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∞-OPERADS AS ANALYTIC MONADS

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Abstract. We develop an ∞-categorical version of the classical theory of polynomial and analytic functors, initial algebras, and free monads. Using this machinery, we provide a new model for ∞-operads, namely ∞-operads as analytic monads. We justify this definition by proving that the ∞-category of analytic monads is equivalent to that of dendroidal Segal spaces, known to be equivalent to the other existing models for ∞-operads.

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1. Introduction

Operads are a powerful formalism for encoding algebraic operations. They were first introduced in the early seventies for the purpose of describing up-to-homotopy algebraic structures on topological spaces [May72, BV73], and have since become a standard tool also in algebra, geometry, combinatorics, and mathematical physics. Operads are closely related to monads, which were introduced some 10 years earlier, implicitly with Godement's "standard construction" of flasque resolutions [God58],[1] and explicitly by Huber [Hub61]. The notion soon spread from algebraic geometry and homological algebra to universal algebra, logic, and computer science. The relationship between operads and monads was exploited from the very beginning of operad theory [May72], and is a major theme of the present contribution.

Classically, an operad $\mathcal{O}$ consists of a sequence $\mathcal{O}(n)$ of topological spaces, where $\mathcal{O}(n)$ is equipped with an action of the symmetric group Σ_n (this data is called a *symmetric sequence*), together with a unital and associative composition law. The object $\mathcal{O}(n)$ describes the n-ary operations of the operad. Every symmetric sequence $\mathcal{O}$ gives rise to an endofunctor

$$F(X) = \coprod_n (\mathcal{O}(n) \times X^{\times n})_{\Sigma_n};$$

endofunctors of this form are sometimes called *analytic functors* due to their resemblance to power series.[2] When $\mathcal{O}$ is an operad, this endofunctor acquires the structure of a monad, and the algebras for the operad are canonically identified with the algebras for this monad.

From a homotopical viewpoint, these topological operads (and their associated algebras) have certain shortcomings, analogous to those afflicting topological categories when viewed as a model for "categories weakly enriched in spaces" (or ∞-categories). Just as these issues can be avoided by using a better-behaved model for ∞-categories, it is often convenient to work with less rigid notions of "operads weakly enriched in spaces" or *∞-operads*. Indeed, even for well-known topological operads there are advantages to viewing them as ∞-operads. For example, if $\mathbb{E}_n$ denotes the ∞-operad corresponding to the classical operad of little n-discs (introduced by May and Boardman–Vogt to study the algebraic structure of n-fold loop spaces), then Lurie has proved a homotopically meaningful version of Dunn's additivity theorem, $\mathbb{E}_n \otimes \mathbb{E}_m \simeq \mathbb{E}_{n+m}$, where $\otimes$ is the Boardman–Vogt tensor product of ∞-operads (which is well behaved, in contrast to the classical Boardman–Vogt tensor product of topological operads, which is not homotopy-invariant). There are various models for ∞-operads, the approach of Lurie [Lur17] being the most well-developed at the moment.

In this work we introduce a new model for ∞-operads, in terms of monads, and show that it is equivalent to the existing models. As a consequence, we shall see that an ∞-operad can be recovered from its free algebra monad, and obtain a characterization of the monads that arise in this way. One such characterization is expressed by the following slogan:

∞-operads are monads cartesian over the symmetric monad.

Here the symmetric monad means the monad Sym associated to the terminal ∞-operad; its underlying endofunctor on the ∞-category of spaces $\mathcal{S}$ is given by $\mathrm{Sym}(X) \simeq \coprod_n X^n_{h\Sigma_n}$. If T is a monad over Sym, then evaluating the natural transformation at the point we get a space $T(*)$ over $\mathrm{Sym}(*) \simeq \coprod_n B\Sigma_n$. This is precisely the same data as a symmetric sequence: the fibre of $T(*)$ at the point of $B\Sigma_n$ gives the space of n-ary operations with its Σ_n-action. The operad structure on this symmetric sequence is encoded by the monad structure on the endofunctor.

Being cartesian over Sym means we have a map of monads whose underlying natural transformation is cartesian, i.e. its naturality squares are pullbacks. It turns out that such a natural transformation to Sym is unique if it exists, so that being cartesian over Sym is a *property* of a monad. We will see that this property has an inherent characterization as the monad being *analytic*, by which we mean that it is cartesian (i.e. its multiplication and unit transformations are cartesian) and

[1] Godement used (co)monads to construct (co)simplicial resolutions, an essential tool for computations in algebra, geometry and topology.

[2] This should not be confused with the notion of analytic functor used in the setting of Goodwillie calculus.

the underlying endofunctor preserves sifted colimits and wide pullbacks (or equivalently all weakly contractible limits). We can thus reformulate our slogan still more succinctly:

∞-operads are analytic monads.

This generalizes a classical description of operads in sets: by a result of Joyal [Joy86], these are also equivalent to analytic monads.

So far we have only discussed one-object operads, but it is quite often useful to work with the more general notion of operads with many objects (commonly called *coloured operads* or *symmetric multicategories*), which generalizes categories by allowing arrows (operations) with multiple inputs instead of just one input. The term ∞-operad usually denotes the higher-categorical version of this more general notion of operad, and our slogan remains true with this interpretation, provided we consider analytic monads on slices of $\mathcal{S}$:

∞-operads with space of objects I are analytic monads on $\mathcal{S}_{/I}$.

More precisely, in this paper we set up an ∞-category of analytic monads (on all slices of $\mathcal{S}$ simultaneously) and prove that this is equivalent to an existing model of ∞-operads, namely the dendroidal Segal spaces of Cisinski and Moerdijk [CM13a]. This model is known to be equivalent to other models of ∞-operads, including those of Lurie [Lur17] and Barwick [Bar18], as well as to simplicial operads, thanks to results of Cisinski–Moerdijk [CM13a, CM13b], Heuts–Hinich–Moerdijk [HHM16], Barwick [Bar18], and Chu–Haugseng–Heuts [CHH18].

In order to study analytic monads, we first develop a theory of analytic functors between slices of $\mathcal{S}$, which can be viewed as a categorification of power series in many variables. In fact, these analytic functors turn out to be a special case of a more general notion of *polynomial* functors, and our first task is to set up an ∞-categorical framework for polynomial functors. This is in contrast to the situation in ordinary categories, where analytic functors are *not* in general polynomial.

