Combinatorial Characterization of Queer Supercrystals (Survey)

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1 Introduction

The representation theory of Lie algebras is of fundamental importance, and hence combinatorial models for representations, especially those amenable to computation, are of great use. In the 1990s, Kashiwara [1] showed that integrable highest weight representations of the Drinfeld–Jimbo quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra, in the $q \to 0$ limit result in a combinatorial skeleton of the integrable representation. He coined the term crystal bases, reflecting the fact that q corresponds to the temperature of the underlying physical system. Since then, crystal bases have appeared in many areas of mathematics, including algebraic geometry, combinatorics, mathematical physics, representation theory, and number theory. One of the major advances in the theory of crystals for simply-laced Lie algebras was the discovery by Stembridge [2] of local axioms that uniquely characterize the crystal graphs corresponding to Lie algebra representations. These local axioms provide a completely combinatorial approach to the theory of crystals; this viewpoint was taken in [3].

Lie superalgebras [4] arose in physics in theories that unify bosons and fermions. They are essential in modern string theories [5] and appear in other areas of mathematics, such as the projective representations of the symmetric group. The crystal basis theory has been developed for various quantum superalgebras [6–12]. In this paper, we are in particular interested in the queer superalgebra $\mathfrak{q}(n)$ (see for example [13]). A theory of highest weight crystals for the queer superalgebra $\mathfrak{q}(n)$ was recently developed by Grantcharov et al. [7–9]. They provide an explicit combinatorial realization of the highest weight crystal bases in terms of

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semistandard decomposition tableaux and show how these crystals can be derived from a tensor product rule and the vector representation. They also use the tensor product rule to derive a Littlewood–Richardson rule. Choi and Kwon [14] provide a new characterization of Littlewood–Richardson–Stembridge tableaux for Schur P-functions by using the theory of $\mathfrak{q}(n)$-crystals. Independently, Hiroshima [15] and Assaf and Oguz [16, 17] defined a queer crystal structure on semistandard shifted tableaux, extending the type A crystal structure of [18] on these tableaux.

In this paper, we provide a characterization of the queer supercrystals in analogy to Stembridge's [2] characterization of crystals associated to classical simply-laced root systems. Assaf and Oguz [16, 17] conjecture a local characterization of queer crystals in the spirit of Stembridge [2], which involves local relations between the odd crystal operator f_{-1} with the type A_{n-1} crystal operators f_i for $1 \leqslant i < n$. However, we provide a counterexample to [17, Conjecture 4.16], which conjectures that these local axioms uniquely characterize the queer supercrystals. Instead, we define a new graph $G(C)$ on the relations between the type A components of the queer supercrystal C, which together with Assaf's and Oguz' local queer axioms and further new axioms uniquely fixes the queer crystal structure (see Theorem 3). We provide a combinatorial description of $G(C)$ by providing the combinatorial rules for all odd queer crystal operators f_{-i} on certain highest weight elements for $1 \leqslant i < n$. A long version of this paper containing all proofs is available in [19].

2 Queer Supercrystals

An *(abstract) crystal* of type A_n is a nonempty set B together with the maps $e_i, f_i \colon B \to B \sqcup \{0\}$ for $i \in I$ and $\mathrm{wt}\colon B \to \Lambda$, where $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$ is the weight lattice of the root of type A_n and $I = \{1, 2, \ldots, n\}$ is the index set, subject to several conditions. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I$ the simple roots of type A_n, where ϵ_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Then we require:

A1. For $b, b' \in B$, we have $f_i b = b'$ if and only if $b = e_i b'$. Also $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.

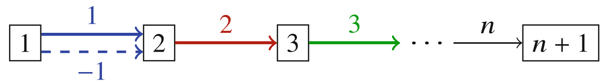
For $b \in B$, we also define $\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\}$ and $\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}$. For further details, see for example [3, Definition 2.13].

There is an action of the symmetric group S_n on a type A_n crystal B given by the operators

$$s_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0, \end{cases} \tag{1}$$

for $b \in B$, where $k = \varphi_i(b) - \varepsilon_i(b)$. An element $b \in B$ is called *highest weight* if $e_i(b) = 0$ for all $i \in I$. For a subset $J \subseteq I$, we say that b is J-highest weight if $e_i(b) = 0$ for all $i \in J$. We are now ready to define an abstract queer crystal.

