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Hyperbolic four-manifolds with vanishing Seiberg-Witten invariants

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Abstract. We show the existence of hyperbolic 4-manifolds with vanishing Seiberg-Witten invariants, addressing a conjecture of Claude LeBrun. This is achieved by showing, using results in geometric and arithmetic group theory, that certain hyperbolic 4-manifolds contain L-spaces as hypersurfaces.

Introduction

In [**9**, Conjecture 1.1], Claude LeBrun asked whether the Seiberg-Witten invariants of hyperbolic 4-manifolds vanish. This question stems from his result that for a hyperbolic 4-manifold, Seiberg-Witten basic classes satisfy much stronger constraints than one would expect; furthermore, it turns out to be related to several problems in low-dimensional topology [**18**, §4]. Here, we show that there exist certain hyperbolic 4-manifolds with vanishing Seiberg-Witten invariants.

Theorem 0.1. *There exist closed arithmetic hyperbolic 4-manifolds with vanishing Seiberg-Witten invariants.*

In the statement, we consider all possible Seiberg-Witten invariants coming from evaluating elements of the cohomology ring $\Lambda^* H^1(X;\mathbb{Z}) \otimes \mathbb{Z}[U]$ of the space of configurations. Theorem 0.1 is proved by exhibiting hyperbolic 4-manifolds admitting separating L-spaces, using the main result of [**6**]; under mild additional conditions, this implies that such manifolds admit finite covers with vanishing Seiberg-Witten invariants. Our construction will show in fact that there are infinitely many commensurability classes of arithmetic hyperbolic 4-manifolds containing representatives with vanishing Seiberg-Witten invariants. Furthermore, by interbreeding as in [**4**], one can also obtain non-arithmetic examples.

1. A vanishing criterion for the Seiberg-Witten invariants

We discuss a vanishing result for the Seiberg-Witten invariants of four-manifolds containing a separating hypersurface. This is well-known to experts, but the exact form we will need is only implicitly stated in [**7**], so we will point it out for the reader's convenience. Most of our discussion is based on formal properties of the invariants, and we will follow closely follow the exposition of [**7**, Chapter 3].

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Consider a spin^c structure $\mathfrak{s}_X$ on a closed oriented 4-manifold X. For a cohomology class $u \in \Lambda^* H^1(Y;\mathbb{Z}) \otimes \mathbb{Z}[U]$, we define the Seiberg-Witten invariant $\mathfrak{m}(u|X,\mathfrak{s}_X)$ to be the evaluation of u on the moduli space of solutions to the Seiberg-Witten equations. This is a topological invariant provided that $b_2^+ \geq 2$. The latter is not a particularly restrictive assumption in our case; hyperbolic 4-manifolds have signature zero by [**2**, Theorem 3] and the Hirzebruch signature formula. Hence

$$\chi(X) = 2(1 - b_1(X) + b_2^+(X)).$$

If $b_2^+(X) \leq 1$, we would have $\chi(X) \leq 4$; on the other hand, in all known examples of closed orientable hyperbolic 4-manifolds $\chi \geq 16$ [**11**, **14**] (recall that by Chern-Gauss-Bonnet, volume and Euler characteristic are proportional).

We discuss a vanishing criterion for $\mathfrak{m}(u|X,\mathfrak{s}_X)$. Let Y be a closed, oriented three-manifold. To this, in [**7**, Section 3.1] it is defined for each spin^c structure $\mathfrak{s}$ on Y the monopole Floer homology groups fitting in the exact triangle of graded $\mathbb{Z}[U]$-modules

$$\cdots \longrightarrow \overline{HM}_*(Y,\mathfrak{s}) \xrightarrow{i_*} \widecheck{HM}_*(Y,\mathfrak{s}) \xrightarrow{j_*} \widehat{HM}_*(Y,\mathfrak{s}) \xrightarrow{p_*} \overline{HM}_*(Y,\mathfrak{s}) \longrightarrow \cdots \tag{1.1}$$

where U has degree -2 (notice that this convention differs from the one in the four-dimensional literature; this is because we identify U with the corresponding capping operation in homology). The reduced Floer group $HM_*(Y,\mathfrak{s})$ is defined to be the image of j_* in $\widehat{HM}_*(Y,\mathfrak{s})$ [**7**, Definition 3.6.3]. We will be particularly interested in the case in which Y is a rational homology sphere. In this case we have an identification of $\mathbb{Z}[U]$-modules (up to grading shift) with Laurent series [**7**, Proposition 35.3.1]

$$\overline{HM}_*(Y,\mathfrak{s}) \cong \mathbb{Z}[U^{-1},U].$$

Definition 1.1 ([**8**]). We say that a rational homology sphere Y is an *L-space* if, up to grading shift, $\widehat{HM}_*(Y,\mathfrak{s}) \cong \mathbb{Z}[U]$ as $\mathbb{Z}[U]$-modules for all spin^c structures $\mathfrak{s}$.

