MONOPOLE FLOER HOMOLOGY, EIGENFORM MULTIPLICITIES AND THE SEIFERT-WEBER DODECAHEDRAL SPACE

FRANCESCO LIN AND MICHAEL LIPNOWSKI

ABSTRACT. We show that the Seifert-Weber dodecahedral space SW is a monopole Floer homology L-space. The proof relies on our approach to study Floer homology using hyperbolic geometry. While SW is significantly larger than previous manifolds studied with this technique, we overcome computational complexity issues inherent to our method by exploiting the many symmetries of SW. In particular, we prove that small eigenvalues on coexact 1-forms on SW have large multiplicity.

The Seifert-Weber dodecahedral space SW is obtained by identifying opposite faces of a dodecahedron by a 3/10 full turn [16]; it was one of the first examples of closed hyperbolic three-manifold to be discovered [18]. Despite its very simple description, it is a quite complicated space from the point of view of three-dimensional topology. For example, the conjecture of Thurston from 1980 that SW is not Haken has been considered a benchmark problem in computational topology and took 30 years to settle [3].

In the present paper, we look at SW from the point of view of monopole Floer homology [8]. Recall that $H_1(\mathsf{SW}) = (\mathbb{Z}/5\mathbb{Z})^3$, so that SW is a rational homology sphere.

Theorem 0.1. *The Seifert-Weber dodecahedral space* SW *is an L-space, i.e. its reduced Floer homology* $HM_*(\mathsf{SW})$ *vanishes.*

More is true, in fact: our proof will show that, for all spin^c-structures on SW (equipped with the hyperbolic metric), small perturbations of the Seiberg-Witten equations on M admit no irreducible solutions, and therefore SW is a *minimal L-space* in the sense of [10].

As a direct consequence of Theorem 0.1, we obtain that the Seifert-Weber dodecahedral space does not admit coorientable taut foliations [9]. Furthermore, as SW is also an arithmetic hyperbolic three-manifold of the simplest type with $H_1(\mathsf{SW}, \mathbb{Z}/2\mathbb{Z}) = 0$ (see Remark 1.2), the construction of [1] can be directly adapted to provide more examples of hyperbolic 4-manifolds with vanishing Seiberg-Witten invariants.

Our approach to Theorem 0.1 builds on the ideas of our previous work [10], where we focused our attention on much smaller manifolds. There, we showed that a hyperbolic rational homology sphere Y for which the first eigenvalue on coexact 1-forms λ_1^* is strictly larger than 2 is an L-space. We then developed numerical techniques (based on the Selberg trace formula) to provide explicit lower bounds on λ_1^* in terms of the volume and closed geodesics of Y. More specifically, taking as input the volume and the list of complex lengths of geodesics with length at most R (as computed for example by SnapPy [4]), we determine an explicit function $J_{R,t}(Y)$ which is an upper bound to the multiplicity of t^2 as an eigenvalue of Δ on coexact 1-forms. In particular, if $J_{R,t} < 1$ then t^2 is not an eigenvalue; using this, we showed (choosing $R = 6.5$) that several manifolds with small volume ($\leqslant 2.029..$) have $\lambda_1^* > 2$, and are therefore L-spaces.

The volume of SW is ≈ 11.119, about 5 times greater than the census examples we considered in [10]. Correspondingly, by Weyl's law one expects coexact 1-form eigenvalues to be about 5 times as abundant. Based on this influence of volume, compared to the small census manifolds Y considered in [10], one also heuristically expects the resolutions of $J_{R,t}(\mathsf{SW})$ and $J_{R',t}(Y)$ to be comparable only when R is significantly larger than R'. Using SnapPy, we computed the length spectrum up to cutoff $R = 8$ in about 5 hours. The function $J_{R,t}(\mathsf{SW})$ for $R = 8$ determined by our method has the form

$$J_{8,t}(\mathsf{SW}) = \frac{1}{\langle A^{-1}c_t, c_t\rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product on $\mathbb{R}^{41}$ and:

- A is a symmetric positive definite 41×41 matrix (independent of t) whose entries are determined via the trace formula,
- $c_t := \left(\frac{\sin(\delta t)}{\delta t}\right)^2 \cdot \begin{pmatrix} 1 \\ \cos(\delta t) \\ \vdots \\ \cos(40\delta t) \end{pmatrix} \in \mathbb{R}^{41}$ with $\delta = 8/(2 \cdot 40 + 4)$;

its plot can be found in Figure 1. Unfortunately, the lower bound provided by $J_{8,t} < 1$ is

$$\lambda_1^* > 1.9188\ldots$$

which is insufficient for our purposes.

On the other hand, the graph of $J_{8,t}$ is peaked just barely above height 6 in the narrow interval $[1.427877\ldots, 1.430337\ldots]$; this strongly suggests that

$$\lambda_1^* \in [(1.427877\ldots)^2, (1.430337\ldots)^2] = [2.03883\ldots, 2.04586\ldots]$$

and that the corresponding eigenspace has dimension 6. As we expect $J_{R,t}$ to approximate better and better the indicator function of the spectrum (with multiplicities) for large R, one could in principle prove that $\lambda_1^* > 2$ by showing that $J_{R,t} < 1$ for $t < \sqrt{2}$ by computing the length spectrum for some larger value of R. Unfortunately, this is unfeasible at a practical level because the amount of time required to compute the length spectrum to some cutoff grows at least exponentially with the cutoff, see Table 1.

Remark 0.1. It should be pointed out that the computations for SW are *extremely* fast (even though not enough for our purposes). For example, the computation at cutoff $R = 6$ only took 12 seconds, while for most of the other three-manifolds we tested before it took around $15 - 20$ minutes.

We instead took a more conceptual approach. Our main result is the following:

Claim 0.2. *Any eigenvalue $\lambda^* \leqslant 64$ of the Hodge Laplacian on coexact 1-forms on SW has multiplicity at least 4.*

From this, we can prove Theorem 0.1 by looking again at the function $J_{8,t}(\mathsf{SW})$.

