

L^∞ NORM ERROR ESTIMATES FOR HDG METHODS APPLIED TO THE POISSON EQUATION WITH AN APPLICATION TO THE DIRICHLET BOUNDARY CONTROL PROBLEM*

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Abstract. We prove quasi-optimal L^∞ norm error estimates (up to logarithmic factors) for the solution of Poisson’s problem in two dimensional space by the standard hybridizable discontinuous Galerkin (HDG) method. Although such estimates are available for conforming and mixed finite element methods, this is the first proof for HDG. The method of proof is motivated by known L^∞ norm estimates for mixed finite elements. We show two applications: the first is to prove optimal convergence rates for boundary flux estimates, and the second is to prove that numerically observed convergence rates for the solution of a Dirichlet boundary control problem are to be expected theoretically. Numerical examples show that the predicted rates are seen in practice.

Key words. L^∞ error analysis, HDG method, Dirichlet boundary control

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1. Introduction. In this paper we derive L^∞ norm estimates for the standard hybridizable discontinuous Galerkin (HDG) method applied to a diffusion problem. The problem is posed on a bounded convex polyhedral domain $\Omega \subset \mathbb{R}^2$. We seek to approximate the solution $(u, \mathbf{q})$ of the following elliptic system:

$$\begin{aligned} (1.1a) \quad & c\mathbf{q} + \nabla u = 0 \quad \text{in } \Omega, \\ (1.1b) \quad & \nabla \cdot \mathbf{q} = f \quad \text{in } \Omega, \\ (1.1c) \quad & u = g \quad \text{on } \partial\Omega. \end{aligned}$$

We assume the data is given as follows: the diffusivity $c \in (W^{1,\infty}(\Omega))^{2 \times 2}$ is a uniformly bounded positive definite symmetric matrix-valued function, and the functions c, f, g and the domain Ω are such that the solution $(u, \mathbf{q}) \in L^\infty(\Omega) \times \mathbf{L}^\infty(\Omega)$ (see Remark 3.2). In particular, we shall prove quasi-optimal L^∞ error estimates (up to logarithmic factors) for the HDG approximation to u and $\mathbf{q}$. We also verify that, after a standard procedure, the postprocessed solution denoted u_h^* is superconvergent in the L^∞ norm.

Quasi-optimal L^∞ norm estimates on general quasi-uniform meshes for the conforming finite element method were first proved by Scott [34] in 1976. The method of proof is based on weighted L^2 norms and was extended in [33, 13, 11, 12, 14, 36] to

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mixed methods for elliptic equations and in [16, 15] to the Stokes equations. Another technique was developed in the series of papers by Schatz and Wahlbin [30, 31, 32]. They use dyadic decomposition of the domain and require local energy estimates together with sharp pointwise estimates for the corresponding components of the Green's matrix. For smooth domains such a technique was successfully used in [7] for mixed methods, in [19, 24] for discontinuous Galerkin (DG) methods, and in [6] for local DG methods. This technique was also applied on a nonsmooth domain for the Stokes equations; see Guzmán and Leykekhman [20].

The HDG method for elliptic equations was devised by Cockburn, Gopalakrishnan, and Lazarov [9] and was analyzed using a special projection in [10]. Since there is a strong relation between the HDG method and mixed finite element methods (see [9]), it is reasonable to ask if similar $L^\infty(\Omega)$ norm estimates on general quasi-uniform meshes could be obtained for the HDG method. To the best of our knowledge, there is no such result in the literature. In section 3, we give quasi-optimal L^∞ norm estimates for the flux variable $\mathbf{q}$ and scalar variable u (see Theorem 3.1). One advantage of the HDG method is that we can obtain superconvergent rates of convergence for a postprocessed approximation u_h^* to u in the $L^2(\Omega)$ norm [9]. In Theorem 3.1, we show that the postprocessed solution also enjoys superconvergent rates in the L^∞ norm. We present a numerical test in Example 6.1 (see Table 1) to confirm our theoretical results from Theorem 3.1. As mentioned in [26], we can use our L^∞ norm estimates to improve $L^2(\Gamma)$ norm estimates on an interface Γ ; see Theorem 4.1. The numerical test in Example 6.1 (see Table 2) confirms the theoretical result from Theorem 4.1. It is worthwhile to mention that a standard analysis of convergence on an interface (usually via the trace theorem) only gives a suboptimal convergence rate.

The optimal $L^2(\Gamma)$ norm estimates on an interface Γ or on the boundary of the domain have many applications. One example [26] is where some complex problems require the use of a variety of models in different parts of the computational domain, which in turn are coupled through the normal flux across common interfaces. On the level of numerical methods, this entails a need to understand and quantify the discretization error in the normal flux at interfaces [26]. Another example appears in the problem of Dirichlet boundary control of PDEs with $L^2(\partial\Omega)$ -regularization, where the normal derivative naturally arises in the discrete optimality system. Hence, the estimation of the error in the normal derivative plays an essential role in the error analysis of the Dirichlet boundary control of PDEs; see [21, 27, 1, 29, 37] for more details. In recent papers where HDG methods have been successfully applied to the Dirichlet boundary control of PDEs [5, 23, 22, 17, 18], the analysis for the control is optimal in the sense of regularity and suboptimal for other variables. Furthermore, numerical experiments show that the discrete control can achieve optimal convergence with respect to the polynomial degree if the control is smooth enough. However, the analysis in the above-mentioned HDG papers is suboptimal in this situation. In section 5, we use the improved L^2 norm estimates on the boundary in Theorem 4.1 to obtain an optimal convergence rate for both the control and the other variables; see Theorem 5.1. The numerical test in Example 6.2 confirms our theoretical result.

2. HDG formulation and preliminary material. In this section, we shall give the HDG formulation of (1.1) and introduce some standard auxiliary projections. Our main result in this section is to extend the L^2 norm estimates for the auxiliary projections used in the error analysis of HDG to L^p norms ($1 \leq p \leq \infty$); see Theorem 2.6. This is one essential step of the paper. Although our final L^∞ norm estimates require the domain to be two dimensional and convex, it is worth mentioning that

we do not need these restrictions in Theorem 2.6. Hence, in the present section, we assume that $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) and do not assume convexity.

Throughout the paper we adopt the standard notation $W^{m,p}(D)$ for Sobolev spaces on a bounded domain $D \subset \mathbb{R}^d$ ($d = 2, 3$) with norm $\|\cdot\|_{W^{m,p}(D)}$ and seminorm $|\cdot|_{W^{m,p}(D)}$:

$$\|u\|_{W^{m,p}(D)}^p = \sum_{|i| \leq m} \int_D |D^i u|^p d\mathbf{x}, \quad |u|_{W^{m,p}(D)}^p = \sum_{|i|=m} \int_D |D^i u|^p d\mathbf{x},$$

where i is a multi-index and D^i is the corresponding partial differential operator of order $|i|$. We denote $W^{m,2}(D)$ by $H^m(D)$ with norm $\|\cdot\|_{H^m(D)}$ and seminorm $|\cdot|_{H^m(D)}$. Specifically, $H_0^1(D) = \{v \in H^1(D) : v = 0 \text{ on } \partial D\}$. We denote the L^2 -inner products on $L^2(D)$ and $L^2(S)$ by

$$(v, w)_D = \int_D v w \, d\mathbf{x} \quad \forall v, w \in L^2(D), \quad \langle v, w \rangle_S = \int_S v w \, d\mathbf{x} \quad \forall v, w \in L^2(S),$$

where $S \subset \partial D$. Finally, we define the space $\mathbf{H}(\text{div}, \Omega)$ as

$$\mathbf{H}(\text{div}, \Omega) = \{\mathbf{v} \in [L^2(\Omega)]^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$$

Let $\mathcal{T}_h$ be a shape-regular simplicial family of triangulations of Ω so that there exists a constant ζ , independent of h and K , such that

$$\zeta_K := \frac{h_K}{\rho_K} \leq \zeta \quad \forall K \in \mathcal{T}_h,$$

where h_K is the diameter of K and ρ_K is the diameter of the largest inscribed circle in K .

We denote by $\partial\mathcal{T}_h$ the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element K of the mesh $\mathcal{T}_h$, let $F = \partial K \cap \partial\Omega$ denotes the boundary face of K having nonzero $d-1$ dimensional Lebesgue measure. Let $\mathcal{F}_h^\partial$ be the set of boundary faces and $\mathcal{F}_h$ denote the set of all faces. We define the following mesh dependent inner product, spaces, and norms by

$$\begin{aligned} (w, v)_{\mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} (w, v)_K, & \langle \zeta, \rho \rangle_{\partial\mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}, \\ H^s(\mathcal{T}_h) &= \prod_{K \in \mathcal{T}_h} H^s(K), & L^s(\partial\mathcal{T}_h) &= \prod_{K \in \mathcal{T}_h} L^s(\partial K), \\ \|w\|_{\mathcal{T}_h} &= \left(\sum_{K \in \mathcal{T}_h} \|w\|_{L^2(K)}^2 \right)^{1/2}, & \|w\|_{L^p(\partial\mathcal{T}_h)} &= \left(\sum_{K \in \mathcal{T}_h} \|w\|_{L^p(K)}^p \right)^{1/p}, \quad 1 \leq p < \infty. \end{aligned}$$

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most k on a domain D . We introduce the discontinuous finite element spaces used in the HDG method as follows:

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v}_h \in [L^2(\Omega)]^d : \mathbf{v}_h|_K \in [\mathcal{P}^k(K)]^d \quad \forall K \in \mathcal{T}_h\}, \\ W_h &:= \{w_h \in L^2(\Omega) : w_h|_K \in \mathcal{P}^k(K) \quad \forall K \in \mathcal{T}_h\}, \\ \widehat{W}_h &:= \{\widehat{w}_h \in L^2(\mathcal{F}_h) : \widehat{w}_h|_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h; \widehat{w}_h|_F = 0 \quad \forall F \in \mathcal{F}_h^\partial\}. \end{aligned}$$

2.1. HDG formulation. To simplify the presentation, we assume the Dirichlet boundary condition is homogeneous, i.e., $g = 0$. Then the HDG method of Cockburn, Gopalakrishnan, and Lazarov [9] seeks the flux $\mathbf{q}_h \in \mathbf{V}_h$, the scalar variable $u_h \in W_h$, and its numerical trace $\widehat{u}_h \in \widehat{W}_h$ satisfying

$$\begin{aligned} (2.1a) \quad & (c\mathbf{q}_h, \mathbf{v}_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{v}_h)_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = 0, \\ (2.1b) \quad & -(\mathbf{q}_h, \nabla w_h)_{\mathcal{T}_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w_h \rangle_{\partial\mathcal{T}_h} = (f, w_h)_{\mathcal{T}_h}, \\ (2.1c) \quad & \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \widehat{w}_h \rangle_{\partial\mathcal{T}_h} = 0 \end{aligned}$$

for all $(\mathbf{v}_h, w_h, \widehat{w}_h) \in \mathbf{V}_h \times W_h \times \widehat{W}_h$. The numerical flux on $\partial\mathcal{T}_h$ is defined by [10]

$$(2.1d) \quad \widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \quad \text{on } \partial\mathcal{T}_h,$$

where the stabilization parameter $\tau \in L^\infty(\mathcal{F}_h)$ is uniformly positive and bounded. For simplicity, we consider the stabilization function τ to be constant on the boundary of each element.

After computing the solution $(\mathbf{q}_h, u_h, \widehat{u}_h)$ of (2.1), we can use the following element-by-element postprocessing to find $u_h^*|_K \in \mathcal{P}^{k+1}(K)$ such that for all $(z_h, w_h) \in [\mathcal{P}^{k+1}(K)]^\perp \times \mathcal{P}^0(K)$,

$$\begin{aligned} (2.2a) \quad & (\nabla u_h^*, \nabla z_h)_K = -(c\mathbf{q}_h, \nabla z_h)_K, \\ (2.2b) \quad & (u_h^*, w_h)_K = (u_h, w_h)_K, \end{aligned}$$

where $[\mathcal{P}^{k+1}(K)]^\perp = \{z_h \in \mathcal{P}^{k+1}(K) : (z_h, 1)_K = 0\}$.

To shorten lengthy equations, we define the following HDG bilinear form $\mathcal{B} : \mathbf{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h) \rightarrow \mathbb{R}$ by

$$(2.3) \quad \begin{aligned} \mathcal{B}(\mathbf{q}, u, \widehat{u}; \mathbf{v}, w, \widehat{w}) = & (c\mathbf{q}, \mathbf{v})_{\mathcal{T}_h} - (u, \nabla \cdot \mathbf{v})_{\mathcal{T}_h} + \langle \widehat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ & - (\nabla \cdot \mathbf{q}, w)_{\mathcal{T}_h} - \langle \tau(u - \widehat{u}), w - \widehat{w} \rangle_{\partial\mathcal{T}_h} + \langle \mathbf{q} \cdot \mathbf{n}, \widehat{w} \rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

By the definition of $\mathcal{B}$ in (2.3), we can rewrite the HDG formulation of system (2.1), as follows: find $(\mathbf{q}_h, u_h, \widehat{u}_h) \in \mathbf{V}_h \times W_h \times \widehat{W}_h$ such that

$$(2.4) \quad \mathcal{B}(\mathbf{q}_h, u_h, \widehat{u}_h; \mathbf{v}_h, w_h, \widehat{w}_h) = -(f, w_h)_{\mathcal{T}_h}$$

for all $(\mathbf{v}_h, w_h, \widehat{w}_h) \in \mathbf{V}_h \times W_h \times \widehat{W}_h$. Moreover, the exact solution $(\mathbf{q}, u)$ also satisfies (2.4), i.e.,

$$(2.5) \quad \mathcal{B}(\mathbf{q}, u, u; \mathbf{v}_h, w_h, \widehat{w}_h) = -(f, w_h)_{\mathcal{T}_h}$$

for all $(\mathbf{v}_h, w_h, \widehat{w}_h) \in \mathbf{V}_h \times W_h \times \widehat{W}_h$.

From [4, Lemma 2] we recall the following stability result.

LEMMA 2.1. *For any $(\mathbf{q}_h, u_h, \widehat{u}_h) \in \mathbf{V}_h \times W_h \times \widehat{W}_h$, we have*

$$\mathcal{B}(\mathbf{q}_h, u_h, \widehat{u}_h; \mathbf{q}_h, -u_h, -\widehat{u}_h) = (c\mathbf{q}_h, \mathbf{q}_h)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\mathcal{T}_h}.$$

The following lemma shows that the bilinear form $\mathcal{B}$ is symmetric and is proved by integration by parts. We do not provide details.