Fig. 1 $\mathfrak{q}(n+1)$-queer crystal of letters $\mathcal{B}$



Definition 1 ([8, Definition 1.9]) An *abstract* $\mathfrak{q}(n+1)$*-crystal* is a type A_n crystal B together with the maps $e_{-1}, f_{-1} \colon B \to B \sqcup \{0\}$ satisfying the following conditions:

Q1. $\mathrm{wt}(B) \subset \Lambda$;
Q2. $\mathrm{wt}(e_{-1}b) = \mathrm{wt}(b) + \alpha_1$ and $\mathrm{wt}(f_{-1}b) = \mathrm{wt}(b) - \alpha_1$;
Q3. for all $b, b' \in B$, $f_{-1}b = b'$ if and only if $b = e_{-1}b'$;
Q4. if $3 \leqslant i \leqslant n$, we have (a) the crystal operators e_{-1} and f_{-1} commute with e_i and f_i and (b) if $e_{-1}b \in B$, then $\varepsilon_i(e_{-1}b) = \varepsilon_i(b)$ and $\varphi_i(e_{-1}b) = \varphi_i(b)$.

Given two $\mathfrak{q}(n+1)$-crystals B_1 and B_2, Grantcharov et al. [8, Theorem 1.8] provide a crystal on the tensor product $B_1 \otimes B_2$, which we state here in reverse convention. It consists of the type A_n tensor product rule (see for example [3, Section 2.3]) and the *tensor product rule* for $b_1 \otimes b_2 \in B_1 \otimes B_2$

$$e_{-1}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes e_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ e_{-1}b_1 \otimes b_2 & \text{otherwise,} \end{cases} \tag{2}$$

and similarly for f_{-1}. *Queer supercrystals* are connected components of $\mathcal{B}^{\otimes \ell}$, where $\mathcal{B}$ is the $\mathfrak{q}(n+1)$-queer crystal of letters depicted in Fig. 1.

In addition to the queer crystal operators $f_{-1}, f_1, \ldots, f_n$ and $e_{-1}, e_1, \ldots, e_n$, we define crystal operators $f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i}$ and $e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}$ for $1 < i \leqslant n$, where $s_{w_i} = s_2 \cdots s_i s_1 \cdots s_{i-1}$ with s_i as in (1). By [8, Theorem 1.14], with all operators e_i, f_i for $i \in \{\pm 1, \pm 2, \ldots, \pm n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique highest weight vector.

The operators f_i for $i \in I_0$ have an easy combinatorial description on $b \in \mathcal{B}^{\otimes \ell}$ given by the *signature rule*, which can be directly derived from the tensor product rule (see for example [3, Section 2.4]). One can consider b as a word in the alphabet $\{1, 2, \ldots, n+1\}$. Consider the subword of b consisting only of the letters i and $i+1$. Pair any consecutive letters $i+1, i$ in this order, remove this pair, and repeat. Then f_i changes the rightmost unpaired i to $i+1$; if there is no such letter $f_i(b) = 0$. Similarly, e_i changes the leftmost unpaired $i+1$ to i; if there is no such letter $e_i(b) = 0$.

Remark 1 From (2), one may also derive a simple combinatorial rule for f_{-1} and e_{-1}. Consider the subword v of $b \in \mathcal{B}^{\otimes \ell}$ consisting of the letters 1 and 2. The crystal operator f_{-1} on b is defined if the leftmost letter of v is a 1, in which case it turns it into a 2. Otherwise $f_{-1}(b) = 0$. Similarly, e_{-1} on b is defined if the leftmost letter of v is a 2, in which case it turns it into a 1. Otherwise $e_{-1}(b) = 0$.

We now give explicit descriptions of $\varphi_{-i}(b)$ and $f_{-i}b$ for J-highest-weight elements $b \in \mathcal{B}^{\otimes \ell}$ for certain $J \subseteq I_0 := \{1, 2, \ldots, n\}$ (see Proposition 1 and

Theorem 1). We will need these results in Sect. 4 when we characterize certain graphs on the type A components of the queer crystal.