Proof of Theorem 0.1. Recall that $J_{8,t}(\mathsf{SW})$ provides an upper bound for the multiplicity of the eigenvalue t^2. We have that $J_{8,t}(\mathsf{SW}) < 4$ for $t \leqslant 1.414380\ldots$ (see Figure 2). This implies that $\lambda_1^* > (1.414380\ldots)^2 = 2.0004717\ldots > 2$, and Theorem 0.1 follows from Theorem 0.3 of [10]. $\square$

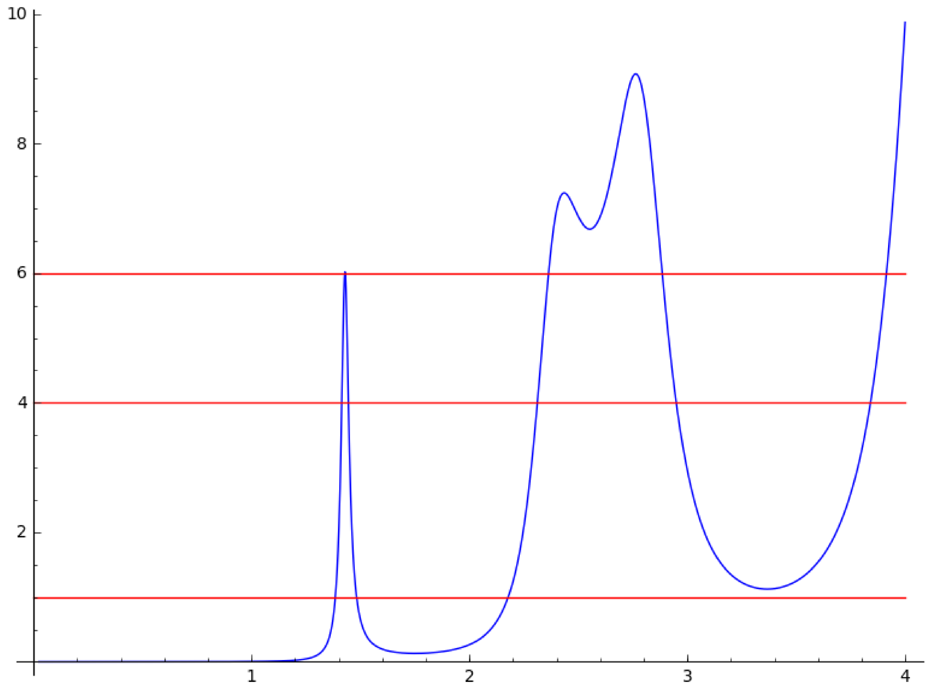

FIGURE 1. The graph of $t \mapsto J_{8,t}(\mathsf{SW})$ for $t \in [0, 4]$.

Length cutoff R	$\sqrt{2}$ − height 1 crossing point	Running time in CPU seconds
6.0	0.207732 ...	12
6.5	0.152766...	61
7.0	0.096385...	348
7.5	0.051723...	2255
8.0	0.028979...	19112

TABLE 1. Differences between $\sqrt{2}$ and point where the graphs of $t \mapsto J_{R,t}(\mathsf{SW})$ cross height 1 for various length cutoffs R. Assuming the actual value of $\sqrt{\lambda_1^*}$ to be about 1.428.., this suggest that a computation of length spectrum at cutoff $R = 9.5$ could prove $\lambda_1^* > 2$. Being quite optimistic (e.g. assuming that memory limitations do not affect the running time), such a computation would take at least several months. Here we used an Intel Core i7 2.7GHz with 10GB of allocated memory.

The rest of this paper is dedicated to the proof of Claim 0.2. Determining eigenvalue multiplicities is in general a very delicate problem, especially in the context of numerical computations. There are two key observations about SW underlying Claim 0.2:

(1) SW is a very symmetric manifold. In particular, its isometry group is isomorphic to the symmetric group S_5 [13]. Here the alternating subgroup A_5 corresponds to the orientation preserving isometries of the dodecahedron;

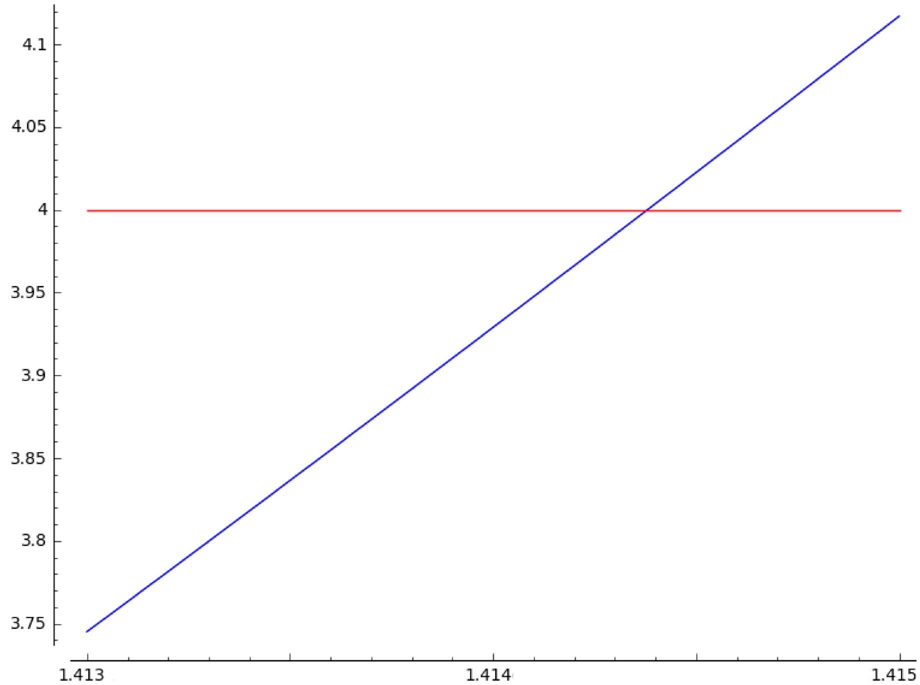

FIGURE 2. The graph of $t \mapsto J_{8,t}(\mathsf{SW})$ for $t \in [1.413, 1.415]$.

(2) while the orbifold SW/A_5 is not implemented in SnapPy (which currently only has infrastructure to handle orbifolds with cyclic singularities), it admits a very explicit arithmetic description; in particular, one can use number theoretic techniques to compute its length spectrum (see Chapter 30 of [17]).

By generalizing the techniques of [10] to the case of orbifolds, we can use the computation of the length spectrum (together with some additional geometric data) to show that the orbifold SW/A_5 satisfies $\lambda_1^* > 64$; this implies that the λ^*-eigenspace of SW for every $\lambda^* \leqslant 64$, viewed as a representation of the isometry group S_5, does not contain copies of the trivial representation of A_5. From this we will be able to conclude Claim 0.2 via the classification of irreducible representations of S_5.

In fact, a more detailed analysis involving the trace formula allows one to confirm the multiplicity and narrow window for the first eigenvalue suggested by Figure 1.

Proposition 0.3. *There is exactly one eigenvalue t^2, for the Laplacian acting on coexact 1-forms on* SW, *satisfying $|t| < 2.3124$. It occurs with multiplicity 6 and lies in the window $|t| \in [1.4278772, 1.4303375]$.*

The proof will also show that the first eigenspace is isomorphic to the unique 6-dimensional irreducible representation of S_5.