To make sense of this, let us explain what we actually mean by a polynomial functor. To any map of spaces $f\colon I \to J$ there is associated a string of three adjoint functors $f_! \dashv f^* \dashv f_*$,

$$\mathcal{S}_{/I} \underset{f_*}{\overset{f_!}{\underset{\longleftarrow}{\overset{f^*}{\rightleftarrows}}}} \mathcal{S}_{/J},$$

where $f_!$ is given by composition with f and f^* by pullback along f.[3] Alternatively, we may identify $\mathcal{S}_{/I}$ with $\mathrm{Fun}(I, \mathcal{S})$; then f^* is given by precomposition with f, and $f_!$ and f_* are respectively the left and right Kan extension functors along f. A *polynomial functor*[4] is a functor that is built as a composite of functors of these three kinds. The terminology "polynomial functor" has some drawbacks, such as the odd fact that "analytic" is a special case of "polynomial", but we keep this terminology due to its long history in logic, computer science and category theory (see §1.2 for some pointers).

The description of polynomial functors in terms of these fundamental adjoints can be formulated as a representability property: a polynomial functor $P\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ has a unique description as $t_! p_* s^*$ for a diagram of spaces

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J;$$

moreover, P is analytic precisely when the homotopy fibres of the map p are finite discrete spaces. Many questions about polynomial functors can be handled by manipulating these representing diagrams, and our description of ∞-operads as analytic monads allows us to leverage this calculus of polynomial functors in the setting of ∞-operads. This combinatorial interpretation of ∞-operads does not have a direct analogue in the 1-categorical setting. In sets, the endofunctors corresponding

[3] The fundamental nature of these three operations is witnessed by the fact that they correspond precisely to *substitution*, *dependent sums*, and *dependent products*, the most basic building blocks of type theory [HoTT].

[4] The notion of "polynomial functor" we consider here should not be confused with the notion of "polynomial functor" introduced by Eilenberg and Mac Lane and subsequently used in the study of functor homology, nor with the notion occurring in Goodwillie's calculus of functors.

to most operads are not polynomial — this is only true for the so-called Σ-*free* operads, i.e. those for which the actions of the symmetric groups are all free.

In this paper, for the sake of emphasizing the key ideas, we consider polynomial and analytic monads over (slices of) $\mathcal{S}$ only, but it is an attractive feature of the polynomial formalism that it is readily adaptable to more general contexts. In particular, it would seem to be a natural setting for notions of operads with non-discrete arities, as required in certain situations beyond spaces.

1.1. **Overview of Results.**

Polynomial Functors. To carry out our programme, we first develop the higher-categorical version of the basic theory of polynomial functors, roughly corresponding to the results of Gambino–Kock [GK13] in the case of ordinary categories. In view of the broad spectrum of applications of ordinary polynomial functors, we expect that this theory will be of independent interest, and hope that it can serve as a starting point for further developments.

Our first main result is the following classification of polynomial functors:

Theorem. *The following are equivalent for a functor $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$:*

(i) F is a polynomial functor.
(ii) F is of the form $t_! p_* s^*$ for a diagram of spaces

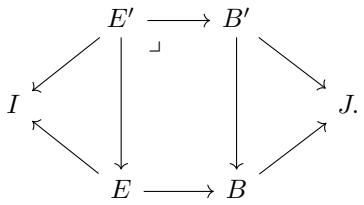
$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{s} J.$$

(iii) F is accessible and preserves weakly contractible limits.
(iv) F is a local right adjoint.

Here a weakly contractible limit means a limit of a diagram indexed by an ∞-category whose classifying space is contractible, and a local right adjoint functor is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for every object $x \in \mathcal{C}$ the induced functor $\mathcal{C}_{/x} \to \mathcal{D}_{/Fx}$ is a right adjoint.

This characterization is the higher-categorical version of classical theorems due to Lamarche, Taylor, Johnstone–Carboni, and Weber (as synthesized in [GK13]). Its proof takes up §2.1–§2.2.

For our purposes, the relevant morphisms between polynomial functors $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ are the cartesian natural transformations. We show in §2.3 that these are represented by diagrams of the form



The interplay between the polynomial functors and the diagrams that represent them (called polynomial diagrams) is a key aspect of the theory: some features are most easily handled in terms of functors and some more easily in terms of representing diagrams. To exploit this fully we need to describe polynomial functors with varying source and target in terms of diagrams. To define such a general ∞-category of polynomial functors we start by constructing a double ∞-category of "colax squares" in which the vertical arrows are right adjoints. We have delegated its definition, in terms of lax natural transformations as studied in [Hau20], to Appendix A, where we also discuss the naturality of the procedure of taking "mates"; the aim is to ensure coherence of all the Beck-Chevalley transformations once and for all in a uniform way. With this in place, we can define an ∞-category PolyFun of polynomial functors and cartesian transformations, and a (much simpler) ∞-category Poly of polynomial diagrams (a subcategory of the ∞-category of diagrams in $\mathcal{S}$ of shape $\bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$). The main result of §2.4 is then that there is an equivalence of ∞-categories

$$\mathrm{Poly} \xrightarrow{\sim} \mathrm{PolyFun}$$

over the source and target projections to $\mathcal{S} \times \mathcal{S}$.

In §2.5 we exploit this equivalence to show that the colimit of a diagram of polynomial functors and cartesian transformations is a polynomial functor, corresponding to the (pointwise) colimit of the corresponding diagrams (Proposition 2.5.4). We end the section in §2.6 by studying slices $\mathrm{PolyFun}_{/P}$ for P a polynomial functor; we prove that these ∞-categories are all ∞-topoi (Theorem 2.6.1). Note that PolyFun itself is not even accessible (see Remark 3.1.15).

Analytic Functors. In §3 we study the special case of *analytic* functors, which we characterize by the equivalent conditions of the following theorem:

Theorem. *Let* $\mathbf{E}\colon \mathcal{S} \to \mathcal{S}$ *denote the polynomial functor* $X \mapsto \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n}$, *represented by the diagram*

$$* \leftarrow \coprod_n n_{h\Sigma_n} \to \coprod_n B\Sigma_n \to *.$$

The following are equivalent for a functor $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$*:*

(i) F *is a polynomial functor with a morphism to* $\mathbf{E}$ *(which is unique if it exists).*

(ii) F *is a polynomial functor, represented by a diagram*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{s} J$$

where the map p *has finite discrete fibres.*

(iii) F *preserves sifted colimits and weakly contractible limits.*

We are mostly interested in endofunctors. For a functor $F\colon \mathcal{S} \to \mathcal{S}$, condition (ii) implies that F is of the form

$$X \mapsto \coprod_n (B_n \times X^{\times n})_{h\Sigma_n},$$

so our notion of analytic functors does indeed generalize the standard definition for endofunctors of Set. We observe that analytic endofunctors of $\mathcal{S}$ are equivalent to symmetric sequences, and that they can also be characterized as the left Kan extensions of homotopical species, meaning functors $\iota\mathrm{Fin} \to \mathcal{S}$ where $\iota\mathrm{Fin}$ denotes the groupoid of finite sets and bijections — this is an ∞-categorical version of a theorem of Joyal. More generally, analytic endofunctors of $\mathcal{S}_{/I}$ are equivalent to I-coloured symmetric sequences (or symmetric I-collections), defined as functors $\mathbf{E}(I) \times I \to \mathcal{S}$.