LEMMA 2.2. *For any $(\mathbf{q}, u, \widehat{u}; \mathbf{v}, w, \widehat{w}) \in \mathbf{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \times L^2(\partial\mathcal{T}_h)$, we have*

$$(2.6) \quad \mathcal{B}(\mathbf{q}, u, \widehat{u}; \mathbf{v}, w, \widehat{w}) = \mathcal{B}(\mathbf{v}, w, \widehat{w}; \mathbf{q}, u, \widehat{u}).$$

2.2. Preliminary material. Recall the HDG projection $\Pi_h(\mathbf{q}, u) := (\Pi_V \mathbf{q}, \Pi_W u) \in \mathbf{V}_h \times W_h$ (see [10, equation (2.1a)–(2.1c)]) that satisfies the following equations:

$$(2.7a) \quad (\Pi_V \mathbf{q}, \mathbf{v}_h)_K = (\mathbf{q}, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in [\mathcal{P}^{k-1}(K)]^d,$$

$$(2.7b) \quad (\Pi_W u, w_h)_K = (u, w_h)_K \quad \forall w \in \mathcal{P}^{k-1}(K),$$

$$(2.7c) \quad \langle \Pi_V \mathbf{q} \cdot \mathbf{n} + \tau \Pi_W u, \hat{w}_h \rangle_F = \langle \mathbf{q} \cdot \mathbf{n} + \tau u, \hat{w}_h \rangle_F \quad \forall \hat{w}_h \in \mathcal{P}^k(F)$$

for all faces F of the simplex K . If $k = 0$, then (2.7a) and (2.7b) are vacuous and Π_h is defined solely by (2.7c). Note that although we denoted the first component of the projection by $\Pi_V \mathbf{q}$, it depends not just on $\mathbf{q}$ but on both $\mathbf{q}$ and u , as we see from (2.7). The same is true for $\Pi_W u$. Hence the notation $(\Pi_V \mathbf{q}, \Pi_W u)$ for $\Pi_h(\mathbf{q}, u)$ is somewhat misleading, but its convenience outweighs this disadvantage.

It is worthwhile mentioning that the domain of the projection Π_h is a subspace of $[L^2(\Omega)]^d \times L^2(\Omega)$ on which the right-hand sides of (2.7) are well defined. We do not require that the two components $(\mathbf{q}, u)$ satisfy (1.1a).

The well-posedness of (Π_V, Π_W) and its approximation properties are given in the following Lemma 2.3. The proof can be found in [10, Appendix].

LEMMA 2.3. *Suppose $\tau|_{\partial K}$ is positive and is a constant on each face $F \subset \partial K$. Then the system (2.7) is uniquely solvable for $\Pi_V \mathbf{q}$ and $\Pi_W u$. Furthermore, there is a constant C independent of K and τ such that*

$$(2.8a) \quad \|\Pi_W u - u\|_{L^2(K)} \leq Ch_K^{\ell_u+1} |u|_{H^{\ell_u+1}(K)} + C \frac{h_K^{\ell_q+1}}{\tau_K^{\max}} |\nabla \cdot \mathbf{q}|_{H^{\ell_q}(K)},$$

$$(2.8b) \quad \|\Pi_V \mathbf{q} - \mathbf{q}\|_{L^2(K)} \leq Ch_K^{\ell_q+1} |\mathbf{q}|_{H^{\ell_q+1}(K)} + Ch_K^{\ell_u+1} \tau_K^* |u|_{H^{\ell_u+1}(K)}$$

for $\ell_u, \ell_q \in [0, k]$. Here $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$, where F^* is a face of K at which $\tau|_{\partial K}$ is maximum.

Besides the projections Π_V and Π_W , in the analysis we also need to introduce the standard local L^2 projection operators $\Pi_\ell^o : L^2(K) \rightarrow \mathcal{P}^\ell(K)$ and $\Pi_k^o : L^2(F) \rightarrow \mathcal{P}^k(F)$ satisfying

$$(2.9a) \quad (\Pi_\ell^o w, w_h)_K = (w, w_h)_K \quad \forall w_h \in \mathcal{P}^\ell(K),$$

$$(2.9b) \quad \langle \Pi_k^o w, \hat{w}_h \rangle_F = \langle w, \hat{w}_h \rangle_F \quad \forall \hat{w}_h \in \mathcal{P}^k(F).$$

We use Π_ℓ^o to denote the local vector L^2 projection operator; the definition componentwise is the same as local scalar L^2 projection operator. The next lemma gives the approximation properties of Π_ℓ^o , and its proof can be found in [35, Theorem 3.3.3, Theorem 3.3.4].

LEMMA 2.4. *Let $\ell \geq 0$ be an integer and $\rho \in [1, +\infty]$. If $(\ell + 1)\rho < d$, then we require d, ρ , and ℓ to also satisfy $2 \leq \frac{d\rho}{d-(\ell+1)\rho}$. For $j \in \{0, 1, \dots, \ell + 1\}$, if s_j satisfies*

$$(2.10) \quad \begin{cases} \rho \leq s_j \leq \frac{d\rho}{d-(\ell+1-j)\rho} & (\ell + 1 - j)\rho < d, \\ \rho \leq s_j < \infty & (\ell + 1 - j)\rho = d, \\ \rho \leq s_j \leq \infty & (\ell + 1 - j)\rho > d, \end{cases}$$

then there exists a constant C which is independent of K such that

$$(2.11a) \quad \|\nabla^j (\Pi_\ell^o u - u)\|_{L^{s_j}(K)} \leq Ch_K^{\ell+1-j+\frac{d}{s_j}-\frac{d}{\rho}} |u|_{W^{\ell+1,\rho}(K)},$$

$$(2.11b) \quad \|\nabla^j (\Pi_\ell^o u - u)\|_{L^{s_j}(\partial K)} \leq Ch_K^{\ell+1-j+\frac{d-1}{s_j}-\frac{d}{\rho}} |u|_{W^{\ell+1,\rho}(K)}.$$

In the analysis, we also need the following standard inverse inequality [35, Theorem 3.4.1].

LEMMA 2.5 (inverse inequality). *Let $k \geq 0$ be an integer and $\mu, \rho \in [1, +\infty]$; then there exists C depend on k, μ, ρ , and d such that*

$$(2.12) \quad |u_h|_{t,\mu,K} \leq Ch_K^{\frac{d}{\mu} - \frac{d}{\rho} - t + s} |u_h|_{s,\rho,K} \quad \forall u_h \in \mathcal{P}_k(K), \quad t \geq s.$$

In the analysis, not only are the L^2 approximation properties of (Π_V, Π_W) important for us, but the L^∞ approximation of these projection operators plays an essential role. We provide these estimates in the next theorem.

THEOREM 2.6. *Let $k \geq 0$ be an integer and $\rho \in [1, +\infty]$. If $(k+1)\rho < d$, then we need d, ρ , and k to satisfy $2 \leq \frac{d\rho}{d-(k+1)\rho}$. For $j \in \{0, 1, \dots, k+1\}$, if s_j satisfies (2.10), then*

(2.13a)

$$\|\Pi_W u - u\|_{L^{s_j}(K)} \leq \frac{C}{\tau} h_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |\nabla \cdot \mathbf{q}|_{W^{k,\rho}(K)} + Ch_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |u|_{W^{k+1,\rho}(K)},$$

(2.13b)

$$\|\Pi_V \mathbf{q} - \mathbf{q}\|_{L^{s_j}(K)} \leq Ch_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |\mathbf{q}|_{W^{k+1,\rho}(K)} + Ch_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |u|_{W^{k+1,\rho}(K)}.$$

Proof. First, we prove (2.13a). In the proof of [10, Proposition A.2.] we have

$$(2.14) \quad \begin{aligned} \|\Pi_W u - \Pi_k^o u\|_{L^2(K)} &\leq C \frac{h_K}{\tau} (\|\nabla \cdot \mathbf{q} - \Pi_{k-1}^o(\nabla \cdot \mathbf{q})\|_{L^2(K)}) \\ &\quad + C (\|u - \Pi_k^o u\|_{L^2(K)} + h_K \|\nabla(u - \Pi_k^o u)\|_{L^2(K)}). \end{aligned}$$

Then, using the local inverse estimate in Lemma 2.5,

$$\begin{aligned} \|\Pi_W u - u\|_{L^{s_j}(K)} &\leq \|\Pi_W u - \Pi_k^o u\|_{L^{s_j}(K)} + \|\Pi_k^o u - u\|_{L^{s_j}(K)} \\ &\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \|\Pi_W u - \Pi_k^o u\|_{L^2(K)} + \|\Pi_k^o u - u\|_{L^{s_j}(K)} && \text{by (2.12)} \\ &\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \left(\frac{h_K}{\tau} \|\nabla \cdot \mathbf{q} - \Pi_{k-1}^o(\nabla \cdot \mathbf{q})\|_{L^2(K)} \right. \\ &\quad \left. + \|u - \Pi_k^o u\|_{L^2(K)} + h_K \|\nabla(u - \Pi_k^o u)\|_{L^2(K)} \right) && \text{by (2.14)} \\ &\quad + \|\Pi_k^o u - u\|_{L^{s_j}(K)} \\ &\leq \frac{C}{\tau} h_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |\nabla \cdot \mathbf{q}|_{W^{k,\rho}(K)} + Ch_K^{k+1+\frac{d}{s_j}-\frac{d}{\rho}} |u|_{W^{k+1,\rho}(K)} && \text{by (2.11a).} \end{aligned}$$

Next, we prove (2.13b). Because we do not have an estimate like (2.14) for Π_V , we introduce the single face HDG projection $\mathbf{B}_V$ defined on sufficiently smooth vector functions $\mathbf{q}$ such that $\mathbf{B}_V \mathbf{q} \in [\mathcal{P}^k(K)]^d$ satisfies [8, equation (3.10)]

$$(2.15a) \quad (\mathbf{B}_V \mathbf{q}, \mathbf{v}_h)_K = (\mathbf{q}, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in [\mathcal{P}^{k-1}(K)]^d,$$

$$(2.15b) \quad \langle \mathbf{B}_V \mathbf{q} \cdot \mathbf{n}, \mu_h \rangle_{F_i} = \langle \mathbf{q} \cdot \mathbf{n}, \mu_h \rangle_{F_i} \quad \forall \mu_h \in \mathcal{P}^k(F_i), i = 1, \dots, d.$$

By [8, equation (3.13) of Lemma 3.2, Lemma 3.3] we have

$$(2.16a) \quad \|\mathbf{B}_V \mathbf{q}\|_{L^2(K)} \leq \|\mathbf{q}\|_{L^2(K)} + h_K^{1/2} \|\mathbf{q} \cdot \mathbf{n}\|_{L^2(\partial K)},$$

$$(2.16b) \quad \|\mathbf{B}_V \mathbf{q} - \mathbf{q}\|_{\mathbf{H}^s(K)} \leq Ch_K^{k+1-s} |\mathbf{q}|_{\mathbf{H}^{k+1}(K)}$$

for $0 \leq s \leq k+1$. Note that $\mathbf{B}_V \mathbf{q}_h = \mathbf{q}_h$ for any $\mathbf{q}_h \in [\mathcal{P}^k(K)]^d$. Then, using the local inverse estimate in Lemma 2.5,

$$\begin{aligned}
\|\mathbf{B}_V \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} &\leq \|\mathbf{B}_V \mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} + \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \\
&\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \|\mathbf{B}_V \mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathbf{L}^2(K)} + \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \quad \text{by (2.12)} \\
&= Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \|\mathbf{B}_V(\mathbf{q} - \Pi_k^o \mathbf{q})\|_{\mathbf{L}^2(K)} + \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \\
&\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \left(\|\mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathbf{L}^2(K)} + h_K^{1/2} \|\mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathbf{L}^2(\partial K)} \right) \quad \text{by (2.16a)} \\
&\quad + \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \\
&\leq Ch_K^{k+1 + \frac{d}{s_j} - \frac{d}{\rho}} |\mathbf{q}|_{\mathbf{W}^{k+1, \rho}(K)} \quad \text{by (2.11a),}
\end{aligned}$$

where in the last inequality we used (2.11a) and (2.11b). In the proof of [10, Proposition A.3.] we find that

$$(2.17) \quad \|\mathbf{B}_V \mathbf{q} - \Pi_V \mathbf{q}\|_{\mathbf{L}^2(K)} \leq C \left(h_K^{1/2} \|u - \Pi_k^o u\|_{\mathbf{L}^2(K)} + \|\Pi_W u - \Pi_k^o u\|_{\mathbf{L}^2(K)} \right).$$

Then, again using the local inverse estimate in Lemma 2.5,

$$\begin{aligned}
\|\Pi_V \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} &\leq \|\mathbf{B}_V \mathbf{q} - \Pi_V \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} + \|\mathbf{B}_V \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \\
&\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \|\mathbf{B}_V \mathbf{q} - \Pi_V \mathbf{q}\|_{\mathbf{L}^2(K)} + \|\mathbf{B}_V \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \quad \text{by (2.12)} \\
&\leq Ch_K^{\frac{d}{s_j} - \frac{d}{2}} \left(h_K^{1/2} \|u - \Pi_k^o u\|_{\mathbf{L}^2(\partial K)} + \|\Pi_W u - \Pi_k^o u\|_{\mathbf{L}^2(K)} \right) \quad \text{by (2.17)} \\
&\quad + \|\mathbf{B}_V \mathbf{q} - \mathbf{q}\|_{\mathbf{L}^{s_j}(K)} \\
&\leq Ch_K^{k+1 + \frac{d}{s_j} - \frac{d}{\rho}} |\mathbf{q}|_{\mathbf{W}^{k+1, \rho}(K)} + Ch_K^{k+1 + \frac{d}{s_j} - \frac{d}{\rho}} |u|_{W^{k+1, \rho}(K)},
\end{aligned}$$

where in the last inequality we split $\Pi_W u - \Pi_k^o u = \Pi_W u - u + u - \Pi_k^o u$ and used (2.11a), (2.11b), and (2.13b). $\square$

3. L^∞ norm estimates. In the rest of this paper, we restrict the domain Ω to two dimensional space, i.e., $d = 2$. Furthermore, we assume the following.

(A) The domain is convex, and the triangular mesh $\mathcal{T}_h$ is quasi-uniform.

Now, we state the main result of our paper.

