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Fluctuation theorem and extended thermodynamics of turbulence

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Turbulent flows are out-of-equilibrium because the energy supply at large scales and its dissipation by viscosity at small scales create a net transfer of energy among all scales. This energy cascade is modelled by approximating the spectral energy balance with a nonlinear Fokker–Planck equation consistent with accepted phenomenological theories of turbulence. The steady-state contributions of the drift and diffusion in the corresponding Langevin equation, combined with the killing term associated with the dissipation, induce a stochastic energy transfer across wavenumbers. The fluctuation theorem is shown to describe the scale-wise statistics of forward and backward energy transfer and their connection to irreversibility and entropy production. The ensuing turbulence entropy is used to formulate an extended turbulence thermodynamics.

1. Introduction

It is perhaps not coincidental that one of the most influential experiments in the history of thermodynamics is also a turbulence experiment. In 1849, James Prescott Joule used a stirrer to show that the shaft work on a fluid ends up increasing its internal energy, thereby demonstrating the equivalence of heat and work. Dealing

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with the generation of turbulent kinetic energy (K) and its subsequent dissipation rate (ϵ) by viscosity, the Joule experiment also offers a modern link between thermodynamics, a theory of the macroscopic effects of microscopic fluctuations, and non-equilibrium fluctuations, of which turbulence is a quintessential example.

In turbulence, the thermodynamic fluctuations are typically so small to allow a mathematical description of fluids as a continuum. For this reason, turbulence is conveniently described by the Navier–Stokes equations assuming local thermodynamic equilibrium [1]. Turbulent fluctuations are thus of macroscopic nature and technically outside the scope of traditional thermodynamics [2]. However, the random-like nature of turbulence [3–5] invites a thermodynamic formalism to the problem of turbulence; previous attempts along these lines include the eddy thermodynamics of Richardson [6], Blackadar [7] and others [8,9], as well as the Onsager analysis of 2D turbulence [10]. An assertion that 'turbulence can be defined by a statement of impotence reminiscent of the second law of thermodynamics' [11] continues to draw research attention. More recently, a number of studies have argued that the fluctuation theorem (FT) derived for small systems [12,13] can be partly applied to describe macroscopic fluctuations so as to explore their time reversibility at multiple scales, including turbulence.

The statistical properties of turbulence differ from systems near thermal equilibrium because the flux of energy per unit mass is supplied at scales much larger than the scales at which energy is dissipated by the action of viscosity, resulting in an energy flux (cascade) across all scales. Such a transport is linked to multiple processes, including vortex stretching, self-amplification of the strain-rate and viscous diffusion [14,15]. One of the defining features of the turbulence cascade is that the probability of forward and backward transitions between two energetic states at a given scale are not identical (i.e. a scale-wise 'detailed balance' is not applicable [16,17]).

With these premises, the objective of this work is to illustrate that non-equilibrium thermodynamics and in particular the FT can be employed to describe the behaviour of the turbulent energy cascade. While the net transfer of energy from large to small scales is prevalent, it is shown here that back-scatter of energy, with its connection to time-scale irreversibility, obeys the statistics predicted by the FT [13,17–21]. To provide a physical context, a turbulent flow conceptually analogous to Joule's original experiment is used, where work is done on the fluid system to generate, in a narrow band of scales, turbulent kinetic energy, which is then dissipated as heat. Thus a constant power is externally supplied at a pre-selected scale much larger than the Kolmogorov length scale where viscous effects are significant.

The focus here is on steady-state conditions where the energy cascade develops in a manner in which the energy injection rate at a large scale is balanced by the viscous dissipation rate at small scales. The FT is shown to describe the forward and backward probabilities of energy packets moving scale-wise in time through the energy cascade. For analytical tractability and to illustrate connections with the FT, simplified closure schemes for the energy transfer rate across scales are employed. These closure schemes offer plausible expressions for the energy cascade that are consistent with a wide range of experiments and theories on locally homogeneous and isotropic turbulence. The resulting stochastic differential equation is analysed within the framework of stochastic thermodynamics (e.g. [20,22,23]) to link the forward and backward probabilities to entropy production; this approach may also offer a framework for the interpretation of turbulent experiments [24,25].

2. Spectral energy balance

For a homogeneous, isotropic turbulent flow of an incompressible Newtonian fluid, the spectral energy balance per unit mass of fluid is [3,4,26]

$$\frac{\partial E(k,t)}{\partial t} = p(k,t) + \vartheta(k,t) - \eta(k,t), \tag{2.1}$$