The combinatorics of trees enter all approaches to operads, explicitly or otherwise. In the polynomial formalism, the interplay between trees and operads is particularly intimate, since following [Koc11] we can define trees as certain polynomial endofunctors

$$A \leftarrow N' \to N \to A,$$

where A is the set of edges, N is the set of nodes, and N' is the set of nodes with a marked incoming edge. We thus have a full subcategory $\mathbf{\Omega}_{\mathrm{int}}$ of trees inside the ∞-category AnEnd of analytic endofunctors. In §3.3 we use this to show that analytic endofunctors can be described in terms of trees — more precisely, we prove that the restricted Yoneda embeddings give equivalences of ∞-categories

$$\mathrm{AnEnd} \simeq \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}}) \simeq \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}}).$$

Here $\mathbf{\Omega}_{\mathrm{el}}$ is the full subcategory of *elementary* trees, which are the corollas and the trivial tree (the edge without nodes), and $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}})$ is the full subcategory of presheaves on $\mathbf{\Omega}_{\mathrm{int}}$ that satisfy a Segal condition, which can be interpreted as a sheaf condition for the covers of trees by elementary subtrees.

Initial Algebras and Free Monads. Our comparison result relies on understanding the free monad on an analytic endofunctor. As a first step, we need to know that these free monads actually exist, which is the main result of §4. We follow the classical approach using initial Lambek algebras, which goes back to Adámek [Adá74]; a standard reference for the classical case is Kelly [Kel80]. For a finitary endofunctor P, i.e. an endofunctor that preserves filtered colimits, we show in §4.1 that the ∞-category of Lambek algebras has an initial object, constructed inductively; we present the construction in terms of a bar-cobar adjunction for Lambek algebras, which appears to be new.

In §4.2 we use the initial algebra construction to exhibit a left adjoint to the forgetful functor $\mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}$, where $\mathrm{alg}_P(\mathcal{C})$ is the ∞-category of Lambek algebras for P, and show that the resulting adjunction is monadic (Proposition 4.2.4). The monad induced by the adjunction is the free monad on P, i.e. characterized by a universal property (see Proposition 4.2.8). The forgetful functor from finitary monads on $\mathcal{C}$ to finitary endofunctors thus has a left adjoint, taking an endofunctor to its free monad. We observe that, at least if we restrict to endofunctors that preserve sifted colimits, this adjunction is itself monadic (Corollary 4.2.15).

Then, in §4.3 we give a more explicit description of the underlying endofunctor of the free monad as the colimit of a sequence of functors, which we will later exploit to understand the free monad on an analytic endofunctor in terms of trees.

In §4.4 we extend our results to obtain a monadic left adjoint to the forgetful functor from monads that preserve sifted colimits to endofunctors of varying ∞-categories. This requires an ∞-category of monads over varying ∞-categories, which is studied in [Hau20]; we recall the results we need from there in §A.5.

Analytic Monads. In §5 we apply our results on free monads in the special case of analytic monads. In §5.1 we show that the free monad on an analytic endofunctor exists and is again analytic, and the natural transformations of the monad structure are cartesian. In §5.2 we then show that the free monad on an analytic endofunctor has an explicit description in terms of trees, giving:

Theorem. *The forgetful functor* $\mathrm{AnMnd} \to \mathrm{AnEnd}$ *from analytic monads to analytic endofunctors has a left adjoint, taking an analytic endofunctor to its free monad, and the resulting adjunction is monadic. If P is an analytic endofunctor given by the diagram*

$$I \leftarrow E \to B \to I$$

then the underlying endofunctor of the free monad on P is represented by

$$I \leftarrow \mathrm{tr}'(P) \to \mathrm{tr}(P) \to I,$$

where $\mathrm{tr}(P)$ is the ∞-groupoid of P-trees, i.e. trees with a morphism to P in AnEnd*, and $\mathrm{tr}'(P)$ is the ∞-groupoid of P-trees with a marked leaf.*

This is an ∞-categorical version of a result from [Koc11].

Comparison with ∞-Operads. We are now ready to establish the main result of the paper, namely the equivalence between analytic monads and ∞-operads.

Let $\mathbf{\Omega}$ be the full subcategory of AnMnd on the free monads on trees; this is the polynomial description (cf. [Koc11]) of the dendroidal category of Moerdijk and Weiss [MW07].

Theorem. *The restricted Yoneda functor* $\mathrm{AnMnd} \to \mathcal{P}(\mathbf{\Omega})$ *is fully faithful, and its essential image is $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega})$. We thus have an equivalence of ∞-categories*

$$\mathrm{AnMnd} \simeq \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}).$$

Here $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega})$ is the ∞-category of presheaves whose restriction to $\mathbf{\Omega}_{\mathrm{int}}$ lies in $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}})$; these are precisely the dendroidal Segal spaces.

The proof is inspired by the Nerve Theorem of Weber [Web07]. The main ingredients are the monadicity of the free monad adjunction, the interpretation of analytic endofunctors as presheaves on $\mathbf{\Omega}_{\mathrm{el}}$, and the explicit description of the free monad in terms of trees.

1.2. **Related Work.**

Operads. A number of categorical descriptions exist for operads in Set. While it is well known that *non-symmetric* operads are equivalent to monads cartesian over the free-monoid monad (and are hence automatically polynomial), it is not true that a non-symmetric operad can be recovered from is monad alone [Lei04] — the cartesian natural transformation is a structure, not a property. Leinster took this as the starting point for a theory of generalized operads, defined as monads cartesian over a fixed cartesian monad. Symmetric operads are not an instance of this notion, though: they ought

to be cartesian over the free-commutative-monoid monad Sym, but while the free-algebra monad of a symmetric operad does admit a canonical monad map to Sym, neither the monads nor the map are cartesian in general.

The symmetric case can be handled with the notion of weakly cartesian natural transformation, introduced by Joyal [Joy86]; see Weber [Web04] for a systematic treatment. Joyal proved that an endofunctor is analytic if and only if it admits a weakly cartesian natural transformation to Sym, and showed that the category of analytic functors and weakly cartesian natural transformations is equivalent to that of symmetric sequences (or species). This equivalence is monoidal: composition of analytic functors corresponds to the composition product of symmetric sequences, which goes back to Kelly [Kel05]. Kelly had observed that operads are monoids in symmetric sequences, so it follows that operads are analytic monads. The characterization of operads as weakly cartesian over Sym also follows.

