# Non-arithmetic hybrid lattices in $\mathrm{PU}(2,1)$

Joseph Wells
Arizona State University

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**Abstract**

We explore hybrid subgroups of certain non-arithmetic lattices in $\mathrm{PU}(2,1)$. We show that all of Mostow's lattices are virtually hybrids; moreover, we show that some of these non-arithmetic lattices are hybrids of two non-commensurable arithmetic lattices in $\mathrm{PU}(1,1)$.

## 1 Introduction

One key notion in the study of lattices in a semisimple Lie group $G$ is that of *arithmeticity* (which we will not define here; see [Mor15] for a standard reference). When $G$ arises as the isometry group of a symmetric space $X$ of non-compact type, the combined work of Margulis [Mar84], Gromov–Schoen [GS92], and Corlette [Cor92] imply that non-arithmetic lattices only exist when $X = \mathbf{H}^n_{\mathbb{R}}$ or $\mathbf{H}^n_{\mathbb{C}}$ (real and complex hyperbolic space, respectively); equivalently, up to finite index, when $G = \mathrm{PO}(n,1)$ or $\mathrm{PU}(n,1)$. Due to their exceptional nature, it has been a major challenge to find and understand non-arithmetic lattices in these Lie groups.

Given two arithmetic lattices $\Gamma_1, \Gamma_2$ in $\mathrm{PO}(n,1)$ with common sublattice $\Gamma_{1,2} \leq PO(n-1,1)$, Gromov and Piatestki-Shapiro showed in [GPS87] that one can produce a new "hybrid" lattice $\Gamma$ in $\mathrm{PO}(n,1)$ by way of a technique that they call "interbreeding" or "hybridization". In particular, when $\Gamma_1$ and $\Gamma_2$ are not commensurable, $\Gamma$ is shown to be non-arithmetic. It has been asked whether an analogous technique can exist for lattices in $\mathrm{PU}(n,1)$.

Hunt proposed one possible analog (see the references contained in [Pau12]) where one starts with two arithmetic lattices $\Gamma_1, \Gamma_2$ in $\mathrm{PU}(n,1)$ and embeddings $\iota_i : \mathrm{PU}(n,1) \hookrightarrow \mathrm{PU}(n+1,1)$ such that (1) $\iota_1(\Gamma_1)$ and $\iota_2(\Gamma_2)$ stabilize totally geodesic complex hypersurfaces in $\mathbf{H}^{n+1}_{\mathbb{C}}$, (2) these hypersurfaces are orthogonal to one another, and (3) $\iota_1(\Gamma_1) \cap \iota_2(\Gamma_2)$ is a lattice in $\mathrm{PU}(n-1,1)$. The hybrid subgroup is then $H(\Gamma_1,\Gamma_2) := \langle \iota_1(\Gamma_1), \iota_2(\Gamma_2) \rangle$.

In [Pau12] Paupert produces an infinite family of hybrids that are non-discrete. In [PW] Paupert and the author used the same hybridization technique to produce both arithmetic lattices and thin subgroups in the Picard modular groups. In this note, we explore a more general hybrid construction in the context of the lattices $\Gamma(p,t) \subset \mathrm{PU}(2,1)$ originally produced by Mostow in [Mos80] (see Section 3 for explanation of notation). We obtain the following results:

**Theorem.** 1. *All of Mostow's lattices* $\Gamma(p,t)$ *are virtually hybrids.*

2. *The non-arithmetic lattices* $\Gamma(4,1/12)$*,* $\Gamma(5,1/5)$*, and* $\Gamma(5,11/30)$ *are virtually hybrids of two non-commensurable arithmetic lattices in* $\mathrm{PU}(1,1)$.

The second part of this theorem highlights the similarity of these hybrids and those hybrids of Gromov–Piatetski-Shapiro, specifically in that the hybridization procedure can produce a non-arithmetic lattice from two non-commensurable arithmetic lattices. The author would like to thank Julien Paupert for many insightful discussions and suggested edits.

# 2 Complex hyperbolic geometry and hybrids

We give a brief overview of relevant definitions in complex hyperbolic geometry; the reader can see [Gol99] for a standard source.