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where $E(k,t)$ is the turbulent kinetic energy per unit wavenumber k, $p(k,t)$ is the production spectrum, here assumed to be concentrated at $k=k_i$, $\vartheta(k,t)$ is the energy transfer spectrum, $\eta(k,t)$ is the viscous dissipation spectrum assuming the fluid viscosity ν is constant and k is the wavenumber or inverse eddy-size. The normalizing property $\int_0^\infty E(k,t)\,\mathrm{d}k = K$ defines the turbulent kinetic energy K. Equation (2.1) makes no other assumptions about the velocity statistics other than homogeneity and isotropy. Because ϑ is a scale-wise transport and cannot contribute to the production or destruction of K, it satisfies the integral constraint $\int_0^\infty \vartheta\,\mathrm{d}k = 0$. It can be expressed as the gradient of an energy flux J,

$$\vartheta(k,t) = -\frac{\partial J}{\partial k}, \tag{2.2}$$

while the scale-wise viscous dissipation rate is given by

$$\eta(k,t) = 2\nu k^2 E(k,t). \tag{2.3}$$

Integrating equation (2.1) over k yields the energy balance of K,

$$\frac{\mathrm{d}K}{\mathrm{d}t} = w - \epsilon, \tag{2.4}$$

where $w = \int_0^\infty p(k,t)\,\mathrm{d}k$ is the rate of work done on the fluid to produce turbulence and ϵ is the dissipation rate of K,

$$\epsilon = \int_0^\infty \eta(k,t)\,\mathrm{d}k. \tag{2.5}$$

The concomitant balance for internal energy U is then $\mathrm{d}U/\mathrm{d}t = \epsilon - q$, where q is the heat loss to the environment. Because of fluid incompressibility, temperature fluctuations resulting from dissipation have no feedback on the dynamics of the turbulence, including the energy cascade. Finally, the entropy balance is given by $\mathrm{d}S/\mathrm{d}t = -q/T + \sigma$, where T is the absolute temperature and $\sigma = \epsilon/T$ is the entropy production. It is assumed that q is immediately delivered to a surrounding environment, acting as a thermal bath at the same temperature, thus ensuring isothermal conditions.

A closure of minimal complexity for the spectral energy balance that preserves both direct energy cascade and an inverse cascade (or back-scatter) may be obtained by representing the contributions to J as a scale-wise drift and a diffusion term, linked by a time scale of eddy relaxation $\tau(k,t)$. A flexible form for such a closure is proposed here as

$$J = \alpha\frac{kE}{\tau} - \frac{k^2}{\tau}\frac{\partial E}{\partial k}, \tag{2.6}$$

where the coefficient α is to be determined depending on models for τ [27–30]. Substituting J and η from equations (2.6) and (2.3) into the spectral energy balance in equation (2.1) yields

$$\frac{\partial E}{\partial t} = [p - 2\nu k^2 E] - \frac{\partial}{\partial k}\left(\alpha\frac{kE}{\tau} - \frac{k^2}{\tau}\frac{\partial E}{\partial k}\right). \tag{2.7}$$

Depending on the choices made about τ, a general class of nonlinear diffusion models for J can be recovered. Here, a $\tau(k,t)$ that is linked to $E(k,t)$ is adopted,

$$\tau(k,t) = [k^3 E(k,t)]^{-1/2}. \tag{2.8}$$

Other choices for $\tau(k,t)$ that accommodate non-local interactions can also be used in this framework. One common non-local closure for the energy flux is the so-called Heisenberg model [31] that can be re-casted to yield an eddy time scale given by [32]

$$\tau(k,t) = \left[\int_0^k p^2 E(p,t)\,\mathrm{d}p\right]^{-1/2}. \tag{2.9}$$

However, there are issues with the original Heisenberg model related to the direction of energy transfer and equipartition of energy that have already been identified and discussed elsewhere

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[33]. For this reason, the focus here is maintained on the use of equation (2.8) for the generic time scale. For the inertial subrange scales in steady state, the Kolmogorov [34] scaling (hereafter referred to as K41 scaling) given by $E(k) = C_o\epsilon^{2/3}k^{-5/3}$ is expected to hold resulting in $\tau(k) = C_o{}^{-1/2}\epsilon^{-1/3}k^{-2/3}$ (i.e. Onsager's relaxation time [10]), where $C_o = 1.55$ is the Kolmogorov constant. Due to the dissipative anomaly [35], the $\lim_{\nu\to 0}\epsilon$ is finite, so that in this limit $\eta_K = (\nu^3/\epsilon)^{1/4} \to 0$, $(\eta_K)^{-1} \to \infty$. An estimate of the total time for energy to be passed from a finite k_i to an infinitely high wavenumber can be determined using

$$\int_{1/k_i}^{\infty} \tau(k)\frac{\mathrm{d}k}{k} = \frac{3}{2}\frac{1}{\sqrt{C_o}}k_i^{2/3}\epsilon^{1/3} < \infty. \tag{2.10}$$

Equation (2.10) implies that the steps in the energy cascade rapidly accelerate such that (if not interrupted by the action of viscosity at a finite wavenumber) the time for energy to be passed to an infinitely high wavenumber is finite. This finding, originally put forth by Onsager [10], foreshadows the finite time singularity in the inviscid limit for such classes of $\tau(k)$ models [36].