Our approach to Theorem 0.1 is based on many of the beautiful geometric and arithmetic properties of SW. One would expect that the same result can be achieved by other means, given that there is an algorithmic (yet impractical) way to determine whether a given rational homology sphere is an L-space [15]. On the other hand, even though SW is a cyclic 5-fold branched cover of the Whitehead link (the one corresponding to the homomorphism $H_1(S^3\setminus L) \to \mathbb{Z}/5\mathbb{Z}$ sending one meridian to 1 and the other to 2), it does not admit a simple surgery description and is not the double branched cover of any link in S^3 (see Remark 1.1), so at least the most efficient computational tools available seem not to be directly applicable to it.[1]

Remark 0.2. In particular, SW provides an example of an L-space which is neither asymmetric nor the branched double cover of a link in S^3 (cfr. the asymmetric L-spaces found in [6]).

Plan of the paper. In §1 we discuss several geometric and arithmetic properties of SW which will be relevant for our purposes. In §2, we generalize our techniques from [10] to the case of orbifolds and apply it to the case of SW/A_5. In §3 we prove Claim 0.2, and in §4 we extend the analysis to prove Proposition 0.3. Finally, in §5 we discuss a closely related tetrahedral orbifold.

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1. The geometry and arithmetic of SW

1.1. **The isometry group.** We follow closely the discussion of [13], to which we refer for additional details. As mentioned in the introduction, SW is obtained by identifying opposite faces of a dodecahedron by a 3/10 full turn. Under this identification, all vertices of the dodecahedron get identified and the 30 edges get identified in groups of five. The hyperbolic metric is realized by considering the regular dodecahedron D in $\mathbb{H}^3$ with dihedral angles of $2\pi/5 = 72^\circ$. The barycentric subdivision of D is made of 120 copies of the the tetrahedron T with totally geodesic faces in Figure 3.

Denote by Γ the group generated by reflections across the faces of T, and by $\Gamma^+ < \Gamma$ the index two subgroup consisting of orientation-preserving isometries. This has presentation

$$\Gamma^+ = \langle a, b, c | a^2 = b^2 = c^5 = (bc)^2 = (ca)^3 = (ab)^5 = 1\rangle \tag{1}$$

where a, b and c are the rotations around the axes VF, EF and EV of angles π, π and $2\pi/5$ (the latter with orientation as in Figure 3); see Section 4.7 of [12]. In terms of reflections, defining for a vertex P of T the reflection R_P across the face opposite to P, we have

$$a = R_E R_O, \quad b = R_V R_O, \quad c = R_F R_O.$$

[1]Nathan Dunfield has recently pointed out to us that, even though SW is significantly larger than the manifolds he studied in [5], his approach can be adapted to provide an alternative proof of Theorem 0.1. In particular, he cleverly found a sequence of suitable surgeries to bootstrap the computations of [5] to the case of SW.

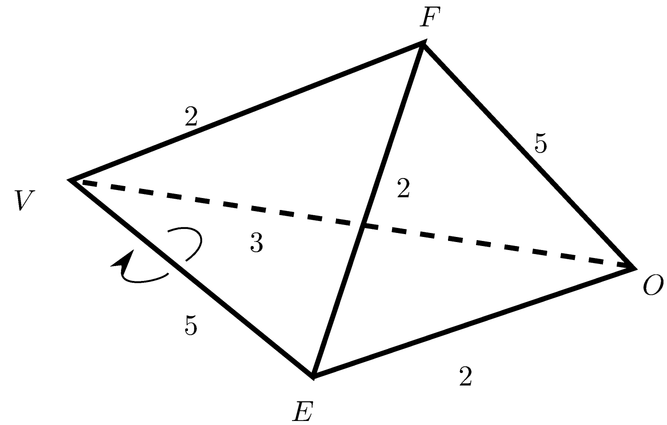

FIGURE 3. A schematic picture of the tetrahedron T. An edge labeled n has dihedral angle π/n. The vertices O, V, E and F are respectively the center of the dodecahedron D, one of its vertices, the center of an edge and the center of a face. In Coxeter notation this is the tetrahedron $[5,3,5]$. The arrow denotes the orientation for the order 5 rotation c.

Following [13], the surjective homomorphism

$$\begin{gathered}\varphi : \Gamma^+ \twoheadrightarrow A_5 \\ a \mapsto (23)(45) \quad b \mapsto (12)(35) \quad c \mapsto (12345)\end{gathered}$$

has kernel identified with $\pi_1(\mathsf{SW})$. In particular, we see that there is a natural action of A_5 on SW by isometries (corresponding to the isometry group of D, which is isomorphic to A_5), and the quotient orbifold SW/A_5 has fundamental group Γ^+. Furthermore, the fundamental domain for the action of Γ^+ on $\mathbb{H}^3$ is given by doubling T along any face; looking at the identifications, we see that the quotient SW/A_5 is homeomorphic to S^3, and the orbifold locus is described again by Figure 3, when thought of as a labeled trivalent graph in S^3. In particular the isotropy groups of the vertices V, O are isomorphic to A_5 and the isotropy groups of E, F are isomorphic to the dihedral group with 10 elements D_{10}.

The tetrahedron T admits an incidence and edge label-preserving symmetry sending V to O and E to F; because there is a unique hyperbolic tetrahedron with given dihedral angles, this symmetry is realized by an isometry ι. Geometrically, ι is the rotation by π along the geodesic connecting the midpoints of the segments EF and VO (such a geodesic is necessarily orthogonal to the edges at the endpoints). Since ι maps faces to faces, it normalizes Γ and hence the orientation-preserving subgroup Γ^+. Thus, the subgroup Λ of $\mathrm{Isom}^+(\mathbb{H}^3)$ generated by Γ^+ and ι is the semidirect product of Γ^+ and the order two subgroup generated by ι. Furthermore, φ can be extended to a homomorphism

$$\tilde{\varphi} : \Lambda \to S_5$$

by sending ι to $x = (12)$. Indeed, extending to Λ is equivalent to the relation $\varphi(\iota g \iota^{-1}) = x\varphi(g)x^{-1}$ for all $g \in \Gamma^+$. It is readily checked that this relation holds for every g among the generators a, b, c of Γ^+,[2] which suffices.

In particular, ι normalizes the kernel of φ, and hence induces an isometry of SW. This isometry switches O and V, and will be referred to as the *inside-out* isometry. The main theorem of [13] is then the following.

Theorem 1.1 ([13]). *The isometry group of* SW *is isomorphic to* S_5*, and is generated by* ι *and the orientation-preserving isometry group of the dodecahedron* $\cong A_5$*.*

Remark 1.1. In [13], the author also determines the action of S_5 on the first homology $H_1(\mathsf{SW})$. In particular, for the order 2 elements in S_5 the quotient has non trivial H_1, and is therefore not S^3. By geometrization, SW is not the branched double cover of a link in S^3.

Remark 1.2. From our description it is clear that $\pi_1(\mathsf{SW})$ is commensurable to a tetrahedral group (see Section 4.7.2 of [12]), and is therefore arithmetic of the simplest type by [11].