Definition 2 The *initial k-sequence* of a word $b = b_1 \dots b_\ell \in \mathcal{B}^{\otimes \ell}$, if it exists, is the sequence of letters $b_{p_k}, b_{p_{k-1}}, \dots, b_{p_1}$, where b_{p_k} is the leftmost k and b_{p_j} is the leftmost j to the right of $b_{p_{j+1}}$ for all $1 \leqslant j < k$.

Let $i \in I_0$ and $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight with $\mathrm{wt}(b)_{i+1} > 0$, where $\mathrm{wt}(b)_{i+1}$ is the $(i+1)$-st entry in $\mathrm{wt}(b) \in \mathbb{Z}^{n+1}$. Then note that b has an initial $(i+1)$-sequence, say $b_{p_{i+1}}, b_{p_i}, \dots, b_{p_1}$. Also let $b_{q_i}, b_{q_{i-1}}, \dots, b_{q_1}$ be the initial i-sequence of b. Note that $p_{i+1} < p_i < \cdots < p_1$ and $q_i < q_{i-1} < \cdots < q_1$ by the definition of initial sequence. Furthermore either $q_j = p_j$ or $q_j < p_{j+1}$ for all $1 \leqslant j \leqslant i$.

Proposition 1 *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight for $i \in I_0$. Then $\varphi_{-i}(b) = 1$ if and only if $\mathrm{wt}(b)_i > 0$ and either $\mathrm{wt}(b)_{i+1} = 0$ or $p_j \neq q_j$ for all $j \in \{1, 2, \dots, i\}$.*

Example 1 Take $b = 1331242312111$ and $i = 3$. Then $p_4 = 6$, $p_3 = 8$, $p_2 = 10$, $p_1 = 11$ and $q_3 = 2, q_2 = 5, q_1 = 9$. We indicate the chosen letters p_j by underlines and q_j by overlines: $b = 1\overline{3}31\overline{2}\underline{4}2\underline{3}\overline{1}\underline{2}\underline{1}11$. Since no letter has a both an overline and underline (meaning $p_j \neq q_j$ for all j), we have $\varphi_{-3}(b) = 1$.

Recall that in a queer crystal B an element $b \in B$ is *highest-weight* if $e_i(b) = 0$ for all $i \in I_0 \cup I_-$, where $I_0 = \{1, 2, \dots, n\}$ and $I_- = \{-1, -2, \dots, -n\}$.

Proposition 2 ([8, Prop.1.13]) *Let $b \in \mathcal{B}^{\otimes \ell}$ be highest weight. Then $\mathrm{wt}(b)$ is a strict partition.*

Next, we provide an explicit description of $f_{-i}(b)$ for $i \in I_0$, when b is $\{1, 2, \dots, i\}$-highest weight. Recall that the sequence $b_{q_i}, b_{q_{i-1}}, \dots, b_{q_1}$ is the leftmost sequence of letters $i, i-1, \dots, 1$ from left to right. Set $r_1 = q_1$ and recursively define $r_j < r_{j-1}$ for $1 < j \leqslant i$ to be maximal such that $b_{r_j} = j$. Note that by definition $q_j \leqslant r_j$. Let $1 \leqslant k \leqslant i$ be maximal such that $q_k = r_k$.

Theorem 1 *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$ (see Proposition 1). Then $f_{-i}(b)$ is obtained from b by changing $b_{q_j} = j$ to $j - 1$ for $j = i, i-1, \dots, k+1$ and $b_{r_j} = j$ to $j + 1$ for $j = i, i-1, \dots, k$.*

Example 2 Let us continue Example 1 with $b = 1331242312111$ and $i = 3$. We overline b_{q_j} and underline b_{r_j}, so that $b = 1\overline{3}\underline{3}1\overline{2}4\underline{2}3\overline{\underline{1}}2111$. From this we read off $q_3 = 2, q_2 = 5, q_1 = 9$, $r_3 = 3, r_2 = 7, r_1 = 9$, $k = 1$ and $f_{-3}(b) = 1241143322111$.

