Such a numerical calculation would provide a very non-trivial check for our conjecture (for a recent discussion of entanglement entropy in matrix models in a different context see [20]).

Some of the above discussion is best understood for the $c = 1$ model which is dual to $1 + 1$ dimensional string theory [11]. Here the space in the string theory arises from the space of

eigenvalues of the $N \times N$ Hermitian matrix M while the eigenvalues themselves are coordinates of N fermions. The ‘bulk’ description arises from second quantization of these fermions. The fermion field can be bosonized yielding collective field theory of the density of eigenvalues. The fluctuations of the collective field are related to the ‘massless tachyon’ of the two dimensional string theory, which is the only dynamical mode. The bulk entanglement entropy of an interval has the usual meaning in this second quantized language and was computed in [12]. This calculation has been more recently revisited and improved in [13]. The leading answer for the entanglement entropy of an interval is *finite*; the UV cutoff discussed earlier is provided here by the position dependent string coupling. The fact that the string coupling enters as the cutoff is consistent with the conjecture that the Newton constant provides the UV cutoff. The finiteness can be ultimately traced to the fact that we are dealing with $N \times N$ matrices. In the bosonic formulation, this manifests itself in the fact that there are at most N independent single trace operators of the form $\text{Tr } M^n$. (Note that $\text{Tr } M^n$ for $n > N$ is expressible as a sum of products of the lower single traces.) Since n is a quantum number conjugate to the emergent space direction, this means that the collective field should really be thought of as living on a lattice with spacing $\sim 1/N$. This becomes clearer in a basis formed by the characters of the permutation group which are in one to one correspondence to fermion wavefunctions [17], or in the formalism of bosonization of a finite number of fermions in [18]. Matrix quantum mechanics is equivalent to the first quantized formulation—the bulk entanglement then relates to an appropriate subalgebra of operators. This becomes an example of ‘target space entanglement’ mentioned above.

Our considerations also apply to field theories, which arise for example as duals in the Dp brane case with $p \leq 3$. In this case there is the usual notion of entanglement entropy associated with a spatially localized region. However one can also consider a notion of target space entropy which arises when one restricts to observables which can only access some region of target space, without imposing any restriction along the spatial directions in which the field theory lives. We show that it is the latter type of target space entanglement which is dual to the bulk entanglement when we consider spatial regions in the bulk extending fully along those in which the field theory lives with restrictions only in the spatial directions transverse to the field theory ones. Upto the kind of ambiguity mentioned above which one faces in the $D0$ brane case, the mapping of the bulk entanglement to the field theory target space entanglement is precise, and we find once again that our proposed bulk entanglement entropy equation (2) scales like N^2 when exposed in terms of the dimensionless variables of the field theory. It is worth pointing out that for field theories we can consider a more general notion of entanglement where we impose a restriction on both the spatial region and within that region a further restriction on the region of target space that can be accessed. This generalized entanglement would interpolate between the usual spatially localized entanglement, which has a dual interpretation as an RT surface, and the target space entanglement we have been discussing here. We leave an exploration of this interesting idea for the future.

As was pointed out above the formula we suggest for bulk entanglement, equation (2), is only at leading order and would have corrections, due to both α' and string loop effects. Since the definition of target space entropy in the field theory is a general one these corrections could be computed on the boundary side. We can also consider the weakly coupled limit in the field theory where the dual spacetime is highly curved with a curvature of order the string scale; the boundary theory definition would still hold in this case and would allow us to make sense of the entanglement.

Finally, our conjecture (2) considers the Einstein frame area: this is natural from the presumed connection to the Bekenstein–Hawking formula. We can also consider another possibility, viz that the entanglement entropy is proportional to the area in the string frame metric. In

fact we find that in this case we get a result which can be obtained from the holographic dual, provided we use the string length l_s as the UV cutoff. However, in this case the result scales as N^0 rather than N^2 . This is, in a sense, less natural to expect. However it is still possible. A detailed numerical calculation which we allude to will determine which of these alternatives is correct.

This note is organized as follows. In section 2 we calculate the bulk entanglement across a simple co-dimension two surface in the geometry of N coincident $D0$ branes following the proposal (2), and show that when the parameters which appear in the setup are expressed in terms of appropriate scales of the $D0$ brane theory, the answer scales as N^2 . In section 3 we put forward our proposal for the target space entanglement which corresponds to the calculation of section 2. Section 4 extends the supergravity calculation of section 2 to Dp branes for $p < 3$. Section 5 discusses the target space entanglement proposal for the Dp brane field theory. Section 6 contains discussions of our results and their extensions. The appendices deal with the definition and evaluation of target space entanglement for the case of a single matrix relevant for $c = 1$ case, and a proof that this notion is identical to the notion of entanglement in the second quantized formulation which is commonly used.

While this work was in progress, the paper [21] appeared, which discussed the possible relevance of areas of *extremal* surfaces in BFSS/gravity duality. Our work differs in an essential way: our ultimate aim to understand the meaning of entropy and entanglement across *any* surface in the bulk, regardless of whether it is extremal or not. (In fact in recent literature the phrase ‘bulk entanglement entropy’ is sometimes used to talk about the entanglement across an extremal surface obtained in the context of a quantum correction to entanglement entropy of the dual field theory associated with a spatial region of the boundary field theory [3, 22]). In this paper we put forward a proposal for a set of simple surfaces in various geometries. The fact that the cutoff in the entanglement entropy in a theory of gravity should be the Newton constant has been previously argued in [23] from several viewpoints, in particular from the point of view of a derivation of Einstein equation as an equation of state. Our proposal is based on a consistency with a holographic description⁴. We also note that [24] had in fact proposed that the entanglement entropy in a theory of gravity saturates the Bekenstein bound and gave some supporting evidence for the proposal which are different from ours⁵. Our proposal goes somewhat further: we also conjecture that for general states, the Bekenstein bound is saturated by the total entropy which includes an entanglement and a disorder part.

2. Bulk entanglement for $D0$ brane geometries

The simplest setup is the background produced by a stack of N coincident $D0$ branes. We begin by considering the extremal limit at temperature $T = 0$. The string frame metric and the dilaton in the near horizon region are given by [25]

$$\begin{aligned} ds_{\text{string}}^2 &= -H_0(r)^{-1/2} dt^2 + H_0(r)^{1/2} [dx_1^2 + \cdots + dx_9^2], \\ e^{-2\phi} &= H_0(r)^{-3/2}, \\ H_0(r) &= \frac{R^7}{r^7}, \\ r^2 &= x_1^2 + \cdots + x_9^2. \end{aligned} \tag{6}$$

⁴ We thank T Jacobson for bringing this to our attention.

⁵ We thank Gary Horowitz for bringing this reference to our attention after the arXiv version appeared.

Here the scale R is given by

$$R^7 = \frac{(2\pi)^7}{7\Omega_8} l_s^7 (g_s N). \quad (7)$$

l_s is the string length, g_s is the string coupling and Ω_8 is the volume of an eight dimensional unit sphere. The string frame curvature of this solution becomes large when

$$r = r_0 \equiv (g_s N)^{1/3} l_s, \quad (8)$$

so that supergravity description is valid for $r \ll r_0$. However when $r = r_1 \sim (g_s N)^{1/7} l_s$ the dilaton becomes large, so that for such small r the M -theory description takes over.

Consider now dividing the nine dimensional bulk into two parts by an eight dimensional plane at $x_1 = d$. We choose d to be in the region of validity of IIA supergravity or M theory, i.e. $d \ll (g_s N)^{1/3} l_s$. When in addition $d \gg (g_s N)^{1/7} l_s$, the induced string frame metric on the surface $x_1 = d$ (at a given time t) is

$$ds_{\text{induced}}^2 = H_0(\bar{r})^{1/2} [d\rho^2 + \rho^2 d\Omega_7^2], \quad (9)$$

where we have defined

$$\begin{aligned} \rho^2 &= x_2^2 + \cdots + x_9^2, \\ \bar{r}^2 &= d^2 + \rho^2. \end{aligned} \quad (10)$$

The Einstein frame area of this eight dimensional surface is then given by

$$A_d(T=0) = \Omega_7 \int_0^{\rho_0} d\rho \rho^7 H_0(\bar{r})^{1/2} = \Omega_7 R^{7/2} \int_0^{\rho_0} d\rho \frac{\rho^7}{(d^2 + \rho^2)^{7/4}}, \quad (11)$$

where we have used the following relation

$$ds_{\text{Einstein}}^2 = e^{-\phi/2} ds_{\text{string}}^2.$$

We have imposed an IR cutoff on the integral at some ρ_0 . We have in mind taking

$$d \ll \rho_0 < r_0. \quad (12)$$

The result of the integral is then

$$A_d(T=0) = \frac{2}{9} \Omega_7 R^{7/2} \rho_0^{9/2} [1 + O(d^2/\rho_0^2)] \quad (13)$$

If we take the IR cutoff to be $\rho_0 \sim r_0$ and $d \sim r_1$ we see that in the regime $(g_s N) \gg 1$ the area A_d behaves as (using (7))

$$A_d(T=0) \sim (g_s N)^2 l_s^8 [1 + O((g_s N)^{-21/2})] \quad (14)$$

According to our proposal for the bulk entanglement, equation (2) the entanglement entropy of the region $x_1 > d$ with its complement is

$$S_{\text{EE}}(d) = \frac{A_d}{4G_N} \quad (15)$$

As we will see soon, equation (15), where the area term is cut-off by $1/G_N$, can be expressed in terms of dimensionless quantities in the $D0$ brane quantum mechanics, and will scale like N^2 .

Let us note in passing that one could imagine taking $d = 0$. This would necessitate including the small r region of the bulk theory where the dilaton is large. This is the region described by M-theory, one expects equation (15) to continue to hold in this case as well, since the RHS is invariant under a change of duality frames.

From equation (13) we see that A_d and therefore $S_{EE}(d)$ is dependent on the bulk IR cutoff ρ_0 . We would like to get rid of this dependence so that the result can be compared in a precise way with the matrix theory. One way to do so is to consider the difference between the entanglement entropy in a finite temperature $D0$ brane black hole background and the extremal $D0$ brane solution considered above. The near-extremal black $D0$ brane string frame metric is given by [25]

$$ds_{\text{string}}^2 = -H_0(r)^{-1/2} f(r) dt^2 + H_0(r)^{1/2} \left[\frac{dr^2}{f(r)} + r^2 d\Omega_8^2 \right] \\ f(r) = 1 - \left(\frac{r_H}{r} \right)^7 \quad (16)$$

while the dilaton and the one form gauge field remain the same. The horizon is now at $r = r_H$. The Hawking temperature for this solution is given by

$$T = \frac{7}{4\pi R} \left(\frac{r_H}{R} \right)^{5/2} \quad (17)$$

Before proceeding let us note that the black hole geometry equation (16) admits an extension, analogous to the well-known Kruskal extension for a Schwarzschild black hole, which has two time-like boundaries. This double sided geometry is dual to a pure state—the thermo-field double state—in a system consisting of two non-interacting $D0$ brane systems. We can consider the conjecture equation (2) for a bulk subregion in this extended geometry and its dual description as the thermo-field double state, see the comments after equation (2) in the introduction. Here we will only consider a bulk region on one side and that too lying outside the horizon. For our purposes therefore we do not have to worry about the full extended geometry, and the single sided geometry, described by the metric in equation (16), will be sufficient. The case of more general bulk regions is very interesting and left for the future.

Consider now an $x_1 = d$ surface in the geometry equation (16), with

$$d \gg r_H. \quad (18)$$

The area of this surface is

$$A_d(T) = \Omega_7 R^{7/2} \int_0^{\rho_0} d\rho \, \rho^7 \frac{1}{(d^2 + \rho^2)^{7/4}} \left[(f(\bar{r})^{-1} - 1) \frac{\rho^2}{d^2 + \rho^2} + 1 \right]^{1/2} \quad (19)$$

We will consider low temperatures so that $r_H < \rho_0$. If $\rho_0 \sim r_0$ this translates to

$$(RT) \ll (g_s N)^{10/21} \quad (20)$$

In that case one can expand the integrand in powers of $r_H/\sqrt{d^2 + \rho^2}$. To lowest order one gets

$$A_d(T) = \Omega_7 R^{7/2} \int_0^{\rho_0} d\rho \left[\frac{\rho^{7/2}}{(1 + \frac{d^2}{\rho^2})^{7/4}} + \frac{r_H^7}{2} \frac{\rho^{-7/2}}{(1 + \frac{d^2}{\rho^2})^{25/4}} + \dots \right] \quad (21)$$

Using (11), the difference of areas in the large ρ_0 limit becomes

$$A_d(T) - A_d(0) = \frac{1}{2} \Omega_7 R^{7/2} r_H^7 \int_0^{\rho_0} d\rho \frac{\rho^{-7/2}}{(1 + \frac{d^2}{\rho^2})^{25/4}} + \dots \quad (22)$$

The integral on the right-hand side is finite in the limit of large ρ_0 , so that we can replace the upper limit of integration by ∞ . The leading result is then

$$A_d(T) - A_d(0) = C_0 \frac{\Omega_7 R^{7/2} r_H^7}{d^{5/2}} + \dots, \quad C_0 = \frac{2048}{69615} \quad (23)$$

Here the \dots represent subleading terms in the r_H/d expansion.

As promised, the difference (23) is insensitive to the IR cutoff ρ_0 . The resulting difference of the entropies, to leading order, using equation (2), equation (1), can be expressed as

$$S(d, T) - S_{EE}(d, T = 0) = C_0 \frac{\Omega_7 R^{7/2} r_H^7}{4G_N d^{5/2}} \quad (24)$$

Before going on, let us mention one more way in which the dependence on ρ_0 in equation (13) can be made to cancel. Consider the supergravity background when the $D0$ branes are not at the origin of the Coulomb branch. In this case the Harmonic function in equation (6) is replaced by

$$H = \frac{R^7}{N} \sum_{i=1}^N \frac{1}{|\vec{r} - \vec{r}_i|^7} \quad (25)$$

where \vec{r}_i is the location of the i th brane in the 9 transverse directions. The area of the surface $x_1 = d$ in this case is given by equation (11) with H replaced by equation (25). Taking the difference of the area in the geometry when the branes are at the origin of the coulomb branch and away from the origin then gives,

$$\Delta A = R^{\frac{7}{2}} \int dx_2 dx_3 \dots dx_9 \left[\frac{1}{r^{7/2}} - \left(\frac{1}{N} \sum_i \frac{1}{|\vec{r} - \vec{r}_i|^7} \right)^{1/2} \right] \quad (26)$$

Here r is given in terms of ρ by equation (10). At large ρ the two terms in the brackets will cancel to leading order. The second term in the square brackets due to the non-trivial Harmonic function can be expanded in a multipole expansion, the first correction to the leading term is due to the dipole and goes like $\frac{1}{\rho^{9/2}}$, etc. The measure in the integral does like $\rho^7 d\rho$, so if the dipole term is present the integral will still blow up as $\rho \rightarrow \infty$. If fact one needs multipole contributions upto a fairly high order to vanish so that the leading contribution from the difference in the two terms in the bracket goes like $\frac{1}{\rho^{17/2}}$. While this is not elegant it can be arranged by choosing a suitable distribution of branes, and the resulting difference in area and hence entanglement entropies will then be finite.

2.1. Comparison with $D0$ brane matrix theory

In the proposal which follows we will identify each of the terms $S_{EE}(d, T = 0)$ and $S(d, T)$ in (24) with quantities in the $D0$ brane matrix theory. However, as explained above, an unambiguous comparison will be possible for the difference of these quantities with the difference

of corresponding quantities in the $D0$ brane theory. Using equations (24), (7) and (17) and the relation

$$G_N = 8\pi^6 g_s^2 l_s^8 \quad (27)$$

this can be written as

$$S(d, T) - S_{EE}(d, T = 0) = B_0 N^2 T_0^{14/5} d_0^{-5/2} \quad (28)$$

where

$$B_0 = \frac{480 \cdot 2^{2/5} 15^{9/10} \pi^{13/2} \Gamma\left(\frac{5}{4}\right)}{49 \cdot 7^{4/5} \Gamma\left(\frac{25}{4}\right)} \quad (29)$$

We have defined

$$T = T_0 \Lambda \quad (30)$$

with

$$\Lambda = \frac{(g_s N)^{1/3}}{l_s} \quad (31)$$

and

$$d = d_0 (g_s N)^{1/3} l_s \quad (32)$$

We aim to reproduce this behaviour from the theory of $D0$ branes. The theory of $D0$ branes does not have any dimensionless parameter—there is only one scale which is the dimensional 't Hooft coupling $\lambda = g_{YM}^2 N$. In terms of the bulk parameters this is given by

$$g_{YM}^2 N = (g_s N)/l_s^3 = \Lambda^3. \quad (33)$$

This allows us to define a dimensionless temperature T_0 given in equation (30). Also, the transverse radial coordinate r is proportional to the energy scale of the dual theory. This means that we should define a dimensionless distance d_0 as given by equation (32). We note that the size of the ground state wave function in this system is also given by $(g_s N)^{1/3} l_s$ [26, 28], and this is also the length scale r_0 , equation (8) beyond which the supergravity approximation breaks down; these observations agree with taking the dimensionless distance to be d_0 as above.