Let $H$ be a Hermitian matrix of signature $(n,1)$ and let $\mathbb{C}^{n,1}$ denote $\mathbb{C}^{n+1}$ endowed with the Hermitian form $\langle \cdot, \cdot \rangle$ coming from $H$. Let $V_-$ denote the set of points $z \in \mathbb{C}^{n,1}$ for which $\langle z, z\rangle < 0$, and let $V_0$ denote the set of points for which $\langle z, z\rangle = 0$. Given the usual projectivization map $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \to \mathbb{CP}^n$, *complex hyperbolic n-space* is $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_-)$ with distance $d$ coming from the Bergman metric

$$\cosh^2 \frac{1}{2} d(\mathbb{P}(x), \mathbb{P}(y)) = \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle\langle y, y\rangle}$$

The ideal boundary $\partial_\infty \mathbf{H}^n_{\mathbb{C}}$ is then identified with $\mathbb{P}(V_0)$.

## 2.1 Complex hyperbolic isometries

Let $U(n, 1)$ denote the group of unitary matrices preserving $H$. The holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n, 1) = \mathrm{U}(n, 1)/\,\mathrm{U}(1)$, and the full isometry group is generated by $\mathrm{PU}(n, 1)$ and the antiholomorphic involution $z \mapsto \overline{z}$. Any holomorphic isometry of $\mathbf{H}^n_{\mathbb{C}}$ is one of the following three types:

- *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$.
- *parabolic* if it has exactly one fixed point in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).
- *loxodromic* if it has exactly two fixed points in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).

Given a vector $v \in \mathbb{C}^{n,1}$ with $\langle v, v\rangle > 0$ and complex number $\zeta$ with unit modulus, the map

$$R_{v,\zeta}(x) : x \mapsto x + (\zeta - 1)\frac{\langle x, v\rangle}{\langle v, v\rangle} v$$

is an an isometry of $\mathbf{H}^n_{\mathbb{C}}$ called a *complex reflection*, and its fixed point set $v^\perp \subset \mathbf{H}^n_{\mathbb{C}}$ is a totally geodesic subspace called a $\mathbb{C}^{n-1}$*-plane* (or a *complex line* when $n = 2$). We refer to $v$ as a *polar vector* for the subspace $\mathbb{P}(v^\perp) \cap \mathbf{H}^n_{\mathbb{C}}$; abusing notation slightly we will denote such a projective subspace simply by $v^\perp$.

## 2.2 Complex hyperbolic hybrid construction

The lack of totally geodesic real hypersurfaces in $\mathbf{H}^n_{\mathbb{C}}$ presents an issue in finding a suitable complex-hyperbolic analog of the Gromov–Piatetski-Shapiro hybrid groups. Below we present a slightly more general notion of a hybrid group than that originally introduced by Hunt (see [Pau12] and the references therein).

**Definition.** Let $\Gamma_1, \Gamma_2 < \mathrm{PU}(n,1)$ be lattices. We define a *hybrid of* $\Gamma_1, \Gamma_2$ to be any group $H(\Gamma_1, \Gamma_2)$ generated by discrete subgroups $\Lambda_1, \Lambda_2 < \mathrm{PU}(n+1,1)$ stabilizing totally geodesic hypersurfaces $\Sigma_1, \Sigma_2$ (respectively) such that

1. $\Sigma_1$ and $\Sigma_2$ are orthogonal,
2. $\Gamma_i = \Lambda_i|_{\Sigma_i}$, and
3. $\Lambda_1 \cap \Lambda_2$ is a lattice in $\mathrm{PU}(n-1,1)$.

*Remark.* The groups explored by Paupert and the author in [Pau12] and [PW] are still hybrids in this new sense as well.

# 3 Mostow's lattices

In [Mos80], Mostow constructed the first known non-arithmetic lattices in $\mathrm{PU}(2,1)$ among a family of groups generated by complex reflections. These groups, denoted $\Gamma(p,t)$, are defined as follows: Let $p = 3,4,5$, $t$ be a real number satisfying $|t| < 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $\alpha = \frac{1}{2\sin(\pi/p)}$, $\varphi = e^{\pi i t/3}$, and $\eta = e^{\pi i/p}$. Define a Hermitian form $\langle x, y\rangle = x^T H \overline{y}$ where

$$H = \begin{pmatrix} 1 & -\alpha\varphi & -\alpha\overline{\varphi} \\ -\alpha\overline{\varphi} & 1 & -\alpha\varphi \\ -\alpha\varphi & -\alpha\overline{\varphi} & 1 \end{pmatrix}.$$