THEOREM 3.1. *Let $(\mathbf{q}, u)$ and $(\mathbf{q}_h, u_h, \hat{u}_h)$ be the solution of (1.1) and (2.1), respectively. We assume that (A) holds. First, if $u \in L^\infty(\Omega)$, $\mathbf{q} \in \mathbf{L}^\infty(\Omega)$, and $f \in L^2(\Omega)$, then we have the following stability bounds:*

$$(3.1a) \quad \|u_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + C\|f\|_{L^2(\Omega)}, \quad k = 0,$$

$$(3.1b) \quad \|u_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + C(|\log h|^{1/2} + 1)\|\mathbf{q}\|_{\mathbf{L}^\infty(\Omega)}, \quad k \geq 1,$$

$$(3.1c) \quad \|\mathbf{q}_h\|_{\mathbf{L}^\infty(\Omega)} \leq \|\mathbf{q}\|_{\mathbf{L}^\infty(\Omega)} + C\|f\|_{L^2(\Omega)}, \quad k \geq 1.$$

Second, if $(\mathbf{q}, u) \in \mathbf{W}^{k+1, \infty}(\Omega) \times W^{k+1, \infty}(\Omega)$, then we have the following error estimates:

$$(3.1d)$$

$$\|\mathbf{q} - \mathbf{q}_h\|_{\mathbf{L}^\infty(\Omega)} \leq Ch^{k+1}(|\log h|^{1/2} + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1, \infty}(\Omega)} + |u|_{W^{k+1, \infty}(\Omega)}), \quad k \geq 1,$$

$$(3.1e)$$

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^{k+1}(|\log h| + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1, \infty}(\Omega)} + |u|_{W^{k+1, \infty}(\Omega)}), \quad k \geq 1.$$

Furthermore, if $(\mathbf{q}, u) \in \mathbf{W}^{k+1,\infty}(\Omega) \times W^{k+2,\infty}(\Omega)$, then we have the following error estimate for the postprocessed solution:

$$(3.1f) \quad \|u - u_h^*\|_{L^\infty(\Omega)} \leq Ch^{k+2}(1 + |\log h|)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+2,\infty}(\Omega)}), \quad k \geq 1,$$

where u_h^* was defined in (2.2).

Remark 3.2. The assumption $u \in L^\infty(\Omega)$ and $\mathbf{q} \in \mathbf{L}^\infty(\Omega)$ hold in Theorem 3.1 if $f \in H^s(\Omega)$, $s > 0$, $g = 0$, $c = 1$, and Ω is convex in 2D. It is well known [2] that, in this case,

$$\|u\|_{H^{2+s}(\Omega)} \leq C(s)\|f\|_{H^s(\Omega)} \quad \forall s < \min\{1, \pi/\omega - 1\},$$

where ω is the largest interior angle of the boundary of Ω . By the Sobolev embedding theorem [35, Theorem 1.1.2] we have

$$H^{s+2}(\Omega) \hookrightarrow W^{t,\infty}(\Omega) \quad \forall t < s + 1,$$

in particular for $t = 1$.

The remainder of this section will be devoted to proving the above result.

3.1. Proof of Theorem 3.1. We start the proof of Theorem 3.1 by defining suitable regularized Green's functions. We follow the notation of Girault, Nocketto, and Scott [16] to define by $\delta_\star \geq 0$ the usual mollifier in $\mathcal{D}(\mathbb{R}^2)$ such that $\text{supp}(\delta_\star) \subset B(0, 1)$ and $\int_{\mathbb{R}^2} \delta_\star(x) d\mathbf{x} = 1$. Then for any point $\mathbf{x}_0 \in \Omega$ and real number $\rho_0 > 0$ such that the ball $B(\mathbf{x}_0, \rho_0)$ is contained in Ω , we define the mollifier by

$$(3.2) \quad \delta(\mathbf{x}) = \frac{1}{\rho_0^2} \delta_\star \left(\frac{\mathbf{x} - \mathbf{x}_0}{\rho_0} \right).$$

LEMMA 3.3 (see [16, Lemma 1.1]). *Suppose the triangular mesh $\mathcal{T}_h$ is quasi-uniform. Let φ_h be a polynomial in $\mathcal{P}^k$ on each K , $\mathbf{x}_M$ be a point of $\bar{\Omega}$ where $|\varphi_h(\mathbf{x})|$ attains its maximum, K be an element containing $\mathbf{x}_M$, and $B \subset K$ be the disk of radius ρ_K inscribed in K . Then there exists a smooth function δ_M supported in B such that*

$$(3.3a) \quad \int_{\Omega} \delta_M d\mathbf{x} = 1,$$

$$(3.3b) \quad \|\varphi_h\|_{L^\infty(\Omega)} = \left| \int_B \delta_M \varphi_h d\mathbf{x} \right|$$

and for any number t with $1 < t \leq \infty$, there exists a constant C such that

$$(3.3c) \quad \|\delta_M\|_{L^t(B)} \leq Ch^{2/t-2},$$

$$(3.3d) \quad \|\nabla \delta_M\|_{L^t(B)} \leq Ch^{2/t-3}.$$

Proof. The proof of (3.3a)–(3.3c) can be found in [16, Lemma 1.1] where it is shown that there exists polynomial $P_M \in \mathcal{P}^k(K)$ such that

$$\delta_M = \delta P_M.$$

Since $\|\delta\|_{L^\infty(\mathbb{R}^2)} \leq C/\rho_K^2$ and $\|\nabla \delta\|_{L^\infty(\mathbb{R}^2)} \leq C/\rho_K^3$, by (2.12) and the assumption that the triangle mesh is quasi-uniform, we have

$$\begin{aligned} \|\nabla \delta_M\|_{L^t(B)} &= \|P_M \nabla \delta + \delta \nabla P_M\|_{L^t(B)} \\ &\leq \|\nabla \delta\|_{L^\infty(B)} \|P_M\|_{L^t(B)} + \|\delta\|_{L^\infty(B)} \|\nabla P_M\|_{L^t(B)} \\ &\leq Ch^{2/t-3}. \end{aligned}$$

□

The main idea behind the proof of L^∞ norm estimates is to use the so-called smooth δ_M function, which was described in Lemma 3.3. Given a scalar function δ_1 and a vector δ_2 of the above type, we define two regularized Green's functions for problem (1.1) in mixed form:

$$(3.4) \quad \begin{aligned} c\Phi_1 + \nabla\Psi_1 &= \mathbf{0} && \text{in } \Omega, \\ \nabla \cdot \Phi_1 &= \delta_1 && \text{in } \Omega, \\ \Psi_1 &= 0 && \text{on } \partial\Omega \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} c\Phi_2 + \nabla\Psi_2 &= \delta_2 && \text{in } \Omega, \\ \nabla \cdot \Phi_2 &= 0 && \text{in } \Omega, \\ \Psi_2 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Next, we define the weight σ by

$$(3.6) \quad \sigma(\mathbf{x}) = (|\mathbf{x} - \mathbf{x}_0|^2 + h^2)^{1/2},$$

where $\mathbf{x}_0 \in \Omega$. In this paper, we are going to estimate four quantities: $\|u\|_{L^\infty(\Omega)}$, $\|u - u_h\|_{L^\infty(\Omega)}$, $\|\mathbf{q}\|_{L^\infty(\Omega)}$, and $\|\mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)}$. Without loss of generality, we assume that $\mathbf{x}_0$ is a point where the maximum of one of these four quantities is attained. This function was introduced by Nitsche [28] for conforming finite element methods and was applied by Gastaldi and Nochetto [14] for Raviart–Thomas mixed methods.

We need two auxiliary results before starting the proof of Theorem 3.1. The first concerns bounds on the regularized Green's function (Ψ_1, Ψ_2) .

LEMMA 3.4. *Let Ψ_1 and Ψ_2 be the solution of (3.4) and (3.5), respectively. If assumption (A) holds, then we have*

$$(3.7a) \quad \|\Psi_1\|_{H^2(\Omega)} \leq Ch^{-1}, \quad \|\sigma D^2\Psi_1\|_{L^2(\Omega)} \leq C|\log h|^{1/2}, \quad \|D^2\Psi_1\|_{L^1(\Omega)} \leq C|\log h|,$$

$$(3.7b)$$

$$\|\Psi_2\|_{H^2(\Omega)} \leq Ch^{-2}, \quad \|\sigma D^2\Psi_2\|_{L^2(\Omega)} \leq Ch^{-1}, \quad \|D^2\Psi_2\|_{L^1(\Omega)} \leq Ch^{-1}|\log h|^{1/2}.$$

Proof. The proof of (3.7a) can be found in [36, equation (3.6)]. By the elliptic regularity of Poisson problem and (3.3d) we have

$$\|\Psi_2\|_{H^2(\Omega)} \leq C\|\nabla \cdot \delta_2\|_{L^2(\Omega)} \leq Ch^{-2}.$$

The remaining estimations in (3.7b) can be found in [11, Lemma 3.2] and [36, equation (3.12d)]. $\square$

In the second auxiliary lemma we summarize some properties of the function which will be used later.

LEMMA 3.5 (see [36, equation (2.13)]). *For any $\alpha \in \mathbb{R}$ there is a constant C independent of α such that the function σ has the following properties:*

$$(3.8a) \quad \frac{\max_{\mathbf{x} \in K} \sigma(\mathbf{x})^\alpha}{\min_{\mathbf{x} \in K} \sigma(\mathbf{x})^\alpha} \leq C \quad \forall K \in \mathcal{T}_h,$$

$$(3.8b) \quad |\nabla^k(\sigma(\mathbf{x})^\alpha)| \leq C\sigma(\mathbf{x})^{\alpha-k},$$

$$(3.8c) \quad \int_{\Omega} \sigma(\mathbf{x})^{-2} d\mathbf{x} \leq C|\log h|.$$

We split the proof of Theorem 3.1 into four steps. First, we shall obtain the L^1 norm approximations error of the solution of (3.4) and (3.5). Second, we prove the L^∞ norm stability of $\mathbf{q}_h$ and u_h . Next, we obtain the L^∞ norm error estimates of $\mathbf{q} - \mathbf{q}_h$ and $u - u_h$. Finally, we obtain the L^∞ norm error estimates of the postprocessed solution u_h^* .

Step 1: L^1 norm error estimates for the regularized Green's functions. Let $(\Phi_{1,h}, \Psi_{1,h}, \hat{\Psi}_{1,h})$ and $(\Phi_{2,h}, \Psi_{2,h}, \hat{\Psi}_{2,h})$ be the HDG solution of (3.4) and (3.5), respectively, i.e.,

$$(3.9a) \quad \mathcal{B}(\Phi_{1,h}, \Psi_{1,h}, \hat{\Psi}_{1,h}; \mathbf{v}_h, w_h, \hat{w}_h) = (\delta_1, w_h)_{\mathcal{T}_h},$$

$$(3.9b) \quad \mathcal{B}(\Phi_{2,h}, \Psi_{2,h}, \hat{\Psi}_{2,h}; \mathbf{v}_h, w_h, \hat{w}_h) = (\delta_2, \mathbf{v}_h)_{\mathcal{T}_h}$$

for all $(\mathbf{v}_h, w_h, \hat{w}_h) \in \mathbf{V}_h \times W_h \times \hat{W}_h$. The existence and uniqueness of these solutions follow by standard HDG theory [9].

Our goal in this step is to prove the upcoming Lemma 3.8. To start we summarize some relevant results in the following:

$$(3.10a) \quad \mathcal{B}(\Pi_V \Phi_1, \Pi_W \Psi_1, \Pi_k^\partial \Psi_1; \mathbf{v}_h, w_h, \hat{w}_h) = (c(\Pi_V \Phi_1 - \Phi_1), \mathbf{v}_h)_{\mathcal{T}_h} - (\delta_1, w_h)_{\mathcal{T}_h},$$

$$(3.10b) \quad \mathcal{B}(\Pi_V \Phi_2, \Pi_W \Psi_2, \Pi_k^\partial \Psi_2; \mathbf{v}_h, w_h, \hat{w}_h) = (c(\Pi_V \Phi_2 - \Phi_2), \mathbf{v}_h)_{\mathcal{T}_h} + (\delta_2, \mathbf{v}_h)_{\mathcal{T}_h},$$

$$(3.10c)$$

$$\mathcal{B}(\Pi_V \Phi_1 - \Phi_{1,h}, \Pi_W \Psi_1 - \Psi_{1,h}, \Pi_k^\partial \Psi_1 - \hat{\Psi}_{1,h}; \mathbf{v}_h, w_h, \hat{w}_h) = (c(\Pi_V \Phi_1 - \Phi_1), \mathbf{v}_h)_{\mathcal{T}_h},$$

$$(3.10d)$$

$$\mathcal{B}(\Pi_V \Phi_2 - \Phi_{2,h}, \Pi_W \Psi_2 - \Psi_{2,h}, \Pi_k^\partial \Psi_2 - \hat{\Psi}_{2,h}; \mathbf{v}_h, w_h, \hat{w}_h) = (c(\Pi_V \Phi_2 - \Phi_2), \mathbf{v}_h)_{\mathcal{T}_h},$$

$$(3.10e) \quad \|\Pi_V \Phi_1 - \Phi_{1,h}\|_{L^2(\Omega)} \leq C, \quad \|\Pi_W \Psi_1 - \Psi_{1,h}\|_{L^2(\Omega)} \leq Ch^{\min\{k,1\}},$$

$$(3.10f) \quad \|\Pi_V \Phi_2 - \Phi_{2,h}\|_{L^2(\Omega)} \leq Ch^{-1}, \quad \|\Pi_W \Psi_2 - \Psi_{2,h}\|_{L^2(\Omega)} \leq Ch^{\min\{k,1\}-1}.$$

The proof of (3.10a) and (3.10c) can be found in [3, Lemma 3.6], and the proof of (3.10b) and (3.10d) is similar. The proof of (3.10e) and (3.10f) can be found in [10, Theorem 3.1 and Theorem 4.1] and the regularity of regularized Green's functions in Lemma 3.4.

We now present a series of lemmas providing convergence estimates for the projections used in our analysis.