An alternative way of overcoming the subtleties consists in observing that while Sym is not cartesian on Set, it *is* cartesian as a 2-monad on Cat, as is important in Kelly's theory of clubs [Kel74]. This was exploited by Weber [Web15b] to give a characterization of symmetric Set-operads as polynomial monads cartesian over Sym in a certain 2-categorical sense. Weber's work was an important starting point for us.

It is a pleasant feature of the ∞-categorical setting that these various approaches are unified in clean statements, as expressed in the slogans of the introduction. These hold true already over 1-groupoids, but to get a good description of analytic functors in 1-groupoids one is forced to pass to 2-groupoids, etc., giving the usual infinite ladder — only for ∞-groupoids do we get a nice self-contained theory.

∞-Operads. As we mentioned above, our description of ∞-operads as analytic monads can be interpreted as an ∞-categorical version of the classical description of (I-coloured) operads as associative algebras in (I-coloured) symmetric sequences. Another version of such a description of ∞-operads, which also works for enriched ∞-operads, was recently obtained by the second author [Hau19] by describing the composition product using an extension of Day convolution to double ∞-categories. Alternatively, the composition product can be constructed using free symmetric monoidal ∞-categories (extending to ∞-categories the construction of [Tri]); this approach is implemented in the thesis of Brantner [Bra17], though it has not yet been compared to other models for ∞-operads.

Polynomial Functors. The theory of polynomial functors has roots in topology, representation theory, combinatorics, logic and computer science. For instance, the 1-categorical version of our Theorem 2.2.3 grew out of work on Girard's linear logic and domain theory, and some of the basic results on polynomial functors were first established in connection with semantics for generic data types and polymorphic functions [AAG03] (see [GK13] for further background and references, and [Koc12] for analytic functors in that context). Moerdijk and Palmgren [MP00] showed that initial algebras for polynomial functors are semantics for W-types in (extensional) Martin-Löf type theory, the fundamental example being the natural numbers as initial algebra for $X \mapsto 1 + X$, cf. [Law64, Lam68]. With the homotopy interpretation of type theory [HoTT], a full-blown intentional interpretation has recently been given by Awodey–Gambino–Sojakova [AGS17] as *homotopy* initial algebras. (Generalized) ∞-operads are expected to serve as semantics for the so-called higher inductive types (see [LS20]). The polynomial approach to ∞-operads might play some role in fleshing out the semantics side of those ideas.

For their role in encoding both substitution and induction/recursion, polynomial functors have also become an important tool for handling the intricate combinatorial structures that arise in higher category theory. For example, polynomial monads were used to give a purely combinatorial description of opetopes [KJBM10], and Batanin and Berger [BB17] have exploited polynomial monads to give unified constructions of Quillen model structures on categories of algebras. Their paper has many references to related developments.

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2. Polynomial Functors

2.1. **Polynomial Functors.** We write $\mathcal{S}$ for the ∞-category of spaces (also known as ∞-groupoids or homotopy types); in the model of ∞-categories as quasicategories this can be explicitly defined as the coherent nerve of the simplicial category of Kan complexes.

If $f\colon I \to J$ is a map of spaces, then f induces three adjoint functors between the slice ∞-categories $\mathcal{S}_{/I}$ and $\mathcal{S}_{/J}$: Composition with f gives a functor

$$f_!\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$$

which is left adjoint to the functor

$$f^*\colon \mathcal{S}_{/J} \to \mathcal{S}_{/I}$$

given by pullback along f. The functor f^* also has a right adjoint

$$f_*\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$$

since $\mathcal{S}$ is locally cartesian closed. If we interpret the slice ∞-categories $\mathcal{S}_{/I}$ as functor ∞-categories $\mathrm{Fun}(I, \mathcal{S})$ using the straightening equivalence, then the functor $f^*\colon \mathrm{Fun}(J, \mathcal{S}) \to \mathrm{Fun}(I, \mathcal{S})$ is given by precomposition with f, and $f_!$ and f_* are given by left and right Kan extension along f.

Definition 2.1.1. A *polynomial functor* is a functor $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ of the form $t_! p_* s^*$ corresponding to a diagram of spaces

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J.$$

Remark 2.1.2. In this paper we only consider polynomial functors in the context of the ∞-category of spaces, since this is the appropriate setting for ∞-operads. It is possible to consider polynomial functors in the more general setting of an arbitrary ∞-topos (or a locally cartesian closed ∞-category, as treated in [GK17]), and we expect that most of our results can be generalized to this context. However, this would require working in the setting of internal ∞-categories, which has not yet been sufficiently developed. For example, instead of natural transformations between polynomial functors we must use the analogue of so-called *strong* natural transformations (cf. [GK13]), or equivalently fibred natural transformations (cf. [KK13]). In ordinary category theory, polynomial functors have also been considered [Web15a] in general categories with pullbacks, at the price of having to impose an exponentiability condition separately on the middle maps in the diagrams.

A basic fact about polynomial functors is that they compose (cf. Theorem 2.1.8 below). This result amounts to being able to rewrite any composite of upper-star, lower-star and lower-shriek functors in the normal form of the definition. This is achieved through Beck-Chevalley transformations and distributivity, which we proceed to discuss. This works essentially as in the 1-categorical case [GK13]. Our treatment follows [Web15a, §2.2].

Definition 2.1.3. A natural transformation $\phi\colon F \to G$ of functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is *cartesian* if for every morphism $f\colon C \to C'$ in $\mathcal{C}$ the commutative square

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FC' \\ {\scriptstyle \phi_C}\downarrow & & \downarrow{\scriptstyle \phi_{C'}} \\ GC & \xrightarrow[Gf]{} & GC' \end{array}$$

is cartesian.

Remark 2.1.4. If $\mathcal{C}$ has a terminal object $*$, then by the 2-of-3 property of pullback squares a natural transformation ϕ as above is cartesian if and only if for every object $c \in \mathcal{C}$ the naturality square

$$\begin{array}{ccc} Fc & \longrightarrow & F* \\ {\scriptstyle \phi_c}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ Gc & \longrightarrow & G* \end{array}$$

is cartesian.