For any pair $(p,t)$ as above, the group $\Gamma(p,t)$ is generated by the three complex reflections of order $p$,

$$R_1 = \begin{pmatrix} \eta^2 & -i\eta\overline{\varphi} & -i\eta\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -i\eta\varphi & \eta^2 & -i\eta\overline{\varphi} \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\eta\overline{\varphi} & -i\eta\varphi & \eta^2 \end{pmatrix},$$

and these reflections satisfy the braid relations $R_iR_jR_i = R_jR_iR_j$. The mirror for the reflection $R_i$ is given by $e_i^\perp$ where $e_i$ is the standard $i^{\text{th}}$ basis vector. When $|t| < \frac{1}{2} - \frac{1}{p}$, Mostow refers to these groups has having *small phase shift*. We'll similarly refer to $|t| = \frac{1}{2} - \frac{1}{p}$ as having *critical phase shift* and $|t| > \frac{1}{2} - \frac{1}{p}$ as having *large phase shift*. Since the groups $\Gamma(p,t)$ and $\Gamma(p,-t)$ are isomorphic, we restrict our focus to the cases where $t \geq 0$.

*Remark* (Tables 1 and 2 in [Mos80]). For $p = 3, 4, 5$, there are only finitely-many values of $t$ for which $\Gamma(p,t)$ is discrete, and they are given in Table 1. If $\Gamma(p,t)$ is discrete, we'll refer to the pair $(p,t)$ as *admissible*.

| $p$ | $t < 1/2 - 1/p$ | $t = 1/2 - 1/p$ | $t > 1/2 - 1/p$ |
|---|---|---|---|
| 3 | 0, 1/30, 1/18, 1/12, 5/42 | 1/6 | 7/30, 1/3 |
| 4 | 0, 1/12, 3/20 | 1/4 | 5/12 |
| 5 | 1/10, 1/5 | | 11/30, 7/10 |

Table 1: Values of $p$ and $t$ for which $\Gamma(p,t)$ is discrete.

**Theorem 1** (Theorem 17.3 in [Mos80])**.** *For each admissible pair $(p,t)$, the group $\Gamma(p,t)$ is a lattice in $\mathrm{PU}(2,1)$, and the following are non-arithmetic:* $\Gamma(3,5/42)$, $\Gamma(3,1/12)$, $\Gamma(3,1/30)$, $\Gamma(4,3/20)$, $\Gamma(4,1/12)$, $\Gamma(5,1/5)$, $\Gamma(5,11/30)$.

Following the notation in [DFP05], we examine closely related groups $\tilde{\Gamma}(p,t) = \langle R_1, J\rangle$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$J$ has order 3 and $R_{i+1} = JR_iJ^{-1}$ (where $i = 1, 2, 3$ and indices are taken modulo 3). It is sufficient to study these groups $\tilde{\Gamma}(p,t)$ due to the following result:

**Proposition 2** (Lemma 16.1 in [Mos80])**.** *For each admissible pair $(p,t)$, the group $\Gamma(p,t)$ has index dividing 3 in $\tilde{\Gamma}(p,t)$. The two groups are equal precisely when $k = \frac{1}{2} - \frac{1}{p} - \frac{1}{t}$ and $\ell = \frac{1}{2} - \frac{1}{p} + \frac{1}{t}$ are both integers and 3 does not divide both $k$ and $\ell$.*

# 4 Hybrids in Mostow's lattices

As Deraux–Falbel–Paupert show in [DFP05], when $\tilde{\Gamma}(p,t)$ has small phase shift, a fundamental domain for this group can be constructed by coning over two polytopes that intersect in a right-angled hexagon (see Figure 1) whose walls are determined by the polar vectors $v_{ijk}$. Taking lifts to $\mathbb{C}^{2,1}$ these vectors are given explicitly below:

$$v_{123} = \begin{pmatrix} -i\eta\overline{\varphi} \\ 1 \\ i\overline{\eta}\varphi \end{pmatrix}, \qquad v_{231} = \begin{pmatrix} i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \\ 1 \end{pmatrix}, \qquad v_{312} = \begin{pmatrix} 1 \\ i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \end{pmatrix},$$
$$v_{321} = \begin{pmatrix} i\overline{\eta\varphi} \\ 1 \\ -i\eta\varphi \end{pmatrix}, \qquad v_{132} = \begin{pmatrix} -i\eta\varphi \\ i\overline{\eta\varphi} \\ 1 \end{pmatrix}, \qquad v_{213} = \begin{pmatrix} 1 \\ -i\eta\varphi \\ i\overline{\eta\varphi} \end{pmatrix}.$$

Geometrically, $v_{ijk}^{\perp}$ is the mirror for the complex reflection $J^{\pm 1}R_jR_k$ for $k = i \pm 1 \pmod 3$. When $\tilde{\Gamma}(p,t)$ has critical phase shift, the hexagon degenerates into an ideal triangle (see Figure 2) and $JR_jR_k$ is parabolic (hence $\tilde{\Gamma}(p,t)$ is non-cocompact). When $\tilde{\Gamma}(p,t)$ has large phase shift, the ideal vertices sit inside $\mathbf{H}^2_{\mathbb{C}}$ (see Figure 3) and $JR_jR_k$ is elliptic.