The time scale in equation (2.8) is also singled out because it recovers the well-studied Leith's nonlinear diffusion approximation [33,37,38],

$$\frac{\partial}{\partial k}\left[k^{11/2}\sqrt{E}\left(\alpha\frac{E}{k^3} - \frac{1}{k^2}\frac{\partial E}{\partial k}\right)\right] = \frac{\partial E}{\partial t} - [p - 2\nu k^2 E]. \tag{2.11}$$

When $\alpha = 2$, the conventional form of Leith's model becomes evident [26,37]. The latter recovers the so-called warm cascade condition (i.e. a steady equipartitioned energy spectrum, $\forall k : E \propto k^2$) originally derived by Lee [39] under specific conditions [33,38,40]. Leith's model was also derived from the so-called direct-interaction approximation when a number of simplifications are made [33].

Returning to stationary conditions, which will be the focus of the rest of the paper, and considering for the moment a range of wavenumbers away from the production and viscous subranges, the left-hand side of equation (2.11) is equal to zero, and the solution to the spectral budget reduces to

$$E(k) = \left(C_1 k^{-5/2} + C_2 k^{(3/2)\alpha}\right)^{2/3}. \tag{2.12}$$

If p is injected at $k = k_i$, then for $k > k_i$ $C_1 = C_o^{3/2}\epsilon$, necessitating $C_2 = 0$ to recover K41 inertial subrange scaling (also referred to as the cold cascade). For $k < k_i$, $C_1 = 0$ and $C_2 = C_o^{3/2}\epsilon k_i^{-11/2}$ is set by the continuity of $E(k)$ at k_i to achieve a warm cascade for $\alpha = 2$ [40]. The $E(k) \sim k^{+2}$ is also compatible with the well-known Saffman spectrum [36,41,42], a scaling law derived from considerations (continuity and smoothness) of how $E(k)$ is approached as $k \to 0$.

In the presence of viscous dissipation, the spectral budget equation is not analytically solvable; however, we numerically confirmed that

$$E_*(k) \approx (C_o^{3/2}\epsilon k^{-5/2} + C_2 k^3)^{2/3} f_\eta(k\eta_K) \tag{2.13}$$

reasonably approximates the spectral energy budget, as shown in figure 1. Here, $f_\eta(k\eta_K) = \exp[-\beta(k\eta_K)^{4/3}]$ is the Pao correction [44] reshaping the $k^{-5/3}$ spectrum for $k\eta_K > 0.1$ [45]. The $E_*(k)$ from equation (2.13) implies that $\tau(k)$ in equation (2.8) increases within the viscous subrange when $k\eta_K > \beta^{-3/4}$, which is not physically plausible. The increase in $\tau(k)$ is expected when $E_*(k)$ decreases faster than k^{-3} with increasing k. Hence, an amendment proposed by Batchelor [46] was used in the calculations featured in figure 1 whereby the straining rate ($\propto \tau(k)^{-1}$) at k is assumed to be uniform beyond wavenumbers commensurate with $1/\eta_K$. This amendment revises the model for $\tau(k)$ as

$$\tau_*(k) = \begin{cases} C_2^{-1/2}k^{-5/2}f_\eta(k\eta_K)^{-1/2} & k\eta_K < k_i\eta_K \\ C_o^{-1/2}\epsilon^{-1/3}k^{-2/3}f_\eta(k\eta_K)^{-1/2} & k_i\eta_K < k\eta_K \le \beta^{-3/4} \\ \sqrt{\dfrac{e\beta}{C_o}}\tau_K & k\eta_K > \beta^{-3/4}, \end{cases} \tag{2.14}$$

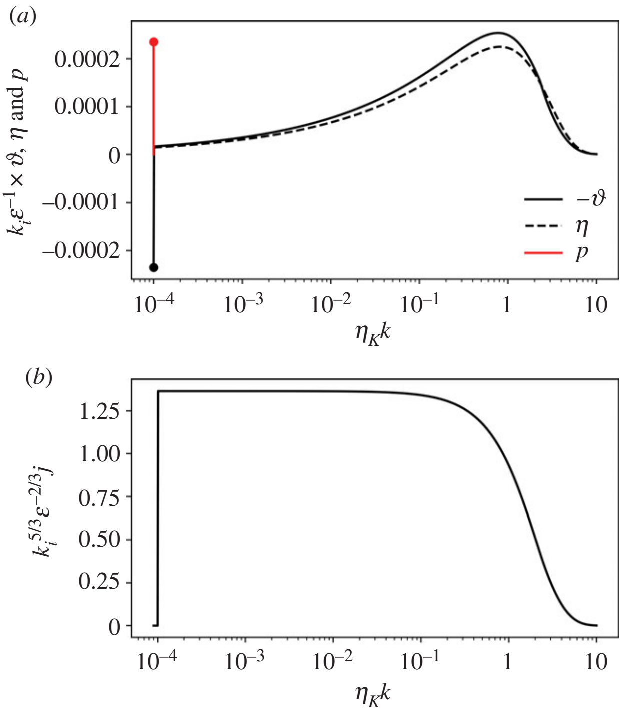

Figure 1. (*a*) The balance between production p, energy transfer $\vartheta = -\mathrm{d}J/\mathrm{d}k$ and viscous dissipation η across scale based on the empirical spectrum in equation (2.13) for $\eta_K k_i = 10^{-4}$. (*b*) The energy flux J across scales (note the similarity with the results in [43]). Here, $\beta = 0.33$ results in an acceptable spectral energy balance closure at steady state. This numerical value of β differs from the original Pao constant because of the choices made when deriving τ. (Online version in colour.)

