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On 3-braids and L-space knots

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Abstract We classify closed 3-braids which are L-space knots.

Keywords 3-braid · L-space · Jones polynomial

1 Introduction

A rational homology 3-sphere Y is an L-space if $|H_1(Y;\mathbb{Z})| = \text{rank}\ \widehat{HF}(Y)$, where $\widehat{HF}$ denotes the 'hat' version of Heegaard Floer homology, and the name stems from the fact that lens spaces are L-spaces. Besides lens spaces, examples of L-spaces include all connected sums of manifolds with elliptic geometry [Ozsváth and Szabó(2005)].

A prominent source of L-spaces arises from surgeries on knots. Suppose that K is a knot in S^3: if K admits a non-trivial surgery to an L-space, then K is an *L*-space knot. Examples include torus knots and, more generally, Berge knots in S^3. Various properties of L-space knots have been studied in the previous years; the two particularly pertinent to our work are about the Alexander polynomial $\triangle_K(t)$ of an L-space knot K:

- The absolute value of a nonzero coefficient of $\triangle_K(t)$ is 1. The set of nonzero coefficients alternates in sign [Ozsváth and Szabó(2005), Corollary 1.3].
- If g is the maximum degree of $\triangle_K(t)$ in t, then the coefficient of the term t^{g-1} is nonzero and therefore ± 1 [Hedden and Watson(2018)].

The purpose of this manuscript is to study which 3-braids, that close to form a knot, admit L-space surgeries. To state our result we need to prepare a definition: a twisted $(3, q)$ torus knot, denoted $K(3, q; 2, r)$ where $q \equiv \pm 1 \pmod 3$, is the

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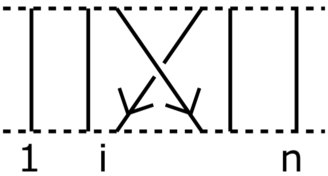

Fig. 1 The generator σ_i.

closure of a 3-braid made up of a $(3, q)$ torus braid with r full twist(s) on two adjacent strands. We prove that:

Theorem 1 *The only knots with 3-braid representatives that admit L-space surgeries are $K(3, q; 2, r)$ where $rq > 0$.*

Our proof of Theorem

2 The Jones polynomial, 3-braids, and the Alexander polynomial

We will first derive an expression of the Alexander polynomial of a closed 3-braid in terms of the Jones polynomial. Let B_n be the n-strand braid group. The *Burau representation* of B_n is a map ψ from B_n to $n-1 \times n-1$ matrices with entries in $\mathbb{Z}[t, t^{-1}]$.

$$\psi : B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}]).$$

For $n = 3$, ψ is defined explicitly on the generators σ_1, σ_2 (see Figure

Let a be an element of B_3, $\hat{a}$ be the closed braid, and e_a be the exponent sum of a. In general for $a \in B_n$, $\hat{a}$ being a knot implies that $n - 1 + e_a$ is even, since if $n - 1 + e_a$ is odd, a quick argument by visual inspection shows that $\hat{a}$ has more than one component. Thus for $a \in B_3$, where $\hat{a}$ is a knot, $2 \pm e_a$ is even and therefore e_a is even. The Jones polynomial $J_{\hat{a}}(t)$ of $\hat{a}$ can be written in terms of ψ [Jones(1985)]:

$$J_{\hat{a}}(t) = (-\sqrt{t})^{-e_a}(t + t^{-1} + \text{trace } \psi(a)). \qquad (1)$$

The sign change on e_a as compared to [Birman(1985), Eq. (5)], where this form of the equation is from, is due to the difference in convention on the Jones polynomial. When $n = 3$, the Alexander polynomial of $\hat{a}$ may also be written in terms of the trace of ψ [Birman(1985), Eq. (7)]:

$$(t^{-1} + 1 + t)\triangle_{\hat{a}}(t) = (-1)^{-e_a}(t^{-e_a/2} - t^{e_a/2}\text{trace } \psi(a) + t^{e_a/2}), \qquad (2)$$

with similar adjustments on the signs. Rearranging Equations (

This expression allows us to compute certain coefficients of the Alexander polynomial from the Jones polynomial for closed 3-braids. To organize the computation, we will use Schreier's [Schreier(1924)] normal form for each representative of a conjugacy class.