1.2. **Covolume of centralizers.** For later applications, for a non-trivial element $\gamma \in \Gamma^+$ we need to understand the volume $\mathrm{vol}(\Gamma^+_\gamma \backslash G_\gamma)$, where G_γ and Γ^+_γ are the centralizers of γ in $G = \mathrm{PSL}_2(\mathbb{C})$ and Γ^+ respectively. In closed hyperbolic manifold groups, all elements are hyperbolic and the centralizer of γ_0^n with γ_0 primitive is the cyclic group generated by γ_0, and therefore the quantity of interest is simply the translation length $\ell(\gamma_0)$. For the closed hyperbolic orbifold group Γ^+, we need to consider two special classes of elements:

- elliptic elements;
- *bad* hyperbolic elements, i.e. hyperbolic elements whose axis is the fixed axis for some elliptic element.

To determine these quantities, we will refer to the picture of SW/A_5 in Figure 3 (considered again as S^3 with orbifold locus).

Up to conjugacy, the elliptic elements of order 5 are rotations around FO and VE. As these two rotations are exchanged by the inside-out isometry, we need only perform the computations on the non-trivial conjugacy classes of powers of γ_f, a rotation of $2\pi/5$ around the geodesic f in $\mathbb{H}^3$ connecting F and O. Notice that γ_f and γ_f^{-1} are conjugate in Γ^+ because in the isotropy group of F, which is isomorphic to D_{10}, every order two element is conjugate to its inverse. Therefore in Γ^+ there are exactly 4 conjugacy classes of order 5 elliptics, namely γ_f, γ_f^2 and their conjugates under ι. Now, G_{γ_f} consists of the elliptic elements with axis f and the hyperbolic elements with axis f which preserve the endpoints of f. Furthermore, $\Gamma^+_{\gamma_f}$ is the abelian group generated by γ_f and a primitive hyperbolic element h_f in Γ^+ with axis f (and which preserves endpoints). The image of f in SW/A_5 is $|OF|$, a geodesic of mirrored-arc type, and therefore the translation length of h_f is $2|OF|$. This quantity can be determined by looking at the geometry of the triangle with vertices E, F and O as follows. Direct geometric considerations with the dodecahedron D show that

$$\angle OFE = \pi/2, \qquad \angle OEF = \pi/5.$$

[2]For example, because ι conjugates R_P to $R_{\iota(P)}$, it conjugates $a = R_E R_O$ to $R_F R_V = R_F R_0 R_0 R_V = cb^{-1}$. So indeed,

$$\varphi(\iota a \iota^{-1}) = \varphi(cb^{-1}) = (13)(45) = x\varphi(a)x^{-1}.$$

The angle $\angle EOF$ can be computed via formulas of spherical trigonometry using the fact that a small sphere centered at O intersects T in a geodesic triangle with angles $\pi/2$, $\pi/3$ and $\pi/5$. We have

$$\angle EOF = \arctan\left(\frac{\sqrt{5}-1}{2}\right).$$

Since OFE spans a geodesic hyperbolic right triangle, we obtain

$$\cosh|OF| = \frac{\cos(\angle OEF)}{\sin(\angle EOF)} = 1.5388\dots$$

so that $|OF| = 0.9963...$ and

$$\mathrm{vol}(\Gamma^+_{\gamma_f}\backslash G_{\gamma_f}) = \frac{1}{5}\cdot\ell(h_f) = \frac{1}{5}\cdot 2\cdot|OF| = 0.3985\dots$$

The same computation holds also for γ_f^2.

There is exactly one conjugacy class of order 3 elliptic elements corresponding to the rotation by $2\pi/3$ around the geodesic v in $\mathbb{H}^3$ connecting O and V, which we denote by γ_v. We see that γ_v is conjugate to γ_v^{-1} by looking at the isotropy group of O: this is isomorphic to A_5, and every order 3 element in A_5 is conjugate to its inverse. Denoting by h_v a primitive hyperbolic element in $\Gamma^+_{\gamma_v}$, computations analogous to the case of γ_f show that

$$\mathrm{vol}(\Gamma^+_{\gamma_v}\backslash G_{\gamma_v}) = \frac{1}{3}\cdot\ell(h_v) = \frac{1}{3}\cdot 2\cdot|OV| = 1.2685\dots$$

The case of the order 2 elements is somewhat more complicated. The isotropy groups of E and F are isomorphic to D_{10}, and therefore the three edges of T labeled with 2 are the image in SW/A_5 of fixed point set of a single order 2 elliptic element. Therefore, there is only one conjugacy class of such elements, given by the π rotation around the geodesic γ_e in $\mathbb{H}^3$ connecting E and O. Denoting by h_e a primitive hyperbolic element in $\Gamma^+_{\gamma_e}$, we have

$$\ell(h_e) = 2(|OE|+|EF|+|FV|) = 7.5836\dots$$

Furthermore:

(a) the centralizer G_{γ_e} has an extra connected component corresponding to hyperbolic elements with axis e that switch the endpoints.

(b) The centralizer $\Gamma^+_{\gamma_e}$ contains the group generated by γ_e and h_e as an index 2 subgroup; more specifically, it contains an extra involution commuting with γ_e. This is given by another order two elliptic element in the isotropy group of O, corresponding to the fact that order 2 elements in A_5 have centralizer isomorphic to the Klein four group.

Putting things together, we obtain

$$\mathrm{vol}(\Gamma^+_{\gamma_e}\backslash G_{\gamma_e}) = 2\cdot\frac{1}{4}\ell(h_e) = 3.7918\dots$$

where the factors of 2 and 1/4 take into account (a) and (b) respectively.

The case of a bad hyperbolic element h is simpler, as in this case the quantity $\mathrm{vol}(\Gamma^+_h\backslash G_h)$ is the length $\ell(h)$ divided by the order of the subgroup of elliptics having the same axis at h. Only the case of order 2 elements require some extra thought, and follows from the fact that the extra involution described in (b) above does not commute with the hyperbolic element.

1.3. **Arithmetic description.** The group Γ^+ is a tetrahedral group (see also Remark 1.2), and admits the following arithmetic description (we refer the reader to Chapter 8 of [12] for the relevant notions). Consider the number field $k = \mathbb{Q}\left(\sqrt{-1-2\sqrt{5}}\right)$, which has exactly one complex place. Consider the quaternion algebra A over k ramified exactly at the two real places, and let $\mathcal{O}$ be a maximal order in A (all of them are conjugate in this case, cfr. Section 6.7 of [12]). Under the complex embedding, we get the inclusion of the norm one elements

$$\rho : \mathcal{O}^1 \hookrightarrow \mathrm{SL}_2(\mathbb{C}),$$

and by projectivizing we obtain the arithmetic group $P\rho(\mathcal{O}^1) \subset \mathrm{PSL}_2(\mathbb{C})$. We have the following:

$$\Gamma^+ \cong P\rho(\mathcal{O}^1). \tag{2}$$

The proof of this statement can be easily adapted from the analogous result for the tetrahedral group with Coxeter symbol $[3,5,3]$ in Section 11.2.5 of [12]. First of all, (1) readily implies that $\Gamma^+ = (\Gamma^+)^{(2)}$. Furthermore, Γ^+ is arithmetic with invariant trace field and quaternion algebras k and A (see Appendix 13.1 of [12]). By Corollary 8.3.3 in [12], $\Gamma^+ = (\Gamma^+)^{(2)} \subset P\rho(\mathcal{O}^1)$ for some maximal order $\mathcal{O}$, and the equality follows because the two groups have the same covolume.