LEMMA 3.6. *For any integer $k \geq 0$, $K \in \mathcal{T}_h$, and $\alpha \in \mathbb{R}$, let Π_k° be the standard L^2 projection (see (2.9a)); then for $\mathbf{v} \in \mathbf{H}^{k+1}(K)$ we have*

$$(3.11a) \quad \|\sigma^\alpha(\mathbf{v} - \Pi_k^\circ \mathbf{v})\|_{L^2(K)} \leq Ch_K^{k+1} \|\sigma^\alpha \nabla^{k+1} \mathbf{v}\|_{L^2(K)}.$$

Furthermore, let $w \in H^{k+1}(K)$; then $(\sigma^2 \mathbf{v}, \sigma^2 w)$ is in the domain of Π_h and we have

$$(3.11b) \quad \|\sigma^\alpha(\mathbf{v} - \Pi_V \mathbf{v})\|_{L^2(K)} \leq Ch_K^{k+1} (\|\sigma^\alpha \nabla^{k+1} \mathbf{v}\|_{L^2(K)} + \|\sigma^\alpha \nabla^{k+1} w\|_{L^2(K)}),$$

$$(3.11c) \quad \|\sigma^\alpha \nabla(w - \Pi_W w)\|_{L^2(K)} \leq Ch_K^k (\|\sigma^\alpha \nabla^{k+1} w\|_{L^2(K)} + \|\sigma^\alpha \nabla^{k+1} w\|_{L^2(K)}).$$

Proof. We only prove (3.11b) because the proofs of (3.11a) and (3.11c) are similar.

$$\begin{aligned} \|\sigma^\alpha(\mathbf{v} - \Pi_V \mathbf{v})\|_{L^2(K)} &\leq \max_{\mathbf{x} \in K} \{\sigma^\alpha\} \|\mathbf{v} - \Pi_V \mathbf{v}\|_{L^2(K)} \\ &\leq Ch_K^{k+1} \max_{\mathbf{x} \in K} \{\sigma^\alpha\} (|\nabla^{k+1} \mathbf{v}|_{L^2(K)} + |\nabla^{k+1} w|_{L^2(K)}) \quad \text{by (2.8b)} \\ &\leq Ch_K^{k+1} \min_{\mathbf{x} \in K} \{\sigma^\alpha\} (|\nabla^{k+1} \mathbf{v}|_{L^2(K)} + |\nabla^{k+1} w|_{L^2(K)}) \quad \text{by (3.8a)} \\ &\leq Ch_K^{k+1} (\|\sigma^\alpha \nabla^{k+1} \mathbf{v}\|_{L^2(K)} + \|\sigma^\alpha \nabla^{k+1} w\|_{L^2(K)}). \quad \square \end{aligned}$$

LEMMA 3.7. Let $(\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h$; then for any integer $k \geq 0$, we have

$$(3.12a) \quad \|\sigma^{-1}(\sigma^2 \mathbf{v}_h - \Pi_k^o(\sigma^2 \mathbf{v}_h))\|_{L^2(K)} \leq Ch \|\mathbf{v}_h\|_{L^2(K)},$$

$$(3.12b) \quad \|\sigma^{-1}(\sigma^2 \mathbf{v}_h - \Pi_V(\sigma^2 \mathbf{v}_h))\|_{L^2(K)} \leq Ch(\|\mathbf{v}_h\|_{L^2(K)} + \|w_h\|_{L^2(K)}).$$

Proof. Notice that $\mathbf{v}_h|_K \in [\mathcal{P}^k(K)]^2$, i.e., $\nabla^{k+1} \mathbf{v}_h = 0$. Then by Lemma 3.6 we have

$$\begin{aligned} \|\sigma^{-1}(\sigma^2 \mathbf{v}_h - \Pi_k^o(\sigma^2 \mathbf{v}_h))\|_{L^2(K)} &\leq Ch^{k+1} \|\sigma^{-1}(\nabla^{k+1}(\sigma^2 \mathbf{v}_h))\|_{L^2(K)} \\ &= Ch^{k+1} \left\| \sigma^{-1} \sum_{j=1}^{k+1} \nabla^j(\sigma^2) \nabla^{k+1-j} \mathbf{v}_h \right\|_{L^2(K)} \\ &\leq Ch \|\mathbf{v}_h\|_{L^2(K)}, \end{aligned}$$

where we applied (3.8b) and Lemma 2.5 to the above inequality. This proves (3.12a), and the proof of (3.12b) is the same; hence we omit the details here. $\square$

LEMMA 3.8. Let (Φ_1, Ψ_1) and $(\Phi_{1,h}, \Psi_{1,h}, \hat{\Psi}_{1,h})$ be the solution of (3.4) and (3.9a), respectively, and (Φ_2, Ψ_2) and $(\Phi_{2,h}, \Psi_{2,h}, \hat{\Psi}_{2,h})$ be the solutions of (3.5) and (3.9b). Let

$$(3.13) \quad \mathcal{E}_h^{\Phi_1} = \Pi_V \Phi_1 - \Phi_{1,h}, \quad \mathcal{E}_h^{\Psi_1} = \Pi_W \Psi_1 - \Psi_{1,h}, \quad \mathcal{E}_h^{\hat{\Psi}_1} = \Pi_k^\partial \Psi_1 - \hat{\Psi}_{1,h}.$$

If assumption (A) holds and $k \geq 1$, then we have

$$(3.14a) \quad \|\sigma \mathcal{E}_h^{\Phi_1}\|_{L^2(\Omega)} + \|\sigma \sqrt{\tau}(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1})\|_{L^2(\partial \mathcal{T}_h)} \leq Ch(|\log h|^{1/2} + 1),$$

$$(3.14b) \quad \|\sigma \mathcal{E}_h^{\Phi_2}\|_{L^2(\Omega)} + \|\sigma \sqrt{\tau}(\mathcal{E}_h^{\Psi_2} - \mathcal{E}_h^{\hat{\Psi}_2})\|_{L^2(\partial \mathcal{T}_h)} \leq C.$$

Proof. On the one hand, by the definition of $\mathcal{B}$ in (2.3) we obtain

$$\begin{aligned} &\mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1}, -\sigma^2 \mathcal{E}_h^{\Psi_1}, -\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}) \\ &= (c \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} - (\mathcal{E}_h^{\Psi_1}, \nabla \cdot (\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} + \langle \mathcal{E}_h^{\hat{\Psi}_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\nabla \cdot \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Psi_1})_{\mathcal{T}_h} + \langle \tau(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}), \sigma^2 \mathcal{E}_h^{\Psi_1} - \sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} \rangle_{\partial \mathcal{T}_h} - \langle \mathcal{E}_h^{\Phi_1} \cdot \mathbf{n}, \sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} \rangle_{\partial \mathcal{T}_h} \\ &= (c \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} - (\mathcal{E}_h^{\Psi_1}, \sigma^2 \nabla \cdot \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} - (\mathcal{E}_h^{\Psi_1}, 2\sigma \nabla \sigma \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + \langle \mathcal{E}_h^{\hat{\Psi}_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\nabla \cdot \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Psi_1})_{\mathcal{T}_h} + \langle \tau(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}), \sigma^2 \mathcal{E}_h^{\Psi_1} - \sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} \rangle_{\partial \mathcal{T}_h} - \langle \mathcal{E}_h^{\Phi_1} \cdot \mathbf{n}, \sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

This gives

$$(3.15) \quad \begin{aligned} &\mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1}, -\sigma^2 \mathcal{E}_h^{\Psi_1}, -\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}) \\ &= (c \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + \|\sigma \sqrt{\tau}(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1})\|_{L^2(\partial \mathcal{T}_h)}^2 - (\mathcal{E}_h^{\Psi_1}, 2\sigma \nabla \sigma \cdot \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h}. \end{aligned}$$

We use definition (2.7) with $\mathbf{q} = \sigma^2 \mathcal{E}_h^{\Phi_1}$, $u = -\sigma^2 \mathcal{E}_h^{\Psi_1}$. On the other hand, by the error equation (3.10c) we get

$$\begin{aligned} &\mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1}, -\sigma^2 \mathcal{E}_h^{\Psi_1}, -\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}) \\ &= \mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}), -\sigma^2 \mathcal{E}_h^{\Psi_1} - \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}), \\ &\quad -(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} - \Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}))) \end{aligned}$$

$$\begin{aligned}
& + \mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}), \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}), -\Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1})) \\
& = \mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}), -\sigma^2 \mathcal{E}_h^{\Psi_1} - \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}), \\
& \quad - (\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} - \Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}))) \\
& + (\Pi_V \Phi_1 - \Phi_1, \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h}.
\end{aligned}$$

Next, we use the definition of $\mathcal{B}$ in (2.3) again to get

$$\begin{aligned}
& \mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}), -\sigma^2 \mathcal{E}_h^{\Psi_1} - \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}), -(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} - \Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}))) \\
& = (c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} - (\mathcal{E}_h^{\Psi_1}, \nabla \cdot (\sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1})))_{\mathcal{T}_h} \\
& \quad + \langle \mathcal{E}_h^{\hat{\Psi}_1}, (\sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1})) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + (\nabla \cdot \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}))_{\mathcal{T}_h} \\
& \quad + \langle \tau(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}), (\sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1})) - (\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} - \Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1})) \rangle_{\partial \mathcal{T}_h} \\
& \quad - \langle \mathcal{E}_h^{\Phi_1} \cdot \mathbf{n}, \sigma^2 \mathcal{E}_h^{\hat{\Psi}_1} - \Pi_k^\partial(\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}) \rangle_{\partial \mathcal{T}_h} + (c(\Pi_V \Phi_1 - \Phi_1), \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1})) \\
& = (c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} + (\nabla \mathcal{E}_h^{\Psi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} \\
& \quad - \langle \mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}, (\sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1})) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + (\nabla \cdot \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}))_{\mathcal{T}_h} \\
& \quad + \langle \tau(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}), \sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}) \rangle_{\partial \mathcal{T}_h} + (c(\Pi_V \Phi_1 - \Phi_1), \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h},
\end{aligned}$$

where we used integration by parts in the above equation. Notice that $(\sigma^2 \mathcal{E}_h^{\Phi_1}, -\sigma^2 \mathcal{E}_h^{\Psi_1})$ is in the domain of Π_h (see (2.7)); then by (2.7c) and the fact that $(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1})|_F \in \mathcal{P}^k(F)$ for all $F \in \mathcal{F}_h$, we have

$$\begin{aligned}
& \langle \mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}, (\Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}) - \sigma^2 \mathcal{E}_h^{\Phi_1}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
& + \langle \tau(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1}), \sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}) \rangle_{\partial \mathcal{T}_h} = 0.
\end{aligned}$$

Furthermore, by (2.7a)–(2.7b) and the fact that $\nabla \mathcal{E}_h^{\Psi_1}|_K \in [\mathcal{P}^{k-1}(K)]^2$ and $\nabla \cdot \mathcal{E}_h^{\Phi_1}|_K \in \mathcal{P}^{k-1}(K)$, we have

$$\begin{aligned}
& (\nabla \mathcal{E}_h^{\Psi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} = 0, \\
& (\nabla \cdot \mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Psi_1} + \Pi_W(-\sigma^2 \mathcal{E}_h^{\Psi_1}))_{\mathcal{T}_h} = 0.
\end{aligned}$$

This gives

$$\begin{aligned}
(3.16) \quad & \mathcal{B}(\mathcal{E}_h^{\Phi_1}, \mathcal{E}_h^{\Psi_1}, \mathcal{E}_h^{\hat{\Psi}_1}; \sigma^2 \mathcal{E}_h^{\Phi_1}, -\sigma^2 \mathcal{E}_h^{\Psi_1}, -\sigma^2 \mathcal{E}_h^{\hat{\Psi}_1}) \\
& = (c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} + (c(\Pi_V \Phi_1 - \Phi_1), \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1})).
\end{aligned}$$

Comparing with (3.15) and (3.16) we have

$$\begin{aligned}
& (c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + \|\sigma \sqrt{\tau}(\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\hat{\Psi}_1})\|_{L^2(\partial \mathcal{T}_h)}^2 \\
& = (\mathcal{E}_h^{\Psi_1}, 2\sigma \nabla \sigma \cdot \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + (c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} \\
& \quad + (c(\Pi_V \Phi_1 - \Phi_1), \Pi_V(\sigma^2 \mathcal{E}_h^{\Phi_1}) - \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + (c(\Pi_V \Phi_1 - \Phi_1), \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} \\
& = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the first term I_1 , we use (3.8b), Young's inequality, (3.10e), and $k \geq 1$ to get

$$|I_1| \leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2 \mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + C\|\mathcal{E}_h^{\Psi_1}\|_{L^2(\Omega)}^2 \leq \frac{1}{4}\|\sigma \mathcal{E}_h^{\Phi_1}\|_{L^2(\Omega)}^2 + Ch^2.$$

For the second term I_2 , we use Young's inequality, (3.12b), and (3.10e) to get

$$\begin{aligned}
 |I_2| &= \left| (c\sigma\mathcal{E}_h^{\Phi_1}, \sigma^{-1}(\sigma^2\mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2\mathcal{E}_h^{\Phi_1})))_{\mathcal{T}_h} \right| \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + C \left\| \sigma^{-1}(\sigma^2\mathcal{E}_h^{\Phi_1} - \Pi_V(\sigma^2\mathcal{E}_h^{\Phi_1})) \right\|_{L^2(\Omega)}^2 \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + Ch^2(\|\mathcal{E}_h^{\Phi_1}\|_{L^2(\Omega)}^2 + \|\mathcal{E}_h^{\Psi_1}\|_{L^2(\Omega)}^2) \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + Ch^2.
 \end{aligned}$$

For the third term I_3 , we use Young's inequality, (3.12b), (3.10e), (3.11b), and (3.7a) to get

$$\begin{aligned}
 |I_3| &= \left| (\sigma(c(\Pi_V\Phi_1 - \Phi_1)), \sigma^{-1}(\Pi_V(\sigma^2\mathcal{E}_h^{\Phi_1}) - \sigma^2\mathcal{E}_h^{\Phi_1}))_{\mathcal{T}_h} \right| \\
 &\leq \frac{1}{2}\|\sigma(\Pi_V\Phi_1 - \Phi_1)\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\| \sigma^{-1}(\Pi_V(\sigma^2\mathcal{E}_h^{\Phi_1}) - \sigma^2\mathcal{E}_h^{\Phi_1}) \right\|_{L^2(\Omega)}^2 \\
 &\leq Ch^2\|\sigma D^2\Psi_1\|_{L^2(\Omega)}^2 + Ch^2(\|\mathcal{E}_h^{\Phi_1}\|_{L^2(\Omega)}^2 + \|\mathcal{E}_h^{\Psi_1}\|_{L^2(\Omega)}^2) \\
 &\leq Ch^2(1 + |\log h|).
 \end{aligned}$$

For the last term I_4 , we use Young's inequality, (3.11b), and (3.7a) to get

$$\begin{aligned}
 |I_4| &= \left| (\sigma(c(\Pi_V\Phi_1 - \Phi_1)), \sigma\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} \right| \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + C\|\sigma(\Pi_V\Phi_1 - \Phi_1)\|_{L^2(\Omega)}^2 \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + Ch^2\|\sigma D^2\Psi\|_{L^2(\Omega)}^2 \\
 &\leq \frac{1}{4}(c\mathcal{E}_h^{\Phi_1}, \sigma^2\mathcal{E}_h^{\Phi_1})_{\mathcal{T}_h} + Ch^2(1 + |\log h|).
 \end{aligned}$$

Summing the estimates for $\{I_k\}_{k=1}^4$ gives (3.14a). The proof of (3.14b) is similar to the proof of (3.14a). $\square$

By (3.11b) we have the following corollary.