Theorem 2 *[Schreier(1924)] Let $b \in B_3$ be a braid on three strands, and C be the 3-braid $(\sigma_1\sigma_2)^3$. Then b is conjugate to a braid in exactly one of the following forms:*

1. $C^k\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, *where* $k\in\mathbb{Z}$ *and* p_i, q_i, *and* s *are all positive integers,*
2. $C^k\sigma_1^p$, *for* $k,p\in\mathbb{Z}$,
3. $C^k\sigma_1\sigma_2$, *for* $k\in\mathbb{Z}$,
4. $C^k\sigma_1\sigma_2\sigma_1$, *for* $k\in\mathbb{Z}$, *or*
5. $C^k\sigma_1\sigma_2\sigma_1\sigma_2$, *for* $k\in\mathbb{Z}$.

It suffices to study the 3-braids among the conjugacy representatives above to determine which closed 3-braid is an L-space knot. It is straightforward to check that $C^k\sigma_1^p$ and $C^k\sigma_1\sigma_2\sigma_1$ represent links for any $k,p\in\mathbb{Z}$. Also, since $C=(\sigma_1\sigma_2)^3$, we get that, for any $k\in\mathbb{Z}$, $C^k\sigma_1\sigma_2$ and $C^k\sigma_1\sigma_2\sigma_1\sigma_2$ represent the $(3,3k+1)$ and $(3,3k+2)$ torus knots, respectively. Thus we will only need to study class (

Recall that if a nontrivial knot K is an L-space knot, then the absolute value of a nonzero coefficient of the Alexander polynomial $\triangle_K(t)$ is 1, and the nonzero coefficients alternate in sign. Moreover, let g be the maximum degree of $\triangle_K(t)$ in t, then the coefficients of the term t^{g-1} is nonzero and therefore ±1. The symmetric Alexander polynomial has the two possible forms given below for an L-space knot:

$$t^g - t^{g-1} + \cdots + \text{terms in-between} - t^{-(g-1)} + t^{-g}$$

or

$$-t^g + t^{g-1} - \cdots - \text{terms in-between} + t^{-(g-1)} - t^{-g}.$$

Thus by a quick computation, we can conclude the following statement.

Lemma 1 *Suppose that a nontrivial knot K is an L-space knot, then the product*

$$\triangle_K(t)\cdot(t^{-1}+1+t)$$

is a symmetric polynomial with coefficients in $\{-1,0,1\}$*, which do not necessarily alternate in sign, and the second coefficient and the second-to-last coefficient are both zero.*

The conjugacy representatives of class (1) in Theorem

The braid $a=\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$ is called an *alternating 3-braid.* The first three coefficients and the last three coefficients of the Jones polynomial for the closure of this class of 3-braids, as well as the degree, are explicitly calculated in [Futer et al.(2010)Futer, Kalfagianni, and Purcell]. We assemble below the results we will need.

Definition 1 For an alternating 3-braid $a=\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, let

$$\mathbf{p}:=\sum_{i=1}^{s}p_i,\ \text{and}\ \mathbf{q}:=\sum_{i=1}^{s}q_i,$$

so the exponent sum $e_a=\mathbf{p}-\mathbf{q}$. We will call s the *index* of the alternating braid.

We have the following lemma from [Futer et al.(2010)Futer, Kalfagianni, and Purcell], phrased in terms of the notations in this paper with item (c) replaced by a result within the proof.

Lemma 2 *[Futer et al.(2010)Futer, Kalfagianni, and Purcell, Lemma 6.2] Suppose that a link $\hat{a}$ is the closure of an alternating 3-braid a:*

$$a = \sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s},$$

with $p_i, q_i, s > 0$ and $\mathbf{p} > 1$ and $\mathbf{q} > 1$, then the following holds.

(a) *The highest and lowest powers, $M(\hat{a})$ and $m(\hat{a})$ of $J_{\hat{a}}(t)$ in t are*

$$M(\hat{a}) = \frac{3\mathbf{q} - \mathbf{p}}{2} \text{ and } m(\hat{a}) = \frac{\mathbf{q} - 3\mathbf{p}}{2}.$$

(b) *The first two coefficients α, β in $J_{\hat{a}}(t)$ from $M(\hat{a})$, and the last two coefficients β', α' in $J_{\hat{a}}(t)$ from $m(\hat{a})$ are*

$$\alpha = (-1)^{\mathbf{p}}, \beta = (-1)^{\mathbf{p}+1}(s - \epsilon_{\mathbf{q}}), \beta' = (-1)^{\mathbf{q}+1}(s - \epsilon_{\mathbf{p}}), \alpha' = (-1)^{\mathbf{q}},$$

where $\epsilon_{\mathbf{p}} = 1$ if $\mathbf{p} = 2$ and 0 if $\mathbf{p} > 2$, and similarly for $\epsilon_{\mathbf{q}}$.