where $\tau_K = (\nu/\epsilon)^{1/2}$ is the Kolmogorov time scale. The amendment of the relaxation time scale at $k\eta_K > \beta^{-3/4}$ arrests the tendency towards the finite-time singularity attributed to Onsager's relaxation time [10].

3. Fluctuation theorem

The spectral budget in equation (2.7) may be written in symbolic form as $\mathcal{F}\{E\} = 0$, where $\mathcal{F}\{\}$ denotes the operator defined by the PDE equation (2.7). The operator $\mathcal{F}\{\}$ corresponds to that defining a nonlinear Fokker–Planck equation (FPE) with scale-wise separated source/sink terms. Accordingly, we may represent $E(k,t)$ as a scaled probability density function (PDF), namely $E(k,t) = K\langle\delta(k_t - k)\rangle$, where $\langle\ \rangle$ denotes an ensemble average over the realizations of the velocity fluctuations. Here, k_t is a time-dependent random variable describing the trajectory of an 'energy packet' that moves in the sample-space defined by the wavenumbers $k \in [0, \infty)$, and $\delta(\cdot)$ is the Dirac distribution. This representation in terms of a scaled PDF satisfies the original definition of $E(k,t)$ as the energy spectrum, with $\int E(k,t)\,\mathrm{d}k = \int K\langle\delta(k_t - k)\rangle\,\mathrm{d}k = K$, and $K\langle\delta(k_t - k)\rangle\,\mathrm{d}k$ denoting the kinetic energy at wavenumber k. Furthermore, due to the structure of $\mathcal{F}\{\}$, k_t is governed by a stochastic process [47] involving a deterministic advection term, a diffusion term and a killing term linked to the action of viscous dissipation. The latter term has the effect of absorbing (i.e. stopping) the trajectories according to a state-dependent Poisson process with a rate $2\nu k_t^2$ [48,49].

If the steady-state solution to $\mathcal{F}\{E\}$ is known, as in our case (equation 2.13), the corresponding drift and diffusion terms defining the stochastic process for k_t can be formulated as a function of

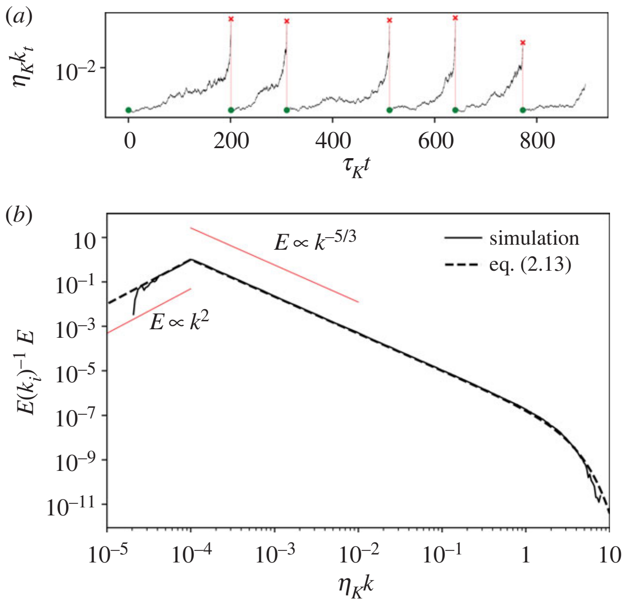

Figure 2. (*a*) A numerical realization of the stochastic process given by equation (3.1). The trajectories are terminated following a state-dependent Poisson process with rate $2\nu k_t^2$ and initiated at $\eta_K k_i = 10^{-4}$. (*b*) The steady-state PDF from the numerical simulation and the approximate spectrum in equation (2.13) are compared. For reference, the two lines show $k^{-5/3}$ (K41 or cold cascade) and k^2 (warm cascade or Saffman spectrum). (Online version in colour.)