COROLLARY 3.9. *Let (Φ_1, Ψ_1) and $(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h})$ be the solution of (3.4) and (3.9a), respectively, and (Φ_2, Ψ_2) and $(\Phi_{2,h}, \Psi_{2,h}, \widehat{\Psi}_{2,h})$ be the solutions of (3.5) and (3.9b). If assumption (A) holds and $k \geq 1$, then we have*

$$(3.17a) \quad \|\sigma(\Phi_1 - \Phi_{1,h})\|_{L^2(\Omega)} \leq Ch(|\log h|^{1/2} + 1),$$

$$(3.17b) \quad \|\sigma(\Phi_2 - \Phi_{2,h})\|_{L^2(\Omega)} \leq C.$$

Next, we give a weighted L^2 estimation of $\mathcal{E}_h^{\Psi_1}$.

LEMMA 3.10. *Let (Φ_1, Ψ_1) and $(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h})$ be the solution of (3.4) and (3.9a), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$(3.18) \quad \|\sigma\nabla(\Psi_1 - \Psi_{1,h})\|_{L^2(\Omega)} \leq C.$$

Proof. For any $K \in \mathcal{T}_h$, we take $\mathbf{v}_h = \nabla\mathcal{E}_h^{\Psi_1}$ on K and $\mathbf{v}_h = \mathbf{0}$ otherwise, $(w_h, \widehat{w}_h) = (0, 0)$ in (3.10c), use the notation in (3.13), and use integration by parts to get

$$\begin{aligned} & (c\mathcal{E}_h^{\Psi_1}, \nabla \mathcal{E}_h^{\Psi_1})_K + (\nabla \mathcal{E}_h^{\Psi_1}, \nabla \mathcal{E}_h^{\Psi_1})_K - \langle \mathcal{E}_h^{\Psi_1} - \widehat{\mathcal{E}}_h^{\Psi_1}, \nabla \mathcal{E}_h^{\Psi_1} \cdot \mathbf{n} \rangle_{\partial K} \\ & = (c(\Pi_V \Phi_1 - \Phi_1), \nabla \mathcal{E}_h^{\Psi_1})_K. \end{aligned}$$

By the Cauchy–Schwarz inequality and inverse inequality we obtain

$$\|\nabla \mathcal{E}_h^{\Psi_1}\|_{L^2(K)} \leq C(\|c\mathcal{E}_h^{\Phi_1}\|_{L^2(K)} + h^{-1/2}\|\mathcal{E}_h^{\Psi_1} - \widehat{\mathcal{E}}_h^{\Psi_1}\|_{L^2(\partial K)} + \|\Pi_V \Phi_1 - \Phi_1\|_{L^2(K)}).$$

Using the property of the weight function σ in (3.8a) we get

$$\begin{aligned} \|\sigma \nabla \mathcal{E}_h^{\Psi_1}\|_{L^2(K)} & \leq \sigma_{\max} \|\nabla \mathcal{E}_h^{\Psi_1}\|_{L^2(K)} \leq C \sigma_{\min} \|\nabla \mathcal{E}_h^{\Psi_1}\|_{L^2(K)} \\ & \leq C(\|\sigma(c\mathcal{E}_h^{\Phi_1})\|_{L^2(K)} + h^{-1/2}\|\sigma(\mathcal{E}_h^{\Psi_1} - \widehat{\mathcal{E}}_h^{\Psi_1})\|_{L^2(\partial K)} + \|\Pi_V \Phi_1 - \Phi_1\|_{L^2(K)}). \end{aligned}$$

Summing over all elements and using (3.14a) we have

$$\|\sigma \nabla \mathcal{E}_h^{\Psi_1}\|_{\mathcal{T}_h} \leq C(\|\sigma(c\mathcal{E}_h^{\Phi_1})\|_{\mathcal{T}_h} + h^{-1/2}\|\sigma(\mathcal{E}_h^{\Psi_1} - \widehat{\mathcal{E}}_h^{\Psi_1})\|_{\partial \mathcal{T}_h} + \|\Pi_V \Phi_1 - \Phi_1\|_{\mathcal{T}_h}) \leq C.$$

Our desired result (3.18) follows by (3.11c). $\square$

LEMMA 3.11. *Let (Φ_1, Ψ_1) and $(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h})$ be the solutions of (3.4) and (3.9a), respectively, and let (Φ_2, Ψ_2) , $(\Phi_{2,h}, \Psi_{2,h}, \widehat{\Psi}_{2,h})$ be the solution of (3.5) and (3.9b). If assumption (A) holds and $k \geq 1$, then we have*

$$(3.19a) \quad \|\Phi_{1,h} - \Phi_1\|_{L^1(\Omega)} + \|c\Phi_1 - \Pi_{k-1}(c\Phi_1)\|_{L^1(\Omega)} \leq Ch(|\log h| + 1),$$

$$(3.19b) \quad \|\Phi_{2,h} - \Phi_2\|_{L^1(\Omega)} + \|c\Phi_2 - \Pi_{k-1}(c\Phi_2)\|_{L^1(\Omega)} \leq C(|\log h|^{1/2} + 1),$$

$$(3.19c) \quad \|\nabla(\Psi_1 - \Psi_{1,h})\|_{L^1(\Omega)} \leq C(|\log h|^{1/2} + 1).$$

Proof. By the Cauchy–Schwarz inequality and (3.14a) we have

$$\|\Phi_1 - \Phi_{1,h}\|_{L^1(\Omega)} = \int_{\Omega} \sigma^{-1}(\sigma|\Phi_1 - \Phi_{1,h}|) \, d\mathbf{x} \leq Ch(1 + |\log h|).$$

Next, by (2.11a) we have

$$(3.20) \quad \|c\Phi_1 - \Pi_{k-1}(c\Phi_1)\|_{L^1(\Omega)} \leq Ch\|\nabla^2 \Psi_1\|_{L^1(\Omega)} \leq Ch|\log h|.$$

Then (3.19a) follows. The proofs of (3.19b) and (3.19c) are similar to the proof of (3.19a). $\square$

Step 2: Proof of (3.1a)–(3.1c) in Theorem 3.1.

Proof. We only prove (3.1a) and (3.1b) since the proof of (3.1c) is similar. We choose δ_1 so that $\|u_h\|_{L^\infty(\Omega)} = (\delta_1, u_h)_{\mathcal{T}_h}$; then

$$\begin{aligned} -(\delta_1, u_h)_{\mathcal{T}_h} &= \mathcal{B}(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}; \mathbf{q}_h, u_h, \widehat{u}_h) && \text{by (3.9a)} \\ &= \mathcal{B}(\mathbf{q}_h, u_h, \widehat{u}_h; \Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}) && \text{by (2.6)} \\ &= \mathcal{B}(\mathbf{q}, u, u; \Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}) && \text{by (2.5)} \\ &= \mathcal{B}(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}; \mathbf{q}, u, u) && \text{by (2.6)} \\ &= \mathcal{B}(\Phi_{1,h} - \Phi_1, \Psi_{1,h} - \Psi_1, \widehat{\Psi}_{1,h} - \Psi_1; \mathbf{q}, u, u) + \mathcal{B}(\Phi_1, \Psi_1, \Psi_1; \mathbf{q}, u, u) \\ &= \mathcal{B}(\Phi_{1,h} - \Phi_1, \Psi_{1,h} - \Psi_1, \widehat{\Psi}_{1,h} - \Psi_1; \mathbf{q}, u, u) - (\delta_1, u)_{\mathcal{T}_h}. && \text{by (3.9a)} \end{aligned}$$

By the definition of $\mathcal{B}$ in (2.3) we have

$$\begin{aligned} \mathcal{B}(\Phi_{1,h} - \Phi_1, \Psi_{1,h} - \Psi_1, \widehat{\Psi}_{1,h} - \Psi_1; \mathbf{q}, u, u) \\ = (c(\Phi_{1,h} - \Phi_1), \mathbf{q})_{\mathcal{T}_h} - (\Psi_{1,h} - \Psi_1, \nabla \cdot \mathbf{q})_{\mathcal{T}_h} + \langle \widehat{\Psi}_{1,h} - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ - (\nabla \cdot (\Phi_{1,h} - \Phi_1), u)_{\mathcal{T}_h} + \langle (\Phi_{1,h} - \Phi_1) \cdot \mathbf{n}, u \rangle_{\partial \mathcal{T}_h} \\ = -(\Psi_{1,h} - \Psi_1, \nabla \cdot \mathbf{q})_{\mathcal{T}_h} + \langle \widehat{\Psi}_{1,h} - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

This implies

$$(3.21) \quad (\delta_1, u_h)_{\mathcal{T}_h} = (\delta_1, u)_{\mathcal{T}_h} + (\Psi_{1,h} - \Psi_1, \nabla \cdot \mathbf{q})_{\mathcal{T}_h} - \langle \widehat{\Psi}_{1,h} - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}.$$

Next, we split the proof into two cases.

(1) If $k = 0$, we have

$$(\delta_1, u_h)_{\mathcal{T}_h} = (\delta_1, u)_{\mathcal{T}_h} + (\Psi_{1,h} - \Psi_1, \nabla \cdot \mathbf{q})_{\mathcal{T}_h} = (\delta_1, u)_{\mathcal{T}_h} + (\Psi_{1,h} - \Psi_1, f)_{\mathcal{T}_h},$$

where we used $\langle \widehat{\Psi}_{1,h}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$ and $\langle \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$. Using the fact that $\|\delta_1\|_{L^1(\Omega)} = 1$ together with the estimates in (3.10e) gives

$$\|u_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} + |(\Psi_{1,h} - \Psi_1, f)_{\mathcal{T}_h}| \leq \|u\|_{L^\infty(\Omega)} + C\|f\|_{L^2(\Omega)}.$$

This completes the proof of (3.1a).

(2) If $k \geq 1$, we have

$$\begin{aligned} (\delta_1, u_h)_{\mathcal{T}_h} &= (\delta_1, u)_{\mathcal{T}_h} - (\nabla(\Psi_{1,h} - \Psi_1), \mathbf{q})_{\mathcal{T}_h} + \langle \Psi_{1,h} - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} - \langle \widehat{\Psi}_{1,h} - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= (\delta_1, u)_{\mathcal{T}_h} - (\nabla(\Psi_{1,h} - \Psi_1), \mathbf{q})_{\mathcal{T}_h} + \langle \Psi_{1,h} - \Pi_W \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \Pi_W \Psi_1 - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} - \langle \widehat{\Psi}_{1,h} - \Pi_k^\partial \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= (\delta_1, u)_{\mathcal{T}_h} - (\nabla(\Psi_{1,h} - \Psi_1), \mathbf{q})_{\mathcal{T}_h} - \langle \mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\widehat{\Psi}_1}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + \langle \Pi_W \Psi_1 - \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$

where we used integration by parts in the first equality and the fact that $\langle \widehat{\Psi}_{1,h}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$ and $\langle \Pi_k^\partial \Psi_1, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$ in the second equality. Using the fact that $\|\delta_1\|_{L^1(\Omega)} = 1$ together with the estimates in (3.10e) gives

$$\begin{aligned} \|u_h\|_{L^\infty(\Omega)} &\leq C\|\mathbf{q}\|_{L^\infty(\Omega)} (\|\nabla(\Psi_{1,h} - \Psi_1)\|_{L^1(\Omega)} + \|\mathcal{E}_h^{\Psi_1} - \mathcal{E}_h^{\widehat{\Psi}_1}\|_{L^1(\mathcal{T}_h)} + \|\Pi_W \Psi_1 - \Psi_1\|_{L^1(\partial \mathcal{T}_h)}) \\ &\quad + \|u\|_{L^\infty(\Omega)} \\ &\leq \|u\|_{L^\infty(\Omega)} + C(|\log h|^{1/2} + 1)\|\mathbf{q}\|_{L^\infty(\Omega)}. \end{aligned}$$

This completes the proof of (3.1b). $\square$

Step 3: Proof of (3.1d)–(3.1e) in Theorem 3.1. We choose δ_1 and δ_2 such that $\|\Pi_W u - u_h\|_{L^\infty(\Omega)} = (\delta_1, \Pi_W u - u_h)_{\mathcal{T}_h}$, $\|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)} = (\delta_2, \Pi_V \mathbf{q} - \mathbf{q}_h)_{\mathcal{T}_h}$.