(c) *[Futer et al.(2010)Futer, Kalfagianni, and Purcell, Eq. (14) in the proof of Lemma 6.2] Let γ, γ' denote the third and the third-to-last coefficient of $J_{\hat{a}}(t)$, respectively. We have*

$$(-1)^{\mathbf{p}}\gamma = \frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{q}=3},$$

and

$$(-1)^{\mathbf{q}}\gamma' = \frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{p}=3},$$

where $\delta_{\mathbf{q}=3}$ is zero if $\mathbf{q} \neq 3$ and 1 otherwise, and $\delta_{\mathbf{p}=3}$ is similarly defined.

Note that item (c) implies the original statement in item (c) of [Futer et al.(2010)Futer, Kalfagianni, and Purcell, Lemma 6.2].

The next result writes the Jones polynomial of the closure of a generic 3-braid in terms of the Jones polynomial of the closure of an alternating braid. Again, the statement is phrased in terms of the notations in this paper.

Lemma 3 *[Futer et al.(2010)Futer, Kalfagianni, and Purcell, Lemma 6.3] If b is a generic 3-braid of the form*

$$b = C^k a,$$

where a is an alternating 3-braid, and let $J_{\hat{b}}(t)$ denote the Jones polynomial of the closure $\hat{b}$, then

$$J_{\hat{b}}(t) = t^{-6k}J_{\hat{a}}(t) + (-\sqrt{t})^{-e_a}(t + t^{-1})(t^{-3k} - t^{-6k}).$$

By Equation (

We are now ready to determine which closed 3-braids are L-space knots. The proof is outlined as follows: We will consider generic 3-braids and first rule out the cases where the index s, as defined in Definition

The case $\mathbf{s} = \mathbf{2}$*:* Note that $\mathbf{q} \geq 2, \mathbf{p} \geq 2$ so the first, last, second, and penultimate coefficients are distinct by Lemma 2 (a) since $M(\hat{a}) - m(\hat{a}) \geq 4$. If we have a strict inequality $\mathbf{p} > 2$ or $\mathbf{q} > 2$, then the third and third-to-last coefficients are also distinct.

If $\mathbf{p} > 2$, then from Lemma

The case $\mathbf{s} = \mathbf{1}$*:* Assuming that $\mathbf{p} > 3$ and $\mathbf{q} \geq 1$ or $\mathbf{q} > 3$ and $\mathbf{p} \geq 1$, then the first, last, second, penultimate, third, and third-to-last coefficients are distinct since $M(\hat{a}) - m(\hat{a}) > 4$ by Lemma 2 (a). The absolute values of the third coefficient γ and the third-to-last coefficient γ' of $-t^{e_a} J_{\hat{a}}(t)$ have the form

$$|\gamma| = \left| \left(\frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{q}=3} \right) \right|,$$

and

$$|\gamma'| = \left| \left(\frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{p}=3} \right) \right|,$$

by Lemma

Setting $(\mathbf{q} + \mathbf{p})/2 - 2$ equal to $(\mathbf{p} - \mathbf{q})/2 - 1$ or $(\mathbf{p} - \mathbf{q})/2 + 1$ gives $\mathbf{q} = 1$ or $\mathbf{q} = 3$. Setting $(\mathbf{q} + \mathbf{p})/2 - 2$ equal to $-3k - (\mathbf{p} - \mathbf{q})/2$ or $3k + (\mathbf{p} - \mathbf{q})/2$ gives $\mathbf{p} = -3k + 2$ or $\mathbf{q} = 3k + 2$. Similarly, if $\mathbf{p} > 3$ and $\mathbf{q} > 1$, then we must have $\mathbf{p} = 3$, $\mathbf{p} = 1$, $\mathbf{q} = 3k + 2$, or $\mathbf{p} = -3k + 2$.