k by substituting directly into $\mathcal{F}\{E\}$ the expression for $E(k)$. In particular, the Langevin equation that ensures that the steady-state form of $K\langle\delta(k_t - k)\rangle$ satisfies equation (2.13) is

$$\mathrm{d}k_t = \frac{k_t}{\tau_*^2(k_t)}\left(4\tau_*(k_t) - k_t\frac{\mathrm{d}\tau_*}{\mathrm{d}k}(k_t)\right)\mathrm{d}t + b(k_t)\,\mathrm{d}W, \tag{3.1}$$

where $\mathrm{d}W$ is the Wiener increment and $b(k) = \sqrt{2}k\tau_*^{-1/2}$. This equation is subjected to a unit rate of birth at $k_t = k_i$ and a state-dependent killing term with rate $2\nu k_t^2$ [49]. Moreover, since the FPE operator $\mathcal{F}\{\}$ defined by equation (2.7) is written in the transport form, the interpretation of the multiplicative term must be in the form of the so-called Hanggi–Klimontovich prescription [50–52]. With this formal correspondence, the time reversal statistics of equation (3.1) can now be analysed [20,52] for steady-state homogeneous and isotropic turbulence with energy injected at k_i and transported on average towards higher wavenumbers where dissipation takes place. Since the drift, diffusion and killing term are obtained only as a function of k using the steady-state solution of the spectrum, this Langevin equation and its analysis cannot be extended to transient cases.

For steady-state conditions, however, a tractable illustration of the FT for fully developed turbulence fluctuations can be constructed. This construction entails linking the turbulent entropy balance at k to the statistics of the forward and backward energy cascades. Figure 2a shows a numerical realization of this process in which energy packets are injected at $k_i\eta_K = 10^{-4}$ after the termination of trajectories by dissipation. It is important to note that, in general, the energy injection at k_i is only related statistically to the dissipation, and that the immediate re-injection of trajectories after trajectory termination by dissipation is only done here for the convenience of simulation and visualization. Such a scheme however preserves the steady-state PDF and thus can be suitably used to obtain the desired turbulent statistics of the problem. Accordingly, as shown in figure 2*b*, the steady-state form of the scaled PDF $K\langle\delta(k_t - k)\rangle$ corresponds to the empirical spectrum in equation (2.13).

The injection of energy at lower k and the dissipation sink at higher k produce a non-zero average current and a non-equilibrium steady-state (NESS) current of energy towards smaller scales. A stochastic fluctuating velocity at k corresponding to k_t may be defined for $\mathrm{d}t > 0$ as [52]

$$\dot{k}_t|k = \frac{(k_{t+\mathrm{d}t/2} - k_{t-\mathrm{d}t/2})|_{k_t=k}}{\mathrm{d}t}, \tag{3.2}$$

whose mean (the current velocity) is given by [52]

$$v_{\mathrm{NESS}}(k) = \frac{J(k)}{E(k)} \approx 2k\tau_*^{-1} - k^2 E_*^{-1}\tau_*^{-1}\frac{\mathrm{d}E_*}{\mathrm{d}k}, \tag{3.3}$$

and obeying a fundamental FT-type symmetry [20] (figure 3*a*). The degree of irreversibility of the NESS resulting from the cascade towards dissipation may be given by the rate of energy transfer to smaller scales. According to the formalism of stochastic thermodynamics [23,53], the non-zero current velocity may be associated with a positive 'turbulent entropy' production rate Σ [52],

$$\Sigma(k) = 2\left[\frac{v_{\mathrm{NESS}}(k)}{b(k)}\right]^2 \approx \left(k\frac{\mathrm{d}E_*}{\mathrm{d}k} - 2E_*\right)^2 E_*^{-2}\tau_*^{-1}. \tag{3.4}$$

In the inertial subrange, where $E(k) = C_o\epsilon^{2/3}k^{-5/3}$ (i.e. $f_\eta(k\eta_K) \approx 1$), $J = (11/3)C_o^{3/2}\epsilon$ (a constant with $\partial J/\partial k = 0$), $v_{\mathrm{NESS}} = \sqrt{C_o}\epsilon^{1/3}k^{5/3}$ and $\Sigma = (121/9)\sqrt{C_o}\epsilon^{1/3}k^{2/3}$. These scaling laws have also been confirmed by simulations (figure 3*b*,*c*) of the stochastic process in equation (3.1), equations (3.3) and (3.4) using the model spectrum and relaxation time scale in equations (2.13) and (2.14).

Figure 4*a* shows the scale-wise distribution of the normalized (by the Kolmogorov time scale $\tau_K = (\nu/\epsilon)^{1/2}$) turbulent entropy production rate Σ from equation (3.4) for a range of Re defined as $Re = l_i u/\nu$, where $l_i = 1/k_i$ and $u = (\epsilon/k_i)^{1/3}$ is a velocity.