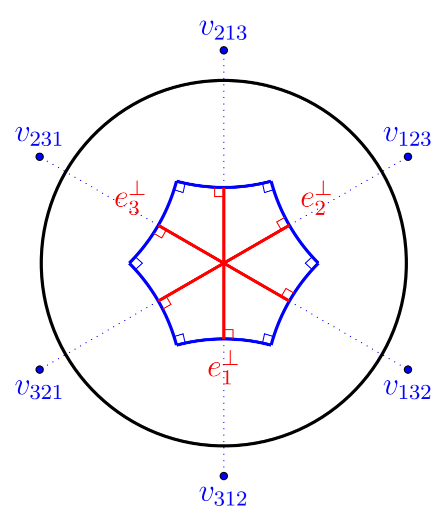


Figure 1: Core polygon when $0 \le t < \frac{1}{2} - \frac{1}{p}$

The following are readily checked:

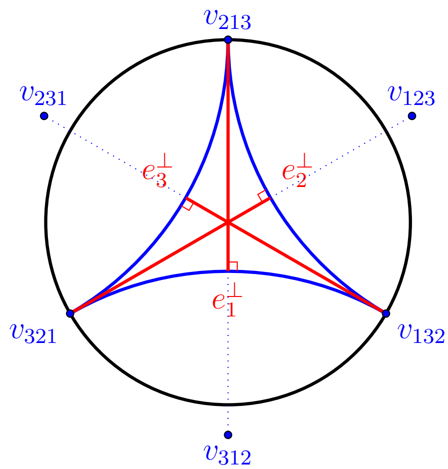


Figure 2: Core polygon when $t = \frac{1}{2} - \frac{1}{p}$

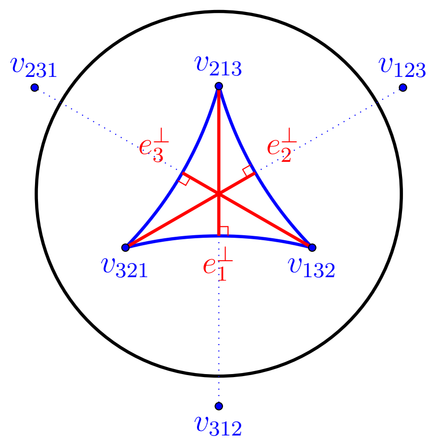


Figure 3: Core polygon when $t > \frac{1}{2} - \frac{1}{p}$

**Proposition 3** (Proposition 2.13(3) in [DFP05]). *$v_{ijk} \perp v_{jik}$ and $v_{ijk} \perp v_{ikj}$.*

**Proposition 4.** *$e_i \perp v_{jik}$ and $e_i \perp v_{kij}$.*

For the hybrid construction, we use the projective subspaces (considered as projective subspaces of $\mathbf{H}^2_{\mathbb{C}}$) corresponding to $e_1^\perp$ and $v_{312}^\perp$; this is sufficient as the remaining subspaces are obtained by successive applications of $J$. In homogeneous coordinates, one sees that

$$e_1^\perp = \{[z, \varphi z/\alpha - \varphi^2, 1]^T \ : \ z \in \mathbb{C}\} \quad \text{and}$$
$$v_{312}^\perp = \{[z, i\eta\overline{\varphi}, 1]^T \ : \ z \in \mathbb{C}\}.$$

Let $\Lambda_{ijk} \leq \tilde{\Gamma}(p,t)$ be the subgroup stabilizing $v_{ijk}^\perp$ and let $\Lambda_i$ the subgroup stabilizing $e_i^\perp$. These groups are naturally identified with subgroups of $\mathrm{PU}(1,1)$, and so we let $\Gamma_{ijk}$ and $\Gamma_i$ be lifts of these groups (respectively) into $\mathrm{SU}(1,1)$.