LEMMA 3.12. *Let $(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h})$ and $(\Phi_{2,h}, \Psi_{2,h}, \widehat{\Psi}_{2,h})$ be the HDG solutions of (3.9a) and (3.9b), respectively. Then we have*

$$\begin{aligned} -(\delta_1, \Pi_W u - u_h)_{\mathcal{T}_h} &= (c(\Phi_{1,h} - \Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (c\Phi_1 - \Pi_{k-1}(c\Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h}, \\ (\delta_2, \Pi_V \mathbf{q} - \mathbf{q}_h)_{\mathcal{T}_h} &= (c(\Phi_{2,h} - \Phi_2), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (c\Phi_2 - \Pi_{k-1}(c\Phi_2), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h}. \end{aligned}$$

Proof. We take $(\mathbf{v}_h, w_h, \widehat{w}_h) = (\Pi_V \mathbf{q} - \mathbf{q}_h, \Pi_W u - u_h, \Pi_k^\partial u - \widehat{u}_h)$ in (3.9a) to get

$$\begin{aligned}
 -(\delta_1, \Pi_W u - u_h)_{\mathcal{T}_h} &= \mathcal{B}(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}; \Pi_V \mathbf{q} - \mathbf{q}_h, \Pi_W u - u_h, \Pi_k^\partial u - \widehat{u}_h) \\
 &= \mathcal{B}(\Pi_V \mathbf{q} - \mathbf{q}_h, \Pi_W u - u_h, \Pi_k^\partial u - \widehat{u}_h; \Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}) && \text{by (2.6)} \\
 &= \mathcal{B}(\Pi_V \mathbf{q} - \mathbf{q}, \Pi_W u - u, \Pi_k^\partial u - u; \Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}) && \text{by (2.5)} \\
 &= \mathcal{B}(\Phi_{1,h}, \Psi_{1,h}, \widehat{\Psi}_{1,h}; \Pi_V \mathbf{q} - \mathbf{q}, \Pi_W u - u, \Pi_k^\partial u - u) && \text{by (2.6)} \\
 &= (c\Phi_{1,h}, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} - (\Psi_{1,h}, \nabla \cdot (\Pi_V \mathbf{q} - \mathbf{q}))_{\mathcal{T}_h} \\
 &\quad + \langle \widehat{\Psi}_{1,h}, (\Pi_V \mathbf{q} - \mathbf{q}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} - (\nabla \cdot \Phi_{1,h}, \Pi_W u - u)_{\mathcal{T}_h} \\
 &\quad - \langle \tau(\Psi_{1,h} - \widehat{\Psi}_{1,h}), \Pi_W u - u \rangle_{\partial \mathcal{T}_h} + \langle \Phi_{1,h} \cdot \mathbf{n}, \Pi_k^\partial u - u \rangle_{\partial \mathcal{T}_h},
 \end{aligned}$$

where we used the definition of $\mathcal{B}$ in the last step. Next, by (2.7a)–(2.7c) we have

$$\begin{aligned}
 -(\delta_1, \Pi_W u - u_h)_{\mathcal{T}_h} &= (c\Phi_{1,h}, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} - (\Psi_{1,h}, \nabla \cdot (\Pi_V \mathbf{q} - \mathbf{q}))_{\mathcal{T}_h} \\
 &\quad + \langle \widehat{\Psi}_{1,h}, (\Pi_V \mathbf{q} - \mathbf{q}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} - \langle \tau(\Psi_{1,h} - \widehat{\Psi}_{1,h}), \Pi_W u - u \rangle_{\partial \mathcal{T}_h} \\
 &= (c\Phi_{1,h}, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (\nabla \Psi_{1,h}, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} \\
 &\quad - \langle \Psi_{1,h} - \widehat{\Psi}_{1,h}, (\Pi_V \mathbf{q} - \mathbf{q}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} - \langle \tau(\Psi_{1,h} - \widehat{\Psi}_{1,h}), \Pi_W u - u \rangle_{\partial \mathcal{T}_h} \\
 &= (c\Phi_{1,h}, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} \\
 &= (c(\Phi_{1,h} - \Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (c\Phi_1, \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} \\
 &= (c(\Phi_{1,h} - \Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (c\Phi_1 - \Pi_{k-1}(c\Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h}.
 \end{aligned}$$

This gives the proof of the first identity; we omit the proof of the second identity since it follows along the same lines. $\square$

LEMMA 3.13. *Let $(\mathbf{q}, u)$ and $(\mathbf{q}_h, u_h)$ be the solution of (1.1) and (2.1), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$(3.22a) \quad \|\Pi_W u - u_h\|_{L^\infty(\Omega)} \leq Ch^{k+2}(|\log h| + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+1,\infty}(\Omega)}),$$

$$(3.22b) \quad \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)} \leq Ch^{k+1}(|\log h|^{1/2} + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+1,\infty}(\Omega)}).$$

Proof. By Lemmas 3.11 and 3.12 we have

$$\begin{aligned}
 \|\Pi_W u - u_h\|_{L^\infty(\Omega)} &= |(c(\Phi_{1,h} - \Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h} + (c\Phi_1 - \Pi_{k-1}(c\Phi_1), \Pi_V \mathbf{q} - \mathbf{q})_{\mathcal{T}_h}| \\
 &\leq C\|\Pi_V \mathbf{q} - \mathbf{q}\|_{L^\infty(\Omega)} (\|\Phi_{1,h} - \Phi_1\|_{L^1(\Omega)} + \|c\Phi_1 - \Pi_{k-1}(c\Phi_1)\|_{L^1(\Omega)}) \\
 &\leq Ch^{k+2}(|\log h| + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+1,\infty}(\Omega)}). \quad \square
 \end{aligned}$$

As a consequence, a simple application of the triangle inequality and Lemma 3.13 and Theorem 2.6 gives convergence rates for $\|\mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)}$ and $\|u - u_h\|_{L^\infty(\Omega)}$. This completes the proof of (3.1d)–(3.1e) in Theorem 3.1.

Step 4: Proof of (3.1f) in Theorem 3.1.

Proof. First, for all $w_0 \in \mathcal{P}^0(K)$, we have

$$(3.23) \quad (\Pi_W u - \Pi_{k+1}^\circ u, w_0)_K = (\Pi_W u - u, w_0)_K + (u - \Pi_{k+1}^\circ u, w_0)_K = 0.$$

Let $e_h = u_h^* - u_h + \Pi_W u - \Pi_{k+1}^\circ u$; by (2.2) we obtain

$$\begin{aligned}
 \|\nabla e_h\|_{L^2(K)}^2 &= (\nabla(u_h^* - u_h), \nabla e_h)_K + (\nabla(\Pi_W u - \Pi_{k+1}^\circ u), \nabla e_h)_K \\
 &= (-\nabla u_h - \mathbf{q}_h, \nabla e_h)_K + (\nabla(\Pi_W u - \Pi_{k+1}^\circ u), \nabla e_h)_K \\
 &= (\nabla(\Pi_W u - u_h)_K - (\mathbf{q}_h - \mathbf{q}) + \nabla(u - \Pi_{k+1}^\circ u), \nabla e_h)_K.
 \end{aligned}$$

Using Lemma 2.5, this implies that

$$(3.24) \quad \|\nabla e_h\|_{L^2(K)} \leq C(h_K^{-1}\|\Pi_W u - u_h\|_{L^2(K)} + \|\mathbf{q}_h - \mathbf{q}\|_{L^2(K)} + \|\nabla(u - \Pi_{k+1}u)\|_{L^2(K)}).$$

By (2.2b) and (3.23) we get $(e_h, 1)_K = 0$, i.e., $\Pi_0^o e_h = 0$. Then standard estimates for the L^2 projection given in (3.24) show that

$$\begin{aligned} \|e_h\|_{L^2(K)} &= \|e_h - \Pi_0^o e_h\|_{L^2(K)} \\ &\leq Ch_K \|\nabla e_h\|_{L^2(K)} \\ &\leq C(\|\Pi_W u - u_h\|_{L^2(K)} + h_K \|\mathbf{q}_h - \mathbf{q}\|_{L^2(K)} + h_K \|\nabla(u - \Pi_{k+1}u)\|_{L^2(K)}). \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\Pi_{k+1}^o u - u_h^*\|_{L^2(K)} &\leq C\|\Pi_W u - u_h\|_{L^2(K)} + Ch\|\mathbf{q}_h - \mathbf{\Pi}_V \mathbf{q}\|_{L^2(K)} \\ &\quad + Ch\|\mathbf{\Pi}_V \mathbf{q} - \mathbf{q}\|_{L^2(K)} + Ch\|\nabla(u - \Pi_{k+1}^o u)\|_{L^2(K)}. \end{aligned}$$

We use the above inequality and Lemma 2.5 to get

$$\begin{aligned} \|\Pi_{k+1}^o u - u_h^*\|_{L^\infty(K)} &\leq Ch^{-1}\|\Pi_{k+1}^o u - u_h^*\|_{L^2(K)} \\ &\leq Ch^{-1}\|\Pi_W u - u_h\|_{L^2(K)} + C\|\mathbf{q}_h - \mathbf{\Pi}_V \mathbf{q}\|_{L^2(K)} \\ &\quad + C\|\mathbf{\Pi}_V \mathbf{q} - \mathbf{q}\|_{L^2(K)} + C\|\nabla(u - \Pi_{k+1}^o u)\|_{L^2(K)} \\ &\leq C\|\Pi_W u - u_h\|_{L^\infty(K)} + Ch\|\mathbf{q}_h - \mathbf{\Pi}_V \mathbf{q}\|_{L^\infty(K)} \\ &\quad + C\|\mathbf{\Pi}_V \mathbf{q} - \mathbf{q}\|_{L^2(K)} + C\|\nabla(u - \Pi_{k+1}^o u)\|_{L^2(K)}. \end{aligned}$$

Now let K^* denote the element in which $\|\Pi_{k+1}^o u - u_h^*\|_{L^\infty(\Omega)} = \|\Pi_{k+1}^o u - u_h^*\|_{L^\infty(K^*)}$. Then

$$\begin{aligned} \|\Pi_{k+1}^o u - u_h^*\|_{L^\infty(\Omega)} &= \|\Pi_{k+1}^o u - u_h^*\|_{L^\infty(K^*)} \\ &\leq C(\|\Pi_W u - u_h\|_{L^\infty(K^*)} + h\|\mathbf{q}_h - \mathbf{\Pi}_V \mathbf{q}\|_{L^\infty(K^*)}) \\ &\quad + C(\|\mathbf{\Pi}_V \mathbf{q} - \mathbf{q}\|_{L^2(K^*)} + \|\nabla(u - \Pi_{k+1}^o u)\|_{L^2(K^*)}) \\ &\leq C(\|\Pi_W u - u_h\|_{L^\infty(\Omega)} + h\|\mathbf{q}_h - \mathbf{\Pi}_V \mathbf{q}\|_{L^\infty(\Omega)}) \\ &\quad + C(\|\mathbf{\Pi}_V \mathbf{q} - \mathbf{q}\|_{L^2(K^*)} + \|\nabla(u - \Pi_{k+1}^o u)\|_{L^2(K^*)}). \end{aligned}$$

By the estimates in Lemmas 2.4, 2.5, and 3.13 and Theorem 2.6 and the triangle inequality we get our claimed result. $\square$

4. Quasi-optimal estimates on interfaces. Let Γ be a finite union of line segments such that Ω is decomposed into finitely many Lipschitz domains by Γ . We stress that, while Ω is assumed to be convex, the subdomains need not be convex. Define $\mathcal{F}_h^\Gamma$ by

$$\mathcal{F}_h^\Gamma = \{F \in \mathcal{F}_h : \text{measure}(F \cap \Gamma) > 0\}.$$

We assume, furthermore, that the triangle mesh $\mathcal{T}_h$ resolves Γ . Hence, Γ can be written as the union of $\mathcal{O}(h^{-1})$ edges in $\mathcal{F}_h$, i.e., $\bar{\Gamma} = \bigcup_{F \in \mathcal{F}_h^\Gamma \subset \mathcal{F}_h} \bar{F}$.

THEOREM 4.1. *Assume Γ has the above properties, and let $(\mathbf{q}, u)$ and $(\mathbf{q}_h, u_h)$ be the solution of (1.1) and (2.1), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$(4.1a) \quad \|\mathbf{q} - \mathbf{q}_h\|_{L^2(\Gamma)} \leq Ch^{k+1}(|\log h|^{1/2} + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+1,\infty}(\Omega)}),$$

$$(4.1b) \quad \|u - u_h\|_{L^2(\Gamma)} \leq Ch^{k+1}(|\log h| + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+1,\infty}(\Omega)}).$$

Furthermore, we have the following error estimate for the postprocessed solution:

$$(4.1c) \quad \|u - u_h^*\|_{L^2(\Gamma)} \leq Ch^{k+2}(|\log h| + 1)(|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+2,\infty}(\Omega)}),$$

where u_h^* is defined in (2.2).

Remark 4.2. The result proves the observation seen in numerical experiments that the flux on Γ converges at an optimal rate. The best theoretical estimate known to us before our paper is $\mathcal{O}(h^{k+1/2})$.

Proof. We only prove (4.1a) since the proofs of (4.1b) and (4.1c) are very similar. We define tubular neighborhoods of Γ by

$$\mathbb{S}_h := \{K \in \mathcal{T}_h | \Gamma \cap \partial K \neq \emptyset\}.$$

Then the number of the elements in $\mathbb{S}_h$ is order of $\mathcal{O}(h^{-1})$.

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_h\|_{L^2(\Gamma)}^2 &\leq \|\Pi_V \mathbf{q} - \mathbf{q}\|_{L^2(\Gamma)}^2 + \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^2(\Gamma)}^2 \\ &\leq \sum_{K \in \mathbb{S}_h} \|\Pi_V \mathbf{q} - \mathbf{q}\|_{L^2(\partial K)}^2 + \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^2(\partial K)}^2 \\ &\leq C \sum_{K \in \mathbb{S}_h} \left(h^{2k+3} |\nabla^{k+1} \mathbf{q}|_{L^\infty(\Omega)}^2 + h^{-1} \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^2(K)}^2 \right) \quad \text{by (2.13b)} \\ &\leq C \sum_{K \in \mathbb{S}_h} \left(h^{2k+3} |\nabla^{k+1} \mathbf{q}|_{L^\infty(\Omega)}^2 + h^{-1} h^2 \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^\infty(K)}^2 \right) \quad \text{by (2.12)} \\ &\leq C \sum_{K \in \mathbb{S}_h} h^{2k+3} |\nabla^{k+1} \mathbf{q}|_{L^\infty(\Omega)}^2 + C \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)}^2 \sum_{K \in \mathbb{S}_h} h^{-1} h^2 \\ &\leq Ch^{2k+2} |\nabla^{k+1} \mathbf{q}|_{L^\infty(\Omega)}^2 + C \|\Pi_V \mathbf{q} - \mathbf{q}_h\|_{L^\infty(\Omega)}^2 \\ &\leq Ch^{2k+2} (|\log h|^{1/2} + 1)^2 (|\mathbf{q}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |u|_{W^{k+2,\infty}(\Omega)})^2 \quad \text{by (3.22b)}. \end{aligned}$$

This completes the proof of (4.1a). $\square$

5. Dirichlet boundary control problem. In this section, we consider an elliptic Dirichlet boundary control problem. Let $u_d \in L^2(\Omega)$ denote a given desired state for the solution, and let $\gamma > 0$ be a given regularization parameter. The problem is to solve the following optimization problem:

$$(5.1a) \quad \min_{g \in L^2(\partial\Omega)} J(g), \quad J(g) := \frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|g\|_{L^2(\partial\Omega)}^2,$$

where u is the solution of the Poisson equation with nonhomogeneous Dirichlet boundary conditions

$$(5.1b) \quad -\Delta u = f \quad \text{in } \Omega,$$

$$(5.1c) \quad u = g \quad \text{on } \partial\Omega.$$

The function g is called the control, and computing the optimal g is the desired result of the above problem.