4. An extended turbulence thermodynamics

The term $\Sigma(k) \geq 0$ can be shown to be the source term in the balance equation for the turbulent entropy [23], $S_t(k) = -\ln(E(k)/E_0)$, where E_0 is a normalization reference value. Figure 4*b* shows the scale-wise distribution of S_t for a range of Re where the normalization constant is set to $E(K_i)$. Thus, the asymmetry in the turbulence cascade that transports energy backward contributing to larger eddies (the so-called back-scatter) is also linked to a turbulent entropy production. Since $S_t(k)$ can be interpreted as a scale-wise 'turbulence entropy', it may be used to define an extended turbulence thermodynamics, together with the corresponding portion of TKE at that scale, thereby completing the program started by Richardson [6]. In this manner, such an entropy provides a measure of the number of turbulent states at wavenumber k to be linked to the corresponding portion of energy, $E(k)$. Neglecting momentarily the exponential cut-off at the viscous subrange while extending K41 up to $1/\eta_K$, the TKE can be linked to the area under the spectrum given in equation (2.12), namely

$$K = \int_0^{1/\eta_K} E\,\mathrm{d}k = \frac{3}{2}C_o\nu^2k_i^2Re^2\left(\frac{11}{9} - Re^{-1/2}\right), \tag{4.1}$$

where $Re = l_i u/\nu$ is a Reynolds number formed from a characteristic length $l_i = 1/k_i$ and large-scale velocity $u = (\epsilon/k_i)^{1/3}$. This definition ensures that $l_i/\eta_K = Re^{3/4}$, consistent with expectations for many turbulent flows [14]. The integrated entropy $S_I = \int S_t(k)\,\mathrm{d}k$ can also be obtained as

$$S_I = -\int_0^{1/\eta_K} \ln(E/E_0)\,\mathrm{d}k = \frac{11}{3}k_i - k_iRe^{3/4}\left[\frac{5}{3} + \ln(C_o\nu^2k_iRe^{3/4})\right] + \text{const.} \tag{4.2}$$

One objection that can be raised here is that S_t is not an entropy in the sense of Clausius, but an extended entropy related to macroscopic turbulent fluctuations. This needs to be borne in mind and motivates the appellation of extended turbulence thermodynamics. In analogy to classical thermodynamics, the TKE corresponds to an internal kinetic energy of turbulence, so

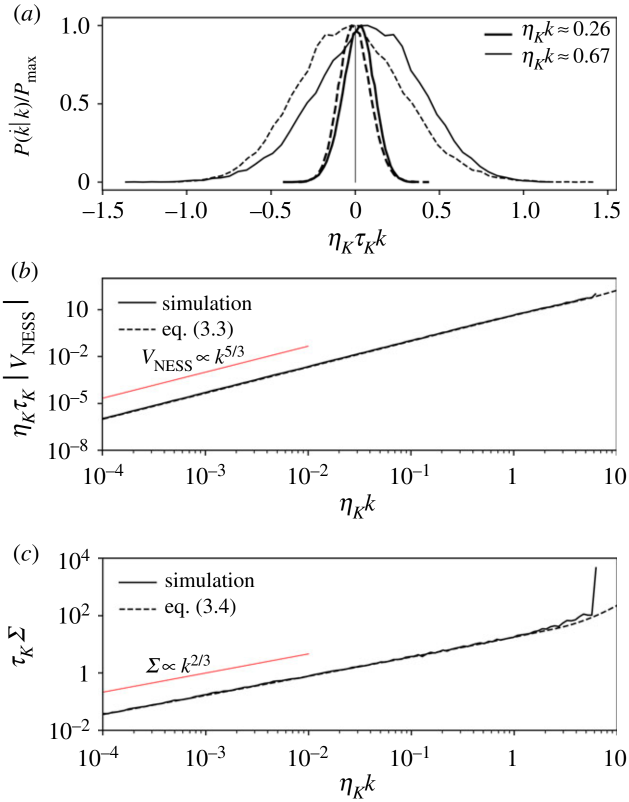

Figure 3. (*a*) The PDF of the current velocity $\dot{k}_t|k$ computed from numerical simulations at two scales defined by $\eta_K k$, where $\dot{k}$ is the sample-space variable conjugate to $\dot{k}_t$. The solid and dashed lines correspond to the forward and backward currents. The current is non-dimensionalized using the Kolmogorov length and time scales η_K and τ_K. (*b*) and (*c*) The average current velocity $v_{\rm NESS}$ and entropy production Σ from numerical simulation and the solutions in equations (3.3) and (3.4). (Online version in colour.)

that $\mathrm{d}S_I/\mathrm{d}K = 1/T_I$, with T_I being analogous to an effective temperature for the turbulent system. When such a definition is combined with equation (4.1),

$$T_I = \frac{\mathrm{d}K}{\mathrm{d}\,Re}\left(\frac{\mathrm{d}S_I}{\mathrm{d}\,Re}\right)^{-1} = 18C_o\nu^2 k_i \frac{Re^{5/4}(Re^{-1/2} - \frac{44}{27})}{8 + 3\ln(C_o\nu^2 k_i Re^{3/4})}. \tag{4.3}$$