Lemma 2.1.5. *Suppose $\mathcal{C}$ is an ∞-category with pullbacks. For any morphism $f\colon S \to T$ in $\mathcal{C}$ we have a functor $f_!\colon \mathcal{C}_{/S} \to \mathcal{C}_{/T}$ given by composition with f, with right adjoint f^* given by pullback along f. The counit and unit transformations $f_!f^* \to \mathrm{id}$ and $\mathrm{id} \to f^*f_!$ for the adjunction $f_! \dashv f^*$ are cartesian.*

Proof. It suffices to check that the naturality squares for the map to the terminal object is cartesian in both cases. For the counit transformation at $q\colon Y \to T$ this is obvious, since the naturality square is

$$\begin{array}{ccc} S \times_T Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow[f]{} & T. \end{array}$$

For $p\colon X \to S$ in $\mathcal{C}_{/S}$, consider the diagram

$$\begin{array}{ccccc} X & \rightrightarrows & S \times_T X & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ S & \longrightarrow & S \times_T S & \longrightarrow & S \\ & \searrow & \downarrow & & \downarrow{\scriptstyle f} \\ & & S & \xrightarrow[f]{} & T. \end{array}$$

Here the top left square is the naturality square for the unit at p. In the right column the bottom square and the composite square are cartesian, hence so is the top right square. The composite in the top row is also cartesian, whence the top left square is cartesian, as required. □

Lemma 2.1.6. *For a commutative diagram of spaces*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[v]{} & D \end{array}$$

the following are equivalent:

(i) The square is cartesian.

(ii) The Beck-Chevalley transformation

$$u_!g^* \to u_!g^*v^*v_! \simeq u_!u^*f^*v_! \to f^*v_!$$

is an equivalence.

(iii) The Beck-Chevalley transformation

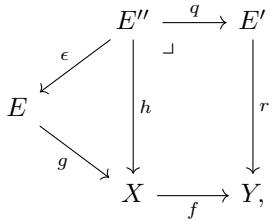
$$v^* f_* \to g_* g^* v^* f_* \simeq g_* u^* f^* f_* \to g_* u^*$$

is an equivalence.

Proof. (i)⇔(ii): To see that the Beck-Chevalley transformation in (ii) is cartesian we use Lemma 2.1.5: this implies that the unit $\mathrm{id} \to v^* v_!$ and counit $u_! u^* \to \mathrm{id}$ are cartesian transformations, and the functors $u_!$, g^*, f^*,$v_!$ all preserve pullbacks (for the left adjoints this follows for instance from Lemma 2.2.7). By Remark 2.1.4 it is therefore a natural equivalence if and only if the map $u_! g^*(\mathrm{id}_C) \to f^* v_!(\mathrm{id}_C)$ in $\mathcal{S}_{/B}$ is an equivalence. Here $u_! g^*(\mathrm{id}_C) \simeq u_!(\mathrm{id}_A) \simeq u$ and $f^* v_!(\mathrm{id}_C) \simeq f^*(v)$, and the map $u \to f^* v$ is given by the natural map from A to the pullback of v along f. Since the forgetful functor $\mathcal{S}_{/B} \to \mathcal{S}$ is conservative, we see that the square is indeed cartesian if and only if this map is an equivalence.

(ii)⇔(iii) follows since the two transformations are mates: (iii) is obtained from (ii) by taking right adjoints, and (ii) from (iii) by taking left adjoints. □

Proposition 2.1.7. *Given maps of spaces $f\colon X \to Y$ and $g\colon E \to X$, put $r := f_* g\colon E' \to Y$ and $h := f^* r\colon E'' \to X$, to get a commutative diagram*



where ϵ is the counit for the adjunction $f^* \dashv f_*$ and q is the pullback of f along r. Then the natural transformation $\delta\colon r_! q_* \epsilon^* \to f_* g_!$, defined as the composite

$$r_! q_* \epsilon^* \to r_! q_* \epsilon^* g^* g_! \simeq r_! q_* h^* g_! \xleftarrow{\sim} r_! r^* f_* g_! \to f_* g_!$$

is an equivalence.

Proof. The natural transformation δ is cartesian, since by Lemma 2.1.5 it is a composite of cartesian transformations. It therefore suffices to show that the component of δ at id_E is an equivalence. We have $r_! q_* \epsilon^*(\mathrm{id}_E) \simeq r_!(\mathrm{id}_{E'}) \simeq r$ (since $q_* \epsilon^*$ preserves the terminal object) and $f_* g_!(\mathrm{id}_E) \simeq f_*(g)$ which by definition is also r. Tracing through the maps constituting the transformation reveals that the actual map from r to r is the diagonal followed by a projection, which is the identity. □

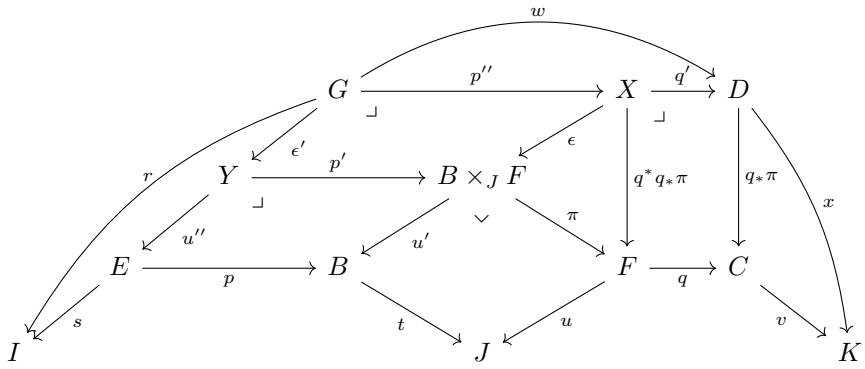
We can now give an explicit description of the composite of polynomial functors:

Theorem 2.1.8. *Suppose $P\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ and $Q\colon \mathcal{S}_{/J} \to \mathcal{S}_{/K}$ are polynomial functors, represented by diagrams of spaces*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$

$$J \xleftarrow{u} F \xrightarrow{q} C \xrightarrow{v} K,$$

respectively. Consider the commutative diagram of spaces



where ϵ is the counit map $q^*q_*\pi \to \pi$ for the adjunction $q^* \dashv q_*$, and the squares are all pullbacks. Then the composite $Q \circ P \colon \mathcal{S}_{/I} \to \mathcal{S}_{/K}$ is the polynomial functor represented by the diagram

$$I \xleftarrow{r} G \xrightarrow{w} D \xrightarrow{x} K.$$

Proof. We have natural equivalences

$$\begin{aligned}
v_!q_*u^*t_!p_*s^* &\simeq v_!q_*\pi_!u'^*p_*s^* && \text{(using the Beck-Chevalley equivalence } u^*t_! \simeq \pi_!(u')^*\text{)}\\
&\simeq v_!(q_*\pi)_!q'_*\epsilon^*u'^*p_*s^* && \text{(using the distributivity equivalence } q_*\pi_! \simeq (q_*\pi)_!q'_*\epsilon^*\text{)}\\
&\simeq x_!q'_*p''_*\epsilon'^*u''^*s^* && \text{(using the Beck-Chevalley equivalence } \epsilon^*u'^*p_* \simeq p''_*\epsilon'^*u''^*\text{)}\\
&\simeq x_!w_*r^*. && \square
\end{aligned}$$

Corollary 2.1.9. *The composite of two polynomial functors is again a polynomial functor.* □

Remark 2.1.10. This corollary also follows from the characterization of polynomial functors we will prove below in Theorem 2.2.3, but we will also need the explicit formula for the composition given by Theorem 2.1.8.