As another example, take $b = 545423321211$ in the $\mathfrak{q}(6)$-crystal $\mathcal{B}^{\otimes 12}$ and $i = 5$. Again, we overline b_{q_j} and underline b_{r_j}, so that $b = \overline{5}\overline{4}\underline{5}\underline{4}2\overline{3}\underline{3}\overline{\underline{2}}\overline{\underline{1}}211$. This means that $q_5 = 1$, $q_4 = 2$, $q_3 = 6$, $q_2 = 8$, $q_1 = 9$, $r_5 = 3$, $r_4 = 4$, $r_3 = 7$, $r_2 = 8$, $r_1 = 9$, $k = 2$, and $f_{-5}(b) = 436522431211$.

Corollary 1 *Let $b \in \mathcal{B}^{\otimes \ell}$ be J-highest weight for $\{1, 2, \ldots, i\} \subseteq J \subseteq I_0$ and $\varphi_{-i}(b) = 1$ for some $i \in I_0$. Then:*

1. *Either $f_{-i}(b) = f_i(b)$ or $f_{-i}(b)$ is J-highest weight.*
2. *$f_{-i}(b)$ is I_0-highest weight only if $b = f_{i+1} f_{i+2} \cdots f_{h-1} u$ for some $n + 1 \geqslant h > i$ and u a I_0-highest weight element.*

3 Local Axioms

In [17, Definition 4.11], Assaf and Oguz give a definition of regular queer crystals. In essence, their axioms are rephrased in the following definition, where $\tilde{I} := I_0 \cup \{-1\}$.

Definition 3 (Local Queer Axioms) Let C be a graph with labeled directed edges given by f_i for $i \in I_0$ and f_{-1}. If $b' = f_j b$ for $j \in \tilde{I}$ define e_j by $b = e_j b'$.

LQ1. The subgraph with all vertices but only edges labeled by $i \in I_0$ is a type A_n Stembridge crystal.
LQ2. $\varphi_{-1}(b), \varepsilon_{-1}(b) \in \{0, 1\}$ for all $b \in C$.
LQ3. $\varphi_{-1}(b) + \varepsilon_{-1}(b) > 0$ if $\mathrm{wt}(b)_1 + \mathrm{wt}(b)_2 > 0$.
LQ4. Assume $\varphi_{-1}(b) = 1$ for $b \in C$.

(a) If $\varphi_1(b) > 2$, we have $f_1 f_{-1}(b) = f_{-1} f_1(b)$, $\varphi_1(b) = \varphi_1(f_{-1}(b)) + 2$, and $\varepsilon_1(b) = \varepsilon_1(f_{-1}(b))$.
(b) If $\varphi_1(b) = 1$, we have $f_1(b) = f_{-1}(b)$.

LQ5. Assume $\varphi_{-1}(b) = 1$ for $b \in C$.

(a) If $\varphi_2(b) > 0$, we have $f_2 f_{-1}(b) = f_{-1} f_2(b)$, $\varphi_2(b) = \varphi_2(f_{-1}(b)) - 1$, and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$.
(b) If $\varphi_2(b) = 0$, we have

$$\begin{aligned} \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1 = 0, & \text{or} \quad \varphi_2(b) &= \varphi_2(f_{-1}(b)) = 0, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)), & \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)) + 1. \end{aligned}$$

LQ6. Assume that $\varphi_{-1}(b) = 1$ and $\varphi_i(b) > 0$ with $i \geqslant 3$ for $b \in C$. Then $f_i f_{-1}(b) = f_{-1} f_i(b)$, $\varphi_i(b) = \varphi_i(f_{-1}(b))$, and $\varepsilon_i(b) = \varepsilon_i(f_{-1}(b))$.

Axioms **LQ4** and **LQ5** are illustrated in Fig. 2.