An integrated positive turbulence-entropy production for the integrated turbulence entropy, S_I, can be obtained assuming that molecular dissipation primarily acts in the neighbourhood of $k = 1/\eta_K$. For $k < k_i$, where $E \approx C_2 k^2$ and $f_\eta(k\eta_K) \approx 1$, $v_{\rm NESS} \approx 0$ and $\Sigma \approx 0$. As a result,

$$\Sigma_I = \int_0^{1/\eta_K} E\,\Sigma\,\mathrm{d}k = \frac{121}{12} C_o^{3/2}\nu^3 k_i^4 Re^3 \ln Re. \tag{4.4}$$

In the limit of high Re, the above relations approach to $K \propto Re^2$, $\Sigma_I \propto Re^3$, $S_I \propto Re^{3/4-\alpha_1}$ and $T_I \propto Re^{5/4+\alpha_2}$, where α_1 and α_2 are deviations due to the logarithmic terms. At Re ≈ 1, the spectrum within the inertial subrange vanishes with production scales being commensurate to the Kolmogorov microscale ($\eta_K k_i \approx 1$). For this case, $\Sigma_I \approx 0$ (i.e. the detailed balance holds) although the cascade still generates small K, S_I and T_I. The finite values arise due to the approximation extending a K41 spectrum from $1/k_i$ to $1/\eta_K$. The key quantities in this turbulent thermodynamics are plotted in figure 5. The fundamental equation, $S_I = S_I(K)$, shows a downward concavity that ensures entropy production by 'combining' turbulent flows of different TKE, while the

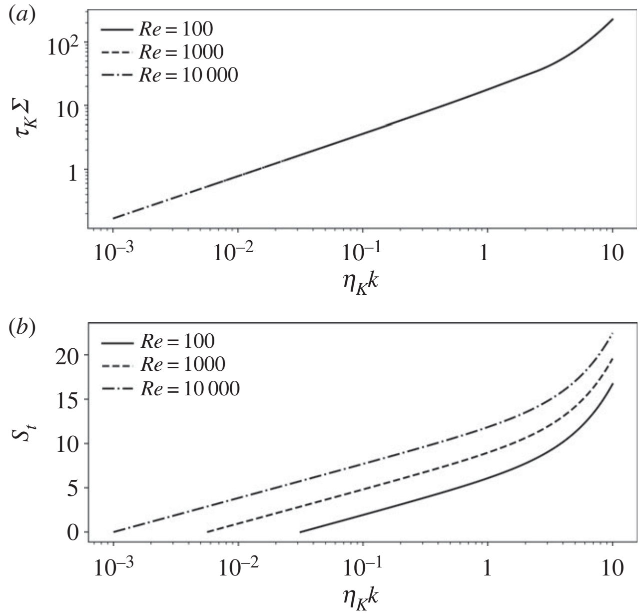

Figure 4. (*a*) The scale-wise distribution of the normalized 'turbulent entropy' production rate Σ. The Kolmogorov time scale $\tau_K = (\nu/\epsilon)^{1/2}$ is used to normalize Σ. (*b*) The turbulent entropy S_t with the normalization constant $E_0 = E(k_i)$. The curves correspond to $Re = 10^2, 10^3$ and 10^4, where $Re = l_i u/\nu$ is Reynolds number defined by the characteristic length $l_i = 1/k_i$ and velocity $u = (\epsilon/k_i)^{1/3}$.

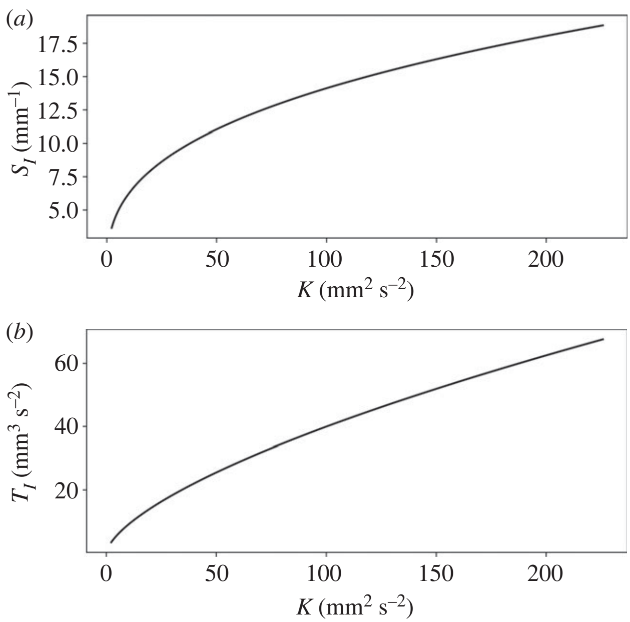

Figure 5. The relation between entropy (S_I) (*a*) and turbulent temperature (T_I) (*b*) with TKE (K). For these results, it was assumed that $k_i = 1$ m and $\nu = 8.95 \times 10^{-7}$ m^2 s^{-1}.

dependence of T_I on K shows a turbulent TKE capacity ($\mathrm{d}T_I/\mathrm{d}K$) that is not constant but decays with the Reynolds number.