**Proposition 5.** *$\Gamma_{312}$ is a lattice in $\mathrm{SU}(1,1)$. It is cocompact for all non-critical phase shift values.*

*Proof.* $v_{312}$ is a positive eigenvector for both $R_1$ and $R_3J$, hence they both stabilize $v_{312}^\perp$. The action of these elements on $v_{312}^\perp$ can be seen below:

$$R_1 : \begin{bmatrix} z \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \eta^2 z + \overline{\varphi}^2 - i\eta\varphi \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix}$$
$$R_3J : \begin{bmatrix} z \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \overline{i\eta\varphi} \\ \frac{z}{i\eta\overline{\varphi}} \\ 1 \end{bmatrix}$$

Let $A$ and $B$ be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of $R_1$ and $R_3J$ on $v_{312}^\perp$, respectively,

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \overline{\varphi}^2 - i\eta\varphi \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{-\overline{\overline{i\eta\varphi}}}}\begin{pmatrix} 0 & i\eta\overline{\varphi} \\ 1 & 0 \end{pmatrix}.$$

One then sees that

$$|\operatorname{Tr}(A)| = \left|1 + e^{i2\pi/p}\right|,$$
$$|\operatorname{Tr}(B)| = 0,$$
$$|\operatorname{Tr}(A^{-1}B)| = \left|1 + e^{i\pi(t-1/2+1/p)}\right|.$$

All of these values are less than or equal to 2 for all admissible $p$ and $t$, so neither $A$ nor $B$ is loxodromic and thus they generate the orientation-preserving subgroup of a Fuchsian triangle group of finite covolume. It follows that $\Gamma_{312}$ is a lattice in $\mathrm{PU}(1,1)$. By computing orders of these elements for admissible $(p,t)$, one obtains Table 2 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [Tak77]. □

| $(p,t)$ | $\triangle(x,y,z)$ | A/NA | $(p,t)$ | $\triangle(x,y,z)$ | A/NA |
|---|---|---|---|---|---|
| $(3,0)$ | $\triangle(2,3,12)$ | A | $(4,0)$ | $\triangle(2,4,8)$ | A |
| $(3,1/30)$ | $\triangle(2,3,15)$ | NA | $(4,1/12)$ | $\triangle(2,4,12)$ | A |
| $(3,1/18)$ | $\triangle(2,3,18)$ | A | $(4,3/20)$ | $\triangle(2,4,20)$ | NA |
| $(3,1/12)$ | $\triangle(2,3,24)$ | A | $(4,1/4)$ | $\triangle(2,4,\infty)$ | A |
| $(3,5/42)$ | $\triangle(2,3,42)$ | NA | $(4,5/12)$ | $\triangle(2,4,12)$ | A |
| $(3,1/6)$ | $\triangle(2,3,\infty)$ | A | $(5,1/10)$ | $\triangle(2,5,10)$ | A |
| $(3,7/30)$ | $\triangle(2,3,30)$ | A | $(5,1/5)$ | $\triangle(2,5,20)$ | A |
| $(3,1/3)$ | $\triangle(2,3,12)$ | A | $(5,11/30)$ | $\triangle(2,5,30)$ | A |
| | | | $(5,7/10)$ | $\triangle(2,5,5)$ | A |

Table 2: Properties of $\Gamma_{312}$

**Proposition 6.** $\Gamma_1$ *is a lattice in* $\mathrm{SU}(1,1)$. *It is cocompact for all non-critical phase shift values.*

*Proof.* $J^{-1}R_1R_2$ and $JR_1R_3$ both stabilize $e_1^{\perp}$:

$$J^{-1}R_1R_2 : \begin{bmatrix} z \\ \frac{\varphi}{\alpha}(z) - \varphi^2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \frac{\alpha\eta^2\varphi^3 z + \alpha\varphi - i\alpha\eta\varphi^4}{\eta^2\varphi^2 z + -i\alpha\eta - i\eta\varphi^3} \\ \frac{\varphi}{\alpha}\left(\frac{\alpha\eta^2\varphi^3 z + \alpha\varphi - i\alpha\eta\varphi^4}{\eta^2\varphi^2 z + -i\alpha\eta - i\eta\varphi^3}\right) - \varphi^2 \\ 1 \end{bmatrix}$$

$$JR_1R_3 : \begin{bmatrix} z \\ \frac{\varphi}{\alpha}(z) - \varphi^2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \frac{(i\eta\varphi^3 + i\alpha\eta)z - i\eta\alpha\varphi^4 - \alpha\eta^2\varphi}{-\varphi^2 z + \alpha\varphi^3} \\ \frac{\varphi}{\alpha}\left(\frac{(i\eta\varphi^3 + i\alpha\eta)z - i\eta\alpha\varphi^4 - \alpha\eta^2\varphi}{-\varphi^2 z + \alpha\varphi^3}\right) \\ 1 \end{bmatrix}$$