From equation (28) we see that the difference of the two bulk entanglement entropies when expressed in terms of the appropriate dimensionless variables of the $D0$ brane matrix theory scales like N^2 . We also note that equation (28) is valid when equation (18) holds, this condition can also be expressed in terms of d_0 and T_0 and becomes,

$$d_0 \gg T_0^{2/5} \quad (34)$$

Finally, equation (28) assumes that the supergravity approximation is valid, this requires,

$$T_0 \ll 1, \quad N \gg 1 \quad (35)$$

It is also worth mentioning that from equations (13) and (2) it follows that the entanglement entropy itself (obtained by ignoring the d dependent contributions) is given by

$$S_{EE} \sim N^2 \left(\frac{\rho_0}{(g_s N)^{1/3} l_s} \right)^{9/2} \quad (36)$$

and also scales like N^2 when ρ_0 is expressed in terms of the appropriate dimensional length scale of the matrix theory.

In the above discussion we have asserted that the entanglement entropy is proportional to area in Einstein frame. It is interesting to see what would happen if this was the area in string frame metric. In that case we get an answer

$$\Delta A_{\text{string-frame}} = A_{\text{string-frame}}(T) - A_{\text{string-frame}}(T=0) \sim T_0^{14/5} d_0^{-13} l_s^8 \quad (37)$$

If we now use a UV cutoff which is the string length l_s we see that $\Delta A_{\text{string-frame}}/l_s^8$ can be again expressed in terms of quantities in the $D0$ brane quantum mechanics. Note however if this is taken to be a candidate for the entanglement entropy, the answer scales as N^0 . This will not connect with the Bekenstein–Hawking formula and appears unnatural since the $D0$ brane theory has N^2 degrees of freedom. However we cannot rule out this possibility without a concrete calculation in the $D0$ brane quantum mechanics.

We now turn to a more detailed discussion of $D0$ brane quantum mechanics.

3. Entanglement in $D0$ brane quantum mechanics

Here we address the question: what is the bulk entanglement in the dual description in terms of $D0$ brane quantum mechanics? Let us begin by reviewing some basics about the $D0$ brane matrix quantum mechanics [27].

3.1. Matrix quantum mechanics: basic facts

The action for this $0 + 1$ dimensional supersymmetric Yang–Mills theory is given by

$$S = \frac{N}{2(g_s N) l_s} \text{Tr} \int dt \left[\sum_{I=1}^9 (D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J=1}^9 [X^I, X^J]^2 \right] + \text{fermions} \quad (38)$$

where $X^I(t)$ are $N \times N$ Hermitian matrix functions of time and D_t stands for the covariant derivative

$$D_t X^I = \partial_t X^I + i[A_t, X^I] \quad (39)$$

This action has a $SU(N)$ gauge symmetry (actually the symmetry is $U(N)$, but the $U(1)$ decouples). We can now fix a gauge $A_t = 0$. As usual, the resulting Gauss law constraint imposes the condition that all physical states are invariant under a $SU(N)$ rotation⁶. The Hamiltonian in this gauge is

$$H = \frac{1}{2} \text{Tr} \left[\frac{(g_s N) l_s}{N} \sum_{I=1}^9 (P^I)^2 + \frac{N}{(g_s N) l_s^5} \sum_{I \neq J=1}^9 [X^I, X^J]^2 \right] + \text{fermions} \quad (40)$$

where P^I denote the conjugate momenta.

This theory does not have any dimensionless parameter. This is seen clearly by rescaling

$$X^I = (g_s N)^{1/3} l_s \tilde{X}^I \quad P^I = \frac{1}{(g_s N)^{1/3} l_s} \tilde{P}^I \quad (41)$$

⁶ More details of this model are discussed in appendix B.

and the Hamiltonian (40) now becomes

$$H = \frac{(g_s N)^{1/3}}{2l_s} \text{Tr} \left[\frac{1}{N} \sum_{I=1}^9 (\tilde{P}^I)^2 + N \sum_{I \neq J=1}^9 [\tilde{X}^I, \tilde{X}^J]^2 \right] + \text{fermions} \quad (42)$$

Thus the theory is characterized by a single energy scale

$$\Lambda = \frac{(g_s N)^{1/3}}{l_s} \quad (43)$$

In this $A_t = 0$ gauge one is left with a time independent $SU(N)$ symmetry which also needs to be modded out. We will do this by diagonalizing one of the matrices, X^1 . The remaining symmetry is now Weyl transformations which permute the eigenvalues of X^1 which we denote by $\lambda_i, i = 1, \dots, N$, and mix up the matrix elements of the other eight matrices X^I in a non-trivial fashion. In the following discussion we will ignore the fermions.

In the lowest energy state, all the nine matrices commute with each other. In this case all the matrices can be diagonalized simultaneously. If the eigenvalues are denoted by $x_i^I, i = 1 \dots N$, these denote the locations of the N D -branes. The origin of this Coulomb branch has $\langle X^I \rangle = 0$ —their dispersion provides the scale of the bound state, which is $(g_s N)^{1/3} l_s$. The supergravity description of this state is the N coincident $D0$ brane solution discussed above.

A generic state may be expressed in the form (the measure is derived in (147) of appendix B; in the following, we have omitted the tilde sign from $\tilde{\Psi}$ in appendix B):

$$|\psi\rangle = \int [d\mu] \Psi(\lambda_i; X_{ij}^2, \dots, X_{ij}^9) |\lambda_i; X_{ij}^2, \dots, X_{ij}^9\rangle + (\text{Weyl transforms}) \quad (44)$$

where we imposed the Weyl symmetry by summing over Weyl transforms (according to (153)). The measure is

$$[d\mu] = \prod_{i=1}^N d\lambda_i \prod_{I=2}^9 [dX^I] \quad (45)$$

Here $[dX^I] = \prod_i dX_{ii}^I \prod_{i < j} dX_{ij}^I dX_{ji}^I$ is the standard Haar measure. Here and in the following whenever we write X^I the index I runs from 2 to 9.

Using the same basis, a generic operator may be expanded as

$$\hat{O} = \int [d\mu] \int [d\mu'] \mathcal{O}(\lambda_i, X_{ij}^I; \lambda'_i, X'^I) |\lambda_i; X^I\rangle \langle \lambda'_i, X'^I| + \text{Weyl transforms} \quad (46)$$

In the low energy description, and at zero temperature, the space of eigenvalues λ^i corresponds to one of the space directions, namely x^1 , in ten-dimensional supergravity.

3.2. Target space entanglement entropy

It is then clear that our calculation of the entanglement entropy across a $x^1 = d$ surface in a particular geometry in the bulk maps to a calculation of the *target space* entanglement in the $D0$ brane quantum mechanics. That is we would like to restrict ourselves to the region $x_1 > d$ and ask what are the operators we can have access to in this region; the von-Neumann entropy of the density matrix associated with this subalgebra of all observables is then the relevant entanglement entropy. Compared to the single particle case briefly discussed in the

introduction there are two extra features of this problem worth mentioning, both have to do with the fact that we are dealing with a system with many degrees of freedom.

In general in a non-relativistic system with many particles, the analysis breaks up into different sectors, each sector being specified by which of the particles are present in the region of interest. The corresponding set of operators in this sector correspond to all the measurements one can perform on these particles and the full sub algebra with which we associate the entropy is then a sum of the algebras of observables in each sector. In fact these sectors are superselection sectors, since the observables in the algebra do not change the particles in the region of interest.

The second feature has to do with statistics. In our case the different eigenvalues of the X_1 matrix correspond to fermion-like degrees of freedom. More precisely the wave function $\Psi(\lambda_i, X_{ij}^l)$, equation (44) has the property that it picks up a minus sign under interchange of any given pair of indices $i \leftrightarrow j$, i.e., under $\lambda_i \leftrightarrow \lambda_j$, $X_{ii}^l \leftrightarrow X_{jj}^l$ and $X_{ij}^l \leftrightarrow X_{ji}^l$, the wave function, $\Psi \rightarrow -\Psi$. This follows from a special case of the general Weyl transformation (153) in appendix B, where we choose to permute a given pair (i, j) . We are interested here in the target space region $x_1 > d$. The different super selection sectors are therefore specified only by the number of eigenvalues of X_1 meeting the condition $\lambda_i > d$, and not any particular choice of these eigenvalues.

On general grounds, it then follows that the density matrix is block diagonal in the different sectors and of the form

$$\tilde{\rho} = \bigoplus_{m=0}^{m=N} \tilde{\rho}_m \quad (47)$$

where ρ_m is the density matrix in the m th sector in which m eigenvalues of X_1 meet the condition $\lambda > d$ (and the remaining the $N - m$ eigenvalues are outside of this region). This is similar to the equation (99) of appendix A which discusses the case of N fermions; the notation $\tilde{\rho}_m$ here is to be identified with $\tilde{\rho}_{m,N-m}$ of that equation.

Note that we can write (47) as

$$\tilde{\rho} = \bigoplus_{m=0}^{m=N} p_m \hat{\rho}_m \quad (48)$$

where p_m is the probability to be in the m th

$$p_m = \text{Tr}_m(\tilde{\rho}_m) \quad (49)$$

and $\hat{\rho}$ is the normalized density matrix in this sector, satisfying the relation

$$\text{Tr}_m \hat{\rho} = 1 \quad (50)$$

The trace in equations (49), (50) is restricted to the m th sector. Note in equation (48) we have also allowed for no eigenvalue being in the region of interest.

It is worth emphasising that the probabilities p_m satisfy the relation

$$\sum_{m=0}^{m=N} p_m = 1, \quad (51)$$

so that it follows from equation (49) that the full density matrix $\tilde{\rho}$ has the standard normalisation

$$\text{Tr}(\tilde{\rho}) = 1. \quad (52)$$

Before proceeding let us also note that the entanglement entropy, defined as the von Neumann entropy of equation (48) (cf (98) of appendix A), is given by

$$S_{\text{EE}} = - \sum_m \text{Tr}_m \tilde{\rho}_m \ln(\tilde{\rho}_m) \quad (53)$$

$$= - \sum_m p_m \ln p_m + \sum_{m=0}^N p_m \text{Tr}_m \hat{\rho} \ln(\hat{\rho}) \quad (54)$$

where the trace Tr_m again denotes the trace within the sector with m of the eigenvalues lying in the region of interest. The structure of the density matrix, equation (48) and entropy, equation (54) are of the general type which arises in the presence of super selection sectors. And on general grounds it follows that the distillable part of the entanglement is only the second term in equation (54), while the first term $-\sum_m p_m \ln p_m$ is a classical piece which cannot be used as a quantum resource for teleportation, etc [32, 33].

We also note that at non-zero temperature the relationship between the eigenvalues of the matrix X^1 and the coordinate x_1 in the background metric is not straightforward. However for regions far from the horizon these two quantities can be taken to be the same; thus since in this note we are dealing with the parametric region $d \gg r_H$ such an identification would be justified.

Let us now digress briefly to make one comment which is worth emphasising. For our proposal, that the entanglement of some region in the bulk corresponds to target space entanglement in the boundary theory equation (2), to be sensible it is important that in a pure state the target space entanglement for a region and its complement are equal. In the specific example we are considering this implies that the target space entanglement corresponding to the region $x_1 > d$ and $x_1 < d$ are equal. It is easy to see that this is the case and in fact the reasoning we give below can be seen to apply immediately to a general bulk region as well, so long as this region can be mapped suitably to a target space constraint in the boundary.

Let us denote, for the discussion in this paragraph only, the density matrix for the region $x_1 > d$ by $\tilde{\rho}^{(>d)}$. The corresponding entanglement entropy is given by

$$S_{\text{EE}}^{(>d)} = - \text{Tr} [\tilde{\rho}^{(>d)} \log \tilde{\rho}^{(>d)}] . \quad (55)$$

Note that this trace has to be taken over all the N super selection sectors described above. Now when m eigenvalues of X^1 take values in the range $\lambda > d$, $N - m$ eigenvalues lie in the complement, $\lambda < d$. Thus the m th super selection for when we are considering the $x_1 > d$ region maps into the $(N - m)$ th sector for the $x_1 < d$ case. A little more analysis also shows that the density matrices $\tilde{\rho}_m^{(>d)}$ for the $x_1 > d$ region and correspondingly $\tilde{\rho}_{N-m}^{(<d)}$ for the $x_1 < d$ region make an equal contribution to their respective entropies, $S_{\text{EE}}^{(>d)}$, $S_{\text{EE}}^{(<d)}$. This follows from the fact that in each sector the Hilbert space admits a tensor product decomposition, and the entanglement entropy for a pure state in a bipartite system is equal for both of its constituent Hilbert spaces⁷. The sectors where $m = 0$ and $m = N$ have to be dealt with as a special case. The equality in these sectors follows simply from the fact that the probability to not find any eigenvalue taking values in the range $\lambda > d$ equals that to find all N in $\lambda < d$ and vice-versa.

⁷ Actually to deal with the complication of fermion statistics correctly we need to embed the Hilbert space in each super selection sector in an extended Hilbert space where the required anti-symmetrisation constraint is not imposed. This extended Hilbert space admits a tensor product decomposition and that is enough to show the equality of the contributions $\tilde{\rho}_m^{(>d)}$ and $\tilde{\rho}_{N-m}^{(<d)}$ make to their respective entanglement entropies.

We now return to the main thread of our discussion. Consider one of the terms in the state expressed in (44), e.g. the first term. This has a given ordering of the eigenvalue labels and the matrix elements of the remaining X^I . The corresponding wavefunction is the probability amplitude that the location of the N $D0$ branes in the x^1 direction are given by the λ_i . The diagonal matrix elements of X^I represent open strings which begin and end on the same $D0$ brane, while the off-diagonal matrix elements represent open strings which stretch between different $D0$ branes. Now suppose in this particular term the first n eigenvalues have $\lambda_i > d$ while the rest have $\lambda_i < d$. We will relabel the index i for the latter set of eigenvalues by the index a . We need to be able to perform measurements which involve the matrix elements X_{ij}^I , $i, j = 1 \dots n$, $I = 2 \dots 9$, while we do not wish to retain the elements X_{ab}^I , $a, b = (n+1) \dots N$, $I = 2 \dots 9$. This leaves us with the off-diagonal blocks X_{ia}^I, X_{ai}^I , $i = 1 \dots n$, $a = n+1 \dots N$, $I = 2 \dots 9$ and its transpose. Of course the labelling of the matrix elements pertains to one specific term in the sum (44). The question we are allowed to ask is *how many* eigenvalues are larger than d , not *which eigenvalues* are larger than d . The sum over Weyl transforms precisely achieves this—ensuring that the $D0$ branes are identical particles.

As explained below (see section 3.3 as well as appendix B), this leads us to two different proposals for the subalgebra of operators whose associated entanglement entropy corresponds to the quantity computed in the bulk. Let us first focus on the sector in which there are n eigenvalues λ_i in the region $x_1 > d$. In our first proposal, the operator subalgebra relevant to this sector consists of operators in the Hilbert space of variables $\{\lambda_i, X_{ij}^I\}$ (see appendix A for a detailed discussion of operator subalgebras in the simple context of free fermions, e.g. (93)) which are of the form

$$\begin{aligned} \hat{O}_n &= \tilde{O}_n \otimes \bar{\mathbf{1}} + \text{Weyl}, \\ \tilde{O}_n &= \int \prod_i \int_d^\infty d\lambda_i \int_d^\infty d\lambda'_i \prod_{ij}^I dX_{ij}^I dX_{ij}^{I'} \tilde{O}_n(\{\lambda_i, \lambda'_i\}; \{X_{ij}^I, X_{ij}^{I'}\}) \\ &\quad \times |\{\lambda_i, X_{ij}^I\}\rangle \langle \{\lambda'_i, X_{ij}^{I'}\}| + \text{Weyl transf} \\ \bar{\mathbf{1}} &= \int [d\mu_n] |\lambda_a, X_{ia}^I X_{ai}^I X_{ab}^I\rangle \langle \{\lambda_a, X_{ia}^I X_{ai}^I X_{ab}^I\}| \\ \int [d\mu_n] &\equiv \int_{-\infty}^d \prod_{a=n+1}^N d\lambda_a \int \prod_{a,b=n+1}^N [dX_{ab}^I] \int \prod_{a=n+1}^N \prod_{i=1}^n [dX_{ia}^I dX_{ai}^I] \end{aligned} \quad (56)$$

The full operator algebra consists of contribution of operators from the various n -sectors.

From the above definition, it is clear that in this proposal we are tracing over not only the $(N-n) \times (N-n)$ block of the matrices X^I , but also the off-diagonal blocks. This means that we are not performing measurements on the open strings which join the $D0$ branes in the $x_1 > d$ region with those in the $x_1 < d$ region.