1.3.1. *Conjugacy class data for $P\rho(\mathcal{O}^1)$ by arithmetic methods.* While the orbifold corresponding to Γ^+ is not implemented in the software SnapPy because of its complicated orbifold singularities, the identification (2) makes it feasible to compute the length spectrum of Γ^+ using techniques from number theory. The method is described in Chapter 30 of [17], and has been implemented in PARI/GP by Aurel Page [14].

Let $\mathcal{O}$ be an order in a quaternion algebra A over a number field k with ring of integers $\mathbb{Z}_k$. Given an element $\gamma \in \mathcal{O}^1$, $K = k(\gamma)$ is a quadratic extension of k that embeds in A. Furthermore, $S = K \cap \mathcal{O}$ is a quadratic $\mathbb{Z}_k$-order that embeds in $\mathcal{O}$. The key facts underlying the method are the following:

- $\mathcal{O}^1$-conjugacy classes in $\mathcal{O}^1$ having the same characteristic polynomial as γ (or equivalently conjugate to $\rho(\gamma)$ in $\mathrm{PSL}_2(\mathbb{C})$) are in bijection with $\mathbb{Z}_k$-algebra embeddings $\varphi : S \hookrightarrow \mathcal{O}$ up to $\mathcal{O}^1$-conjugation.
- γ is primitive exactly when the embedding $S \hookrightarrow \mathcal{O}$ is *optimal*, i.e. after extending φ linearly to K, we have $\varphi(K) \cap \mathcal{O} = \varphi(S)$.
- the optimal embeddings $S \hookrightarrow \mathcal{O}$ up to $\mathcal{O}^1$-conjugation may be parametrized adelically.

These observations allow one to express the multiplicity of a given element $\mathbb{C}\ell(\gamma)$ in the complex length spectrum in terms of the class number of S and purely local information like local embedding numbers; when $\mathcal{O}$ is a maximal order, the latter can be understood in a very explicit form which is directly computable in PARI/GP.[3]

[3]The computations of class numbers were certified using `bnfcertify`, and therefore do not rely on the Generalized Riemann hypothesis.

2. The Selberg trace formula for coexact 1-forms for SW/A_5

The explicit Selberg trace formula for coexact 1-forms on a hyperbolic three-manifold in [10] can be readily generalized to the case of orbifolds. The main complication is the evaluation of the terms in the geometric side corresponding to conjugacy classes which are either elliptic or bad hyperbolic. We have the following.

Theorem 2.1 (Explicit Selberg trace formula for coexact 1-forms on closed hyperbolic 3-orbifolds)**.** *Let O be a closed oriented hyperbolic three-dimensional orbifold, corresponding to a quotient $\mathbb{H}^3/\Gamma$. Denote by $0 < \lambda_1^* \leqslant \lambda_2^* \leqslant \cdots$ the spectrum of the Hodge Laplacian on coexact 1-forms, and set $t_j = \sqrt{\lambda_j^*}$. Let H be an even, smooth, compactly supported, $\mathbb{R}$-valued function on $\mathbb{R}$. Then the following identity holds:*

$$\left(\frac{1}{2}b_1(O) - \frac{1}{2}\right)\widehat{H}(0) + \frac{1}{2}\sum_{j=1}^{\infty}\widehat{H}(t_j) = \frac{\mathrm{vol}(O)}{2\pi}\cdot\big(H(0) - H''(0)\big)$$

$$+\sum_{[\gamma]\neq 1} t(\gamma)\cdot\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\cdot\frac{\cos(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\cdot|1-e^{-\mathbb{C}\ell(\gamma)}|}H\left(\ell(\gamma)\right), \tag{3}$$

where:

- $\widehat{H}(t) := \int_{\mathbb{R}} H(x)e^{ix\cdot t}dx$ *is the Fourier transform of H;*
- $t(\gamma) = \frac{1}{2}$ *if γ is an elliptic element of order 2, and is 1 otherwise.*
- G_γ *and Γ_γ are the centralizers of γ in G and Γ respectively.*

Furthermore, the formula holds for the class of less regular compactly supported functions described in [10].

Remark 2.1. In the trace formula (3), if γ is elliptic, the term $\frac{\cos(\mathrm{hol}(\gamma))}{|1-e^{\mathbb{C}\ell(\gamma)}|\cdot|1-e^{-\mathbb{C}\ell(\gamma)}|}H\left(\ell(\gamma)\right)$ reduces to $\frac{\cos(\mathrm{hol}(\gamma))}{|1-e^{i\cdot\mathrm{hol}(\gamma)}|\cdot|1-e^{-i\cdot\mathrm{hol}(\gamma)}|}H(0)$.

Note that if γ is a good (i.e not bad) hyperbolic element, then

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = \ell(\gamma_0)$$

where γ_0 is a primitive element of which γ is a multiple. In particular, Theorem 2.1 is a direct generalization of Theorem 0.4 of [10]. In the case of the orbifold SW/A_5, we determine these volume terms for the elliptic and bad hyperbolic elements in Section 1.2.

Proof of Theorem 2.1. The proof of the formula for manifolds in [10] adapts directly to the case of orbifolds, provided that there are no elements of order 2. In particular, each of the terms in the second line corresponds to a quantity of the form

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\frac{dg}{dg_\gamma}$$

for some suitable f. When γ is not an elliptic element of order 2, G_γ is the connected group corresponding of hyperbolic elements sharing the axis of γ and fixing its endpoints, and we proved in [10] that the integral term (after additional work) is of the form in the theorem. In the case of an elliptic element of order 2, G_γ has an additional connected component corresponding to elements preserving the same axis but exchanging the two points at infinity. The integrand $g \mapsto f(g^{-1}\gamma g)$ is invariant under the action of the order 2 group $G^0_\gamma\backslash G_\gamma$, and