Let $A$ and $B$ be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of $J^{-1}R_1R_2$ and $JR_1R_3$ on $e_1^\perp$, respectively.

$$A = \frac{1}{\alpha\sqrt{-i\eta\varphi^3}}\begin{pmatrix}\alpha\eta^2\varphi^3 & \alpha\varphi - i\alpha\eta\varphi^4\\ \eta^2\varphi^2 & -i\alpha\eta - i\eta\varphi^3\end{pmatrix}, \qquad B = \frac{1}{\alpha\sqrt{i\eta^3\varphi^3}}\begin{pmatrix}i\eta\varphi^3 + i\alpha\eta & -i\eta\alpha\varphi^4 - \alpha\eta^2\varphi\\ -\varphi^2 & \alpha\varphi^3\end{pmatrix}.$$

One then sees that

$$\begin{aligned}|\operatorname{Tr}(A)| &= \left|1 + e^{\pi i(t+1/2-1/p)}\right|,\\ |\operatorname{Tr}(B)| &= \left|1 + e^{\pi i(t-1/2+1/p)}\right|,\\ |\operatorname{Tr}(AB)| &= \left|-1 + e^{6\pi i/p}\right|.\end{aligned}$$

All of these values are less than or equal to 2 for admissible values of $p$ and $t$, so neither $A$ nor $B$ is loxodromic and thus they generate the orientation-preserving subgroup of a Fuchsian triangle group of finite covolume. It follows that $\Gamma_{312}$ is a lattice in $\mathrm{PU}(1,1)$. By computing orders of these elements for admissible $(p,t)$, one obtains Table 2 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [Tak77]. □

| $(p,t)$ | $\triangle(x,y,z)$ | A/NA | $(p,t)$ | $\triangle(x,y,z)$ | A/NA |
|---|---|---|---|---|---|
| $(3,0)$ | $\triangle(2,12,12)$ | A | $(4,0)$ | $\triangle(4,8,8)$ | A |
| $(3,1/30)$ | $\triangle(2,10,15)$ | NA | $(4,1/12)$ | $\triangle(4,6,12)$ | NA |
| $(3,1/18)$ | $\triangle(2,6,18)$ | A | $(4,3/20)$ | $\triangle(4,5,20)$ | NA |
| $(3,1/12)$ | $\triangle(2,8,24)$ | NA | $(4,1/4)$ | $\triangle(4,4,\infty)$ | A |
| $(3,5/42)$ | $\triangle(2,7,42)$ | NA | $(4,5/12)$ | $\triangle(3,4,12)$ | A |
| $(3,1/6)$ | $\triangle(2,6,\infty)$ | A | $(5,1/10)$ | $\triangle(5,10,10)$ | A |
| $(3,7/30)$ | $\triangle(2,5,30)$ | A | $(5,1/5)$ | $\triangle(4,10,20)$ | NA |
| $(3,1/3)$ | $\triangle(2,4,12)$ | A | $(5,11/30)$ | $\triangle(3,10,30)$ | A |
| | | | $(5,7/10)$ | $\triangle(2,5,10)$ | A |

Table 3: Properties of $\Gamma_1$

**Lemma 7.** *Let $K = \langle JR_1R_3,\ JR_2R_1,\ JR_3R_2\rangle$. For all admissible $p,t$, $K$ is normal in $\tilde{\Gamma}(p,t)$.*

*Proof.* For indices $i, j, k$ with $k = i + 1 \pmod 3$ and $j = i - 1 \pmod 3$, the following equations are readily checked:

$$R_i(JR_iR_j)R_i^{-1} = JR_iR_j, \quad R_k(JR_iR_j)R_k^{-1} = (JR_iR_j)(JR_jR_k)(JR_iR_j)^{-1},$$
$$R_j(JR_iR_j)R_j^{-1} = JR_kR_i, \quad J(JR_iR_j)J^{-1} = JR_kR_i.$$

□

**Lemma 8.** *For each admissible pair $(p, t)$, the group $K$ (as in the previous lemma) has finite index in $\tilde{\Gamma}(p, t)$.*

*Proof.* $\tilde{\Gamma}(p, t)$ is a quotient of the finitely-presented group

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid J^3 = R_i^p = \mathrm{Id},\; R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\; R_{i+1} = JR_iJ^{-1} \right\rangle$$

where $i = 1, 2, 3$ (and indices are taken modulo 3). Let $\mathcal{X}_\Gamma$ be some set of additional relations so that $\tilde{\Gamma}(p, t)$ has the presentation

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid \mathcal{X}_\Gamma,\; , J^3 = R_i^p = \mathrm{Id},\; R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\; R_{i+1} = JR_iJ^{-1} \right\rangle.$$