As the map p_* in equation (1.1) is an isomorphism in degrees low enough [**7**, Section 22.2], for an L-space $HM_*(Y,\mathfrak{s}) = 0$ for all spin^c structures $\mathfrak{s}$

Proposition 1.2. *Let X be a four-manifold given as $X = X_1 \cup_Y X_2$. Suppose that the separating hypersurface Y is an L-space (so that in particular $b_1(Y) = 0$), and that $b_2^+(X_i) \geq 1$. Then all the Seiberg-Witten invariants of X vanish.*

Remark 1.3. A simpler vanishing criterion is the following: if $b_1(X) = 0$ and $b_2^+(X)$ is even, then all Seiberg-Witten invariants are zero. In fact, under this assumption all Seiberg-Witten moduli spaces are odd dimensional [**7**, Theorem 1.4.4], while all classes in our cohomology ring are even dimensional. On the other hand, we are not aware of examples of hyperbolic 4-manifolds satisfying these conditions.

Proof of Proposition 1.2. All we need to do is to discuss the results of [**7**, Chapter 3] while keeping track of the specific spin^c structures. First of all, notice that as $b_1(Y) = 0$, a spin^c structure $\mathfrak{s}_X$ on X is determined by the restrictions $\mathfrak{s}_i = \mathfrak{s}_X|_{X_i}$. This follows from the injectivity of the map $H^2(X;\mathbb{Z}) \to H^2(X_1;\mathbb{Z}) \oplus H^2(X_2;\mathbb{Z})$ in the Mayer-Vietoris sequence, and the fact the these groups classify spin^c structures. Let $\mathfrak{s} = \mathfrak{s}_X|_Y$. It is sufficient to show that $\mathfrak{m}(u|X,\mathfrak{s}_X) = 0$ for classes $u = u_1u_2$ where u_i is a cohomology class in the configuration space of X_i. Recall from [**7**, Section 3.4] that a cobordism W from Y_0 to Y_1 induces a map in homology fitting with the exact triangle; furthermore, if $b_2^+(W) \geq 1$, we have that $\overline{HM}_*(u|W, \mathfrak{s}) = 0$ [**7**, Proposition 3.5.2]. Given data as above, we can define the relative invariant $\psi_{(u_1|X_1,\mathfrak{s}_1)} \in \widehat{HM}_*(Y, \mathfrak{s})$ obtained as follows: let W_1 be the cobordism obtained from X_1 by removing a ball, and consider the induced map

$$\widehat{HM}_*(u_1|W_1, \mathfrak{s}_1) : \widehat{HM}_*(S^3) \equiv \mathbb{Z}[U] \to \widehat{HM}_*(Y, \mathfrak{s}).$$

Then $\psi_{(u_1|X_1,\mathfrak{s}_1)} = \widehat{HM}_*(u_1|W_1, \mathfrak{s}_1)(1)$. On the other hand, we have the commutative diagram

$$\begin{array}{ccc} \widehat{HM}_*(S^3) & \xrightarrow{p_*} & \overline{HM}_*(S^3) \\ {\scriptstyle \widehat{HM}_*(u_1|W_1,\mathfrak{s}_1)}\downarrow & & \downarrow{\scriptstyle \overline{HM}_*(u_1|W_1,\mathfrak{s}_1)} \\ \widehat{HM}_*(Y, \mathfrak{s}) & \xrightarrow{p_*} & \overline{HM}_*(Y, \mathfrak{s}) \end{array}$$

and as $b_2^+(W_1) \geq 1$, the vertical map on the right vanishes; in turn, this implies that $\psi_{(u_1|X_1,\mathfrak{s}_1)} \in \ker(p_*) \cong HM_*(Y, \mathfrak{s})$. Similarly, using the map induced in cohomology by W_2, we obtain an element $\psi_{(u_2|X_2,\mathfrak{s}_2)} \in HM_*(-Y, \mathfrak{s})$; this last group is by Poincaré duality identified with $HM^*(Y, \mathfrak{s})$. The general gluing theorem in [**7**, Equation 3.22], when keeping track of the spin^c structures, is then