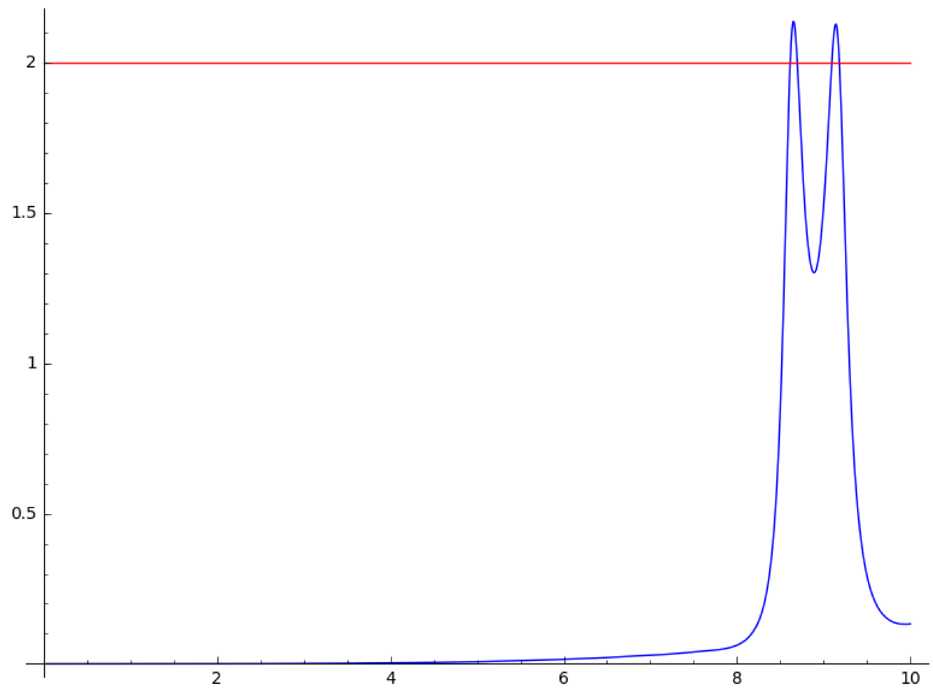

Figure 4. The graph of $t \mapsto J_{8,t}(\mathsf{SW}/A_5)$ for $t \in [0, 10]$.

so integrating over $G_\gamma \backslash G$, which equals the $G^0_\gamma \backslash G_\gamma$ quotient of $G^0_\gamma \backslash G$, introduces the claimed factor of $1/2$. □

3. Proof of Claim 0.2

In [10], we adapted the techniques of Booker and Strombergsson [2] to provide precise lower bounds for λ_1^* of a hyperbolic three manifold Y in terms of its volume and length spectrum. In particular, given the volume and the length spectrum of Y up to cutoff R, we produced a function $J_{R,t}(Y)$ giving an upper bound to the multiplicity of t^2 as an eigenvalue. The same approach (which is based on the Selberg trace formula) can be readily adapted to the case of orbifolds using Theorem 2.1 provided we additionally know exactly the list of elliptic and bad hyperbolic elements, together with the associated quantities $\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)$.

In our case of interest SW/A_5, the length spectrum can be computed via the arithmetic description is §1.3 using the code of Aurel Page [14]. The elliptic conjugacy classes were determined in Section §1.2. Given an elliptic element of order n, for $n = 2, 3, 5$, there are, up to taking inverses, exactly n primitive bad hyperbolic elements sharing the same axis (with holonomies differing by $2\pi k/n$, for $k = 0, 1, \ldots, n-1$); their length was determined in §1.2, and their holonomies are

- $0, \pi$ when $n = 2$;
- $0, 2\pi/3, 4\pi/3$ when $n = 3$;
- $\pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$ when $n = 5$.

This can be seen either directly via a geometric argument, or by looking at the length spectrum, as in our specific case these are exactly the holonomies that appear for the translation lengths of the bad hyperbolics. Finally, the covolumes of the centralizers were determined in §1.2.

Using a cutoff $R = 8$ we obtain the function $J_{8,t}(\mathsf{SW}/A_5)$ in Figure 4. In particular, $J_{8,t}(\mathsf{SW}/A_5) < 1$ for $t \leqslant 8$, so that SW/A_5 has no eigenvalues $\leqslant 64$. Eigenforms on SW/A_5 correspond to eigenforms on SW which are invariant under the action of A_5 by isometries. In particular, we obtain that for any λ^*-eigenspace V_{λ^*} of SW for $\lambda^* \leqslant 64$, there are no copies of the trivial A_5-representation. Now V_{λ^*} is a finite dimensional representation of the full isometry group S_5. The only irreducible representations of S_5 of dimension < 4 are the trivial representation and the sign representation (Section 3.1 of [7]); both of these representations restrict to the trivial representation of A_5. Therefore any V_{λ^*} with $\lambda^* \leqslant 64$ is at least 4-dimensional, and Claim 0.2 follows.

4. The first eigenspace of SW

This section builds to the characterization of the first eigenspace in Proposition 0.3. We begin by proving two lemmas that provide upper and lower bounds for the number of small eigenvalues (counted with multiplicities).

Lemma 4.1. *Every eigenvalue t^2 of the Laplacian acting on coexact 1-forms on SW for which $|t| < 2.3124$ satisfies $|t| \in [1.41, 1.45]$. Furthermore, the number of such eigenvalues (counted with multiplicity) is at most 6.*

Proof. By Claim 0.2, every eigenvalue t^2 satisfying $|t| < 8$ has multiplicity at least 4. Thus, every eigenvalue t^2 must be among those t for which $J_{8,t}(\mathsf{SW}) \geqslant 4$. The value of $J_{8,t}(\mathsf{SW})$ is less than 4 on $[0, 2.3124]\backslash[1.41, 1.45]$. This proves the first part of the Proposition.

For the second part, we recall from [10, §3] that for a given $t_* \in \mathbb{R}$, $J_{8,t_*}(\mathsf{SW})$ is the minimum value $\sum |\widehat{H}(t_n)|^2$ of the optimization problem

$$\begin{aligned} &\text{For } \widehat{H}(t) \in \mathbb{R}\text{-span of } \left\{ \left(\frac{\sin(\delta t)}{\delta t}\right)^2 \cdot \cos(k\delta t) : k = 0, \ldots, 40 \right\} \text{ where } \delta = 8/(2 \cdot 40 + 4) \\ (4)\quad &\text{Minimize } \sum |\widehat{H}(t_n)|^2 \\ &\text{Subject to } \widehat{H}(t_*) = 1. \end{aligned}$$

The functions $H * H$ are linear combinations of shifts of the convolution 4^{th} power of $\mathbf{1}_{[-\delta,\delta]}$ and are supported on $[-8, 8]$ for this particular choice of δ. We refer the reader to [10, §3] for further details.

For a given t_*, the function H_{t_*} solving the optimization problem of (4) is unique, and because $\widehat{H}_{t_*}(t_*) = 1$, by continuity $\widehat{H}_{t_*}(t')$ is close to 1 for t' quite close to t_*. Indeed, for the particular value $t_* = 1.428$, the function $|\widehat{H_{1.428}}|^2$ assumes values between 0.9 and 1.11 on the interval $t \in [1.41, 1.45]$.