We consider two cases: 1) $\mathbf{p} \geq 3$ and $\mathbf{q} > 3$, and 2) $\mathbf{p} > 3$ and $\mathbf{q} \geq 3$. We suppose that $k \neq 0$. By the arguments made in the previous paragraph, we have that $|\gamma|$ or $|\gamma'| = 2$, each of which needs to be canceled out by a term in

$$t^{-3k-(\mathbf{p}-\mathbf{q})/2},\ t^{3k+(\mathbf{p}-\mathbf{q})/2}.$$

If $\mathbf{p} \geq 3$, $\mathbf{q} > 3$, then $|\gamma'| = 2$ and we have that $(\mathbf{q} + \mathbf{p})/2 - 2 = -3k - (\mathbf{p} - \mathbf{q})/2$ or $3k + (\mathbf{p} - \mathbf{q})/2$, which means $\mathbf{p} = -3k + 2$ or $\mathbf{q} = 3k + 2$. We cannot have that $\mathbf{p} = -3k+2$ *and* $\mathbf{q} = 3k+2$ since they are both supposed to be positive. Therefore we suppose that $\mathbf{p} = -3k + 2$ or $\mathbf{q} = 3k + 2$. In the first case, k is negative, and $-(\mathbf{p} + \mathbf{q})/2+2 = 3k+(\mathbf{p} - \mathbf{q})/2$ as well as $(\mathbf{q} + \mathbf{p})/2-2 = -3k-(\mathbf{p}-\mathbf{q})/2$. In the second case, k is positive and $-(\mathbf{p} + \mathbf{q})/2 + 2 = -3k - (\mathbf{p} - \mathbf{q})/2$, $(\mathbf{p} + \mathbf{q})/2 - 2 = 3k + (\mathbf{p} - \mathbf{q})/2$. This means that γ, γ' potentially cancel with the coefficients of $t^{-3k-e_a/2}$ and $t^{3k+e_a/2}$ in Equation (. Either way, we end up having, for $k < 0$,

$$\begin{aligned}&\triangle_K(t) \cdot (t^{-1} + 1 + t) \\ &= \pm t^{-\frac{-3k+2+\mathbf{q}}{2}} \mp t^{-\frac{-3k+2+\mathbf{q}}{2}+1} \pm \underbrace{c t^{-\frac{-3k+2+\mathbf{q}}{2}+2}}_{\text{contribution from } \gamma} \mp \cdots \pm \underbrace{c' t^{\frac{-3k+2+\mathbf{q}}{2}-2}}_{\text{contribution from } \gamma'} \mp t^{\frac{-3k+2+\mathbf{q}}{2}-1} \pm t^{\frac{-3k+2+\mathbf{q}}{2}} \\ &+ \underbrace{t^{\frac{-3k+2-\mathbf{q}}{2}-1} + t^{\frac{-3k+2-\mathbf{q}}{2}+1}}_{\text{do not contribute to the extremal coefficients } \alpha, \alpha', \beta, \beta' \text{ shown in previous line, since it would force } \mathbf{q} = -1, 0, 1 \text{ or } 2},\end{aligned}$$

after cancelling the third and third-to-last coefficients of $J_{\hat{a}}(t)$ with $t^{-3k-e_a/2}$ and $t^{3k+e_a/2}$. Here c, c' are coefficients of the third and third-to-last term of the first line that may cancel with the coefficients on the next line. (If they do not cancel or $\mathbf{p} = 3$ making $|\gamma| = 1$, we can immediately rule out the possibility that $\hat{b}$ is an L-space knot, since the resulting Alexander polynomial has no chance of satisfying the conditions of Lemma

$$\begin{aligned}
&\triangle_K(t) \cdot (t^{-1} + 1 + t) \\
&= \pm t^{-\frac{3k+2+\mathbf{p}}{2}} \mp t^{-\frac{3k+2+\mathbf{p}}{2}+1} \pm \underbrace{c t^{-\frac{3k+2+\mathbf{p}}{2}+2}}_{\text{contribution from } \gamma} \mp \cdots \pm \underbrace{c' t^{\frac{3k+2+\mathbf{p}}{2}-2}}_{\text{contribution from } \gamma'} \mp t^{\frac{3k+2+\mathbf{p}}{2}-1} \pm t^{\frac{3k+2+\mathbf{p}}{2}} \\
&+ \underbrace{t^{\frac{-(3k+2)+\mathbf{p}}{2}-1} + t^{\frac{-(3k+2)+\mathbf{p}}{2}+1}}_{\text{do not contribute to the extremal coefficients } \alpha, \alpha', \beta, \beta' \text{ shown in previous line, since it would force } \mathbf{p} = -1, 0, 1 \text{ or } 2} .
\end{aligned}$$