$$\mathfrak{m}(u|X, \mathfrak{s}_X) = \langle \psi_{(u_1|X_1,\mathfrak{s}_1)}, \psi_{(u_2|X_2,\mathfrak{s}_2)} \rangle,$$

where the angular brackets denote the natural pairing

$$HM_*(Y, \mathfrak{s}) \times HM^*(Y, \mathfrak{s}) \to \mathbb{Z}.$$

Under our assumptions, the group $HM_*(Y, \mathfrak{s})$ vanishes, so this pairing is zero, and the result follows. □

Remark 1.4. In fact, for our purposes of understanding the gluing formula for Seiberg-Witten invariants, it suffices to consider the reduced invariants with rational coefficients, $HM_*(Y, \mathfrak{s}; \mathbb{Q})$. In particular, the previous discussion only relies on the vanishing of this group. Furthermore, via the universal coefficients theorem, this is implied by the vanishing of $HM_*(Y, \mathfrak{s}; \mathbb{Z}/2\mathbb{Z})$, so that our main result actually applies for the reduced Floer homology group with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Our examples will be based on the following.

Corollary 1.5. *Suppose X is a 4-manifold with $b_2^+ \geq 1$ which admits an embedded non-separating L-space Y. Then X admits infinitely many covers which have all vanishing Seiberg-Witten invariants.*

Proof. Consider the double cover $\tilde{X}$ of X formed by gluing together two copies W_1 and W_2 of the cobordism from Y to Y obtained by cutting X along Y, see Figure 1. Consider a properly embedded path $\gamma \subset W_1$ between the two copies of Y, and denote by T its tubular neighborhood. We then have the decomposition $X = (W_1 \setminus T) \cup (W_2 \cup T)$, where the two manifolds are glued along a copy of $Y \# \overline{Y}$; here $\overline{Y}$ denotes Y with the opposite orientation. The latter is an L-space [**10**, Section 4], and both $W_1 \setminus T$ and $W_2 \setminus T$ have $b_2^+ \geq 1$, so the conclusion follows. Finally, to obtain infinitely many examples, for any $N \geq 3$ we cyclically glue together copies W_i for $i = 1, \ldots, N$ of the cobordism from Y to Y obtained by

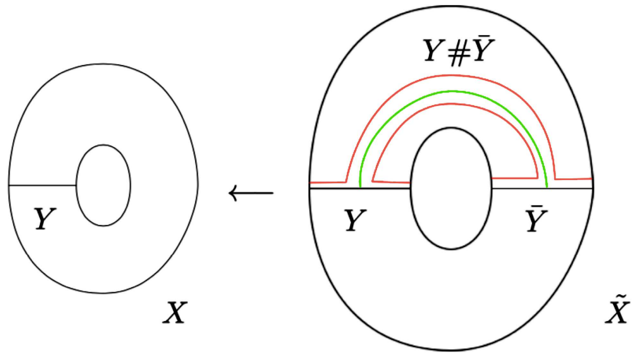

FIGURE 1. A double cover of X contains a separating L-space.

cutting X along Y; this again contains a separating hypersurface diffeomorphic to $Y\#\overline{Y}$ with the required properties. □

2. Geodesic hypersurfaces in arithmetic hyperbolic 4-manifolds

In this section, we will discuss various properties of arithmetic hyperbolic lattices. For the general case of arithmetic lattices, see [**20**], and for the 3-dimensional case, consult [**13**]. We first review the definitions and construction of arithmetic manifolds of simplest type.

DEFINITION 2.1. Let G be a group, $H_1, H_2 \leq G$ be subgroups. We say that H_1 is *commensurable in* G with H_2 if $[H_1 : H_1 \cap H_2] < \infty, [H_2 : H_1 \cap H_2] < \infty$.