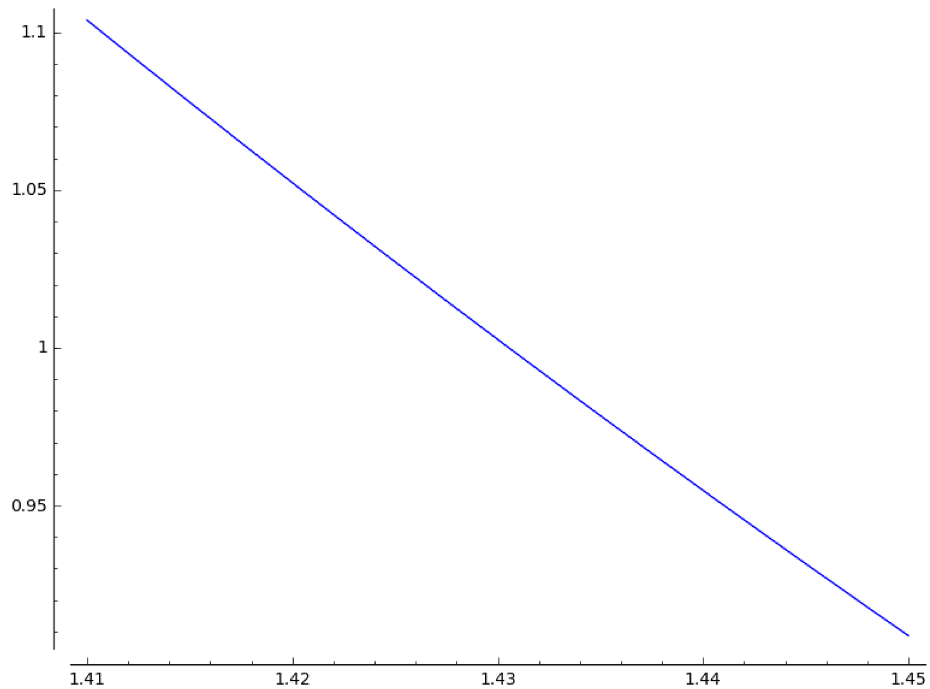

FIGURE 5. The graph of $t \mapsto \widehat{H_{1.428}}(t)$ for $t \in [1.41, 1.45]$.

This implies that the number of $|t_n| \in [1.41, 1.45]$ is at most 6. Indeed,

$$\begin{aligned}
6.0037\ldots &= J_{8,1.428}(\mathsf{SW}) \\
&= \sum_{t_n} |\widehat{H_{1.428}}(t_n)|^2 \\
&\geqslant \sum_{|t_n| \in [1.41,1.45]} |\widehat{H_{1.428}}(t_n)|^2 \\
&\geqslant \#\{n : |t_n| \in [1.41, 1.45]\} \cdot 0.9.
\end{aligned}$$

Since $\#\{n : |t_n| \in [1.41, 1.45]\}$ is a non-negative integer, this implies

$$\#\{n : |t_n| \in [1.41, 1.45]\} \leqslant 6,$$

and the claim follows. □

Lemma 4.2. *The number of eigenvalues t^2 of the Laplacian acting on coexact 1-forms on* SW *for which $|t| \in [1.41, 1.45]$ is at least 6.*

Proof. We will apply the trace formula to test functions $G_{a,b}$ satisfying

$$\widehat{G_{a,b}} = \frac{1}{2} \cdot (h(t-a) + h(t+a)) \text{ where } h(t) = 2 \cdot \left(\frac{\sin(\frac{8}{6} \cdot t)}{\frac{8}{6} t} \right)^6 \cdot \left(1 - \left(\frac{t}{b} \right)^2 \right),$$

where we assume $a > b > 0$ (see Figure 6 for a specific instance). The function $G_{a,b}$ is a linear combination of the 6^{th} convolution power of $\mathbf{1}_{[-\frac{8}{6}, \frac{8}{6}]}$ and its second derivative, all multiplied by the phase $\cos(ax)$. The key properties of the function $\widehat{G_{a,b}}$ are the following:

- it has maximum value close to 1, achieved very close to the points $t = \pm a$;
- it is positive only if $t \in [-a-b, -a+b] \cup [a-b, a+b]$.

This is useful for our purposes because if the value of the spectral side of the trace formula $\sum_{t_n} \widehat{G_{a,b}}(t_n)$ equals $X > 0$, we can reasonably hope that the number of $t_n \in [-a-b, -a+b] \cup [a-b, a+b]$ is at least X.

In the example at hand, we applied the trace formula to the test function $G_{1.33,2.2-1.33}$.[4] This function has maximum approximately 0.99998429.. and we calculate

$$\sum_{t_n} \widehat{G}_{1.33,2.2-1.33}(t_n) = 5.238\ldots$$

The function $\widehat{G}_{1.33,2.2-1.33}$ is only positive for $|t| \in I$ for some interval I containing $[1.41, 1.45]$. All t_n satisfying $|t_n| \leqslant 2.2$ are actually contained in $[1.41, 1.45]$ by Proposition 4.1. It follows that

$$\begin{aligned} 5.238\ldots &= \sum_{t_n} \widehat{G}_{1.33,2.2-1.33}(t_n) \\ &\leqslant \sum_{|t_n| \in [1.41,1.45]} \widehat{G}_{1.33,2.2-1.33}(t_n) \\ &\leqslant \#\{n : |t_n| \in [1.41, 1.45]\} \cdot 1. \qquad (5) \end{aligned}$$

Since $\#\{n : |t_n| \in [1.41, 1.45]\}$ is a non-negative integer, (5) implies that

$$\#\{n : |t_n| \in [1.41, 1.45]\} \geqslant 6$$

and proves our claim. □

Proof of Theorem 0.3. It follows immediately, by combining Lemmas 4.1 and 4.2 that every eigenvalue t^2 for which $|t| < 2.3124$ satisfies $|t| \in [1.41, 1.45]$ and that the total number of such t (counted with multiplicity) exactly equals 6.

Consider $V = \bigoplus_{|t| \in [1.41,1.45]} E_{t^2}$, the direct sum over $|t| \in [1.41, 1.45]$ of the t^2-eigenspaces of the Laplacian for on coexact 1-forms on SW; as the group S_5 acts by isometries on SW, the space V is an S_5-representation. By our main argument proving Claim 0.2, V does not contain either the trivial representation or the sign representation of S_5. By the classification of irreducible representations of S_5, the space V decomposes as a direct sum of S_5-stable subspaces of dimensions 4, 5, or 6. Since $\dim V = 6$, the only possibility is that V is the irreducible 6-dimensional representation of S_5. This implies that there is a single eigenvalue t^2 satisfying $|t| \in [1.41, 1.45]$.