Proposition 3 ([17]) *The queer crystal of words $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 3.*

In [17, Conjecture 4.16], Assaf and Oguz conjecture that every regular queer crystal is a normal queer crystal. In other words, every connected graph satisfying the local queer axioms of Definition 3 is isomorphic to a connected component in some $\mathcal{B}^{\otimes \ell}$. We provide a counterexample to this claim in [19, Figure 3]. This

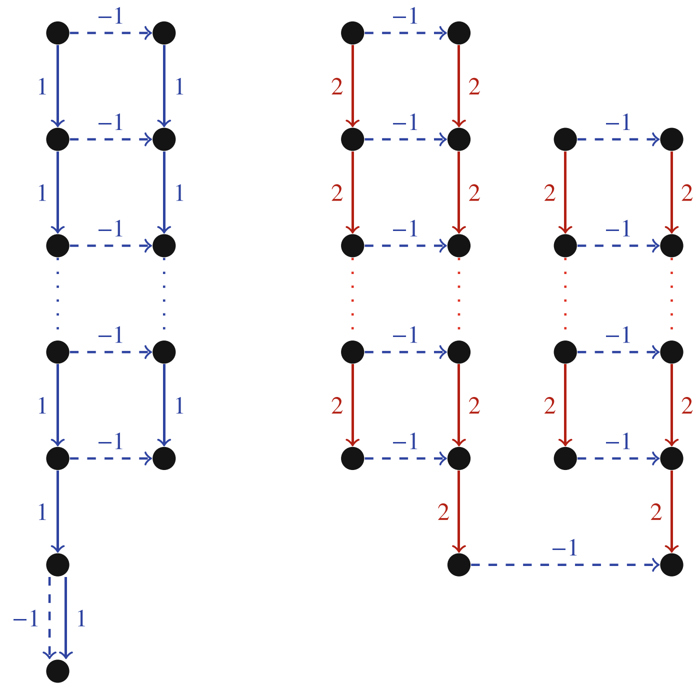


Fig. 2 Illustration of axioms **LQ4** (left) and **LQ5** (right). The (-1)-arrow at the bottom of the right figure might or might not be there

counterexample is based on the I_0-components of the $\mathfrak{q}(3)$-crystal of highest weight $(4, 2, 0)$. In addition to the usual queer crystal, there is another choice of arrows that does not violate the conditions of Definition 3.

The problem with Axiom **LQ5** illustrated in Fig. 2 is that the (-1)-arrow at the bottom of the 2-strings is not closed at the top. Hence, as demonstrated by the counterexample in switching components with the same I_0-highest weights can cause non-uniqueness.

4 Graph on Type A Components

Definition 4 Let C be a crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. We define the *component graph* of C, denoted by $G(C)$, as the following simple directed graph. The vertices of $G(C)$ are the type A_n components of C (typically labeled by their highest weight elements). There is a directed edge from vertex C_1 to vertex C_2, if there is an element b_1 in component C_1 and an element b_2 in component C_2 such that $f_{-1}b_1 = b_2$.

Example 3 Let C be the connected component in the $\mathfrak{q}(3)$-crystal $\mathcal{B}^{\otimes 6}$ with highest weight element $1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1$ of highest weight $(4, 2, 0)$. The graph $G(C)$ is given in Fig. 3 on the left. The graph $G(C')$ for the counterexample C' in [19, Figure 3] is given in Fig. 3 on the right. Since the two graphs are not isomorphic as unlabeled graphs, this confirms that the purple dashed arrows in [19, Figure 3]