One of the conditions from Lemma

For the remaining cases of $\mathbf{p}, \mathbf{q}$, we argue by directly looking at the Alexander polynomial without involving coefficients γ, γ'. When both $\mathbf{p}, \mathbf{q} = 3$, we have that the alternating 3-braid a takes the form $\sigma_1^3 \sigma_2^{-3}$. The Alexander polynomial of this alternating 3-braid is

$$\triangle_a(t) = 3 + \frac{1}{t^2} - \frac{2}{t} - 2t + t^2,$$

obtained by multiplying the Alexander polynomial of the trefoil by itself, since this 3-braid is a connected sum of two (right-hand and left-hand) trefoils. It is clear from the Alexander polynomial that this knot cannot be an L-space knot due to the fact that several of its nonzero coefficients are not ± 1. Now we consider a generic 3-braid $b = C^k a$ with $a = \sigma_1^3 \sigma_2^{-3}$. Since $e_a = 0$, the highest power and the lowest power of the Jones polynomial of $\hat{a}$ are 3 and -3. By Equation (,

$$\triangle_{\hat{b}}(t)(t^{-1} + 1 + t) = -J_{\hat{a}}(t) + \frac{1}{t} + t + t^{-3k} + t^{3k},$$

where

$$J_{\hat{a}}(t) = 3 - \frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - t + t^2 - t^3.$$

When $k \neq 0$, it is clear that the constant term 3 of $J_{\hat{a}}(t)$ will not be canceled out by the terms $\frac{1}{t}, t, t^{-3k}$, or t^{-3k}. Thus none of the closures of braids of the form $C^k \sigma_1^3 \sigma_2^{-3}$ will be an L-space knot.

We may also rule out the case $\mathbf{p} = 2$ or $\mathbf{q} = 2$ since this would give a link rather than a knot. After also ruling out the cases which would make e_a odd, the remaining generic 3-braids whose closure can be an L-space knot are given below.

Table 1 Remaining generic 3-braids whose closures can be L-space knots.

$C^k \sigma_1^1 \sigma_2^{-q}$	for q odd.
$C^k \sigma_1^p \sigma_2^{-1}$	for p odd.

We now claim that $C^k\sigma_1^p\sigma_2^{-1}$, for p odd and $k > 0$, represents an L-space knot. Given $(\sigma_1\sigma_2\sigma_1)^2 = (\sigma_2\sigma_1)^3$ by the braid relations $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, note that:

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1)^{2k}\sigma_1^p\sigma_2^{-1} &\sim (\sigma_2\sigma_1)^{3k}\sigma_1^p\sigma_2^{-1}\\ &\sim (\sigma_2\sigma_1)^{3k-1}\sigma_1^{p+1}.\end{aligned}$$

The latter braid is the twisted torus knot, $K(3, 3k-1; 2, \frac{p+1}{2})$, which is known to be an L-space knot [Vafaee(2014), Theorem 3.1].

Now if $k < 0$ then

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1)^{2k}\sigma_1^p\sigma_2^{-1} &\sim (\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})^{-k}\sigma_1^p\sigma_2^{-1}\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})^{-k}\sigma_1^p\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^{-k}\sigma_1^p\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^{-k}\sigma_1^{-1}\sigma_1\sigma_1^p\\ &\sim (\sigma_2^{-1}\sigma_1^{-1})^{-3k+1}\sigma_1^{p+1}\\ &\sim (\sigma_1\sigma_2)^{3k-1}\sigma_1^{p+1}.\end{aligned}$$

Using the explicit form of the Alexander polynomial of a $(3, 3k-1)$ torus knot with $\frac{p+1}{2}$ full twists in adjacent strings [Morton(2006), Theorem 4], we see that none of these knots can be an L-space knot.

For $k < 0$, a similar argument shows that the closure of $C^k\sigma_1^1\sigma_2^{-q}$, for q odd, also represents an L-space knot. Notice that in this case the knot is isotopic to the closure of $(\sigma_2\sigma_1)^{3k+1}\sigma_2^{-q-1}$, so its mirror image admits a positive L-space surgery. Looking at the Alexander polynomial of the closure of the braid $C^k\sigma_1^1\sigma_2^{-q} \sim (\sigma_1\sigma_2)^{3k+1}\sigma_2^{-q-1}$ for $k > 0$ also shows that it does not represent an L-space knot.

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