The properties of stochastic diffusion also allow the entropy production to be obtained globally in k and for a finite time interval. For this, we define the forward PDF $P_f(k,t) = \langle \delta(k_t - k) \rangle$ and the backward PDF $P_b(k,t) = \langle \delta(k_t^- - k) \rangle$, where k_t^- corresponds to the solution to dk_t under time reversal. Then we may write [52,54]

$$\left\langle \ln \left. \frac{P_f}{P_b} \right|_{k=k_t} \right\rangle = \frac{1}{K} \int_0^t \int_0^\infty E(k,t')\Sigma(k,t')\,\mathrm{d}k\,\mathrm{d}t', \tag{4.5}$$

also in agreement with the FT. Based on this, it should be possible to further extend the former thermodynamic formalism to finite time intervals.

5. Conclusion

The FT has been used to link analytically the shape of the energy spectrum with the imbalance between forward and backward probabilities of energy packets moving scale-wise in time across the energy cascade. The difference between these two aforementioned probabilities is the main cause why the 'detailed balance' or 'microscopic reversibility' (i.e. at equilibrium, each elementary process is in equilibrium with its reverse process) is not applicable to turbulence, and why turbulent fluctuations are out of equilibrium. The analysis here only considered idealized conditions and assumed spatial homogeneity. There are several directions in which this work can be extended, including the analysis of the spatial distribution of forward and backward energy transfer, which can be studied theoretically or using direct numerical simulations of turbulence [43].

Notwithstanding these limitations, the results presented here unfold a connection between the turbulent entropy production rate measuring the effective spreading of energy-packet trajectories in the cascade, the thermodynamic entropy production, and the Reynolds number Re for an externally prescribed injection scale $1/k_i$ (often dictated by boundary conditions or geometry). As first pointed out by Landau & Lifshitz [55] and substantiated in later studies [56], the finite Re is an indicator of the number of degrees of freedom of the turbulence cascade, $N_d \sim (l_i/\eta_K)^3 \sim Re^{9/4}$.

An additional foresight from this analysis is that of the shape of the spectrum at low k. It is shown here that the Saffman spectrum is linked to $v_{\mathrm{NESS}} = 0$ and $\Sigma = 0$ (no scale-wise entropy production and the detailed balance is satisfied as expected for warm cascades). From J in equation (2.6), the condition for $v = J/E \neq 0$ assuming $E(k) \propto k^\gamma$ can now be derived for the large scales ($k < k_i$). With $\gamma > 0$, the condition $-\mathrm{d}J/\mathrm{d}k > 0$ imposed by the energy balance necessitates $\gamma > \alpha$ ($=2$ for the Saffman spectrum and the associated Leith's model) for $k/k_i < 1$. It also follows that $J < 0$ (or $v_{\mathrm{NESS}} < 0$) when $\gamma < \alpha$, a state where the current towards larger scales is caused by the dominance of the back-scatter over the forward drift. From the perspective of $\alpha = 2$ (i.e. Leith's model), the Saffman ($\gamma = 2$) spectrum results in $-\mathrm{d}J/\mathrm{d}k = 0$ whereas the Batchelor [57] spectrum ($\gamma = 4$) yields $-\mathrm{d}J/\mathrm{d}k > 0$ (i.e. forward drift still dominates over back-scattering). However, the Karman spectrum [42,58] often used in reshaping the inertial subrange spectrum at production scales in boundary-layer turbulence yields a non-monotonic $-\mathrm{d}J/\mathrm{d}k$ in the rising limb of $E(k)$ as $k \to k_i$. Recent findings have shown that the inverse energy cascade can indeed exist in 3D flows [59–61]; in this regard, it will be interesting to explore how our extended thermodynamic framework can be related to these statistics and to other interesting thermodynamic-related phenomena such as the turbulent spectral condensation.

These considerations have also led to a new perspective on the turbulence-thermodynamics formalism, linking the emergence of turbulent modes to store disorderly kinetic energy to key macroscopic quantities such as the Reynolds number and the turbulence temperature. One of the main results is the derivation of an effective temperature T_I in analogy with the classic Gibbs' formula. In this regard, it is important to keep in mind that for non-equilibrium macroscopic systems the role played by this macroscopic energy scale is not as straightforward to interpret as that of thermal energy in thermodynamics systems at equilibrium (e.g. [22,62] and references therein).

It will also be of interest to compute S_I and T_I based on an energy spectrum, including intermittency corrections, to assess how intermittency might play a role in the proposed extended thermodynamics. One might even conjecture the existence of an extended global turbulence pressure to link flow configurations to kinetic energy and entering as a natural variable in a Gibbs turbulent free energy to provide a unified criterion for turbulent transition and development. Future investigations along these lines may offer further elements to the picture of turbulence as a non-equilibrium phase-transition phenomenon, of which several aspects are beginning to emerge [63].

Data accessibility. This article has no additional data.

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