DEFINITION 2.2. Consider a non-degenerate quadratic form $q : k^{n+1} \to k$ for a totally real number field $k \subset \mathbb{R}$ with ring of integers $\mathcal{O}_k$. Assume that q is Lorentzian, i.e. has signature $(n, 1)$ over $\mathbb{R}$. Moreover, for each non-trivial embedding $\sigma : k \to \mathbb{R}$, assume that $\sigma \circ q$ is positive definite. Let $O(q; k)$ denote the group of matrices preserving q, i.e. linear transforms $A : k^{n+1} \to k^{n+1}$ such that $q \circ A = q$. Then the subgroup $O(q; \mathcal{O}_k) \subset O(q; k) \subset O(q; \mathbb{R})$ is a lattice, and acts discretely on the hyperboloid of two sheets $\mathcal{H} = \{x \in \mathbb{R}^{n+1} | q(x) = -1\}$. Up to isometry, the group $O(q; \mathbb{R}) \cong O(n, 1; \mathbb{R})$, the orthogonal group associated to the quadratic form $-x_0^2 + x_1^2 + \cdots + x_n^2$. Projectivizing, $PO(q; \mathcal{O}_k)$ acts discretely on hyperbolic space $\mathbb{H}^n$, which is the quotient of the hyperboloid $\mathcal{H}$ by the antipodal map. A hyperbolic orbifold $\mathbb{H}^n/\Gamma$ is said to be of *simplest type* if Γ is commensurable (up to conjugacy) with $PO(q; \mathcal{O}_k)$ for some such q.

Example: Let $q_n : k^{n+1} \to k$ be defined by $q_n(x_0, x_1, \ldots, x_n) = -\sqrt{2}x_0^2 + x_1^2 + \cdots + x_n^2$ over the field $k = \mathbb{Q}(\sqrt{2})$. Let $\sigma : k \to k$ be the Galois automorphism induced by $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $\sigma \circ q_n(x_0, \ldots, x_n) = \sqrt{2}x_0^2 + x_1^2 + \cdots + x_n^2$ is positive definite. Hence $PO(q_n; \mathbb{Z}[\sqrt{2}])$ is a discrete arithmetic lattice acting on $\mathbb{H}^n$. See [**20**, §6.4].

DEFINITION 2.3. Let G be a group. Then $G^{(2)} = \langle g^2 | g \in G \rangle$.

If G is finitely generated, then $G^{(2)}$ is finite-index in G, and $G/G^{(2)}$ is an elementary abelian 2-group.

THEOREM 2.4. *Let M^3 be an orientable hyperbolic arithmetic 3-manifold of simplest type with $H_1(M;\mathbb{Z}/2) = 0$ and not defined over $\mathbb{Q}$. Then M embeds as a totally geodesic non-separating submanifold in a compact arithmetic hyperbolic 4-manifold.*

PROOF. Let $\Gamma = \pi_1(M) \leq Isom^+(\mathbb{H}^3)$. Since M is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, $\Gamma^{(2)} = \Gamma$. By [**6**, Theorem 1.1 (2)], $\mathbb{H}^n/\Gamma^{(2)} \cong M$ embeds as a totally geodesic submanifold of a closed orientable hyperbolic 4-manifold W (the fact that M is not defined over $\mathbb{Q}$ implies that W is compact). Briefly, this is proved by showing that $\Gamma^{(2)} \leq PO(q;k)$ so that it is commensurable with $PO(q;\mathcal{O}_k)$ for some Lorentzian quadratic form $q : k^4 \to k$. Taking the quadratic form $Q_d = dy^2 + q, d \in \mathbb{N}$, we get an embedding of $PO(q;\mathcal{O}_k) < PO(Q_d;\mathcal{O}_k) < PO(Q_d;\mathbb{R}) \cong PO(4,1;\mathbb{R})$. Then a subgroup separability result allows one to embed Γ in a torsion-free lattice $\Lambda < PO(Q_d;k)$ so that $W = \mathbb{H}^4/\Lambda$. By [**1**, Theorem 2], there exists a further finite-sheeted cover $\tilde{W} \to W$, and a lift $M \to \tilde{W}$ such that the lift of M is non-separating in $\tilde{W}$. This is achieved again by a subgroup separability result. □

3. Examples

The *Fibonacci manifold* M_n is the cyclic branched n-fold cover over the figure-eight knot. For $n = 2$ we obtain a lens space, for $n = 3$ the Hantzche-Wendt manifold, while for $n \geq 4$ it is hyperbolic.

For every n the Fibonacci manifold M_n is an L-space. To see this, recall from [**19**] that M_n is the branched double cover over the closure of the 3-braid $(\sigma_1\sigma_2^{-1})^n$ (see Figure 2), which is alternating. Using the surgery exact triangle [**8**], these can be shown to be L-spaces as in the context of Heegaard Floer homology [**15**], with the caveat that in our setting the computation only holds with coefficients in $\mathbb{Z}/2\mathbb{Z}$; on the other hand this is enough for our purposes, see Remark 1.4. Notice also that for $n \neq 0$ modulo 3, the closure is a knot, so that M_n is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.