In our second proposal the subalgebra of operators in the n -sector consists of operators in a Hilbert space of coordinates $\{\lambda_i, X_{ij}^I, X_{ia}^I\}$, and are of the form

$$\begin{aligned} \hat{O}_n &= \tilde{O}_n \otimes \bar{\mathbf{1}} + \text{Weyl}, \\ \tilde{O}_n &= \prod_i \int_d^\infty d\lambda_i \int_d^\infty d\lambda'_i \prod_{ij}^I \int dX_{ij}^I dX_{ij}^{I'} \prod_{ia,ai}^I \int dX_{ia}^I dX_{ia}^{I'} dX_{ai}^I dX_{ai}^{I'} \\ &\quad \times \tilde{O}_n(\{\lambda_i, \lambda'_i\}; \{X_{ij}^I, X_{ij}^{I'}, X_{ia}^I, X_{ia}^{I'}, X_{ai}^I, X_{ai}^{I'}\}) \end{aligned}$$

$$\begin{aligned}
& \times |\{\lambda_i, X_{ij}^I, X_{ia}^I\}\rangle\langle\{\lambda_i', X_{ij}', X_{ia}'\}| + \text{Weyl trans} \\
\bar{\mathbf{1}} &= \int [\mathrm{d}\nu_n] |\{\lambda_a, X_{ab}^I\}\rangle\langle\{\lambda_a, X_{ab}^I\}| \\
\int [\mathrm{d}\nu_n] &\equiv \int_{-\infty}^d \prod_{a=n+1}^N \mathrm{d}\lambda_a \int \prod_{a,b=n+1}^N [\mathrm{d}X_{ab}^I]
\end{aligned} \tag{57}$$

In this proposal we are tracing over only the $(N-n) \times (N-n)$ block of the matrices X^I . This means that our measurements include those made on open strings which join the $D0$ branes in the $x_1 > d$ region with those in the $x_1 < d$ region.

In this paper we will largely focus on the bosonic degrees of freedom in the quantum mechanics and not discuss the fermionic ones. However let us at least mention that the fermionic degrees of freedom θ_A which are also $N \times N$ matrices must be dealt with in the same way as the $X^I, I = 2, \dots, 9$ matrices. This means in the first proposal we only retain the $(\theta_A)_{ij}$ blocks and trace over the $(N-n) \times (N-n)$ blocks $(\theta_A)_{ab}$, as well as the off-diagonal blocks, $(\theta_A)_{ai}, (\theta_A)_{ia}$. In the second proposal we retain the $(\theta_A)_{ij}$ and the $(\theta_A)_{ai}, (\theta_A)_{ia}$ blocks and only trace over the $(\theta_A)_{ab}$ block.

For a given state one can now compute the RDM which correctly reproduces expectation values of either of the set of operators, and from this the von Neumann entropy. The formalism to write this down is explained in the appendix and will also be elaborated in the next subsection. Our conjecture is that one of these will correspond to the bulk entanglement entropy computed in (section 2).

3.3. The two proposals for target space EE

Before proceeding, some more comments are worth making at this stage. We note that some motivation for the two proposals above come from the Coulomb branch solutions. In supergravity it is known that there are solutions in which the $D0$ branes are displaced from the origin and the harmonic function takes the more general form equation (25), with \vec{r}_i specifying the location of the i th brane. These solutions also correspond to bound states at zero energy in the matrix theory. Consider such a solution in which the x_1 coordinate of some of the $D0$ branes lies in the region $x_1 < d$, i.e. outside the region of interest. In defining the entanglement if these branes are to be excluded, then the open strings stretching between these excluded branes should also be dropped. This still leaves the choice of whether the degrees of freedom corresponding to the open strings stretching between the branes inside the region, with $x_1 > d$, and those outside, with $x_1 < d$, should be retained or dropped. Correspondingly, in the density matrix of the matrix theory we have two choices of retaining the off-diagonal degrees of freedom stretching between the eigenvalues with $\lambda_i < d$ and $\lambda_i > d$, as discussed above. See figure 1. For some more details, see appendix B (section B.1).

It is worthwhile to emphasize that even though we draw motivation from a generic point on the Coulomb branch, the state we are discussing is at the origin of the Coulomb branch. The supergravity solution for this is a set of coincident $D0$ branes. However, in the matrix quantum mechanics this state has a non-trivial wavefunction which has a spread of $\sim N^{1/3} l_s$. This means that while the *expectation value* of the matrices vanish in this state, there is a non-zero probability amplitude (described by the wavefunction) for having a configuration described by values of $\lambda_i, \lambda_a, X_{ij}^I, X_{aj}^I, X_{ab}^I$, using the notation described above. The open strings we refer to above are simply a description of this kind of configuration. These comments are also true for a state of the kind in equation (25) where the branes are displaced from the origin but continue to lie in the region of validity of the supergravity approximation; while the expectation value

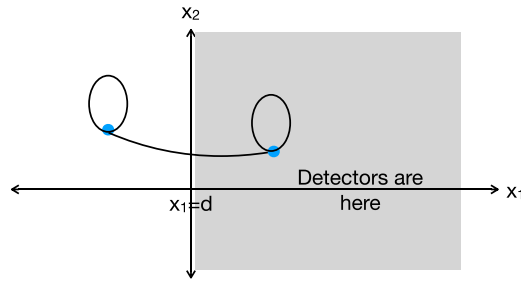


Figure 1. A typical configuration for the case $N = D = 2$, where we have two 2×2 matrices $X = \text{diag}[\lambda_1, \lambda_2]$, $Y = \text{diag}[y_{11}, y_{22}]$; (λ_i, y_{ii}) represent the coordinates of the two $D0$ branes, $i = 1, 2$. The figure depicts the situation in which one of the $D0$ branes, say with coordinates $\mathbf{x}_1 = (\langle \lambda_1 \rangle, \langle y_{11} \rangle)$, is in region $A : x^1 > d$, i.e. $\lambda_1 > d$, while the other $D0$ brane with coordinates $\mathbf{x}_2 = (\langle \lambda_2 \rangle, \langle y_{22} \rangle)$ is in $\bar{A} : x^1 < d$, i.e. $\lambda_2 < d$. The variables λ_1, y_{11} represent an open string beginning and ending on the first $D0$ -brane; they are in the region of interest A and hence must be included in the operator algebra. Similarly the variables λ_2, y_{22} represent an open string beginning and ending on the second $D0$ -brane; they are in the region of interest \bar{A} and hence should be excluded from the operator algebra. y_{12}, y_{21} represent open strings straddling between regions A and \bar{A} . One might wish to exclude the y_{12}, y_{21} from the operator algebra (first proposal), or include them (second proposal). Note that, the two $D0$ branes are actually indistinguishable; as (151), (153) indicate, the situation described above is indistinguishable from the one in which the $D0$ branes are interchanged; hence the above definitions have to take that into account, as was done for case of N fermions in appendix A.

of the matrices do not vanish now, there is a non-zero probability amplitude for various values of λ_i, X_{ij}^I , etc.

In either of the above proposals, there are $O(N^2)$ degrees of freedom which are traced out. It is therefore natural to expect that the entanglement entropy will be proportional to N^2 . We note that the fact there are $O(N)$ sectors in the sum, equation (54) does not alter this estimate. If $O(N)$ sectors contribute and generically each sector gives a contribution of $O(N^2)$ by which we mean that the normalized density matrix $\hat{\rho}_m$ has $\text{Tr}_m \hat{\rho}_m \ln(\hat{\rho}_m) \sim O(N^2)$, then the final result for the second term in equation (54) would still be $O(N^2)$. The first term in equation (54) which is the classical piece is much smaller and can at most be $O(\ln N)$.

When the bulk is a black hole we should consider the $D0$ brane quantum mechanics in a thermal state with the same temperature T . There are now two dimensionful quantities in the calculation. The first is the temperature T and the second is the value of d which has been used to define the subalgebra. It is clear from the discussion from equation (38) to (43) that the energy scale in the 't Hooft limit is given by Λ defined in (43), while the scale which relates supergravity distances with the eigenvalues is $(g_s N)^{1/3} l_s$. Therefore the $D0$ brane quantum mechanics answer for the entanglement entropies will involve the dimensionless temperature $T_0 = T/\Lambda$ and the dimensionless d_0 introduced above. Once this is done, the answer should be simply proportional to N^2 , exactly as in the supergravity calculation. As explained in the previous section, to keep the bulk calculation within the realm of the supergravity approximation one could compare the difference of the entropies at finite and zero temperature, this would allow for a precise test of the coefficient in the area term in equation (2).

While a bound state which corresponds to N $D0$ branes has been shown to exist [29], an explicit analytic form is not known. This makes an analytic check of our proposal difficult. It should be, however, possible to express the target space entanglement entropy discussed above

in a path integral formulation: then numerical calculations along the lines of [19] can be used to provide a concrete check of our proposal.

Before ending this subsection let us also mention that a useful toy model to understand target space entanglement is to consider the case of a single bosonic matrix quantum mechanics with no external potential. In this case the additional X^I are not present and we only have the eigenvalues λ_i . As is well known the λ_i can be considered as the coordinates of N free fermions moving on a line. The above description of the relevant subalgebra of operators is in a first quantized description. In a second quantized description, the Hilbert space becomes a product. The subalgebra of operators pertaining to the subregion $\lambda > 0$ are given by M body operators of the form

$$\begin{aligned} \mathcal{F} = & \int_0^\infty \prod_{i=1}^M [d\lambda_i d\lambda'_i] \psi^\dagger(\lambda_1) \psi^\dagger(\lambda_2) \cdots \psi^\dagger(\lambda_M) \\ & \times F_M(\lambda_1 \cdots \lambda_M; \lambda'_1 \cdots \lambda'_M) \psi(\lambda'_1) \psi(\lambda'_2) \cdots \psi(\lambda'_M) \end{aligned} \quad (58)$$

where $\psi(x), \psi^\dagger(x)$ are the second quantized fermion fields. In the sector where there are n particles in this region the operators which have nonzero expectation values must have $M \leq n$. It can be then shown easily that the functions F_M are in one-to-one correspondence with the matrix elements of operators in the first quantized description in the sector where there are M particles in the sub-region. In fact, for free fermions one may use well known methods to compute the RDM [30] to show that the density matrix obtained in the second quantized description is exactly the same as the first quantized description discussed above. Details of this are provided in appendix A.

3.4. The sector-wise entanglement

Consider the sector where n eigenvalues satisfy the condition $\lambda_i > d$. Let us use the Weyl symmetry and arrange for these to be the first n eigenvalues of X_1 . Then in the first proposal we would also retain in the density matrix the $X_{ij}, (i, j \leq n)$ degrees of freedom and ‘integrate out’ everything else. Starting with a wave function $\Psi(\lambda_i, X_{ij}^I)$ with unit norm,

$$\int d\lambda_i D X_{ij}^I |\Psi(\lambda_i, X_{ij}^I)|^2 = 1, \quad (59)$$

where in the integral $\lambda_i \in [-\infty, \infty]$ and the measure for X_{ij}^I is the standard flat measure for Hermitian matrices, with a range as explained in the appendices A, B. We then get that the density matrix in this sector is given by (cf (47))

$$\tilde{\rho}_n(\lambda_i, X_{ij}^I; \lambda'_{i'}, X_{i'j'}^{II}) = \binom{N}{n} \int D\chi_A \Psi^*(\lambda_i, X_{ij}^I, \chi_A) \Psi(\lambda'_{i'}, X_{i'j'}^{II}, \chi_A) \quad (60)$$

To save clutter we have denoted all variables to be integrated over generically as χ_A . These include $\lambda_i, i > n$ and X_{ij}^I , where one or both indices i, j are greater than n . Note that the range of integration over these variables is as follows $\lambda_i, i > n$ take values $\in [-\infty, d]$, while X_{ij}^I , with i or $j > n$, are to be integrated over their full range (real line for $i = j$ and complex plane for $i \neq j$). It is important to note that the variables being integrated out, χ_a , appear in both Ψ^* and Ψ . The combinatorial factor $\binom{N}{n}$ arises as follows. The case with n eigenvalues of X_1 being greater than d can arise in $\binom{N}{n}$ different ways, by the fermionic symmetry these all give the same contribution to the density matrix resulting in this combinatorial factor.

Note that the density matrix ρ_n is an operator in the space of the degrees of freedom that remain after imposing the target space constraint and once it is known we can in principle calculate its contribution to the entropy, $\text{Tr}_n \rho_n \ln(\rho_n)$. Summing the contributions from the different sectors then gives the full entanglement entropy, equation (53).

In the second proposal after arranging for the first n eigenvalues to be greater than d we retain : $\lambda_i, i \leq n, X_{ij}, i, j < n$. In addition we retain the degrees of freedom, X_{ai}^I, X_{ia}^I , with, $i < n, a > n$; these satisfy the relation $X_{ai}^I = (X_{ia}^I)^*$. The density matrix now depends on these degrees of freedom as well, and equation (60) is replaced by

$$\begin{aligned} \tilde{\rho}_n (\lambda_i, X_{ij}^I, X_{ai}^I; \lambda_{i'}^I, X_{i'j'}^I, X_{a'i'}^I) \\ = \binom{N}{n} \int D\chi_A \Psi^* (\lambda_i, X_{ij}^I, X_{ai}^I, \chi_A) \Psi (\lambda_{i'}^I, X_{i'j'}^I, X_{a'i'}^I, \chi_A) \end{aligned} \quad (61)$$

where now the χ_a variables include : $\lambda_i, i > n, X_{ij}^I, i, j > n$. The range of integration for these variables are as above and the combinatorial factor has the same origin as in the previous case. More details can be found in appendices A and B.

While we have not been explicit about fermionic degrees of freedom here they are to be included in a manner analogous to the X^I degrees of freedom, as was discussed after equation (57) above.

Finally let us note the form for ρ_n if we start not with a wave function Ψ but with a density matrix for the full system, as would be the case when we consider the finite temperature case where

$$\rho = \frac{e^{-H/T}}{\sum_i e^{-H/T}}, \quad (62)$$

where H is the Hamiltonian and the index i denotes sum over all states. The density matrix can now be regarded as a general function $\rho(\lambda_i, X_{ij}^I; \lambda_{i'}^I, X_{i'j'}^I)$, with i, j, i', j' , taking values $1, 2, \dots, N$.

In this case similar reasoning as above shows that for the first proposal equation (60) is replaced by

$$\tilde{\rho}_n (\lambda_i, X_{ij}^I; \lambda_{j'}^I, X_{i'j'}^I) = \binom{N}{n} \int D\chi_A \rho (\lambda_i, X_{ij}^I, \chi_A; \lambda_{j'}^I, X_{i'j'}^I, \chi_A) \quad (63)$$

where χ_a as above denotes the variables, $\lambda_i, i > n$ and X_{ij}^I , where one or both labels, $i, j > n$. Whereas in the second proposal equation (61) is replaced by

$$\tilde{\rho}_n (\lambda_i, X_{ij}^I, X_{ai}^I; \lambda_{j'}^I, X_{i'j'}^I, X_{a'i'}^I) = \binom{N}{n} \int D\chi_A \rho (\lambda_i, X_{ij}^I, X_{ai}^I, \chi_A; \lambda_{j'}^I, X_{i'j'}^I, X_{a'i'}^I, \chi_A) \quad (64)$$

where χ_A now includes, $\lambda_i, i > n$, and X_{ij}^I , with both $i, j > n$.

4. Dp branes ($p < 3$)

The results of section (2) generalize to Dp branes with $p < 3$. The string frame metric and the dilaton for the near horizon geometry of N coincident near-extremal black Dp branes are

$$\begin{aligned} ds^2 &= \left(\frac{R}{r}\right)^{-n/2} [-f(r) dt^2 + dy_1^2 + \dots + dy_p^2] + \left(\frac{R}{r}\right)^{n/2} \left[\frac{dr^2}{f(r)} + r^2 d\Omega_{n+1}^2 \right] \\ e^{-\phi/2} &= \left(\frac{R}{r}\right)^{\frac{n(p-3)}{8}} \end{aligned} \quad (65)$$

where

$$\begin{aligned} n &= 7 - p \quad R^n = (4\pi)^{(n-2)/2} \Gamma(n/2) l_s^n (g_s N) \\ r^2 &= x_1^2 + \cdots + x_{9-p}^2 = x_1^2 + \rho^2 \\ f(r) &= 1 - \left(\frac{r_H}{r}\right)^n \end{aligned} \quad (66)$$

and the temperature is given by

$$T = \frac{n}{4\pi R} \left(\frac{r_H}{R}\right)^{\frac{n-2}{2}} \quad (67)$$

The brane directions y_i each have an extent L . Consider once again a $x_1 = d$ surface where $d > r_H$. The Einstein frame area of this surface is given by

$$A_d(T) = \Omega_n R^{n/2} L^p \int_0^\infty d\rho \frac{\rho^{n/2}}{(1 + \frac{d^2}{\rho^2})^{n/4}} \left[1 + \frac{r_H^n}{\rho^n} \frac{1}{(1 + \frac{d^2}{\rho^2})^{\frac{n}{2}+1}} \right]^{1/2} \quad (68)$$

This integral is divergent at the upper limit. However, as in the case of zero branes, the difference $A_d(T) - A_d(0)$ is finite. Performing a low temperature expansion as in the previous section we obtain the difference of the areas which is once again insensitive to the IR cutoff on ρ and the entropy difference is then given by

$$\Delta S_{EE} = \frac{\Omega_n \Gamma\left(\frac{n-2}{4}\right) \Gamma\left(\frac{n+3}{2}\right) L^p R^{n/2} r_H^n}{4\Gamma\left(\frac{3n}{4} + 1\right) \epsilon^8 d^{\frac{n}{2}-1}} \quad (69)$$

Using the expression for R in (66) and r_H in terms of the temperature in (67) we get

$$\Delta S_{EE} = C_p \frac{(g_s N)^2 l_s^8}{\epsilon^8} (g_s N)^{\frac{6-n}{2(n-2)}} l_s^{\frac{3n^2-18n+32}{2(n-2)}} T^{\frac{2n}{n-2}} L^{7-n} d^{1-\frac{n}{2}} \quad (70)$$

where

$$C_p = (n+1) 2^{\frac{3n^2-4n+12}{2(n-2)}} \pi^{\frac{n(5n-2)}{4(n-2)}} n^{-\frac{2n}{n-2}} \Gamma\left(\frac{n}{2}\right)^{\frac{3n-2}{2(n-2)}} \frac{\Gamma\left(\frac{n-2}{4}\right)}{\Gamma\left(\frac{3n}{4} + 1\right)} \quad (71)$$