2.2. **Local Right Adjoints.** In this subsection we will prove an alternative characterization of polynomial functors. To state this we must first introduce some terminology:

Definition 2.2.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *local right adjoint* if for every $x \in \mathcal{C}$ the induced functor $\mathcal{C}_{/x} \to \mathcal{D}_{/Fx}$ is a right adjoint.

Definition 2.2.2. The inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ of spaces into the ∞-category of small ∞-categories has a left adjoint, which takes an ∞-category $\mathcal{C}$ to the space obtained by inverting all morphisms in $\mathcal{C}$, which we denote $\|\mathcal{C}\|$. We say that $\mathcal{C}$ is *weakly contractible* if $\|\mathcal{C}\|$ is a contractible space.

Theorem 2.2.3. *The following are equivalent for a functor $F \colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$:*

(i) F is a polynomial functor.
(ii) F is accessible and preserves weakly contractible limits.
(iii) F is a local right adjoint.

Remark 2.2.4. Theorem 2.2.3 is specific to the ∞-category $\mathcal{S}$ (and its truncations, such as the category of sets). Even over a presheaf topos it is not true in general that a local right adjoint is always polynomial, as exemplified by the free-category monad on directed graphs [Web07]. The corresponding theorem in ordinary category theory has a long history, see [GK13]. It can be extended to general locally cartesian closed categories with a terminal object by considering local *fibred* right adjoints instead of just local right adjoints, cf. [KK13]. Presumably the fibred viewpoint can be upgraded to the ∞-categorical setting to get a version of Theorem 2.2.3 for presentable locally cartesian closed ∞-categories, but we will not pursue this here.

Before proving Theorem 2.2.3, we need some observations on weakly contractible limits.

Definition 2.2.5. A *conical limit* is a limit indexed by an ∞-category of the form $\mathcal{I}^{\triangleright}$ for some ∞-category $\mathcal{I}$.

Lemma 2.2.6. *Suppose $\mathcal{C}$ has a terminal object. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves conical limits if and only if it preserves all weakly contractible limits.*

Proof. Conical limits are in particular indexed by weakly contractible ∞-categories, so suppose F preserves conical limits and let $\phi\colon \mathcal{I} \to \mathcal{C}$ be a diagram with $\mathcal{I}$ weakly contractible. Since $\mathcal{C}$ has a terminal object, the right Kan extension ϕ' of ϕ along the inclusion $i\colon \mathcal{I} \hookrightarrow \mathcal{I}^{\triangleright}$ exists, and $\phi \simeq \phi'|_{\mathcal{I}}$. Moreover, if ϕ has a limit then so does ϕ' and the limit of ϕ is equivalent to that of ϕ'. Since $\mathcal{I}^{\triangleright}$ is conical, F preserves the limit of ϕ'. But as $\mathcal{I}$ is weakly contractible, the inclusion i is coinitial, hence the limit of $F \circ \phi$ exists and is equivalent to the limit of $F \circ \phi'$. In other words, F preserves the limit of ϕ, as required. □

Lemma 2.2.7. *For any object x in an ∞-category $\mathcal{C}$, the forgetful functor $P\colon \mathcal{C}_{/x} \to \mathcal{C}$ preserves and reflects weakly contractible limits.*

Proof. The limit of a diagram $f\colon \mathcal{I} \to \mathcal{C}_{/x}$ is the limit of the corresponding diagram $f'\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$. If $\mathcal{I}$ is weakly contractible, then the inclusion $\mathcal{I} \hookrightarrow \mathcal{I}^{\triangleright}$ is coinitial, so the limit of f' is the same as the limit of $f'|_{\mathcal{I}}$, which is the image of f under the forgetful functor. □

Lemma 2.2.8. *In the ∞-category $\mathcal{S}$, weakly contractible limits commute with colimits indexed by ∞-groupoids.*

Proof. For $X \in \mathcal{S}$ we have the straightening equivalence $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}$, under which the constant diagram functor $\mathcal{S} \to \mathrm{Fun}(X, \mathcal{S})$ corresponds to taking products with X. Passing to left adjoints, this means that taking X-indexed colimits corresponds under the equivalence to the forgetful functor $\mathcal{S}_{/X} \to \mathcal{S}$, which preserves weakly contractible limits by Lemma 2.2.7. □

Proposition 2.2.9. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is an accessible functor between presentable ∞-categories. Then F is a local right adjoint if and only if it preserves weakly contractible limits.*

Proof. For every $x \in \mathcal{C}$, the induced functor $F_{/x}\colon \mathcal{C}_{/x} \to \mathcal{D}_{/F(x)}$ is accessible, so by the adjoint functor theorem it is a right adjoint if and only if it preserves limits. A limit of a diagram $\mathcal{I} \to \mathcal{C}_{/x}$ is the limit in $\mathcal{C}$ of the associated diagram $\mathcal{I}^{\triangleright} \to \mathcal{C}$, so the functors $F_{/x}$ preserve limits for all x if and only if F preserves all conical limits. By Lemma 2.2.6 this is equivalent to F preserving weakly contractible limits, since $\mathcal{C}$ has a terminal object. □

Lemma 2.2.10. *For any map $f\colon S \to T$ in an ∞-category $\mathcal{C}$, the functor $f_!\colon \mathcal{C}_{/S} \to \mathcal{C}_{/T}$ preserves and reflects weakly contractible limits.*

Proof. In the commutative triangle

$$\begin{array}{ccc} \mathcal{C}_{/S} & \xrightarrow{f_!} & \mathcal{C}_{/T} \\ & \searrow \quad \swarrow & \\ & \mathcal{C}, & \end{array}$$

both forgetful functors to $\mathcal{C}$ preserve and reflect weakly contractible limits. Therefore so does $f_!$. □

Lemma 2.2.11. *Suppose $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is a functor that preserves colimits (equivalently, by the adjoint functor theorem, it is a left adjoint). Then F is of the form $s_! p^*$ for some span*

$$I \xleftarrow{p} U \xrightarrow{s} J.$$

Proof. Using equivalences of the form $\mathcal{S}_{/I} \simeq \mathrm{Fun}(I, \mathcal{S})$ we get an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}_{/I}, \mathcal{S}_{/J}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(I), \mathcal{P}(J)) \simeq \mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{S})) \simeq \mathrm{Fun}(I \times J, \mathcal{S}) \simeq \mathcal{S}_{/I \times J}.$$