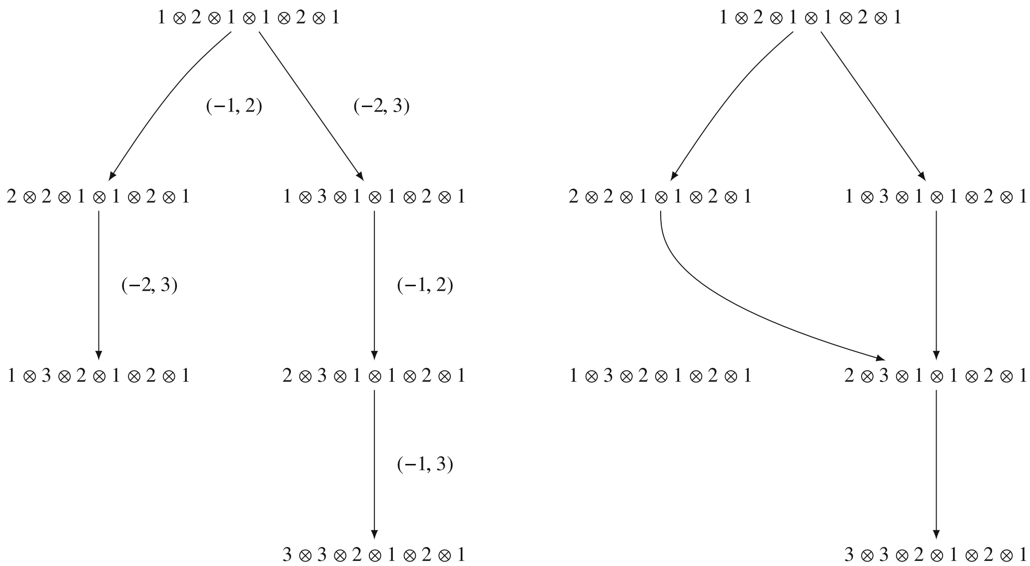


Fig. 3 **Left:** $G(C)$ for the crystals of Example 3. **Right:** $G(C')$ for the crystals of Example 3

do not give the queer crystal even though the induced crystal satisfies the axioms in Definition 3.

Next we show that the arrows in $G(C)$, where C is a connected component in $\mathcal{B}^{\otimes \ell}$, can be modeled by e_{-i} on type A highest weight elements.

Proposition 4 *Let C be a connected component in the $\mathfrak{q}(n+1)$-crystal $\mathcal{B}^{\otimes \ell}$. Let C_1 and C_2 be two distinct type A_n components in C and let u_2 be the I_0-highest weight element in C_2. Then there is an edge from C_1 to C_2 in $G(C)$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$.*

By Proposition 4, there is an edge from component C_1 to component C_2 in $G(C)$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$, where u_2 is the I_0-highest weight element of C_2.

We call the arrow *combinatorial* if $e_{-i}u_2$ is $\{1, 2, \ldots, i\}$-highest weight. Define $f_{(-i,h)} := f_{-i} f_{i+1} f_{i+2} \cdots f_{h-1}$.

Theorem 2 *Let C be a connected component in $\mathcal{B}^{\otimes \ell}$. Then each combinatorial edge in $G(C)$ can be obtained by $f_{(-i,h)}$ for some $i \in I_0$ and $h > i$ minimal such that $f_{(-i,h)}$ applies.*

In [19], we showed that it suffices to know the combinatorial edges to construct all vertices in $G(C)$. By Theorem 2, every combinatorial edge in the graph is labeled by the operator $f_{(-i,h)}$, where f_{-i} is given by the combinatorial rules stated in Theorem 1 and connects an I_0-highest weight element to another I_0-highest weight element. Hence, all vertices of $G(C)$ can be constructed from the $\mathfrak{q}(n+1)$-highest weight element u by the application of these combinatorial arrows.

Remark 2 The construction of the component graph of C with highest weight λ produces a Schur expansion of the Schur-P polynomial $P_\lambda(x_1, \ldots, x_{n+1})$. This expansion is obtained by counting the multiplicities of highest weights for all type A_n components that are present in $G(C)$. For example, the component graph in Example 3 yields the expansion $P_{42} = s_{42} + s_{33} + s_{411} + 2s_{321} + s_{222}$.

5 Characterization of Queer Crystals

Our main theorem gives a characterization of the queer supercrystals.

Theorem 3 *Let C be a connected component of a generic abstract queer crystal (see Definition 1). Suppose that C satisfies the following conditions:*

1. *C satisfies the local queer axioms of Definition 3.*
2. *C satisfies the connectivity axioms of [19, Definition 4.4].*
3. *$G(C)$ is isomorphic to $G(\mathcal{D})$, where $\mathcal{D}$ is some connected component of $\mathcal{B}^{\otimes \ell}$.*

Then the queer supercrystals C and $\mathcal{D}$ are isomorphic.

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