We now need to express the temperature, the p -brane extent and the quantity d in terms of their appropriate scales. The energy scale Λ of the Dp brane theory is provided by the 't Hooft coupling

$$g_{YM}^2 N = \frac{(g_s N)}{l_s^{n-4}} \Rightarrow \Lambda = (g_s N)^{\frac{1}{n-4}} l_s^{-1} \quad (72)$$

This means that we need to express T and the extent L in these units,

$$T = T_0 \Lambda \quad L = L_0 \Lambda^{-1} \quad (73)$$

The transverse distance in the geometry is, however proportional to this energy scale multiplied by l_s^2 . This means that we need to express

$$d = d_0 \Lambda l_s^2 \quad (74)$$

Once again, when expressed in terms of these dimensionless quantities, the result should not involve g_s . This can happen only if the UV cutoff ϵ is proportional to the ten-dimensional Planck scale. Using this cutoff, we are left with a final answer proportional to N^2 ,

$$\Delta S_{EE} = B_p N^2 T_0^{\frac{2n}{n-2}} L_0^{7-n} d_0^{1-\frac{n}{2}} \quad (75)$$

where

$$B_p = (n+1) 2^{\frac{3n^2-14n+32}{2(n-2)}} \pi^{\frac{5n^2-26n+48}{4(n-2)}} n^{-\frac{2n}{n-2}} \Gamma\left(\frac{n}{2}\right)^{\frac{3n-2}{2(n-2)}} \frac{\Gamma\left(\frac{n-2}{4}\right)}{\Gamma\left(\frac{3n}{4}+1\right)} \quad (76)$$

5. Entanglement in Dp brane field theory

The discussion of a candidate subalgebra of operators in the $SU(N)$ Yang–Mills theory living on the Dp brane (for $p < 3$) worldvolume is completely analogous to that for $D0$ brane quantum mechanics. The matrices are now functions of the spatial coordinates on the Dp brane worldvolume ξ . The bosonic fields are now worldvolume gauge fields $A_\mu(\xi)$, $\mu = 1 \dots (p+1)$ and the transverse Higgs fields $X^I(\xi)$ with $I = 1, \dots, 9-p$. We then work in a gauge where one of these Higgs fields, X^1 is chosen to be diagonal with elements $\lambda_i(\xi)$ and consider a division of the space of $\lambda(\xi)$ into two parts, corresponding to $\lambda_i(\xi) > d$ and $\lambda_i(\xi) < d$. As in section (3), there are two choices for the corresponding operator sub-algebra. The generalization for the choice (56) involves an expression

$$\begin{aligned} \hat{O}_n &= \tilde{O}_n \otimes \bar{\mathbf{1}} + \text{Weyl}, \\ \tilde{O}_n &= \prod_{\xi} \prod_i \int_d^\infty d\lambda_i(\xi) \int_d^\infty d\lambda'_i(\xi) \prod_{ij} dX_{ij}^I(\xi) dX_{ij}^{II}(\xi) \\ &\quad \times \tilde{O}_n(\{\lambda_i(\xi), \lambda'_i(\xi)\}; \{X_{ij}^I(\xi), X_{ij}^{II}(\xi)\}) \\ &\quad \times |\{\lambda_i(\xi), X_{ij}^I(\xi)\}\rangle \langle \{\lambda'_i(\xi), X_{ij}^{II}(\xi)\}| + \text{Weyl trans} \\ \bar{\mathbf{1}} &= \int [d\mu_n] |\lambda_a, X_{ia}^I X_{ai}^I X_{ab}^I\rangle \langle \{\lambda_a, X_{ia}^I X_{ai}^I X_{ab}^I\}| \\ \int [d\mu_n] &\equiv \prod_{\xi} \int_{-\infty}^d \prod_{a=n+1}^N d\lambda_a(\xi) \int \prod_{a,b=n+1}^N [dX_{ab}^I(\xi)] \int \prod_{a=n+1}^N \prod_{i=1}^n [dX_{ia}^I(\xi) dX_{ai}^I(\xi)] \quad (77) \end{aligned}$$

This equation should be regarded in the same spirit as (56); the operator $\mathcal{O}(\{\lambda_i(\xi), \lambda'_i(\xi)\}; \{A_{ij}^\mu(\xi) X_{ij}^I(\xi); A_{ij}^{\mu'}(\xi) X_{ij}^{II}(\xi)\})$ belongs to the Hilbert space of the variables $\{\lambda_i(\xi), A_{ij}^\mu(\xi) X_{ij}^I(\xi)\}$. The measure here is again a generalization of (56) with the additional terms involving the gauge fields and the integrals replaced by functional integrals. Note that (77) involves integration over functions, and the restrictions on the ranges of integration are over the values of the function *at each point on the base space* ξ . The subalgebra of operators for our second proposal also follows in a similar fashion.

6. Discussion

In this paper we explored the idea that in any smooth spacetime, to leading order, the Bekenstein bound is saturated, equation (1), leading to the proposal that for a pure state the entanglement

of any co-dimension one region is given by the area of its boundary in units of G_N , equation (2). We have shown that for a special choice of bulk regions the bulk entanglement can be mapped, upto one ambiguity, to the target space entanglement in the boundary theory. Our proposal can therefore be tested precisely using numerical calculations along the lines of [19]. If our proposal lives up to precise tests, this would mean that the UV cutoff which makes the entanglement entropy finite in string theory is the Newton constant, and not the string length. In fact, this is the lesson from the $c = 1$ example in [12, 13].

We have described the bulk region of interest in a coordinate system and used the relationship between bulk coordinates and target space of the matrix theory in this coordinate system. The notion of the region itself and its bounding surface is of course coordinate invariant. In a different coordinate system the map to matrix theory target space will be different, and therefore the target space restriction will be different. The result, however, will remain the same.

One would like to extend our considerations to more general regions in the bulk. A preliminary study suggests that this might be possible. For example in the $D0$ brane case consider a spherical bulk region given by,

$$\sum_{i=1}^9 (x^i)^2 \leq R^2 \quad (78)$$

In the matrix theory the corresponding operator $\sum \text{Tr}(\hat{X}^i)^2$ is Hermitian and one can choose a gauge where it is diagonalized⁸. This suggests that our considerations might be extendable to more general bulk regions as well. Such an extension would be particularly interesting for a region of the type equation (78), since by changing the radius one could then deform the bulk region smoothly from being away from the black hole horizon to lying on it. It is also worth mentioning that if our proposal is correct the entropy contained in the region equation (78) is temperature independent, since its area is independent of T , as can be easily seen from equation (16). The thermal and entanglement contributions to the entropy presumably trade-off against each other keeping the total unchanged.

It is also worth commenting that various positivity properties, e.g. positivity of relative entropy and mutual information [34, 35], should hold for target space entanglement. For example, positivity of relative entropy and its monotonicity under inclusion of algebras are general properties which should also apply to target space entanglement; from these follow positivity of mutual information and strong subadditivity, etc. Using equation (2) these properties can be mapped to properties of areas bounding regions in the bulk. A preliminary analysis suggests that they are true and in some cases the inequalities are in fact saturated. For example, consider two target space regions, $A : d < x_1$ and $B : d_2 < x_1 < d$, with $A \cup B : d_2 < x_1$. Then it is manifestly true that their mutual information $I(A, B)$ is positive, since,

$$I(A, B) = S(A) + S(B) - S(A \cup B) = 2 \frac{A(x_1 = d)}{4G_N} > 0 \quad (79)$$

Similarly considering two overlapping regions with $A \cap B \neq 0$, it is easy to see that the strong subadditivity condition would be saturated

$$S(A) + S(B) = S(A \cup B) + S(A \cap B). \quad (80)$$

See figure 2.

⁸ We are grateful to Shiraz Minwalla for a discussion on this point.

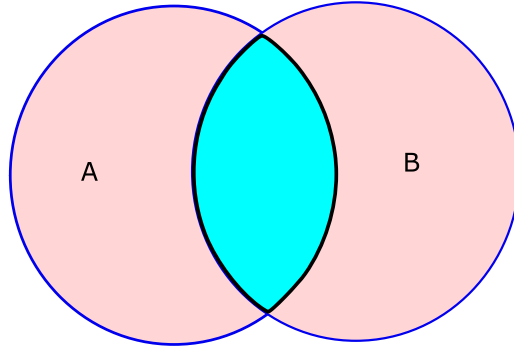


Figure 2. Strong subadditivity: consider regions A and B . The set $A \cup B$ is depicted in pink, whereas $A \cap B$ is depicted in blue. Denoting the area of the boundary of regions A, B etc as $a(A), a(B)$ etc, it is straightforward to see that these areas satisfy the equality $a(A) + a(B) = a(A \cup B) + a(A \cap B)$.

As discussed in the introduction, for field theories, where the degrees of freedom live in both spatially extended regions and time, one can consider a more general notion of entanglement which arises when we consider observables which only access both a spatially localized region and a restricted region in target space. It would be worth exploring this more general notion in the context of AdS/CFT further. Without target space restrictions the bulk dual of the boundary entanglement entropy is the Ryu–Takayanagi surface. With only target space restrictions and no any restrictions along the spatial directions, we have proposed here, for some cases, that the target space entanglement maps to bulk entanglement of an appropriate bulk region. The more general notion combining both spatial and target space restrictions would then interpolate between these two and it will be interesting to understand its bulk dual in more detail. While the Ryu–Takayanagi surface is extremal, our preliminary considerations here suggest that more generally when target space constraints are also included, the bulk surface is not extremal and in fact could be of a quite general type.

We find it very interesting that even for the restricted kind of spatial regions considered here, a precise map of bulk entanglement exists in the boundary theory. Since the notion of bulk locality is not precise in a theory of gravity, this was not *a priori* clear. The boundary theory of course exists for all value of the coupling and all values of N (in terms of the quantities appearing in equation (28) all values of T_0, d_0, N). Thus one could consider how the target space entanglement changes as one goes to weak coupling, and smaller values of N . The α' and string loop corrections correspondingly become important in the bulk, and bulk locality would become a more imprecise notion, but the target space entanglement would continue to be well defined. One could also try to check this by computing α' corrections in the bulk.

It is clearly important to find additional, and more doable, tests for our conjecture, equations (1), (2). One possibility might be to try and investigate this in a semi-classical path integral which attempts to implement the replica trick in the bulk, about a smooth spacetime background⁹.

We end by noting that if, as our preliminary investigation here suggests, the notions of target space entanglement along with its generalization mentioned above which combines spatial and

⁹ We are grateful to Shiraz Minwalla for this suggestion.

target space constraints, can provide a precise notion of bulk entanglement, they would clearly be important for studies related to information loss and more generally black hole physics.

7. A personal note from SRD

I came to know Peter Freund closely during my years as a graduate student at University of Chicago, and we remained in touch ever since. His original style of doing physics has been a major influence in my life, and his enthusiasm has been contagious. I am honoured to contribute this work to his memorial volume.

Acknowledgments

We thank Shaun Hampton, Antal Jevicki, Sinong Liu, Shiraz Minwalla, Suvrat Raju and Ashoke Sen for discussions. SRD would like to thank Tata Institute of Fundamental Research for hospitality during numerous extended visits over the years which led to this work. The work of SRD is partially supported by National Science Foundation Grant NSF/PHY-1818878. AK, GM and SPT acknowledge the support of the Govt. of India, Department of Atomic Energy, under Project No. 12-R & D-TFR-5.02-0200 and support from the Quantum Space-Time Endowment of the Infosys Science Foundation. SPT acknowledges support from a J C Bose Fellowship, Department of Science and Technology, Govt. of India.

Appendix A. Target space entanglement entropy

In this appendix we first present the formalism of target space entanglement entropy in the context of non-relativistic quantum mechanics of N fermions. Then we go on to prove the equivalence of RDM constructed by particle number sector (and the consequent EE) in the first quantized formulation with the standard second quantized theory.

We follow [15, 16] to define the EE in the target space. The key idea here is the algebraic definition of EE which relies on the usage of a theorem (Artin–Wedderburn) which says that given any algebra, there always exists a decomposition of the Hilbert space with the structure of direct sum over tensor products. Once restricted to a particular sector, one can use the usual notions of RDM due to the tensor product structure. However here we do not distinguish between ‘classical’ and ‘quantum’ contribution to the EE.

We introduce the notion of target space EE for a system of N fermions moving on the real line R ; from the first quantized viewpoint, R is the ‘target space’. We would like to define the EE of a target space subregion region $A \subset R$, e.g. A could be the region $x > d$ for some real number d . Given such a region and its complement \bar{A} , the one-particle Hilbert space, \mathcal{H}_1 has the structure of a direct *sum*, rather than product, of the form

$$\mathcal{H}_1 = \mathcal{H}_A + \mathcal{H}_{\bar{A}}, \quad \mathcal{H}_A = \text{span}\{|x_1\rangle, x_1 \in A\}, \quad \mathcal{H}_{\bar{A}} = \text{span}\{|x_1\rangle, x_1 \in \bar{A}\} \quad (81)$$

To study the target space EE, we find it convenient to begin with a discussion of the two-fermion Hilbert space \mathcal{H}_2 (we will come back to the one-particle case later on). The most general two-fermion wavefunction is of the form¹⁰

$$|\psi\rangle = \int dx_1 \int dx_2 \psi(x_1, x_2) |x_1, x_2\rangle_a = \int dx_1 \int dx_2 \psi_a(x_1, x_2) |x_1, x_2\rangle,$$

¹⁰ Unspecified range of integration would mean full range. E.g. $\int dx_1 = \int_{\mathbf{R}} dx_1$.

$$|x_1, x_2\rangle_a \equiv \frac{1}{\sqrt{2!}} (|x_1\rangle \otimes |x_2\rangle - |x_2\rangle \otimes |x_1\rangle), \quad \psi_a(x_1, x_2) \equiv \frac{1}{\sqrt{2!}} (\psi(x_1, x_2) - \psi(x_2, x_1)) \quad (82)$$

The two-particle Hilbert space splits naturally into three sectors, as follows:

$$\mathcal{H}_2 = \mathcal{H}_{2,0} + \mathcal{H}_{1,1} + \mathcal{H}_{0,2} \quad (83)$$

where

$$\begin{aligned} \mathcal{H}_{2,0} &= \text{span}\{|x_1, x_2\rangle_a, \quad x_1, x_2 \in A\} \\ \mathcal{H}_{1,1} &= \text{span}\{|x_1, x_2\rangle_a, \quad x_1 \in A, x_2 \in \bar{A}\} \\ \mathcal{H}_{0,2} &= \text{span}\{|x_1, x_2\rangle_a; \quad x_1, x_2 \in \bar{A}\} \end{aligned}$$

In terms of the wavefunction (82), restricting ranges of the integrals over x_1, x_2 variously to the regions A, \bar{A} give the projection of the wavefunction to the various sectors: thus, e.g.

$$|\psi\rangle_{1,1} = \int_A dx_1 \int_{\bar{A}} dx_2 \psi(x_1, x_2) |x_1, x_2\rangle_a, \quad {}_a\langle x_1, x_2 | \psi \rangle = \psi_a(x_1, x_2) \quad (84)$$

The corresponding projection operators $\Pi_{(p,q)} : \mathcal{H}_2 \rightarrow \mathcal{H}_{(p,q)}$ are given by

$$\begin{aligned} \Pi_{2,0} &= \frac{1}{2} \int_{A,A} dx_1 dx_2 |x_1, x_2\rangle_a {}_a\langle x_1, x_2| \\ \Pi_{1,1} &= \int_{A,\bar{A}} dx_1 dx_2 |x_1, x_2\rangle_a {}_a\langle x_1, x_2| \\ &= \frac{1}{2} \left(\int_A dx_1 \int_{\bar{A}} dx_2 + \int_{\bar{A}} dx_1 \int_A dx_2 \right) |x_1, x_2\rangle_a {}_a\langle x_1, x_2| \\ \Pi_{0,2} &= \frac{1}{2} \int_{\bar{A},\bar{A}} dx_1 dx_2 |x_1, x_2\rangle_a {}_a\langle x_1, x_2| \end{aligned} \quad (85)$$

It is easy to see each of these projection operators squares to itself and they add up to identity in \mathcal{H}_2 .

The generalization of these concepts to an N -fermion Hilbert space is straightforward:

$$\mathcal{H}_N = \oplus_{p,q;p+q=N} \mathcal{H}_{p,q} \quad (86)$$

Here the notation $\mathcal{H}_{p,q}$ denotes a sector in which there are p particles in the region A and q particles in the complementary region \bar{A} . We will denote by $\Pi_{p,q}$ ($p+q=N$) the projection operators $\mathcal{H}_N \rightarrow \mathcal{H}_{p,q}$.