Thus every colimit-preserving functor corresponds to a span, and under this equivalence the span

$$I \xleftarrow{p} U \xrightarrow{s} J$$

is sent to $s_! p^*$. □

Proof of Theorem 2.2.3. The equivalence of (ii) and (iii) is a special case of Proposition 2.2.9. To see that (i) implies (ii), suppose $F \simeq t_! p_* s^*$. The functors $t_!$, p_* and s^* are all accessible, and p_* and s^* preserve all limits, being right adjoints; by Lemma 2.2.10 the functor $t_!$ also preserves weakly contractible limits, which gives (ii). Finally we show that (iii) implies (i). Observe that F factors as

$$\mathcal{S}_{/I} \xrightarrow{F_{/I}} \mathcal{S}_{/F(I)} \xrightarrow{t_!} \mathcal{S}_{/J}$$

where t is the map $F(I) \to J$. By assumption $F_{/I}$ is a right adjoint, so it follows from Lemma 2.2.11 that it is of the form $p_* s^*$ for some span

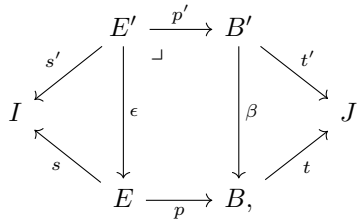
$$I \xleftarrow{s} U \xrightarrow{p} F(I).$$ □

2.3. **Morphisms of Polynomial Functors.** For our purposes the appropriate type of morphism between polynomial functors is a cartesian natural transformation, so we make the following definition:

Definition 2.3.1. The ∞-category $\mathrm{PolyFun}(I, J)$ of polynomial functors is the subcategory of $\mathrm{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$ with objects the polynomial functors and morphisms the cartesian natural transformations between them.

We now wish to identify the cartesian natural transformations with certain diagrams.

Definition 2.3.2. Suppose given a commutative diagram of spaces



where the middle square is cartesian. Let $F' := t'_! p'_* s'^*$ and $F := t_! p_* s^*$ denote the corresponding polynomial functors. Then we define a cartesian natural transformation $\phi\colon F' \to F$ as the composite

$$t'_! p'_* s'^* \simeq t'_! p'_* \epsilon^* s^* \simeq t'_! \beta^* p_* s^* \simeq t_! \beta_! \beta^* p_* s^* \to t_! p_* s^*$$

using the Beck-Chevalley transformation for the cartesian square and the counit for $\beta_! \dashv \beta^*$. Observe that the component of ϕ on the terminal object id_I is essentially β itself (as follows since right adjoints preserve terminal objects): $\phi_{\mathrm{id}_I}\colon F'(\mathrm{id}_I) \to F(\mathrm{id}_I)$ is canonically identified with $\beta\colon B' \to B$ as maps over J.

Our goal in this subsection is to show that in the ∞-category of spaces, this construction gives an equivalence between the space of such diagrams and the mapping space $\mathrm{Map}_{\mathrm{PolyFun}(I,J)}(F', F)$. (Later, we will show that this extends to an equivalence of ∞-categories.) We first prove that every cartesian natural transformation is of this form, which is a consequence of the following observation:

Lemma 2.3.3. *Suppose $\mathcal{C}$ is an ∞-category with a terminal object $*$ and $\mathcal{D}$ is an ∞-category with pullbacks. Then the functor*

$$\mathrm{ev}_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$

is a cartesian fibration, and the cartesian morphisms are precisely the cartesian natural transformations.

Proof. Observe that ev_* has a right adjoint r, taking $d \in \mathcal{D}$ to the constant functor $r(d)\colon \mathcal{C} \to \mathcal{D}$ with value d. Let η denote the unit and ϵ the counit. We can apply the criterion of (the dual of) [Hau17, Corollary 4.52]: ev_* is cartesian if and only if for every functor $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and every morphism $d \to F(*)$ in $\mathcal{D}$, in the pullback square

$$\begin{array}{ccc} F' & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \eta_F} \\ r(d) & \longrightarrow & r(F(*)), \end{array}$$

the composite

$$F'(*) \longrightarrow r(d)(*) \xrightarrow{\epsilon_d} d$$

is an equivalence. But this is obvious since pullbacks in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are computed objectwise, so we have a pullback square

$$\begin{array}{ccc} F'(*) & \longrightarrow & F(*) \\ \downarrow & \lrcorner & \downarrow \\ d & \longrightarrow & F(*). \end{array}$$

Moreover, by [Hau17, Proposition 4.51] a morphism $\phi\colon F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is ev_*-cartesian if and only if the commutative square

$$\begin{array}{ccc} F & \xrightarrow{\phi} & G \\ {\scriptstyle \eta_F}\downarrow & & \downarrow{\scriptstyle \eta_G} \\ rF(*) & \xrightarrow[r(\phi_*)]{} & rG(*) \end{array}$$

is cartesian, i.e. if and only if for every $x \in \mathcal{C}$ the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\phi_x} & G(x) \\ \downarrow & & \downarrow \\ F(*) & \xrightarrow[\phi_*]{} & G(*) \end{array}$$

is cartesian, which is equivalent to ϕ being cartesian by Remark 2.1.4. □

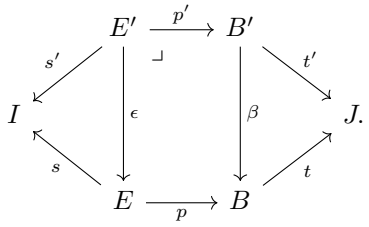
Definition 2.3.4. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{cart}}$ denote the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ containing only the cartesian natural transformations.

Remark 2.3.5. By Lemma 2.3.3 the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{cart}}$ is precisely the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ containing only the cartesian morphisms for the cartesian fibration ev_*. The restriction of this to functor $\mathrm{ev}_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{cart}} \to \mathcal{D}$ is hence a right fibration.