It is well known that the Dirichlet boundary control problem (5.1a)–(1.1b) is equivalent to solving the following optimality system for (u, z, g) :

$$\begin{aligned} (5.2a) \quad & -\Delta u = f \quad \text{in } \Omega, \\ (5.2b) \quad & u = g \quad \text{on } \partial\Omega, \\ (5.2c) \quad & -\Delta z = u - u_d \quad \text{in } \Omega, \\ (5.2d) \quad & z = 0 \quad \text{on } \partial\Omega, \\ (5.2e) \quad & g = \gamma^{-1} \partial_n z \quad \text{on } \partial\Omega. \end{aligned}$$

Define $\mathbf{q} = -\nabla u$ and $\mathbf{p} = -\nabla z$; then the mixed weak form of (5.2a)–(5.2e) is to find $(\mathbf{q}, u, \mathbf{p}, z, g) \in \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega) \times \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$ such that

$$\begin{aligned} (5.3a) \quad & (\mathbf{q}, \mathbf{v}_1) - (u, \nabla \cdot \mathbf{r}) + \langle g, \mathbf{v}_1 \cdot \mathbf{n} \rangle = 0, \\ (5.3b) \quad & (\nabla \cdot \mathbf{q}, w_1) = (f, w_1), \\ (5.3c) \quad & (\mathbf{p}, \mathbf{v}_2) - (z, \nabla \cdot \mathbf{v}_2) = 0, \\ (5.3d) \quad & (\nabla \cdot \mathbf{p}, w_2) - (u, w_2) = -(u_d, w_2), \\ (5.3e) \quad & \langle \gamma g + \mathbf{p} \cdot \mathbf{n}, \xi \rangle = 0 \end{aligned}$$

for all $(\mathbf{v}_1, w_1, \mathbf{v}_2, w_2, \xi) \in \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega) \times \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$.

To give the HDG formulation of the above mixed system (5.3), we need to introduce the following finite element space for the boundary control g :

$$\widehat{W}_h(\partial) := \{\widehat{w}_h \in L^2(\mathcal{F}_h^\partial) : \widehat{w}_h|_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h^\partial\}.$$

By the definition of $\mathcal{B}$ in (2.3) and setting $c = 1$, the HDG formulation of the optimality system (2.1) is to find $(\mathbf{q}_h, u_h, \mathbf{p}_h, z_h, \widehat{u}_h, \widehat{z}_h, g_h) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h \times W_h \times \widehat{W}_h \times \widehat{W}_h \times \widehat{W}_h(\partial)$ such that

$$\begin{aligned} (5.4a) \quad & \mathcal{B}(\mathbf{q}_h, u_h, \widehat{u}_h; \mathbf{v}_1, w_1, \widehat{w}_1) = -\langle g_h, \mathbf{v}_1 \cdot \mathbf{n} + \tau w_1 \rangle_{\mathcal{F}_h^\partial} - (f, w_1)_{\mathcal{T}_h}, \\ (5.4b) \quad & \mathcal{B}(\mathbf{p}_h, z_h, \widehat{z}_h; \mathbf{v}_2, w_2, \widehat{w}_2) = -(u_h - u_d, w_2)_{\mathcal{T}_h}, \\ (5.4c) \quad & \gamma^{-1} \langle \mathbf{p}_h \cdot \mathbf{n} + \tau z_h, \widehat{w}_3 \rangle_{\mathcal{F}_h^\partial} = -\langle g_h, \widehat{w}_3 \rangle_{\mathcal{F}_h^\partial} \end{aligned}$$

for all $(\mathbf{v}_1, \mathbf{v}_2, w_1, w_2, \widehat{w}_1, \widehat{w}_2, \widehat{w}_3) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h \times W_h \times \widehat{W}_h \times \widehat{W}_h \times \widehat{W}_h(\partial)$.

We can now state our main result of this section.

THEOREM 5.1. *Let $(\mathbf{q}, u, \mathbf{p}, z, g)$ and $(\mathbf{q}_h, u_h, \mathbf{p}_h, z_h, g_h)$ be the solution of (5.3) and (5.4), respectively. Assume that $\mathbf{q}, \mathbf{p} \in \mathbf{W}^{k+1, \infty}(\Omega)$, $u, z \in W^{k+1, \infty}(\Omega)$. If assumption (A) holds and $k \geq 1$, then we have*

$$\begin{aligned} & \|g - g_h\|_{L^2(\partial\Omega)} + \|u - u_h\|_{L^2(\Omega)} + \|z - z_h\|_{L^2(\Omega)} + \|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)} + h^{1/2} \|\mathbf{q} - \mathbf{q}_h\|_{L^2(\Omega)} \\ & \leq Ch^{k+1} (|\log h| + 1) (\|\mathbf{p}\|_{\mathbf{W}^{k+1, \infty}(\Omega)} + \|z\|_{W^{k+1, \infty}(\Omega)} + \|\mathbf{q}\|_{\mathbf{H}^{k+1}(\Omega)} + \|u\|_{H^{k+1}(\Omega)}). \end{aligned}$$

Remark 5.2. Numerical experiments for the Dirichlet boundary control problem given in (5.2) always show optimal order convergence rates if the solution is smooth enough. The first work to prove this observation can be found in [25] by May, Rannacher, and Vexler. The proof is based on a duality argument and gives estimates for the control in weaker norms than $L^2(\partial\Omega)$. However, this technique is not straightforward for the HDG method; see [5, 23, 22, 17, 18]. Hence, Theorem 5.1 is the first proof that the HDG method achieves an optimal order convergence rate for the control, state, and dual state, provided we assume the solution of the Dirichlet boundary control problem is smooth enough.

5.1. Proof of Theorem 5.1. We follow the strategy in [23] and introduce an auxiliary problem: find $(\mathbf{q}_h(g), \mathbf{p}_h(g), u_h(g), z_h(g), \widehat{u}_h(g), \widehat{z}_h(g)) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h \times W_h \times \widehat{W}_h \times \widehat{W}_h$ such that

$$(5.5a) \quad \mathcal{B}(\mathbf{q}_h(g), u_h(g), \widehat{u}_h(g); \mathbf{v}_1, w_1, \widehat{w}_1) = -\langle g, \mathbf{v}_1 \cdot \mathbf{n} + \tau w_1 \rangle_{\mathcal{F}_h^\partial} - (f, w_1)_{\mathcal{T}_h},$$

$$(5.5b) \quad \mathcal{B}(\mathbf{p}_h(g), z_h(g), \widehat{z}_h(g); \mathbf{v}_2, w_2, \widehat{w}_2) = -(u - u_d, w_2)_{\mathcal{T}_h}$$

for all $(\mathbf{v}_1, \mathbf{v}_2, w_1, w_2, \widehat{w}_1, \widehat{w}_2) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h \times W_h \times \widehat{W}_h \times \widehat{W}_h$, where $g \in L^2(\partial\Omega)$ is the exact optimal control.

The proof now proceeds in three steps as follows.

Step 1. We first bound the error between the solutions of the auxiliary problem (5.5) and the mixed weak form (5.3a)–(5.3e) of the optimality system. The proof of Lemma 5.3 can be found in [10, Theorem 4.1, Appendix] and Theorem 4.1.

LEMMA 5.3. *Let $(\mathbf{q}, u, \mathbf{p}, z, g)$ and $(\mathbf{q}_h(g), \mathbf{p}_h(g), u_h(g), z_h(g), \widehat{u}_h(g), \widehat{z}_h(g))$ be the solution of (5.3) and (5.5), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$(5.6a) \quad \|\mathbf{q} - \mathbf{q}_h(g)\|_{L^2(\Omega)} \leq Ch^{k+1}(|\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}),$$

$$(5.6b) \quad \|u - u_h(g)\|_{L^2(\Omega)} \leq Ch^{k+1}(|\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}),$$

$$(5.6c) \quad \|\mathbf{p} - \mathbf{p}_h(g)\|_{L^2(\Omega)} \leq Ch^{k+1}(|\mathbf{p}|_{\mathbf{H}^{k+1}(\Omega)} + |z|_{H^{k+1}(\Omega)}),$$

$$(5.6d) \quad \|z - z_h(g)\|_{L^2(\Omega)} \leq Ch^{k+1}(|\mathbf{p}|_{\mathbf{H}^{k+1}(\Omega)} + |z|_{H^{k+1}(\Omega)}),$$

$$(5.6e) \quad \|\mathbf{p}_h(g) - \mathbf{p}\|_{L^2(\partial\Omega)} \leq Ch^{k+1}(|\log h|^{1/2} + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)}),$$

$$(5.6f) \quad \|\tau(z_h(g) - z)\|_{L^2(\partial\Omega)} \leq Ch^{k+1}(|\log h| + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)}).$$

Step 2. Next, we bound the error between the solutions of the auxiliary problem and the HDG problem (2.4). Note that

$$(5.7a) \quad \mathcal{B}(\mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \widehat{u}_h(g) - \widehat{u}_h; \mathbf{v}_1, w_1, \widehat{w}_1) = -\langle g - g_h, \mathbf{v}_1 \cdot \mathbf{n} + \tau w_1 \rangle_{\mathcal{F}_h^\partial},$$

$$(5.7b) \quad \mathcal{B}(\mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \widehat{z}_h(g) - \widehat{z}_h; \mathbf{v}_2, w_2, \widehat{w}_2) = -(u - u_h, w_2)_{\mathcal{T}_h}$$

for all $(\mathbf{v}_1, \mathbf{v}_2, w_1, w_2, \widehat{w}_1, \widehat{w}_2) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h \times W_h \times \widehat{W}_h \times \widehat{W}_h$.

LEMMA 5.4. *Let (u, g) and (u_h, g_h) be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$\begin{aligned} & \|g - g_h\|_{L^2(\partial\Omega)} \\ & \leq Ch^{k+1}(|\log h| + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)} + |\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}), \\ & \|u - u_h\|_{L^2(\Omega)} \\ & \leq Ch^{k+1}(|\log h| + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)} + |\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}). \end{aligned}$$

Proof. First, we take $(\mathbf{v}_1, w_1, \widehat{w}_1) = (\mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \widehat{z}_h(g) - \widehat{z}_h)$, $(\mathbf{v}_2, w_2, \widehat{w}_2) = (\mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \widehat{u}_h(g) - \widehat{u}_h)$ in (5.7) and use Lemma 2.2 to get

$$\langle g - g_h, (\mathbf{p}_h(g) - \mathbf{p}_h) \cdot \mathbf{n} + \tau(z_h(g) - z_h) \rangle_{\mathcal{F}_h^\partial} = (u - u_h, u_h(g) - u_h)_{\mathcal{T}_h}.$$

Since $g + \gamma^{-1}\mathbf{p} \cdot \mathbf{n} = 0$ on $\mathcal{F}_h^\partial$ and $g_h + \gamma^{-1}\mathbf{p}_h \cdot \mathbf{n} + \gamma^{-1}\tau z_h = 0$ on $\mathcal{F}_h^\partial$ we have

$$\begin{aligned}
(u - u_h, u_h(g) - u_h)\tau_h &= \langle g - g_h, \mathbf{p}_h(g) \cdot \mathbf{n} + \tau z_h(g) + \gamma g_h \rangle_{\mathcal{F}_h^\partial} \\
&= \langle g - g_h, \mathbf{p}_h(g) \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n} + \mathbf{p} \cdot \mathbf{n} + \tau z_h(g) + \gamma g_h \rangle_{\mathcal{F}_h^\partial} \\
&= \langle g - g_h, \mathbf{p}_h(g) \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n} + \tau z_h(g) + \gamma g_h - \gamma g \rangle_{\mathcal{F}_h^\partial} \\
&= \langle g - g_h, \mathbf{p}_h(g) \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n} + \tau z_h(g) \rangle_{\mathcal{F}_h^\partial} - \gamma \|g - g_h\|_{L^2(\partial\Omega)}^2.
\end{aligned}$$

Since $z = 0$ on $\mathcal{F}_h^\partial$, we rearrange the above equality and obtain

$$\begin{aligned}
&\gamma \|g - g_h\|_{L^2(\partial\Omega)}^2 + \|u - u_h\|_{L^2(\Omega)}^2 \\
&= \langle (\mathbf{p}_h(g) - \mathbf{p}) \cdot \mathbf{n} + \tau z_h(g), g - g_h \rangle_{\mathcal{F}_h^\partial} - (u - u_h, u_h(g) - u)\tau_h \\
&\leq (\|\mathbf{p}_h(g) - \mathbf{p}\|_{L^2(\partial\Omega)} + \|\tau(z_h(g) - z)\|_{L^2(\partial\Omega)}) \|g - g_h\|_{L^2(\partial\Omega)} \\
&\quad + \|u - u_h\|_{L^2(\Omega)} \|u_h(g) - u\|_{L^2(\Omega)}.
\end{aligned}$$

Our desired result follows by Young's inequality, the triangle inequality, and Lemma 5.3. $\square$

Step 3.