As $K$ is normal, we examine the quotient $\tilde{\Gamma}(p, t)/K$ with presentation

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid \mathcal{X}_\Gamma,\; J^3 = R_i^p = JR_{i+1}R_i = \mathrm{Id},\, R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\, R_{i+1} = JR_iJ^{-1} \right\rangle.$$

where, again, $i = 1, 2, 3$ and the indices are taken modulo 3. Because $\tilde{\Gamma}(p, t)$ is generated by $R_1$ and $J$, many of the relations are superfluous, so the presentation for $\tilde{\Gamma}(p, t)/K$ simplifies a bit to

$$\langle J,\, R_1,\, R_2 \mid \mathcal{X}_\Gamma,\; J^3 = R_1^p = JR_2R_1 = \mathrm{Id},\; R_2 = JR_1J^{-1},\; R_1R_2R_1 = R_2R_1R_2 \rangle.$$

The relation $JR_2R_1 = \mathrm{Id}$ also makes the braid relation $R_1R_2R_1 = R_2R_1R_2$ superfluous, and so the presentation simplifies more to

$$\tilde{\Gamma}(p, t)/K = \left\langle J,\, R_1 \mid \mathcal{X}_\Gamma,\; R_1^p = J^3 = (J^{-1}R_1)^2 = \mathrm{Id} \right\rangle.$$

In this way, one sees that $\tilde{\Gamma}(p, t)/K$ is a quotient of the (orientation-preserving) $(2, 3, p)$-triangle group. These triangle groups are finite when $p = 3, 4, 5$, thus $K$ has finite index in $\tilde{\Gamma}(p, t)$. □

**Theorem 9.** *For each admissible pair $(p, t)$, the hybrid $H(\Gamma_1, \Gamma_{312})$ has finite index in $\tilde{\Gamma}(p, t)$.*

*Proof.* From the previous lemma, it suffices to show that the hybrid

$$H := H(\Gamma_1, \Gamma_{312}) = \langle \Lambda_1, \Lambda_{312} \rangle$$

contains $K$. Indeed, $H$ contains the subgroup $\langle J^{-1}R_1R_2, JR_1R_3, R_1, R_3J \rangle$ by Propositions 5 and 6, from which it immediately follows that $JR_1R_3 \in H$. That $H$ contains the other two generators for $K$ is again a straightforward matrix computation.

$$JR_2R_1 = J(J^{-1}R_3J)R_1 = (R_3J)(R_1), \qquad \text{and}$$
$$JR_3R_2 = JR_3(J^{-1}J)R_2(J^{-1}J) = (R_1)(R_3J).$$

□

By comparing with the table on Page 418 of [MR03], one sees that $\Gamma_1$ and $\Gamma_{312}$ are both arithmetic and non-commensurable in the case that $(p, t) = (5, 11/30)$. Since $H(\Gamma_1, \Gamma_{312})$ has finite index in $\tilde{\Gamma}(5, 11/30)$, it is non-arithmetic and thus

**Corollary 10.** *For $(p, t) = (5, 11/30)$, $H(\Gamma_1, \Gamma_{312})$ is a non-arithmetic lattice obtained by hybridizing two noncommesurable arithmetic lattices.*

# 5 Small phase shift hybrids

In that $\tilde{\Gamma}(p, t)$ has small phase shift, we can instead consider the hybrid with subspaces $v_{312}^{\perp}$ and $v_{321}^{\perp}$. In homogeneous coordinates, one sees that

$$v_{321}^{\perp} = \{[i\overline{\eta}\varphi, z, 1]^T \ : \ z \in \mathbb{C}\}.$$

**Proposition 11.** *$\Gamma_{321}$ is an arithmetic cocompact lattice in $\mathrm{SU}(1,1)$ for all small phase shift values.*

| $(p,t)$ | $\triangle(x,y,z)$ | A/NA | $(p,t)$ | $\triangle(x,y,z)$ | A/NA |
|---|---|---|---|---|---|
| $(3,0)$ | $\triangle(2,3,12)$ | A | $(4,0)$ | $\triangle(2,4,8)$ | A |
| $(3,1/30)$ | $\triangle(2,3,10)$ | A | $(4,1/12)$ | $\triangle(2,4,6)$ | A |
| $(3,1/18)$ | $\triangle(2,3,9)$ | A | $(4,3/20)$ | $\triangle(2,4,5)$ | A |
| $(3,1/12)$ | $\triangle(2,3,8)$ | A | $(5,1/10)$ | $\triangle(2,5,5)$ | A |
| $(3,5/42)$ | $\triangle(2,3,7)$ | A | $(5,1/5)$ | $\triangle(2,4,5)$ | A |