It is straightforward to generalize the above discussion to N fermions in \mathbf{R}^D and the target space region A is defined by a plane, say $A : \{x^1 > d, x^2, \dots, x^D \in \mathbf{R}\}$. $\bar{A} = \mathbf{R}^D - A$. E.g. if we denote the coordinates of the N particles as $\mathbf{x}_i = x_i^I$, $i = 1, 2, \dots, N$, $I = 1, 2, \dots, D$, then the wavefunctions belonging to $\mathcal{H}_{p,q}$ are given by

$$|\psi_{p,q}\rangle = \int_A \prod_{I=1}^D \prod_{i=1}^p d^D \mathbf{x}_i \int_{\bar{A}} \prod_{I=1}^D \prod_{i=p+1}^N d^D \mathbf{x}_i \psi(\{\mathbf{x}_i\}) |\{\mathbf{x}_i\}\rangle_a \quad (87)$$

where the subscript a denotes antisymmetrization as before. The decomposition (86) is again true and the following discussion generalizes in a straightforward fashion with various one-dimensional integrals replaced by the corresponding d -dimensional integrals.

Reduced density matrix (RDM).

We are interested in defining an RDM $\tilde{\rho}$, associated with the region A , in a state ρ in the full Hilbert space (which could be pure or mixed). The RDM should have the property that for observables O which can be measured by detectors in A , we should have, in an appropriate sense,

$$\text{Tr}(\rho O) = \text{Tr}_A(\tilde{\rho} O) \quad (88)$$

In the following we will define each side carefully.

In a QFT, when one is interested in a spatial subregion A of space time (as against target space), one proceeds by noting that the full Hilbert space is a tensor product of the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, which leads to $\tilde{\rho} = \text{Tr}_{\mathcal{H}_{\bar{A}}} \rho$, with Tr_A interpreted as $\text{Tr}_{\mathcal{H}_A}$.

However, there is no such tensor product decomposition for target space subregions. As we saw above, the single-particle Hilbert space \mathcal{H}_1 is a direct *sum*, rather than a product, of subspaces associated with A and \bar{A} . A similar statement is true also for an N -particle Hilbert space. What allows us to proceed is that each given *sector* $\mathcal{H}_{(p,q)}$ in an N -particle Hilbert space, separately, has an (antisymmetric) tensor product of factors associated with A and \bar{A} respectively.

Let us explain the case of the $\mathcal{H}_{1,1} \subset \mathcal{H}_2$ as an illustration. It is easy to see that

$$\mathcal{H}_{1,1} = \mathcal{H}_A \wedge \mathcal{H}_{\bar{A}}; \quad (89)$$

where the antisymmetric tensor product $V \wedge W$ denotes $V \otimes W - W \otimes V$.

Operator algebra.

The operators that map $\mathcal{H}_{1,1} \rightarrow \mathcal{H}_{1,1}$ are of the form

$$\text{Span}\{|x, y\rangle_a \langle x', y'|, \ x, x' \in A, \ y, y' \in \bar{A}\} \quad (90)$$

Among these, operators O which correspond to observables in region A must have the property

$$O|x, y\rangle_a = \int_A dx' \tilde{O}(x, x')|x', y\rangle_a, \quad (91)$$

which do not have any effect on $|y\rangle, y \in \bar{A}$. In fact, the corresponding operator algebra can be obtained by setting $y = y'$ in (90) and integrating over the y coordinate. This gives¹¹

$$\begin{aligned} \mathcal{A}_{1,1} &= \text{Span}\left\{\int_{\bar{A}} dy |x, y\rangle_a \langle x', y|, \ x, x' \in A\right\} \\ &= \text{Span}\{|x\rangle\langle x'| \otimes \mathbf{1}_{\bar{A}} + \mathbf{1}_{\bar{A}} \otimes |x\rangle\langle x'|, \ x, x' \in A\} \\ \mathbf{1}_{\bar{A}} &\equiv \int_{\bar{A}} dy |y\rangle\langle y| \end{aligned} \quad (92)$$

¹¹ The second equality below can be derived as follows. Take a matrix element of the operator inside the ‘span’ in the first line and show that it is that in the second line.

It is easy to show that this operator algebra is closed under multiplication. The operator O , with the action defined in (91) can be identified as an element of (92), with the form:

$$O = \tilde{O}_{1,1} \otimes \mathbf{1}_{\bar{A}} + \mathbf{1}_{\bar{A}} \otimes \tilde{O}_{1,1}, \quad \tilde{O}_{1,1} \equiv \int_{A,A} dx dx' \tilde{O}(x, x') |x\rangle \langle x'|, \quad (93)$$

It is easy to check that this operator satisfies the defining property (91) (note that $\mathbf{1}_{\bar{A}}|x\rangle = 0$ for $x \in A$).

Density matrices.

A general state in the two-fermion Hilbert space is described by a density matrix

$$\rho = \int dx_1 dx_2 \int dx'_1 dx'_2 \rho(x_1, x_2; x'_1, x'_2) |x_1, x_2\rangle_a \langle x'_1, x'_2|$$

The projection of ρ onto the Hilbert space $\mathcal{H}_{1,1}$ is given by

$$\rho_{1,1} = \Pi_{1,1} \rho \Pi_{1,1} = \int_{A,\bar{A}} dx dy \int_{A,\bar{A}} dx' dy' \int_{\bar{A}} dx_2 \rho(x, y; x', y') |x, y\rangle_a \langle x', y'| \quad (94)$$

Although $\mathcal{H}_{(1,1)}$ is not an usual tensor product but an antisymmetrized one (89), one can define a partial trace with respect to $\mathcal{H}_{\bar{A}}$ irrespective of the order of factors: thus

$$\begin{aligned} \tilde{\rho}_{1,1} &= \text{Tr}_{\mathcal{H}_{\bar{A}}} \Pi_{1,1} \rho \Pi_{1,1} \\ &= \int_{\bar{A}} dy_1 \langle y_1 | \left[\int_{A,\bar{A}} dx dy \int_{A,\bar{A}} dx' dy' \int_{\bar{A}} dx_2 \rho(x, y; x', y') |x, y\rangle_a \langle x', y'| \right] |y_1\rangle \\ &= \int_{A,A} dx dx' \int_{\bar{A}} dy \rho(x, y; x', y) |x\rangle \langle x'| \end{aligned} \quad (95)$$

then we get

$$\text{Tr}_{\mathcal{H}_2} (\rho O) = \text{Tr}_{\mathcal{H}_A} (\tilde{\rho}_{1,1} \tilde{O}_{1,1})$$

which is of the form (88), except for the important difference, characteristic of target space EE, that both operators on the LHS are two-particle operators whereas those on the RHS are one-particle operators defined on the one-particle Hilbert space \mathcal{H}_A associated with the factor in (89) associated with region A; the traces on the two sides also pertain to these two-particle and one-particle Hilbert spaces respectively.

In general, as mentioned above, an N -fermion Hilbert space \mathcal{H}_N splits into sectors $\mathcal{H}_{p,q}$, $p + q = N$ (see (86)). In each $\mathcal{H}_{p,q}$, there exists a tensor product decomposition into two Hilbert spaces associated, respectively, with the regions A and \bar{A} :

$$\mathcal{H}_{p,q} = \mathcal{H}_A^p \wedge \mathcal{H}_{\bar{A}}^q, \quad \mathcal{H}_A^p \equiv (\wedge^p \mathcal{H}_A), \quad \mathcal{H}_{\bar{A}}^q \equiv (\wedge^q \mathcal{H}_{\bar{A}}) \quad (96)$$

where $\wedge^p V \equiv V \wedge V \wedge \dots \wedge V$ (p times). For $V = \mathcal{H}_A$ ($\mathcal{H}_{\bar{A}}$), respectively, these represent p fermions in region A (\bar{A}). By definition, $\wedge^0 \mathcal{H}_A = |0\rangle_A = \mathcal{C}$ (zero particles in A) and $\wedge^1 \mathcal{H}_A = \mathcal{H}_A$ (similarly for region \bar{A}). Generalizing (95), it is easy to show that the RDM's in each sector are given by

$$\tilde{\rho}_{p,q} = \text{Tr}_{\mathcal{H}_{\bar{A}}^q} \rho_{p,q}, \quad \rho_{p,q} = \Pi_{p,q} \rho \Pi_{p,q} \quad (97)$$

The target space EE is naturally given by the combined von Neumann entropy of RDM's from all sectors:

$$S = - \sum_{p,q:p+q=N} \text{Tr}_{\mathcal{H}_A^p} \tilde{\rho}_{p,q} \log(\tilde{\rho}_{p,q}) \quad (98)$$

This can be equivalently defined as

$$S = - \text{Tr} \tilde{\rho} \log(\tilde{\rho})$$

where $\tilde{\rho}$ is a formal sum of the sectorwise RDM's

$$\tilde{\rho} = \oplus_{p+q=N} \tilde{\rho}_{p,q} \quad (99)$$

which acts on the sum of the vector spaces \mathcal{H}_A^p (in the notation of (96)).

Explicit calculation ($N = 2$).

Let us work the EE in detail for the two-particle case ($N = 2$). In this case, the various sectors have the tensor decomposition

$$\mathcal{H}_{2,0} = (\wedge^2 \mathcal{H}_A) \otimes \mathcal{C}, \quad \mathcal{H}_{1,1} = \mathcal{H}_A \wedge \mathcal{H}_{\bar{A}}, \quad \mathcal{H}_{2,0} = \mathcal{C} \otimes (\wedge^2 \mathcal{H}_{\bar{A}}) \quad (100)$$

Note that a general 2 particle state $|\psi\rangle$ can be written in multiple ways

$$\begin{aligned} |\psi\rangle &= \int dx_1 \int dx_2 \psi(x_1, x_2) |x_1, x_2\rangle_a \\ &= \int dx_1 \int dx_2 \psi_a(x_1, x_2) |x_1, x_2\rangle \\ &= \frac{1}{\sqrt{2}} \int dx_1 \int dx_2 \psi_a(x_1, x_2) |x_1, x_2\rangle_a \end{aligned} \quad (101)$$

In the first line the kets are anti-symmetric, in the second line the wavefunction ψ is anti-symmetric, while in the last line both the kets and the wavefunction ψ are anti-symmetric. In the following we will use the last representation most often (as the symmetric part of the wavefunction ψ never contributes if the kets are antisymmetrized). The density matrix ρ is given by

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| = \frac{1}{2} \int dx_1 dx_2 \int dx'_1 dx'_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x'_2) |x_1, x_2\rangle_a \langle x'_1, x'_2| \\ &= \frac{1}{2} \int dx_1 dx_2 \int dx'_1 dx'_2 \rho_a(x_1, x_2; x'_1, x'_2) |x_1, x_2\rangle_a \langle x'_1, x'_2| \end{aligned} \quad (102)$$

To proceed, we follow (97) and (100). In the (2, 0) sector the partial trace over \bar{A} is trivial and we need to take care of only the projections, which just restrict the range of the integrals. Thus we get

$$\begin{aligned} \tilde{\rho}_{2,0} &= \text{Tr}_{\bar{A}}(\rho_{2,0}) = \rho_{2,0} \\ &= \frac{1}{2} \int_A dx_1 dx_2 \int_A dx'_1 dx'_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x'_2) |x_1, x_2\rangle_a \langle x'_1, x'_2| \end{aligned} \quad (103)$$

In $(1, 1)$ sector, the projection operator (that restricts to $\mathcal{H}_{(1,1)}$) is given by $\Pi_{1,1} = \int_A dx_1 \int_{\bar{A}} dx_2 |x_1, x_2\rangle_a \langle x_1, x_2|$. First note its action on the ket $|\psi\rangle$

$$\begin{aligned}\Pi_{1,1}|\psi\rangle &= \frac{1}{\sqrt{2}} \int_A dx_1 \int_{\bar{A}} dx_2 |x_1, x_2\rangle_a \langle x_1, x_2| \int dy_1 dy_2 \psi_a(y_1, y_2) |y_1 y_2\rangle_a \\ &= \frac{1}{\sqrt{2}} \int_A dx_1 \int_{\bar{A}} dx_2 |x_1, x_2\rangle_a \int dy_1 dy_2 \psi_a(y_1, y_2) \\ &\quad \times (\delta(x_1 - y_1)\delta(x_2 - y_2) - \delta(x_1 - y_2)\delta(x_2 - y_1)) \\ &= \sqrt{2} \int_A dx_1 \int_{\bar{A}} dx_2 \psi_a(x_1, x_2) |x_1, x_2\rangle_a \\ &= \frac{1}{\sqrt{2}} \left(\int_A dx_1 \int_{\bar{A}} dx_2 + \int_{\bar{A}} dx_1 \int_A dx_2 \right) \psi_a(x_1, x_2) |x_1, x_2\rangle_a\end{aligned}$$

where we have used $\langle x_1, x_2 | y_1 y_2 \rangle_a = \delta(x_1 - y_1)\delta(x_2 - y_2) - \delta(x_1 - y_2)\delta(x_2 - y_1)$. Therefore

$$\rho_{1,1} = 2 \int_A dx_1 dx'_1 \int_{\bar{A}} dx_2 dx'_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x'_2) |x_1, x_2\rangle_a \langle x'_1, x'_2|$$

Now tracing over $\mathcal{H}_{\bar{A}}$

$$\begin{aligned}\tilde{\rho}_{1,1} &= \text{Tr}_{\mathcal{H}_{\bar{A}}}(\rho_{1,1}) = 2 \int_A dx_1 dx'_1 \int_{\bar{A}} dx_2 dx'_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x'_2) \\ &\quad \times \int_{\bar{A}} dz \langle z | (|x_1, x_2\rangle_a \langle x'_1, x'_2|) | z \rangle \\ &= 2 \int_A dx_1 dx'_1 \int_{\bar{A}} dx_2 dx'_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x'_2) |x_1\rangle \langle x'_1| \langle x_2 | x'_2 \rangle \\ &= 2 \int_A dx_1 dx'_1 \int_{\bar{A}} dx_2 \psi_a(x_1, x_2) \psi_a^*(x'_1, x_2) |x_1\rangle \langle x'_1|\end{aligned}\quad (104)$$

In $(0, 2)$ sector, $\Pi_{0,2} = \frac{1}{2} \int_{\bar{A}} dx_1 \int_{\bar{A}} dx_2 |x_1, x_2\rangle_a \langle x_1, x_2|$. After doing the appropriate partial trace over $H(2, \bar{A})$ (see (97)), just gives a number

$$\tilde{\rho}_{0,2} = \text{Tr}_{\bar{A}}(\rho_{0,2}) = \frac{1}{2} \int_{\bar{A}} dx_1 dx_2 \psi_a(x_1, x_2) \psi_a^*(x_1, x_2) \quad (105)$$

As a specific example consider the Slater determinant state given by

$$|\psi\rangle = \frac{1}{\sqrt{2!}} (|u_1, u_2\rangle - |u_2, u_1\rangle) = \frac{1}{\sqrt{2!}} \sum_{i_1, i_2=1}^2 \varepsilon_{i_1 i_2} |u_{i_1} u_{i_2}\rangle \quad (106)$$

where $|u_i\rangle = \int dx u_i(x) |x\rangle$ are single particle wavefunctions. The wavefunction $\psi_a(x_1, x_2) = (u_1(x_1)u_2(x_2) - u_2(x_1)u_1(x_2))/\sqrt{2}$. The corresponding density matrix is

$$\begin{aligned}\rho &= |\psi\rangle \langle \psi| = \frac{1}{2!} (|u_1, u_2\rangle - |u_2, u_1\rangle) (\langle u_1, u_2| - \langle u_2, u_1|) \\ &= \frac{1}{2!} \sum_{i_1, i_2, j_1, j_2=1}^2 \varepsilon_{i_1 i_2} \varepsilon_{j_1 j_2} |u_{i_1} u_{i_2}\rangle \langle u_{j_1} u_{j_2}|\end{aligned}\quad (107)$$

For this particular state the equations (103)–(105) become

$$\tilde{\rho}_{2,0} = \frac{1}{2!} (|u_1, u_2\rangle_{AA} - |u_2, u_1\rangle_{AA}) ({}_{AA}\langle u_1, u_2| - {}_{AA}\langle u_2, u_1|) \quad (108)$$

$$\tilde{\rho}_{1,1} = \sum_{i's, j's=1}^2 \varepsilon_{i_1 i_2} \varepsilon_{j_1 j_2} |u_{i_1}\rangle_A {}_A\langle u_{j_1}| {}_{\bar{A}}\langle u_{j_2}| u_{i_2}\rangle_{\bar{A}} = \begin{bmatrix} {}_A\langle u_1| & {}_A\langle u_2| \\ (1-p_2) & -q_{21\bar{A}} \\ -q_{12\bar{A}} & (1-p_1) \end{bmatrix} \begin{bmatrix} |u_1\rangle_A \\ |u_2\rangle_A \end{bmatrix} \quad (109)$$

$$\tilde{\rho}_{0,2} = \frac{1}{2!} \sum_{i_1, i_2, j_1, j_2=1}^2 \varepsilon_{i_1 i_2} \varepsilon_{j_1 j_2} \langle u_{j_1} u_{j_2} | u_{i_1} u_{i_2} \rangle_{\bar{A}} = (1-p_1)(1-p_2) - |q_{12\bar{A}}|^2 \quad (110)$$

where $|u, v\rangle_{AA} = |u\rangle_A |v\rangle_A$, $|u\rangle_A = P_A |u\rangle \equiv \int_A dx u(x) |x\rangle$, and $\langle u|v\rangle_{\bar{A}} \equiv \int_{\bar{A}} dx u^*(x) v(x)$. Further we have written $\tilde{\rho}_{1,1}$ and $\tilde{\rho}_{0,2}$ in terms of $p_1 = \int_A dx |u_1(x)|^2$, $p_2 = \int_A dx |u_2(x)|^2$ and $q_{12\bar{A}} = \int_{\bar{A}} dx u_1^*(x) u_2(x)$.