By [**5**], M_n is arithmetic when $n = 4, 5, 6, 8, 12$. Of these examples, $n = 4, 5, 8$ are $\mathbb{Z}/2\mathbb{Z}$ homology spheres. The only one of these three which is simplest type and not defined over $\mathbb{Q}$ is M_5. This is example [**13**, 13.7.4(a)(iii)], which has invariant trace field a quartic field. As they point out, this is commensurable with a tetrahedral group [**13**, 13.7.4(a)(i)] which is simplest type and not defined over $\mathbb{Q}$ by [**12**, Theorem 1]. It is defined over a quadratic form over the field $\mathbb{Q}(\sqrt{5})$.

Thus, by Theorem 2.4, M_5 has a non-separating embedding into a closed orientable hyperbolic 4-manifold W. We may assume that $\chi(W) > 2$ (by passing to a 2-fold cover if needed), and hence $b_2^+(W) > 1$. Thus by Corollary 1.5, these embed into a hyperbolic 4-manifold with vanishing Seiberg-Witten invariants. This completes the proof of Theorem 0.1.

REMARK 3.1. One may also get other examples by cutting and doubling or using the interbreeding technique of Gromov-Piatetskii-Shapiro to get non-arithmetic examples. One can isometrically embed this L-space M_5 in infinitely many incommensurable hyperbolic 4-manifolds via the method of [**6**] by taking the forms Q_1 and Q_d in the proof of Theorem 2.4 so that d is square-free in $k = \mathbb{Q}(\sqrt{5})$, and then cut and cross-glue to give a closed non-arithmetic manifold containing M_n as a non-separating hypersurface [**4**, §2.9].

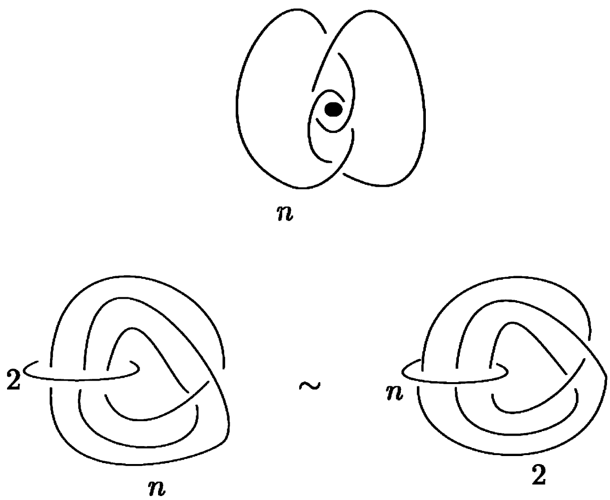

FIGURE 2. In this picture, the numbers indicate the branching. The top picture has an obvious order 2 rotational symmetry along the axis depicted by the big dot. The quotient is the link in S^3 depicted on the bottom left. This is isotopic to the link on the right (which is topologically the same, but with different branchings). Now, the curve with branching 2 is the 3-braid $\sigma_1\sigma_2^{-1}$, so that taking the n-fold branched cover along the other component we see that M_n is the branched double cover over $(\sigma_1\sigma_2^{-1})^n$.

4. Conclusion

We conclude by pointing out some natural questions related to our method.

(1) Can one find an explicit hyperbolic example (such as the Davis manifold or the manifolds described in [**11**]) that satisfies the properties of Proposition 1.2? Recall that the Davis manifold has $b_1 = 24$ and $b_2^+ = 36$ [**16**], so that all moduli spaces have odd dimension.

(2) Can one embed any orientable hyperbolic 3-manifold of simple type as a geodesic hypersurface in an orientable hyperbolic 4-manifold? More generally, can one show that orientable hyperbolic 3-manifolds have quasiconvex embeddings into orientable hyperbolic 4-manifolds?

(3) Can one use bordered Floer theory to compute the Seiberg-Witten invariants of Haken hyperbolic 4-manifolds (in the sense of [**3**])?

(4) Which commensurability classes of compact hyperbolic 3-manifold of the simplest type contain L-spaces? Note that it is not even known if there are infinitely many commensurability classes of arithmetic rational homology 3-spheres.

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