Table 4: Properties of $\Gamma_{321}$

*Proof.* $R_2$ and $JR_3^{-1}$ both stabilize $v_{321}^{\perp}$:

$$R_2 : \begin{bmatrix} i\overline{\eta}\varphi \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} i\overline{\eta}\varphi \\ \eta^2 z + \varphi^2 - i\eta\overline{\varphi} \\ 1 \end{bmatrix}$$

$$JR_3^{-1} : \begin{bmatrix} i\overline{\eta}\varphi \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} i\overline{\eta}\varphi \\ \frac{i\overline{\eta}\varphi}{z} \\ 1 \end{bmatrix}$$

Let $A$ and $B$ be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of $R_2$ and $JR_3^{-1}$ on $v_{321}^{\perp}$, respectively.

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \varphi^2 - i\eta\overline{\varphi} \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{-i\overline{\eta}\varphi}}\begin{pmatrix} 0 & i\overline{\eta}\varphi \\ 1 & 0 \end{pmatrix}.$$

One can check that

$$\begin{aligned} |\operatorname{Tr}(A)| &= |e^{i\pi/p} + e^{-i\pi/p}|, \\ |\operatorname{Tr}(B)| &= 0, \\ |\operatorname{Tr}(A^{-1}B)| &= |e^{i\pi(1/2+1/p-t/3)} - e^{2\pi i t/3}|. \end{aligned}$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B\rangle$ is a cocompact triangle group (and therefore $\Gamma_{321}$ is a cocompact lattice). By computing orders of these elements for $(p,t)$ values in Table 1, one obtains Table 4 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [Tak77]. $\square$

**Theorem 12.** *For* $|t| < \frac{1}{2} - \frac{1}{p}$*, the hybrid* $H(\Gamma_{312}, \Gamma_{321})$ *is the full lattice* $\tilde{\Gamma}(p,t)$*.*

*Proof.* The group $K = \langle R_1, R_3J, R_2, JR_3^{-1}\rangle$ is a subgroup of $H(\Gamma_{312}, \Gamma_{321})$. Since $J = (R_3J)^{-1}(JR_3^{-1})^{-1}$, $K = \langle R_1, J\rangle = \tilde{\Gamma}(p,t)$. □

By comparing with the table on Page 418 of [MR03], one sees that $\Gamma_{312}$ and $\Gamma_{321}$ are both arithmetic and noncommensurable in the cases where $(p,t) = (4, 1/12)$ and $(5, 1/5)$. Thus

**Corollary 13.** $\tilde{\Gamma}(4, 1/12)$ *and* $\tilde{\Gamma}(5, 1/5)$ *are non-arithmetic lattices obtained by interbreeding two noncommensurable arithmetic lattices.*

# References

[Cor92] K. Corlette. Archimedean superrigidity and hyperbolic geometry. *Annals of Mathematics*, 135(1):165–182, 1992.

[DFP05] M. Deraux, E. Falbel, and J. Paupert. New constructions of fundamental polyhedra in complex hyperbolic space. *Acta Math.*, 194(2):155–201, 2005.

[Gol99] W. M. Goldman. *Complex hyperbolic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.

[GPS87] M. Gromov and I. Piatetski-Shapiro. Non-arithmetic groups in lobachevsky spaces. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 66(1):93–103, 1987.

[GS92] M. Gromov and R. Schoen. Harmonic maps into singular spaces and $p$-adic superrigidity for lattices in groups of rank one. *Publications Mathématiques de l'IHÉS*, 76:165–246, 1992.

[Mar84] G. A. Margulis. Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1. *Inventiones mathematicae*, 76(1):93–120, 1984.

[Mor15] D. W. Morris. *Introduction to arithmetic groups*. Deductive Press, 2015.

[Mos80] G. D. Mostow. On a remarkable class of polyhedra in complex hyperbolic space. *Pacific J. Math.*, 86(1):171–276, 1980.

[MR03] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[Pau12] J. Paupert. Non-discrete hybrids in $\mathrm{SU}(2,1)$. *Geom. Dedicata*, 157:259–268, 2012.

[PW] J. Paupert and J. Wells. Hybrid lattices and thin subgroups of Picard modular groups. arXiv: 1806.01438.

[Tak77] K. Takeuchi. Arithmetic triangle groups. *J. Math. Soc. Japan*, 29(1):91–106, 1977.