Finally, following the general prescription (98), the EE in target space is given by

$$S = -\text{Tr}(\tilde{\rho} \log \tilde{\rho}) = -[\text{Tr}(\tilde{\rho}_{2,0} \log \tilde{\rho}_{2,0}) + \text{Tr}(\tilde{\rho}_{1,1} \log \tilde{\rho}_{1,1}) + \text{Tr}(\tilde{\rho}_{0,2} \log \tilde{\rho}_{0,2})]$$

Target space EE for general N .

Now we generalize this to a general N fermion state

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \int dx_1 \dots x_N \psi_a(x_1, \dots, x_N) |x_1 \dots x_N\rangle_a \quad (111)$$

where we define

$$\psi_a(x_1, \dots, x_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma \psi(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \quad (112)$$

$$|x_1 \dots x_N\rangle_a \equiv \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma |x_{\sigma(1)} \dots x_{\sigma(N)}\rangle \quad (113)$$

The corresponding density matrix is

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= \frac{1}{N!} \int dx_1 dx'_1 \dots x_N x'_N \psi_a(x_1, \dots, x_N) \psi_a^*(x'_1, \dots, x'_N) |x_1 \dots x_N\rangle_a {}_a\langle x'_1 \dots x'_N| \\ &= \frac{1}{N!} \int dx_1 dx'_1 \dots x_N x'_N \rho_a(x_1, \dots, x_N; x'_1, \dots, x'_N) |x_1 \dots x_N\rangle_a {}_a\langle x'_1 \dots x'_N| \end{aligned} \quad (114)$$

For N particle state we have $N+1$ sectors namely $(N, 0)$, $(N-1, 1)$, \dots , $(0, N)$ where the first entry is the number of particles in A and second entry in \bar{A} . In the $(k, N-k)$ sector, the projection operator is given by

$$\Pi_{k, N-k} = \frac{1}{N!} \binom{N}{k} \int_A dx_1 \dots dx_k \int_{\bar{A}} dx_{k+1} \dots dx_N |x_1, \dots, x_N\rangle_a {}_a\langle x_1, \dots, x_N| \quad (115)$$

First note its action on $|\psi\rangle$

$$\begin{aligned}\Pi_{k,N-k}|\psi\rangle &= \frac{1}{N!} \binom{N}{k} \int_A dx_1 \dots dx_k \int_{\bar{A}} dx_{k+1} \dots dx_N |x_1, \dots, x_N\rangle_a \\ &\quad \times \frac{1}{\sqrt{N!}} \int dy_1 \dots dy_N \psi_a(y_1, \dots, y_N) {}_a\langle x_1, \dots, x_N | y_1, \dots, y_N \rangle_a \\ &= \frac{1}{\sqrt{N!}} \binom{N}{k} \int_A dx_1 \dots dx_k \int_{\bar{A}} dx_{k+1} \dots dx_N \psi_a(x_1, \dots, x_N) |x_1, \dots, x_N\rangle_a\end{aligned}\quad (116)$$

To go to the last line we have used

$${}_a\langle x_1, \dots, x_N | y_1, \dots, y_N \rangle_a = \sum_{\sigma \in S_N} (-1)^\sigma \delta(x_1 - y_{\sigma(1)}) \dots \delta(x_N - y_{\sigma(N)}) \quad (117)$$

The density matrix restricted to this sector is

$$\begin{aligned}\rho_{k,N-k} &= \Pi_{k,N-k} \rho \Pi_{k,N-k} = \int_A dx_1 dx'_1 \dots dx_k dx'_k \int_{\bar{A}} dx_{k+1} dx'_{k+1} \dots dx_N dx'_N \\ &\quad \times \frac{1}{N!} \binom{N}{k}^2 \psi_a(x_1, \dots, x_N) \psi_a(x'_1, \dots, x'_N)^* |x_1, \dots, x_N\rangle_a {}_a\langle x'_1, \dots, x'_N|\end{aligned}$$

Next we need to trace over the particles in \bar{A} . This is easily done

$$\begin{aligned}\tilde{\rho}_{k,N-k} &= \text{Tr}_{\bar{A}}(\rho_{k,N-k}) = \frac{1}{(N-k)!} \int_{\bar{A}} dz_{k+1} \dots dz_N {}_a\langle z_{k+1} \dots z_N | \rho_{k,N-k} | z_{k+1} \dots z_N \rangle_a \\ &= \frac{1}{N!} \binom{N}{k}^2 \int_A dx_1 dx'_1 \dots dx_k dx'_k \int_{\bar{A}} dx_{k+1} dx'_{k+1} \dots dx_N dx'_N \psi_a(x_1, \dots, x_N) \\ &\quad \times \psi_a^*(x'_1, \dots, x'_N) |x_1, \dots, x_k\rangle_a {}_a\langle x'_1, \dots, x'_k| {}_a\langle x_{k+1}, \dots, x_N | y_{k+1} \dots, y_N \rangle_a \\ &= \frac{1}{k!} \binom{N}{k} \int_A dx_1 dy_1 \dots dx_k dy_k \int_{\bar{A}} dx_{k+1} \dots dx_N \psi_a(x_1, \dots, x_N) \psi_a^*(x'_1, \dots, x'_N) \\ &\quad \times |x_1, \dots, x_k\rangle_a {}_a\langle x'_1, \dots, x'_k|\end{aligned}\quad (118)$$

where we have used

$$\begin{aligned}\frac{1}{(N-k)!} \int_{\bar{A}} dz_{k+1} \dots dz_N {}_a\langle z_{k+1} \dots z_N | x_{k+1}, \dots, x_N \rangle_a {}_a\langle x'_{k+1}, \dots, x'_N | z_{k+1} \dots z_N \rangle_a \\ = |x_1 \dots x_k\rangle_a {}_a\langle x_1 \dots x_k| {}_a\langle y_{k+1} \dots y_N | x_{k+1}, \dots, x_N \rangle_a\end{aligned}\quad (119)$$

More specifically consider the state given by a Slater determinant (of single-particle states u_1, u_2, \dots, u_N)

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \sum_{i'_s} \varepsilon_{i_1 \dots i_N} |u_{i_1} \dots u_{i_N}\rangle \quad (120)$$

with the wavefunction $\psi_a(x_1, \dots, x_N) = \sum_{i'_s} \frac{1}{\sqrt{N!}} \varepsilon_{i_1 \dots i_N} u_{i_1}(x_1) \dots u_{i_N}(x_N)$. Each of the i 's can take values from 1 to N , i.e. $i_n \in \{1, 2, \dots, N\}$. The RDM in the $(k, N-k)$ sector (118) is given by

$$\tilde{\rho}_{k,N-k} = \binom{N}{k} \frac{1}{N!} \sum_{i'_s, j'_s} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \prod_{n=k+1}^N \langle u_{j_n} | u_{i_n} \rangle_{\bar{A}} \quad (121)$$

This formula is very simple to understand. If we worked with the position space wavefunctions $\langle x_1 \dots x_N | \psi \rangle = \psi_a(x_1, \dots, x_N)$, the RDM is simply given by

$$\begin{aligned} \tilde{\rho}(x_1, \dots, x_k; x'_1, \dots, x'_k) &= \binom{N}{k} \int_{\bar{A}} dx_{k+1} \dots dx_N \psi_a(x_1, \dots, x_k, x_{k+1} \dots x_N) \\ &\times \psi_a^*(x'_1, \dots, x'_k, x_{k+1} \dots x_N) \end{aligned} \quad (122)$$

with the factor $\binom{N}{k}$ coming from the number of ways choosing the integration variables. This is the origin of $\binom{N}{k}$ in (118) and (121) while the remaining numerical factor is just for normalization.

The EE is by the general formula given above (98):

$$S = -\text{Tr } \tilde{\rho} \log \tilde{\rho} = -\sum_k \text{Tr } \tilde{\rho}_{k,N-k} \log \tilde{\rho}_{k,N-k} \quad (123)$$

Equivalence of 1st and 2nd quantized entanglement entropy for free theories.

A.1. 2nd quantized theory

The target space subregion $A \subset R$ in the first quantized formalism, can be viewed as a spatial subregion from the viewpoint of the second quantized formalism where the single particle states $|x\rangle$ can be regarded as created from the zero particle state $|0\rangle$ by the second quantized creation operator:

$$|x\rangle = \Psi^\dagger(x)|0\rangle$$

The general Fock space state can be regarded as a linear combination of the antisymmetric states

$$\mathcal{F} \ni |x_1, x_2, \dots, x_N\rangle_a = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S(N)} |x_{\sigma(1)}, \dots, x_{\sigma(N)}\rangle = \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle$$

It is easy to see the tensor product decomposition

$$\mathcal{F} = \mathcal{F}_A \wedge \mathcal{F}_{\bar{A}} \quad (124)$$

which allows one to define RDM's in terms of the usual partial traces.

Note that since each Fock space is a sum of 0, 1, 2, ... particle Hilbert spaces, we can write, using the notations in (96):

$$\mathcal{F}_A = |0\rangle_A \oplus \mathcal{H}_A \oplus \mathcal{H}_A^2 \oplus \dots, \quad \mathcal{F}_{\bar{A}} = |0\rangle_{\bar{A}} \oplus \mathcal{H}_{\bar{A}} \oplus \mathcal{H}_{\bar{A}}^2 \oplus \dots$$

The tensor product (124) thus gets written as a direct sum

$$\begin{aligned} \mathcal{F} &= (|0\rangle) \oplus (\mathcal{H}_A \oplus \mathcal{H}_{\bar{A}}) \oplus (\mathcal{H}_A^2 \oplus (\mathcal{H}_A \wedge \mathcal{H}_{\bar{A}}) \oplus \mathcal{H}_{\bar{A}}^2) \oplus \dots \\ &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \end{aligned} \quad (125)$$

Here $|0\rangle_A \equiv \mathcal{H}_A^0 \equiv C$ is the zero-particle state in A , defined by $\Psi(x)|0\rangle_A = 0$ for all $x \in A$ (similarly for \bar{A}); we have, further used the identities:

$$|0\rangle_A \otimes |0\rangle_{\bar{A}} = |0\rangle, \quad |0\rangle_A \otimes \mathcal{H}_A^p = \mathcal{H}_A^p, \quad \mathcal{H}_A^p \otimes |0\rangle_{\bar{A}} = \mathcal{H}_A^p,$$

Note that a wedge product with zero-particle states such as $|0\rangle_A$ becomes an ordinary tensor product (it amounts to just scalar multiplication by a complex number, see below (96)).

Written in the form (125), we can clearly identify the terms in round brackets as the first quantized Hilbert spaces \mathcal{H}_n with a clear sum of products structure introduced in (86), (96). We will find below that the RDM in the second quantized framework, sector by sector, is the same as that in the first quantized framework.

Computation of RDM.

EE in field theories is well-studied in the literature and we follow the method of [30]. Using the decomposition (124), the RDM ρ_A of a region A , is defined by

$$\rho_A = \text{Tr}_{\mathcal{F}_A} \rho$$

where ρ is the density matrix corresponding to the state of the full system. This is, of course, an operator in \mathcal{F}_A ; however, as shown in [30], for theories with quadratic modular Hamiltonian as in the case for free fermions, ρ_A can be expressed in terms of the exponential of a one-body (particle-number preserving) operator \hat{H}_A , the so-called modular Hamiltonian:

$$\rho_A = K e^{-\hat{H}_A}. \quad (126)$$

where K is a constant ensuring $\text{Tr}_{\mathcal{F}_A} \rho_A = 1$. The modular Hamiltonian, projected onto the one-particle Hilbert space \mathcal{H}_A , (let us call it $\hat{H}_A^{(1)}$) can be expressed in an orthonormal basis of \mathcal{H}_A :

$$\hat{H}_A^{(1)} = \sum_l \epsilon_l |l\rangle \langle l|$$

By definition, $\langle l|l'\rangle = \delta_{ll'}$ and $v_l(x) \equiv \langle x|l\rangle$ has support only in $x \in A$. Defining creation and annihilation operators d_l, d_l^\dagger such that $|l\rangle = d_l^\dagger |0\rangle$, clearly the Fock space operator will be given by $\hat{H}_A = \sum_l \epsilon_l d_l^\dagger d_l$. Using this and (126), we get

$$\rho_A = K \exp \left[-\sum_l \epsilon_l d_l^\dagger d_l \right] = \prod_l \frac{e^{-\epsilon_l d_l^\dagger d_l}}{(1 + e^{-\epsilon_l})}$$

Suppose we restrict to the N -particle sector of the full Fock space \mathcal{F} ; by (125) this sector will have contributions from \mathcal{H}_k , $k = 0, 1, \dots, N$. We find that to describe this situation it is enough to keep only the first N number of λ_i 's non-zero (the corresponding ϵ_i 's finite) while all other λ_i 's can be set equal to zero (the corresponding ϵ_i 's sent to infinity). Therefore one can write the N -particle density matrix as

$$\rho_A^{(N)} = \prod_{l=1}^N \frac{e^{-\epsilon_l d_l^\dagger d_l}}{(1 + e^{-\epsilon_l})} \quad (127)$$

Two particles ($N = 2$).

First we explicitly work out the 2 particle case and then generalize to arbitrary N . The density matrix for $N = 2$ is

$$\rho_A^{(2)} = \frac{e^{-\epsilon_1 d_1^\dagger d_1}}{1 + e^{-\epsilon_1}} \frac{e^{-\epsilon_2 d_2^\dagger d_2}}{1 + e^{-\epsilon_2}} \quad (128)$$

Without loss of generality, consider the following two particle state in the full space

$$|s\rangle = b_2^\dagger b_1^\dagger |0\rangle$$

where b, b^\dagger 's are 'global' fermionic annihilation/creation operators satisfying the standard algebra $\{b_i, b_j^\dagger\} = \delta_{ij}$. The one-particle states $|i\rangle = b_i^\dagger |0\rangle$ are global states, i.e. $u_i(x) \equiv \langle x|i\rangle$ have support in $x \in \mathbf{R} = A \cup \bar{A}$.

The second quantized field $\Psi(x)$ has mode expansions of the form

$$\begin{aligned} \Psi(x) &= \sum_i u_i(x) b_i, \quad x \in \mathbf{R}, \quad u_i(x) = \langle x|b_i^\dagger|0\rangle, \quad \int_{\mathbf{R}} dx u_i^*(x) u_j(x) = \delta_{ij} \\ \Psi(x) &= \sum_l v_l(x) d_l, \quad x \in A, \quad v_l(x) = \langle x|d_l^\dagger|0\rangle, \quad \int_A dx v_l^*(x) v_m(x) = \delta_{lm} \end{aligned} \quad (129)$$

The corresponding formulae for $\Psi^\dagger(x)$ are given by taking Hermitian conjugation of the above equations. If $\rho_A^{(2)}$ is indeed the correct density matrix for the region of interest, the following equations should be true (as long as all the operator insertions are within region A)

$$\begin{aligned} \text{Tr} \left(\rho_A^{(2)} \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x'_1) \Psi(x'_2) \right) &= \langle s | \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x'_1) \Psi(x'_2) | s \rangle \\ \text{Tr} \left(\rho_A^{(2)} \Psi^\dagger(x_1) \Psi(x'_1) \right) &= \langle s | \Psi^\dagger(x_1) \Psi(x'_1) | s \rangle \\ \text{Tr} \left(\rho_A^{(2)} \right) &= \langle s | s \rangle \end{aligned} \quad (130)$$

Using the mode expansions (129), the equation (130) lead to (respectively)

$$\begin{aligned} \lambda_1 \lambda_2 \begin{vmatrix} v_1(x_1) & v_2(x_1) \\ v_1(x_2) & v_2(x_2) \end{vmatrix}^* \begin{vmatrix} v_1(x'_1) & v_2(x'_1) \\ v_1(x'_2) & v_2(x'_2) \end{vmatrix} &= \begin{vmatrix} u_1(x_1) & u_2(x_1) \\ u_1(x_2) & u_2(x_2) \end{vmatrix}^* \begin{vmatrix} u_1(x'_1) & u_2(x'_1) \\ u_1(x'_2) & u_2(x'_2) \end{vmatrix} \\ \lambda_1 v_1(x_1)^* v_1(x'_1) + \lambda_2 v_2(x_1)^* v_2(x'_1) &= u_1(x_1)^* u_1(x'_1) + u_2(x_1)^* u_2(x'_1) \\ \text{Tr} \left(\rho_A^{(2)} \right) &= 1 \end{aligned} \quad (131)$$

where $\lambda_i = e^{-\epsilon_i} / (1 + e^{-\epsilon_i})$. The last equation above just says that our density matrix should be properly normalized. The remaining two can be written more compactly as operator equations and through the use of generalized Kronecker delta functions as

$$\begin{aligned} \sum_{i', j', s=1}^2 \lambda_{i'} \lambda_{j'} \delta_{i' i_2}^{j_1 j_2} |v_{i'}\rangle |v_{j'}\rangle \langle v_{i_1}| \langle v_{j_1}| &= \sum_{i', j', s=1}^2 \delta_{i' i_2}^{j_1 j_2} |u_{i'}\rangle_A |u_{j'}\rangle_A \langle u_{j_1}|_A \langle u_{i_1}|_A \\ \sum_{i_1, j_1=1}^2 \lambda_{i_1} \delta_{i_1}^{j_1} |v_{i_1}\rangle \langle v_{j_1}| &= \sum_{i_1, j_1=1}^2 \delta_{i_1}^{j_1} |u_{i_1}\rangle_A \langle u_{j_1}|_A \end{aligned} \quad (132)$$

where $|u_i\rangle_A = \int_A dx u_i(x) |x\rangle$. This follows since the relation (131) is true for all $x_i, x'_j \in A$, we can multiply by position kets and integrate over region A. The generalized Kronecker delta function $\delta_{i_1 \dots i_n}^{j_1 \dots j_n}$ is defined to be $+1$ (-1) when $i_1 \dots i_n$'s are distinct and even (odd) permutation of $j_1 \dots j_n$'s, otherwise it is 0.