Since the multiplicity of the t^2-eigenvalue equals 6, it follows that $J_{8,t}(\mathsf{SW}) \geqslant 6$. In the range $|t| \in [1.41, 1.45]$,

$$J_{8,t}(\mathsf{SW}) \geqslant 6 \implies |t| \in [1.4278772, 1.4303375].$$

This completes the proof. □

[4] Lest this choice seem arbitrary, note that we computed the spectral side of the trace formula for $G_{c,2.2-c}$ for about 100 different values of c close to $1.42\ldots$ - any eigenvalue parameter provably lies in $[1.41, 1.45]$ by Lemma 4.1. The choice $c = 1.33$ maximized the value of the spectral side of the trace formula within our sample.

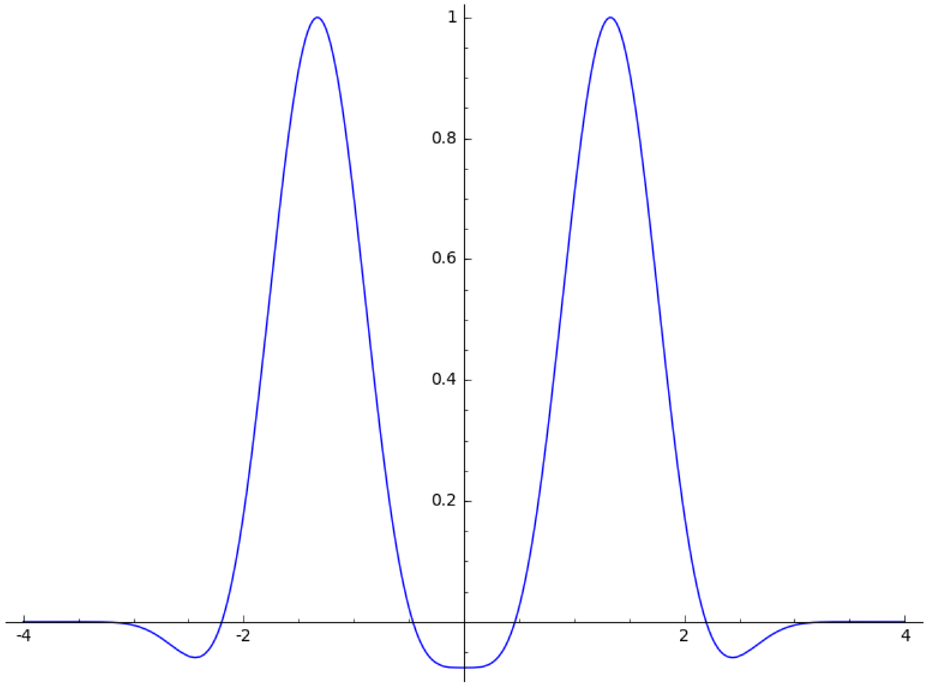

FIGURE 6. The function $\widehat{G_{a,b}}$ for the specific values $a = 1.33$, $b = 2.2 - 1.33$ used in the proof.

5. Another tetrahedral orbifold

We can adapt our discussion to the closely related tetrahedral orbifold T' with Coxeter symbol $[3, 5, 3]$. This is obtained from Figure 3 by switching the label 3 to 5 and viceversa. This is known to be the smallest arithmetic Kleinian group of the form $P\rho(\mathcal{O}^1)$ (see Section 11.7 in [12]). The arithmetic description is provided in Section 11.2.5 of [12]; in particular, it can be identified with $P\rho(\mathcal{O}^1)$ for a maximal order $\mathcal{O}$ in a quaternion algebra over the number field $\mathbb{Q}(\sqrt{3-2\sqrt{5}})$ ramified exactly at the two real places. Our geometric approach from Section 1.2 to determine elliptic and bad hyperbolic elements, and their relevant geometric quantities, can be readily adapted to this case, and we obtain lower bounds for the first eigenvalue on coexact 1-forms as in Figure 7.

Remark 5.1. Both SW/A_5 and T' admit an orientation reversing isometry r corresponding to the fact that they are the index 2 subgroups of orientation preserving isometries in a Coxeter group. Geometrically, r is obtained by reflecting along any of the faces of the tetrahedron. This implies that the eigenspaces of the Laplacian on coexact 1-forms Δ are even dimensional (which is nicely consistent with Figure 4 and 7). This is because Δ acts as $(*d)^2$ on coexact 1-forms, and the action of r on the λ_*-eigenspace of Δ exchanges the $\pm\sqrt{\lambda_*}$ eigenspaces of $*d$.

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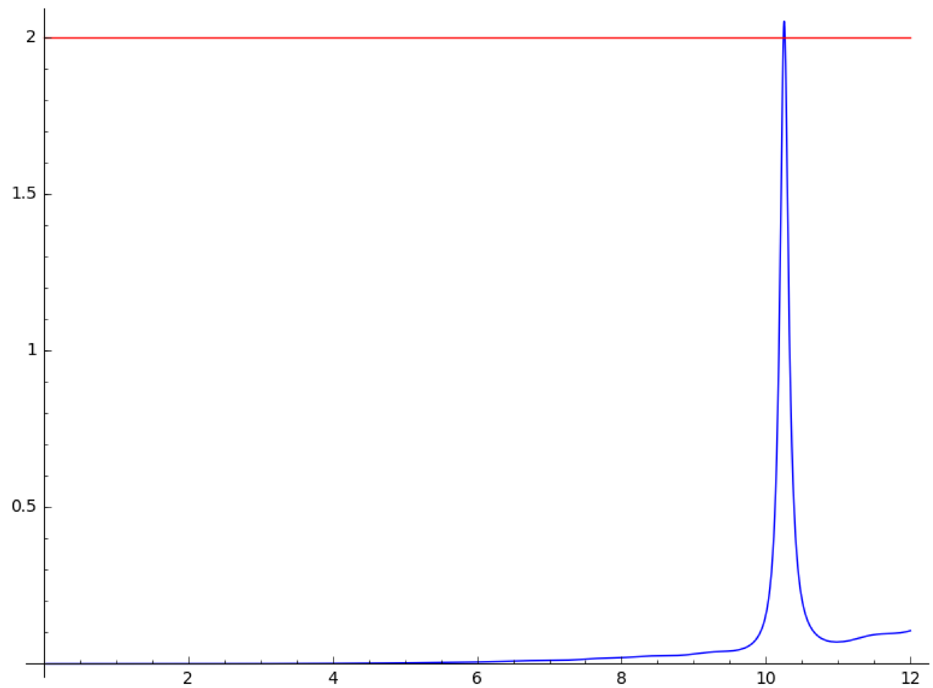

FIGURE 7. The graph of $t \mapsto J_{8,t}(\mathsf{T}')$ for $t \in [0, 10]$.

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Department of Mathematics, Columbia University
E-mail address: flin@math.columbia.edu

Department of Mathematics and Statistics, McGill University
E-mail address: michael.lipnowski@mcgill.ca