LEMMA 5.5. *Let $(\mathbf{p}, z)$ and $(\mathbf{p}_h, z_h)$ be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$\begin{aligned}
&\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)} \\
&\leq Ch^{k+1}(|\log h| + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)} + |\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}), \\
&\|z - z_h\|_{L^2(\Omega)} \\
&\leq Ch^{k+1}(|\log h| + 1)(|\mathbf{p}|_{\mathbf{W}^{k+1,\infty}(\Omega)} + |z|_{W^{k+1,\infty}(\Omega)} + |\mathbf{q}|_{\mathbf{H}^{k+1}(\Omega)} + |u|_{H^{k+1}(\Omega)}).
\end{aligned}$$

Proof. By Lemma 2.1 and letting $(\mathbf{v}_2, w_2, \hat{w}_2) = (\mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \hat{z}_h(g) - \hat{z}_h)$ in the error equation (5.7b), we have

$$\begin{aligned}
\|\mathbf{p}_h(g) - \mathbf{p}_h\|_{L^2(\Omega)}^2 &\leq -(u - u_h, z_h(g) - z_h)\tau_h \\
(5.8) \quad &\leq \|u - u_h\|_{\tau_h} \|z_h(g) - z_h\|_{\tau_h} \\
&\leq \frac{1}{\rho} \|u - u_h\|_{\tau_h}^2 + \rho \|z_h(g) - z_h\|_{\tau_h}^2.
\end{aligned}$$

Here ρ is a positive constant which will be assigned later. Next, we introduce the dual problem of finding (Φ, Ψ) such that

$$\begin{aligned}
c\Phi + \nabla\Psi &= \mathbf{0} && \text{in } \Omega, \\
(5.9) \quad \nabla \cdot \Phi &= z_h(g) - z_h && \text{in } \Omega, \\
\Psi &= 0 && \text{on } \partial\Omega.
\end{aligned}$$

Since the domain Ω is convex, we have the following regularity estimate:

$$(5.10) \quad \|\Phi\|_{\mathbf{H}^1(\Omega)} + \|\Psi\|_{H^2(\Omega)} \leq C_{\text{reg}} \|z_h(g) - z_h\|_{L^2(\Omega)}.$$

On the one hand, we take $(\mathbf{v}_2, w_2, \hat{w}_2) = (\Pi_V \Phi, \Pi_W \Psi, \Pi_k^\partial \Psi)$ in (5.7b) to get

$$\begin{aligned}
(5.11) \quad &\mathcal{B}(\mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \hat{z}_h(g) - \hat{z}_h; \Pi_V \Phi, \Pi_W \Psi, \Pi_k^\partial \Psi) \\
&= \mathcal{B}(\Pi_V \Phi, \Pi_W \Psi, \Pi_k^\partial \Psi; \mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \hat{z}_h(g) - \hat{z}_h) \\
&= (\Pi_V \Phi - \Phi, \mathbf{p}_h(g) - \mathbf{p}_h)\tau_h + \|z_h(g) - z_h\|_{L^2(\Omega)}^2.
\end{aligned}$$

On the other hand, by the error equation (5.7b), we have

(5.12)

$$\mathcal{B}(\mathbf{p}_h(g) - \mathbf{p}_h, z_h(g) - z_h, \hat{z}_h(g) - \hat{z}_h; \Pi_V \Phi, \Pi_W \Psi, \Pi_k^\partial \Psi) = -(u - u_h, \Pi_W \Psi)_{\mathcal{T}_h}.$$

Comparing the above two equalities (5.11), (5.12), and (2.8b) give

$$\begin{aligned} \|z_h(g) - z_h\|_{L^2(\Omega)}^2 &= -(u - u_h, \Pi_W \Psi)_{\mathcal{T}_h} - (\Pi_V \Phi - \Phi, \mathbf{p}_h(g) - \mathbf{p}_h)_{\mathcal{T}_h} \\ &\leq C \|u - u_h\|_{L^2(\Omega)}^2 + \frac{1}{4} \|z_h(g) - z_h\|_{L^2(\Omega)}^2 \\ &\quad + \frac{h^2}{4} \|z_h(g) - z_h\|_{L^2(\Omega)}^2 + C \|\mathbf{p}_h(g) - \mathbf{p}_h\|_{L^2(\Omega)}^2 \\ &\leq C \left(1 + \frac{1}{\rho}\right) \|u - u_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|z_h(g) - z_h\|_{L^2(\Omega)}^2 + C \rho \|z_h(g) - z_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Taking $\rho = \frac{1}{4C}$, then we have

$$(5.13) \quad \|z_h(g) - z_h\|_{L^2(\Omega)} \leq C \|u - u_h\|_{L^2(\Omega)}.$$

Inserting this inequality into (5.8) gives

$$(5.14) \quad \|\mathbf{p}_h(g) - \mathbf{p}_h\|_{L^2(\Omega)} \leq C \|u - u_h\|_{L^2(\Omega)}.$$

Then our desired result follows by (5.13), (5.14), and Lemma 5.5. $\square$

LEMMA 5.6. *Let $(\mathbf{q}, u, \mathbf{p}, z, g)$ and $(\mathbf{q}_h, u_h, \mathbf{p}_h, z_h, g_h)$ be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geq 1$, then we have*

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_h\|_{L^2(\Omega)} &\leq Ch^{k+\frac{1}{2}} (|\log h| + 1) (\|\mathbf{p}\|_{W^{k+1,\infty}(\Omega)} + \|z\|_{W^{k+1,\infty}(\Omega)} + \|\mathbf{q}\|_{H^{k+1}(\Omega)} + \|u\|_{H^{k+1}(\Omega)}). \end{aligned}$$

Proof. On the one hand, by the error equation (5.7a), we have

$$\begin{aligned} \mathcal{B}(\mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \hat{u}_h(g) - \hat{u}_h; \mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \hat{u}_h(g) - \hat{u}_h) \\ = -\langle g - g_h, (\mathbf{q}_h(g) - \mathbf{q}_h) \cdot \mathbf{n} - \tau(u_h(g) - u_h) \rangle_{\mathcal{F}_h} \\ \leq \|g - g_h\|_{L^2(\partial\Omega)} (\|\mathbf{q}_h(g) - \mathbf{q}_h\|_{L^2(\partial\Omega)} + \|u_h(g) - u_h\|_{L^2(\partial\Omega)}) \\ \leq Ch^{-1/2} \|g - g_h\|_{L^2(\partial\Omega)} (\|\mathbf{q}_h(g) - \mathbf{q}_h\|_{L^2(\Omega)} + \|u_h(g) - u_h\|_{L^2(\Omega)}). \end{aligned}$$

On the other hand, by Lemma 2.1, we obtain

$$\begin{aligned} \mathcal{B}(\mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \hat{u}_h(g) - \hat{u}_h; \mathbf{q}_h(g) - \mathbf{q}_h, u_h(g) - u_h, \hat{u}_h(g) - \hat{u}_h) \\ = \|\mathbf{q}_h(g) - \mathbf{q}_h\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}((u_h(g) - u_h) - (\hat{u}_h(g) - \hat{u}_h))\|_{\partial\mathcal{T}_h}. \end{aligned}$$

Comparing the above two inequalities, using Young's inequality and (5.4), gives

$$(5.15) \quad \|\mathbf{q}_h(g) - \mathbf{q}_h\|_{\mathcal{T}_h} \leq Ch^{-1/2} \|g - g_h\|_{L^2(\partial\Omega)}.$$

Then our desired result follows by Lemma 5.5, the triangle inequality, and (5.6a). $\square$

6. Numerical results. In this section, we present two examples to illustrate our theoretical results.

Example 6.1. We first test the convergence rate of the L^∞ norm estimate on a convex domain and the L^2 norm estimate on the boundary. The data is chosen to be

$$\Omega = (0, 1) \times (0, 1), \quad c = 1, \quad u(x, y) = \sin(10x).$$

The source term f is chosen to match the exact solution of (1.1), and the approximation errors are listed in Table 1 for the $L^\infty(\Omega)$ norm error and Table 2 for the $L^2(\partial\Omega)$ norm error. The rates match the theoretical predictions in Theorems 3.1 and 4.1.

Our theoretical result needs the domain to be convex, but it is interesting to observe whether the convergence rate can still hold for a nonconvex domain. For example, we choose the same data as above except the domain is chosen to be an L -shape domain:

$$\Omega = (0, 1) \times (0, 1) \setminus [1/2, 1) \times (0, 1/2].$$

In this case the H^2 regularity of Ψ_1 and Ψ_2 in Lemma 3.4 does not hold. The approximation errors are listed in Table 3 for the $L^\infty(\Omega)$ norm error (the $L^2(\partial\Omega)$ norm error also converges at the quasi-optimal rate: results are not shown). It is obvious that the quasi-optimal convergence rate is still seen for the L -shape domain.

Example 6.2. Lastly, we test the convergence rate for a smooth solution to the Dirichlet boundary control problem. The data and the exact solution are chosen to be

TABLE 1
Example 6.1: $L^\infty(\Omega)$ errors for $\mathbf{q}_h$, u_h , and u_h^ on the convex domain $(0, 1) \times (0, 1)$.*

Degree	$\frac{h}{\sqrt{2}}$	$\ \mathbf{q} - \mathbf{q}_h\ _{L^\infty(\Omega)}$		$\ u - u_h\ _{L^\infty(\Omega)}$		$\ u - u_h^*\ _{L^\infty(\Omega)}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 1$	2^{-1}	1.8881E+01	-	8.1191E+00	-	1.4941E+00	-
	2^{-2}	1.0384E+01	0.86	2.9595E+00	1.46	4.0248E-01	1.89
	2^{-3}	2.9862E+00	1.80	7.7800E-01	1.93	5.9420E-02	2.76
	2^{-4}	7.5737E-01	1.98	2.0046E-01	1.96	7.6187E-03	2.96
	2^{-5}	1.9487E-01	1.96	4.9683E-02	2.01	9.7985E-04	2.96
$k = 2$	2^{-1}	1.8115E+01	-	6.5763E+00	-	7.2961E-01	-
	2^{-2}	3.4370E+00	2.40	1.0994E+00	2.58	6.9452E-02	3.39
	2^{-3}	4.7355E-01	2.86	1.4548E-01	2.92	4.6990E-03	3.89
	2^{-4}	6.2699E-02	2.92	1.7948E-02	3.02	3.1054E-04	3.92
	2^{-5}	7.8798E-03	2.99	2.2918E-03	2.97	1.9522E-05	3.99

TABLE 2
Example 6.1: $L^2(\partial\Omega)$ errors for $\mathbf{q}_h$, u_h , and u_h^ on the convex domain $(0, 1) \times (0, 1)$.*

Degree	$\frac{h}{\sqrt{2}}$	$\ \mathbf{q} - \mathbf{q}_h\ _{L^2(\partial\Omega)}$		$\ u - u_h\ _{L^2(\partial\Omega)}$		$\ u - u_h^*\ _{L^2(\partial\Omega)}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 1$	2^{-1}	9.3751E+00	-	3.9706E+00	-	5.4467E-01	-
	2^{-2}	4.1197E+00	1.19	1.9143E+00	1.05	1.0446E-01	2.38
	2^{-3}	1.1791E+00	1.80	6.1659E-01	1.63	1.4777E-02	2.82
	2^{-4}	3.0648E-01	1.94	1.6398E-01	1.91	1.9370E-03	2.93
	2^{-5}	7.7039E-02	1.99	4.1472E-02	1.98	2.4450E-04	2.99
$k = 2$	2^{-1}	6.4399E+00	-	3.4609E+00	-	1.9906E-01	-
	2^{-2}	9.3121E-01	2.79	5.3204E-01	2.70	1.3075E-02	3.93
	2^{-3}	1.1602E-01	3.00	5.7436E-02	3.21	9.1221E-04	3.84
	2^{-4}	1.4279E-02	3.02	6.5665E-03	3.13	5.8411E-05	3.97
	2^{-5}	1.7866E-03	3.00	8.0200E-04	3.03	3.6752E-06	3.99

TABLE 3

Example 6.1: $L^\infty(\Omega)$ errors for $\mathbf{q}_h$, u_h , and u_h^* on the nonconvex L-shaped domain.

Degree	$\frac{h}{\sqrt{2}}$	$\ \mathbf{q} - \mathbf{q}_h\ _{L^\infty(\Omega)}$		$\ u - u_h\ _{L^\infty(\Omega)}$		$\ u - u_h^*\ _{L^\infty(\Omega)}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 1$	2^{-1}	1.9604E+01	-	8.1190E+00	-	1.4713E+00	-
	2^{-2}	9.9832E+00	0.97	2.9608E+00	1.46	3.7109E-01	1.99
	2^{-3}	2.9810E+00	1.74	7.7748E-01	1.93	5.9410E-02	2.64
	2^{-4}	7.5727E-01	1.98	2.0046E-01	1.96	7.6187E-03	2.96
	2^{-5}	1.9487E-01	1.96	4.9683E-02	2.01	9.8015E-04	2.96
$k = 2$	2^{-1}	1.6115E+01	-	6.4608E+00	-	5.6157E-01	-
	2^{-2}	3.4372E+00	2.23	1.0994E+00	2.55	6.9454E-02	3.02
	2^{-3}	4.7348E-01	2.86	1.4548E-01	2.92	4.7007E-03	3.89
	2^{-4}	6.2862E-02	2.91	1.7948E-02	3.02	3.1283E-04	3.91
	2^{-5}	7.8980E-03	2.99	2.2918E-03	2.97	1.9653E-05	3.99

TABLE 4

Example 6.2, $k = 1$: Errors, observed convergence orders, and expected order (EO) for the control g , state u , adjoint state z , and their fluxes $\mathbf{q}$ and $\mathbf{p}$.

$h/\sqrt{2}$	1/16	1/32	1/64	1/128	1/256	EO
$\ \mathbf{q} - \mathbf{q}_h\ _{L^2(\Omega)}$	2.1856E-02	6.3683E-03	1.9677E-03	6.3980E-04	2.1568E-04	1.50
order	-	1.78	1.69	1.62	1.57	
$\ \mathbf{p} - \mathbf{p}_h\ _{L^2(\Omega)}$	6.3866E-03	1.5958E-03	3.9873E-04	9.9650E-05	2.4911E-05	2.00
order	-	2.00	2.00	2.00	2.00	
$\ u - u_h\ _{L^2(\Omega)}$	8.3560E-03	2.1051E-03	5.2796E-04	1.3218E-04	3.3073E-05	2.00
order	-	1.99	2.00	2.00	2.00	
$\ z - z_h\ _{L^2(\Omega)}$	3.1536E-03	7.9650E-04	2.0006E-04	5.0125E-05	1.2545E-05	2.00
order	-	1.99	1.99	2.00	2.00	
$\ g - g_h\ _{L^2(\partial\Omega)}$	7.2110E-03	1.8119E-03	4.5412E-04	1.1367E-04	2.8425E-05	2.00
order	-	1.99	2.00	2.00	2.00	

$$\Omega = (0, 1) \times (0, 1), \quad \gamma = 1,$$

$$u(x, y) = -\pi(\sin(\pi x) + \sin(\pi y)), \quad z(x, y) = \sin(\pi x) \sin(\pi y).$$

The source term f , the desired state u_d , and the control g are chosen to match the exact solution of (5.2), and the approximation errors are listed in Table 4 when $k = 1$. Results (not shown) for $k = 2$ also confirm the predicted higher order convergence rate in this case. The rates are matched with Theorem 5.1.

7. Conclusion. We have proved quasi-optimal L^∞ norm estimates for the Poisson equation in 2D. Using this result, we obtained quasi-optimal L^2 estimates on an interface. Moreover, we obtained quasi-optimal convergence rates for the Dirichlet boundary control of Poisson's equation, provided the solution is smooth enough.

Our work suggests several interesting directions for further research. First we would like to extend the results to cover L^∞ norm estimates in 3D. In addition the quasi-uniformity assumption on our mesh is restrictive for problems that require adaptive mesh refinement, including those on nonconvex domains. Finally it would be desirable to prove the optimal convergence rate for the Dirichlet boundary control of PDEs without assuming that the solution is smooth.

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