From the structure of (128), it is clear that $\rho_A^{(2)}$ has non-zero matrix elements only in the four-dimensional Hilbert space spanned by

- (a) $d_2^\dagger d_1^\dagger |0\rangle$,
- (b) $d_1^\dagger |0\rangle$, $d_2^\dagger |0\rangle$ and
- (c) $|0\rangle$,

representing, respectively, a two-particle state, two one-particle states and the zero-particle state in \mathcal{H}_A . It is easy to see that these states provide an eigenbasis of (128) with eigenvalues

- (a) $\lambda_1 \lambda_2$,
- (b) $\lambda_1(1 - \lambda_2)$, $\lambda_2(1 - \lambda_1)$, and
- (c) $(1 - \lambda_1)(1 - \lambda_2)$,

respectively.

Using these facts, we can write the density matrix restricted to the two-particle subsector (a), as follows

$$\rho_{A,2}^{(2)} = \lambda_1 \lambda_2 d_2^\dagger d_1^\dagger |0\rangle \langle 0| d_1 d_2 = \lambda_1 \lambda_2 \frac{1}{\sqrt{2!}} (|v_1 v_2\rangle - |v_2 v_1\rangle) \frac{1}{\sqrt{2!}} (\langle v_1 v_2| - \langle v_2 v_1|)$$

Using the first identity in (131) one can write

$$\rho_{A,2}^{(2)} = \frac{1}{\sqrt{2!}} (|u_1 u_2\rangle_A - |u_2 u_1\rangle_A) \frac{1}{\sqrt{2!}} ({}_A \langle u_1 u_2| - {}_A \langle u_2 u_1|)$$

The subscript A is there to remind that this operator has support only in region A . Notice that this precisely matches the first quantized density matrix $\tilde{\rho}_{2,0}$ (108).

Using the eigenvalues mentioned above, the density matrix, restricted to the one-particle subsector (b), can be written as a diagonal matrix in the following basis

$$\rho_{A,1}^{(2)} = \begin{bmatrix} d_1^\dagger |0\rangle & d_2^\dagger |0\rangle \\ \lambda_1(1 - \lambda_2) & 0 \\ 0 & (1 - \lambda_1)\lambda_2 \end{bmatrix} \begin{matrix} \langle 0|d_1 \\ \langle 0|d_2 \end{matrix}$$

Through successive use of the identities (131) we can write

$$\begin{aligned} \rho_{A,1}^{(2)} &= \lambda_1(1 - \lambda_2)|v_1\rangle\langle v_1| + \lambda_2(1 - \lambda_1)|v_2\rangle\langle v_2| \\ &= -\lambda_1\lambda_2 (|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|) + (\lambda_1|v_1\rangle\langle v_1| + \lambda_2|v_2\rangle\langle v_2|) \\ &= -\lambda_1\lambda_2 (|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|) + |v_2\rangle\langle v_2| + |v_1\rangle\langle v_1| - |v_1\rangle\langle v_2| + |v_2\rangle\langle v_1| \\ &\quad + (|u_1\rangle_A {}_A \langle u_1| + |u_2\rangle_A {}_A \langle u_2|) \\ &= -(|u_1\rangle_A {}_A \langle u_1| + |u_2\rangle_A {}_A \langle u_2|) + |u_2\rangle_A {}_A \langle u_2| + |u_1\rangle_A {}_A \langle u_1| - |u_1\rangle_A {}_A \langle u_2| + |u_2\rangle_A {}_A \langle u_1| \\ &\quad - |u_2\rangle_A {}_A \langle u_1| + |u_1\rangle_A {}_A \langle u_2| \end{aligned} \quad (133)$$

where in going from 2nd to 3rd line we have used the 2nd equation in (131) for the second term. In the 3rd line, we have also introduced inner products of v 's (which are orthonormal) so that we could use the 1st equation in (131) leading to the final expression. This matches precisely with the 1st quantized density matrix $\tilde{\rho}_{1,1}$ (109).

The density matrix $\rho_A^{(2)}$, restricted to the zero-particle subsector (c) (let us call it $\rho_{A,0}^{(2)}$) is proportional to $|0\rangle\langle 0|$, and agrees with the corresponding first quantized quantity $\tilde{\rho}_{0,2}$ (this can be directly verified from the two-particle identities (131)).

Thus we see that in our 2 particle example, the density matrices $\rho_{A,2}^{(2)}, \rho_{A,1}^{(2)}, \rho_{A,0}^{(2)}$ match with $\tilde{\rho}_{2,0}, \tilde{\rho}_{1,1}, \tilde{\rho}_{0,2}$ (respectively) in the first quantized language.

Arbitrary N.

Now we move to *arbitrary N*. Similar to the 2 particle example we employ the use of following identities (for $0 \leq n \leq N$)

$$\text{Tr}(\rho_A^{(N)} \Psi^\dagger(x_1) \dots \Psi^\dagger(x_n) \Psi(x'_1) \dots \Psi(x'_n)) = \langle s | \Psi^\dagger(x_1) \dots \Psi^\dagger(x_n) \Psi(x'_1) \dots \Psi(x'_n) | s \rangle \quad (134)$$

which is true as long as all the insertions x_1, \dots, x'_n lie in region A. The state $|s\rangle = b_1^\dagger \dots b_N^\dagger |0\rangle$ is a global N -particle state in the full space. Using the appropriate mode expansion for $\Psi(x)$ and after a bit of algebra, these identities can be written as

$$\begin{aligned} \sum_{i's, j's} \delta_{i_1 \dots i_n}^{j_1 \dots j_n} \lambda_{i_1} \dots \lambda_{i_n} v_{j_1}^*(x_1) \dots v_{j_n}^*(x_n) v_{i_1}(x'_1) \dots v_{i_n}(x'_n) \\ = \sum_{i's, j's} \delta_{i_1 \dots i_n}^{j_1 \dots j_n} u_{j_1}^*(x_1) \dots u_{j_n}^*(x_n) u_{i_1}(x'_1) \dots u_{i_n}(x'_n) \end{aligned} \quad (135)$$

where

$$\lambda_i = \frac{e^{-\epsilon_i}}{1 + e^{-\epsilon_i}}$$

We can also write the identities as an operator equation

$$\sum_{i's, j's} \delta_{i_1 \dots i_n}^{j_1 \dots j_n} \lambda_{i_1} \dots \lambda_{i_n} |v_{i_1} \dots v_{i_n}\rangle \langle v_{j_1} \dots v_{j_n}| = \sum_{i's, j's} \delta_{i_1 \dots i_n}^{j_1 \dots j_n} |u_{i_1} \dots u_{i_n}\rangle_A \langle u_{j_1} \dots u_{j_n}| \quad (136)$$

Further when there only k particles in region A (out of N), one can easily write the $\rho_k^{(N)}$ in region A as $\binom{N}{k} \times \binom{N}{k}$ diagonal matrix in the following basis

$$\rho_{A,k}^{(N)} = \begin{bmatrix} d_1^\dagger \dots d_k^\dagger |0\rangle & \dots & d_{N-k+1}^\dagger \dots d_N^\dagger |0\rangle \\ \prod_{i=1}^k \lambda_i \prod_{j=k+1}^N (1 - \lambda_j) & \dots & 0 \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \prod_{i=1}^{N-k} (1 - \lambda_i) \prod_{j=N-k+1}^N \lambda_j \end{bmatrix} \begin{bmatrix} \langle 0 | d_k \dots d_1 \\ \dots \\ \dots \\ \langle 0 | d_N \dots d_{N-k+1} \end{bmatrix} \quad (137)$$

For example consider $N = 3$ and $k = 2$

$$\rho_{A,2}^{(3)} = \begin{bmatrix} d_1^\dagger d_2^\dagger |0\rangle & d_2^\dagger d_3^\dagger |0\rangle & d_3^\dagger d_1^\dagger |0\rangle \\ \lambda_1 \lambda_2 (1 - \lambda_3) & 0 & 0 \\ 0 & (1 - \lambda_1) \lambda_2 \lambda_3 & 0 \\ 0 & 0 & \lambda_1 (1 - \lambda_2) \lambda_3 \end{bmatrix} \begin{bmatrix} \langle 0 | d_2 d_1 \\ \langle 0 | d_3 d_2 \\ \langle 0 | d_1 d_3 \end{bmatrix} \quad (138)$$

which is a $\binom{3}{2} \times \binom{3}{2}$ matrix. We can write $\rho_k^{(N)}$ in a more compact notation

$$\rho_{A,k}^{(N)} = \frac{1}{k!} \sum_{i's, j's} \delta_{i_1 \dots i_k}^{j_1 \dots j_k} |v_{i_1} \dots v_{i_k}\rangle \langle v_{j_1} \dots v_{j_k}| \lambda_{i_1} \dots \lambda_{i_k} \prod_{m=1, m \neq i_1, \dots, m \neq i_k}^N (1 - \lambda_m) \quad (139)$$

We can rewrite the generalized Kronecker delta function in terms of the Levi-Civita symbols using the following identity

$$\sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} \delta_{j_{k+1} i_{k+1}} \dots \delta_{j_N i_N} = (N-k)! \delta_{i_1 \dots i_k}^{j_1 \dots j_k} \quad (140)$$

Making use of this we write

$$\begin{aligned} \rho_{A,k}^{(N)} &= \frac{1}{k!(N-k)!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |v_{i_1} \dots v_{i_k}\rangle \langle v_{j_1} \dots v_{j_k}| \lambda_{i_1} \dots \lambda_{i_k} \\ &\quad \times \delta_{j_{k+1} i_{k+1}} \dots \delta_{j_N i_N} \prod_{m=1, m \neq i_1, \dots, m \neq i_k}^N (1 - \lambda_m) \\ &= \frac{1}{k!(N-k)!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |v_{i_1} \dots v_{i_k}\rangle \langle v_{j_1} \dots v_{j_k}| \lambda_{i_1} \dots \lambda_{i_k} \\ &\quad \times (1 - \lambda_{i_{k+1}}) \dots (1 - \lambda_{i_N}) \delta_{j_{k+1} i_{k+1}} \dots \delta_{j_N i_N} \\ &= \frac{1}{k!(N-k)!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |v_{i_1} \dots v_{i_k}\rangle \langle v_{j_1} \dots v_{j_k}| \delta_{j_{k+1} i_{k+1}} \dots \delta_{j_N i_N} \\ &\quad \times \left\{ \sum_{l=0}^{N-k} (-1)^l \binom{N-k}{l} \lambda_{i_1} \dots \lambda_{i_{k+l}} \right\} \end{aligned} \quad (141)$$

where in the last line we have opened the product over $(1 - \lambda)$'s and organized the sum in powers of λ_i 's. In a particular l term in the sum, since the v_i 's are orthonormal in region A , we can replace $\delta_{j_{k+1} i_{k+1}} \dots \delta_{j_{k+l} i_{k+l}}$ by $\langle v_{j_{k+1}} | v_{i_{k+1}} \rangle_A \dots \langle v_{j_{k+l}} | v_{i_{k+l}} \rangle_A$ but leave the remaining $\delta_{j_{k+l+1} i_{k+l+1}} \dots \delta_{j_N i_N}$ as it is. After this we use (140) again with the remaining delta functions to go back to the generalized Kronecker delta $\delta_{i_1 \dots i_{k+l}}^{j_1 \dots j_{k+l}}$, this leads to

$$\begin{aligned} \rho_{A,k}^{(N)} &= \frac{1}{k!} \sum_{l=0}^{N-k} \frac{(-1)^l}{l!} \sum_{i's, j's} \delta_{i_1 \dots i_{k+l}}^{j_1 \dots j_{k+l}} |v_{i_1} \dots v_{i_k}\rangle \langle v_{j_1} \dots v_{j_k}| \lambda_{i_1} \dots \lambda_{i_{k+l}} \\ &\quad \times \langle v_{j_{k+1}} | v_{i_{k+1}} \rangle_A \dots \langle v_{j_{k+l}} | v_{i_{k+l}} \rangle_A \\ &= \frac{1}{k!} \sum_{l=0}^{N-k} \frac{(-1)^l}{l!} \left[\sum_{i's, j's} \delta_{i_1 \dots i_{k+l}}^{j_1 \dots j_{k+l}} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \right. \\ &\quad \left. \times \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \dots \langle u_{j_{k+l}} | u_{i_{k+l}} \rangle_A \right] \end{aligned} \quad (142)$$

In the last step we used the identities (136).

A.2. Comparison with the first quantized theory

We will now show that the density matrix within any given sector (121) agrees with its counterpart (142).

To begin, notice that the density matrix within a sector (121) has inner products in region \bar{A} . We would like to write it in terms of region A since that is what naturally appears in the second quantized theory. Using the orthonormality of $u_n(x)$'s

$$\langle u_{j_n} | u_{i_n} \rangle_{\bar{A}} = \delta_{j_n i_n} - \langle u_{j_n} | u_{i_n} \rangle_A$$

We plug this in (121)

$$\begin{aligned} \tilde{\rho}_{k,N-k} &= \frac{\binom{N}{k}}{N!} \sum_{i's, j's} \varepsilon_{u_{i_1} \dots u_{i_N}} \varepsilon_{u_{j_1} \dots u_{j_N}} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \prod_{n=k+1}^N \langle u_{j_n} | u_{i_n} \rangle_{\bar{A}} \\ &= \frac{\binom{N}{k}}{N!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \prod_{n=k+1}^N (\delta_{j_n i_n} - \langle u_{j_n} | u_{i_n} \rangle_A) \\ &= \frac{\binom{N}{k}}{N!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \\ &\quad \times \left[\prod_{n=k+1}^N \delta_{j_n i_n} - (N-1) \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \prod_{n=k+2}^N \delta_{j_n i_n} + \dots + (-1)^l \binom{N-k}{l} \right. \\ &\quad \times \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \langle u_{j_{k+2}} | u_{i_{k+2}} \rangle_A \dots \langle u_{j_{k+l}} | u_{i_{k+l}} \rangle_A \\ &\quad \times \left. \prod_{n=k+l+1}^N \delta_{j_n i_n} + \dots + (-1)^{N-k} \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \dots \langle u_{j_N} | u_{i_N} \rangle_A \right] \\ &= \frac{\binom{N}{k}}{N!} \sum_{i's, j's} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \times \\ &\quad \sum_{l=0}^{N-k} (-1)^l \binom{N-k}{l} \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \dots \langle u_{j_{k+l}} | u_{i_{k+l}} \rangle_A \delta_{j_{k+l+1} i_{k+l+1}} \dots \delta_{j_N i_N} \end{aligned}$$

In the end we get

$$\begin{aligned} \tilde{\rho}_{k,N-k} &= \frac{1}{k!} \sum_{l=0}^{N-k} \frac{(-1)^l}{l!} \sum_{i's, j's} \delta_{i_1 \dots i_{k+l}}^{j_1 \dots j_{k+l}} |u_{i_1} \dots u_{i_k}\rangle_A \langle u_{j_1} \dots u_{j_k}| \\ &\quad \times \langle u_{j_{k+1}} | u_{i_{k+1}} \rangle_A \dots \langle u_{j_{k+l}} | u_{i_{k+l}} \rangle_A \end{aligned} \quad (143)$$

where we have again used (140). The final expression is exactly same as (142). This completes the proof $\tilde{\rho}_{k,N-k} = \rho_{A,k}^{(N)}$.

Appendix B. Multiple matrices

Let us consider the matrix model described by (38). The model is supersymmetric and has bosonic and fermionic matrix variables: X_{ij}^I, χ_{ij} . We will first ignore the fermions in the following discussions (i.e. consider the bosonic model) and briefly discuss them later in the section. In the $A_t = 0$ gauge, the theory has a residual symmetry under the time-independent $SU(N)$

transformation. This is ensured by the Gauss law condition

$$\sum_I [X^I, P_I] \Psi[X] = 0, \quad (144)$$

on the wavefunctions. Equation (144) is equivalent to the singlet condition¹²

$$\Psi[X^I] = \Psi[UX^I U^\dagger]. \quad (145)$$

As a consequence of the $SU(N)$ invariance, we can make one of the matrices, say X^1 , diagonal:

$$X^1 = D = \text{diag}[\lambda_1, \dots, \lambda_N].$$

To do this, we write X^1 in the form $X^1 = V D V^\dagger$, and make a change of variables $X^1 \rightarrow (V, D)$, $X^I \rightarrow \tilde{X}^I = V^\dagger X^I V$, $I = 2, \dots, 9$. The $SU(N)$ amounts to demanding that the wavefunctions are independent of V . The Jacobian of this change of variables is the square of the Vandermonde determinant

$$\Delta(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).$$

In other words,

$$\prod_{I=1, \dots, 9} [dX^I] = \Delta^2(\lambda) \prod_{i=1, \dots, N} d\lambda_i \prod_{I=2, \dots, 9} [d\tilde{X}^I] [dV] \quad (146)$$

The scalar product between two wavefunctions are given by

$$\begin{aligned} & \int \prod_{I=1, \dots, 9} [dX^I] \Psi^*[X^I] \Phi[X^I] \\ &= \text{Vol}(SU(N)) \int \prod_{i=1, \dots, N} d\lambda_i \Delta^2(\lambda) \prod_{I=2, \dots, 9} [d\tilde{X}^I] \Psi^*[D, \tilde{X}^I] \Phi[D, \tilde{X}^I] \\ &= \int [d\mu] \tilde{\Psi}^*[D, X^I] \tilde{\Phi}[D, X^I] \\ & [d\mu] = \prod_{i=1, \dots, N} d\lambda_i \prod_{I=2, \dots, 9} [dX^I] \end{aligned} \quad (147)$$

where

$$\tilde{\Psi}[D, X^I] = C \Delta(\lambda) \Psi[D, X^I], \quad I = 2, \dots, 9. \quad (148)$$

The constant $C = \sqrt{\text{Vol}(SU(N))}$. In the first step, we have used the measure (146) and the singlet condition (145) on the wavefunctions, so that the $SU(N)$ transformation matrix V simply comes out of the integral, yielding a volume factor. In the second step we have absorbed the Vandermonde determinant in each wavefunction, to have a simpler flat measure $[d\mu]$.

Residual symmetry: Weyl transformation.

Even after fixing X^1 diagonal, there is a residual transformation, represented by the Weyl group $S(N) \subset SU(N)$, which permutes the eigenvalues

$$(\lambda_1, \lambda_2, \dots, \lambda_N) \mapsto (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(N)}), \quad \sigma \in S(N). \quad (149)$$

¹² A similar condition also applies to the fermions χ_{ij} .

Under the transformation σ , we also have

$$X^I \mapsto \sigma(X^I), \quad \sigma(X_{ij}^I) = X_{\sigma(i)\sigma(j)}^I, \quad I = 2, \dots, 9. \quad (150)$$

For a simple example, for $N = 2$, we have

$$X^1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad X^2 = \begin{pmatrix} x_{11}^2 & x_{12}^2 \\ x_{21}^2 & x_{22}^2 \end{pmatrix}, \dots$$

The ... represent X^3 onwards which have a similar expression. The Weyl group is $S(2)$ which is generated by the single $SU(2)$ transformation matrix

$$S = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

which represents the permutation $\sigma : (1, 2) \mapsto (2, 1)$. It is easy to compute $\sigma(X^I) := SX^I S^\dagger$ for all $I = 1, \dots, 9$. We find

$$\begin{aligned} \sigma(X^1) &= \sigma \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right] = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \\ \sigma(X^2) &= \sigma \left[\begin{pmatrix} x_{11}^2 & x_{12}^2 \\ x_{21}^2 & x_{22}^2 \end{pmatrix} \right] = \begin{pmatrix} x_{22}^2 & x_{21}^2 \\ x_{12}^2 & x_{11}^2 \end{pmatrix}, \dots \end{aligned} \quad (151)$$

which confirms (149) and (150).

We must ensure that the wavefunctions are also invariant under these residual (Weyl) transformations, as required by (145). In the $N = 2$ case, this condition, in the diagonal X^1 gauge, becomes

$$\Psi[\lambda_1, \lambda_2; x_{11}^2, x_{12}^2, x_{21}^2, x_{22}^2; \dots] = \Psi[\lambda_2, \lambda_1; x_{22}^2, x_{21}^2, x_{12}^2, x_{11}^2; \dots]$$

In the (148) basis, we will have

$$\tilde{\Psi}[\lambda_1, \lambda_2; x_{11}^2, x_{12}^2, x_{21}^2, x_{22}^2; \dots] = -\tilde{\Psi}[\lambda_2, \lambda_1; x_{22}^2, x_{21}^2, x_{12}^2, x_{11}^2; \dots] \quad (152)$$

where the $-$ sign appears because of the Vandermonde determinant $\Delta = (\lambda_1 - \lambda_2)$ which picks up a minus sign under the permutation $(1, 2) \mapsto (2, 1)$.

For more general N , the above equation (152) becomes, for all $\sigma \in S(N)$

$$\tilde{\Psi}[\lambda_i; x_{ij}^2, x_{ij}^3, \dots, x_{ij}^9] = \text{sign}(\sigma) \tilde{\Psi}[\lambda_{\sigma(i)}; x_{\sigma(i)\sigma(j)}^2, x_{\sigma(i)\sigma(j)}^3, \dots, x_{\sigma(i)\sigma(j)}^9] \quad (153)$$

For the case of the single matrix, the above equation simply becomes the statement that the wavefunction $\tilde{\psi}$ represents N fermions. This was the case discussed in appendix A.

B.1. Target space EE for multiple matrices

We will now discuss how to define target space EE for the model of $D0$ branes (38). The variables of the theory are the matrices $\{X_{ij}^I\}/S(N)$ where the quotient represent dividing by the Weyl transformations. In the diagonal X^1 gauge, the wavefunctions, satisfying (153), can be written as

$$\begin{aligned}
\tilde{\Psi} &= \tilde{\Psi}_0(\lambda_1, \dots, \lambda_N; X_{11}^2, X_{12}^2, \dots, X_{NN}^2; \dots) + \text{Weyltransforms} \\
&= \tilde{\Psi}_0(\lambda_1, \dots, \lambda_N; X_{11}^2, X_{12}^2, \dots, X_{NN}^2; \dots) \\
&\quad + \sum_{\sigma} \text{sign}(\sigma) \tilde{\Psi}_0[\lambda_{\sigma(i)}; x_{\sigma(i)\sigma(j)}^2, x_{\sigma(i)\sigma(j)}^3, \dots, x_{\sigma(i)\sigma(j)}^9]
\end{aligned} \tag{154}$$

where the sum over σ denotes all permutations of $S(N)$ (besides the identity). These are the equation (44) in section 3. It is easy to see that the operators in the Hilbert space of such wavefunctions are given by (46).

Let us imagine that we are interested in the target space region $A : x^1 \geq d$.¹³ What is the target space entanglement entropy corresponding to such a region? In particular, how do we generalize the concepts of appendix A to a theory of *matrices*?

Classical moduli space.

Note that there is no easy way to associate configurations of $N \times N$ matrices to regions of target space. *A priori* the simple $SU(N)$ -invariant objects are traces of these matrices and their products. In the diagonal X^1 gauge, the eigenvalues λ_i , are also invariant objects, upto permutation, which can be mapped to points on the x^1 axis. How does one construct a d -dimensional region A defined by the codimension one hypersurface $x^1 > d$?

To get an idea, let us turn to the classical moduli space of the $D0$ brane matrix model (38), which corresponds to solutions of the equation $[X^I, X^J] = 0$. By analogy with higher dimensional gauge theories, we will call this moduli space the ‘Coulomb branch’. In the diagonal gauge for $X^1 = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_N]$, this implies $X^I = \text{diag}[X_{11}^I, X_{22}^I, \dots, X_{NN}^I]$ $I = 2, \dots, 9$. The solutions $\mathbf{x}_i = (\lambda_i, X_{ii}^I)$ represent the coordinates of the N $D0$ branes, $i = 1, 2, \dots, N$. Because of the Weyl invariance, the classical moduli space of $D0$ branes is

$$\mathcal{M} = \frac{\mathbf{R}^D}{S(N)}, \quad D = 9 \tag{155}$$

which is the same as that of N indistinguishable particles in \mathbf{R}^D . Thus one can define a ‘classical sector’ of configurations where k out of the N identical particles are in the D -dimensional region $A \subset \mathbf{R}^D$ (and the remaining $N - k$ in \bar{A}) (see figure 1). How does one proceed to the quantum theory?¹⁴

Note that a *quantization* of the classical configuration space (155) was presented in appendix A which discussed the case of N fermions in d -dimensions (see in particular (86), (87)). Following along the same lines, we could try defining different sectors of the Hilbert space of wavefunctions (154) by projecting onto mutually exclusive subspaces in which k number of \mathbf{x}_i ’s in the region A (the remaining $N - k$ being in \bar{A}), $k = 0, 1, \dots, N$, yielding a similar decomposition as in (86):

$$\mathcal{H} = \bigoplus_{k=0}^N \mathcal{H}_{k,N-k} \tag{156}$$

In the above, we have *defined* $\mathbf{x}_i = (\lambda_i, X_{ii}^I)$. As explained in figure 1, these variables define quantum fluctuations of the coordinates (which are equivalently described in terms of open strings).

¹³ Note that we are using the same notation d as in the supergravity calculations. As indicated in the text (see discussions in section 3, a couple of paragraphs below (46)) in general these two quantities need not be identical; however, the difference between the two can be neglected when d is sufficiently large in an appropriate sense.

¹⁴ Note that ordinarily in a $0 + 1$ dimensional theory, the classical moduli space is not expected to survive under quantization since there is no spontaneous symmetry breaking. However the situation with $D0$ branes is somewhat subtle, especially because of supersymmetry; for an early discussion, see [31] and references therein.

This is not yet a full specification of the quantum theory since we have not said what to do with the extra, *off-diagonal*, variables X_{ij}^I , which were not present in the N -particle problem. As explained in figure 1 these represent open strings connecting different branes. Out of these, there are open strings which connect the k D0 branes which are all in region A (these are not present in the figure since $k = 1$ there). We should definitely include them among our observables (i.e. include them in our operator algebra); similarly the open strings which connect different the $N - k$ D0 branes should be excluded from the operator algebra. The issue is what to do with open strings that straddle between region A and \bar{A} . Depending on the choice we make, we arrive at two proposals (see figure 1):

Proposal 1: we exclude the variables X_{ij}^I straddling between region A and \bar{A} from the operator algebra. This leads to (56) of section 3.

Proposal 2: we take the variables X_{ij}^I straddling between region A and \bar{A} as part of the operator algebra. This leads to (57) of section 3.

ORCID iDs

Sumit R Das  <https://orcid.org/0000-0002-8323-1431>

Gautam Mandal  <https://orcid.org/0000-0002-8065-8931>

References

- [1] Ryu S and Takayanagi T 2006 *Phys. Rev. Lett.* **96** 181602
- [2] Hubeny V E, Rangamani M and Takayanagi T 2007 *J. High Energy Phys.* **JHEP07(2007)062**
- [3] Faulkner T, Lewkowycz A and Maldacena J 2013 *J. High Energy Phys.* **JHEP11(2013)074**
Jafferis D L, Lewkowycz A, Maldacena J and Suh S J 2016 *J. High Energy Phys.* **JHEP06(2016)004**
- [4] Engelhardt N and Wall A C 2015 *J. High Energy Phys.* **JHEP01(2015)073**
- [5] 't Hooft G 1985 *Nucl. Phys. B* **256** 727
- [6] Susskind L 1993 *The Black Hole* ed C Teitelboim pp 118–31 (arXiv:[hep-th/9309145](https://arxiv.org/abs/hep-th/9309145))
- [7] Susskind L and Uglum J 1994 *Phys. Rev. D* **50** 2700
- [8] Jacobson T 1994 arXiv:[gr-qc/9404039](https://arxiv.org/abs/gr-qc/9404039)
Frolov V P, Fursaev D V and Zelnikov A I 1997 *Nucl. Phys. B* **486** 339
- [9] Dabholkar A 1995 *Phys. Lett. B* **347** 222
Dabholkar A 1995 *Nucl. Phys. B* **439** 650
He S, Numasawa T, Takayanagi T and Watanabe K 2015 *J. High Energy Phys.* **JHEP05(2015)106**
Witten E 2019 *J. High Energy Phys.* **JHEP01(2019)126**
Naseer U 2020 arXiv:[2002.12148](https://arxiv.org/abs/2002.12148) [[hep-th](https://arxiv.org/abs/hep-th)]
- [10] Bousso R 2002 *Rev. Mod. Phys.* **74** 825
- [11] For reviews see Klebanov I R 1991 *Proc., String Theory and Quantum Gravity '91 (Trieste)* (Princeton University Press) pp 30–101
Das S R 1992 *Proc., String Theory and Quantum Gravity '92 (Trieste)* pp 172–211
Jevicki A 1994 *Proc., String Theory, Gauge Theory and Quantum Gravity '93 (Trieste)* pp 96–140
and Brown U Providence—BROWN-HET-0918 (93, rec. Sep.) 42 p
Ginsparg P H and Moore G W 1994 *Proc., String Theory, Gauge Theory and Quantum Gravity '93 (Trieste)* Yale Univ. New Haven - YCTP-P23-92 (92,rec.Apr.93) 197 p Los Alamos Nat. Lab. - LA-UR-92-3479 (92,rec.Apr.93) (arXiv:[hep-th/9304011](https://arxiv.org/abs/hep-th/9304011))
Some of the original references which deal with the second quantized fermionic formulation of the $c = 1$ matrix model, relevant in the context of this paper, are
Sengupta A M and Wadia S R 1991 *Int. J. Mod. Phys. A* **06** 1961–84
Gross D J and Klebanov I R 1991 *Nucl. Phys. B* **352** 671–88
Das S R and Jevicki A 1990 *Mod. Phys. Lett. A* **05** 1639–50 the Das–Jevicki second quantized bosonic formulation of $c = 1$ can be obtained from the above fermionic formulation under appropriate assumptions

- Dhar A, Mandal G and Wadia S R 1992 *Mod. Phys. Lett. A* **07** 3129–46 the fermionic formulation is more directly related to another second quantized formulation in terms of constrained bosons in one higher dimension
- [12] Das S R 1995 *Phys. Rev. D* **51** 6901
- [13] Hartnoll S A and Mazenc E 2015 *Phys. Rev. Lett.* **115** 121602
- [14] Graham C R and Karch A 2014 *J. High Energy Phys.* **JHEP04(2014)168**
Mollabashi A, Shiba N and Takayanagi T 2014 *J. High Energy Phys.* **JHEP04(2014)185**
Karch A and Uhlemann C F 2015 *Phys. Rev. D* **91** 086005
Mohammadi Mozaffar M R and Mollabashi A 2016 *J. High Energy Phys.* **JHEP03(2016)015**
Erdmenger J and Gerbershagen M 2020 *J. High Energy Phys.* **JHEP03(2020)082**
- [15] Das S, Mandal G and Trivedi S P unpublished notes
- [16] Mazenc E A and Ranard D 2019 arXiv:1910.07449 [hep-th]
- [17] Jevicki A 1992 *Nucl. Phys. B* **376** 75
- [18] Dhar A and Mandal G 2006 *Phys. Rev. D* **74** 105006
- [19] Hanada M, Hyakutake Y, Ishiki G and Nishimura J 2016 *Phys. Rev. D* **94** 086010
Berkowitz E, Rinaldi E, Hanada M, Ishiki G, Shimasaki S and Vranas P 2016 arXiv:1606.04948 [hep-th]
Berkowitz E, Rinaldi E, Hanada M, Ishiki G, Shimasaki S and Vranas P 2016 *Phys. Rev. D* **94** 094501
Rinaldi E, Berkowitz E, Hanada M, Maltz J and Vranas P 2018 *J. High Energy Phys.* **JHEP02(018)042**
- [20] Alet F, Hanada M, Jevicki A and Peng C 2020 arXiv:2001.03158 [hep-th]
- [21] Anous T, Karczmarek J L, Mintun E, Van Raamsdonk M and Way B 2019 arXiv:1911.11145 [hep-th]
- [22] For recent explicit calculations see e.g. Belin A, Iqbal N and Lokhande S F 2018 *SciPost Phys.* **5** 024
Belin A, Iqbal N and Kruthoff J 2019 arXiv:1912.00024 [hep-th]
- [23] Jacobson T 1995 *Phys. Rev. Lett.* **75** 1260–3
Eling C, Guedens R and Jacobson T 2006 *Phys. Rev. Lett.* **96** 121301
Jacobson T 2012 *Int. J. Mod. Phys. D* **21** 1242006
- [24] Bianchi E and Myers R C 2014 *Class. Quantum Grav.* **31** 214002
- [25] Itzhaki N, Maldacena J M, Sonnenschein J and Yankielowicz S 1998 *Phys. Rev. D* **58** 046004
- [26] Peet A W and Polchinski J 1999 *Phys. Rev. D* **59** 065011
- [27] For reviews see Bigatti D and Susskind L 1999 *NATO Sci. Ser. C* **520** 277
Taylor W 2001 *Rev. Mod. Phys.* **73** 419
- [28] Susskind L 1999 *AIP Conf. Proc.* **493** 98
- [29] Yi P 1997 *Nucl. Phys. B* **505** 307
Sethi S and Stern M 1998 *Commun. Math. Phys.* **194** 675
Sethi S and Stern M 2000 *Adv. Theor. Math. Phys.* **4** 487
- [30] For a review and references to the original literature see Casini H and Huerta M 2009 *J. Phys. A: Math. Theor.* **42** 504007
- [31] Taylor W 1998 arXiv:hep-th/9801182
- [32] Soni R M and Trivedi S P 2016 *J. High Energy Phys.* **JHEP01(2016)136**
- [33] van Acoleyen K, Bultinck N, Haegeman J, Marien M, Scholz V B and Verstraete F 2015 arXiv:1511.04369 [quant-ph]
- [34] Ohya M and Petz D 2004 *Quantum Entropy and Its Use (Text and Monograph in Physics)* (Berlin: Springer) corrected 2nd printing
- [35] Witten E 2018 arXiv:1805.11965 [hep-th]