Lemma 2.3.6. *Suppose $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is a polynomial functor, represented by a diagram*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J.$$

If $\phi\colon F' \to F$ is a cartesian natural transformation, then F' is also a polynomial functor, and ϕ is equivalent to the natural transformation associated to a diagram



Proof. Let $(t'\colon B' \to J) := F'(\mathrm{id}_I)$, then the map ϕ_{id_I} gives a map $\beta\colon B' \to B$ over J. We can then define ϵ as the pullback of β along p and put $s' := s \circ \epsilon$ to get a diagram of this form. The construction of Definition 2.3.2 then gives a natural transformation $\phi'\colon F'' \to F$. Thus we have two cartesian natural transformations to F with the same image in $\mathcal{S}_{/J}$ under evaluation at the terminal object. Thus by Lemma 2.3.3, ϕ and ϕ' are both cartesian morphisms to F with the same image in $\mathcal{S}_{/J}$, and so they must be equivalent — in particular $F' \simeq F''$, which implies that F' is indeed polynomial. □

As a consequence, we get:

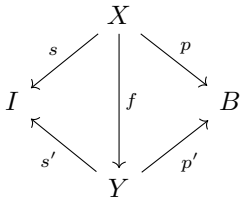
Lemma 2.3.7. *The projection $\mathrm{ev}_{\mathrm{id}_I}\colon \mathrm{PolyFun}(I,J) \to \mathcal{S}_{/J}$ is a right fibration.* □

It remains to understand the fibres of this fibration. Note that, for $\mathcal{C}$ and $\mathcal{D}$ ∞-categories where $\mathcal{C}$ has a terminal object $*$, the fibre of $\mathrm{ev}_*\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathcal{D}$ at $d \in \mathcal{D}$ is the ∞-category $\mathrm{Fun}_*(\mathcal{C},\mathcal{D}_{/d})$ of functors that preserve the terminal object. Restricting to polynomial functors from $\mathcal{S}_{/I}$ to $\mathcal{S}_{/J}$ (with all natural transformations allowed), the fibre at $t\colon B \to J$ is $\mathrm{Fun}^{\mathrm{R}}(\mathcal{S}_{/I},\mathcal{S}_{/B})$, since polynomial functors are local right adjoints by Theorem 2.2.3. This fibre can be described explicitly:

Lemma 2.3.8. *There is a natural equivalence*

$$(\mathcal{S}_{/I\times B})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{R}}(\mathcal{S}_{/I},\mathcal{S}_{/B}),$$

which sends a span $I \xleftarrow{s} X \xrightarrow{p} B$ to the functor p_*s^*, and sends a map of spans



to the natural transformation

$$p'_*s'^* \to p'_*f_*f^*s'^* \simeq p_*s^*$$

induced by the unit of the adjunction $f^* \dashv f_*$.

Proof. This is a reformulation of Lemma 2.2.11, using the natural equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{S}_{/I},\mathcal{S}_{/B}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}_{/B},\mathcal{S}_{/I})^{\mathrm{op}}.$$ □

Remark 2.3.9. Restricting to cartesian natural transformations, we see that the fibre of

$$\mathrm{ev}_{\mathrm{id}_I}\colon \mathrm{PolyFun}(I,J) \to \mathcal{S}_{/J}$$

at $B \to J$ is equivalent to the core ∞-groupoid of $\mathcal{S}_{/I\times B}$.

2.4. **The ∞-Category of Polynomial Functors.** We now wish to construct an ∞-category of all polynomial functors, i.e. to put the ∞-categories $\mathrm{PolyFun}(I,J)$ for varying I and J together into a single ∞-category PolyFun, fibred over $\mathcal{S}\times\mathcal{S}$ (by returning I and J). We will define this using a double ∞-category of colax squares of ∞-categories, constructed in §A.2. We then define a functor from polynomials to polynomial functors with varying source and target, and prove that this is an equivalence.

Definition 2.4.1. In §A.2 we define a double ∞-category $\mathrm{Sq}^{\mathrm{colax}}(\widehat{\mathrm{CAT}}_\infty)^{\mathrm{v=radj}}$ where

- the objects are ∞-categories,
- the vertical morphisms are right adjoints,
- the horizontal morphisms are arbitrary functors,
- the squares (or 2-cells) are colax squares, i.e. diagrams in the $(\infty,2)$-category of ∞-categories of shape

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet. \end{array}$$

We can pull $\mathrm{Sq}^{\mathrm{colax}}(\widehat{\mathrm{CAT}}_\infty)^{\mathrm{v=radj,v\text{-}op}}$ back along the functor $\mathcal{S}^*_{/-}\colon \mathcal{S}^{\mathrm{op}}\to \widehat{\mathrm{Cat}}_\infty^{\mathrm{radj}}$ taking a space I to $\mathcal{S}_{/I}$ and a map $f\colon I\to J$ to $f^*\colon \mathcal{S}_{/J}\to\mathcal{S}_{/I}$, to obtain a double ∞-category where

- the objects are spaces
- the vertical morphisms are maps of spaces,
- the horizontal morphisms are arbitrary functors between slices,
- the squares (or 2-cells) are colax squares using the $(-)^*$-functors for maps of spaces.

We define the double ∞-category POLYFUN to be the sub-double ∞-category of this pullback where the horizontal morphisms are polynomial functors. Thus POLYFUN has

(1) spaces as objects,
(2) maps of spaces as vertical morphisms,
(3) polynomial functors as horizontal morphisms,
(4) diagrams of the form

$$\begin{array}{ccc} \mathcal{S}_{/I} & \xrightarrow{P} & \mathcal{S}_{/J} \\ {\scriptstyle f^*}\uparrow & \searrow & \uparrow{\scriptstyle g^*} \\ \mathcal{S}_{/I'} & \xrightarrow[Q]{} & \mathcal{S}_{/J'} \end{array}$$

as squares.

Remark 2.4.2. Taking mates in the vertical direction should give an equivalence of double ∞-categories

$$\mathrm{Sq}^{\mathrm{colax}}(\widehat{\mathrm{CAT}}_\infty)^{\mathrm{v=radj,v\text{-}op}}\xrightarrow{\sim}\mathrm{Sq}^{\mathrm{lax}}(\widehat{\mathrm{CAT}}_\infty)^{\mathrm{v=ladj}}.$$

Assuming this, our definition of POLYFUN is equivalent to the alternative, and perhaps more standard, definition where the vertical morphisms are the left adjoint functors $f_!\colon \mathcal{S}_{/I}\to\mathcal{S}_{/J}$. We have chosen our convention to match with the correct convention for polynomial monads, where we really do want lax transformations (cf. Remark A.5.8) with direction reversed (which are not the same as colax morphisms of monads) — we thereby avoid unnecessarily using the above-mentioned equivalence of lax and colax squares via mates, which we do not prove here.

Proposition 2.4.3. *The double ∞-category* POLYFUN *is framed, in the sense of Definition A.4.3.*

Proof. In Proposition A.4.5 we prove that the double ∞-category $\mathrm{Sq}^{\mathrm{colax}}(\widehat{\mathrm{CAT}}_\infty)^{\mathrm{v=radj}}$ is framed. But for a vertical morphism of the form f^* the four squares of the framing live in the sub-double ∞-category POLYFUN since the unit and counit transformations for $f_!\dashv f^*$ are cartesian. Thus POLYFUN is also framed. □

Definition 2.4.4. We write PolyFun for the ∞-category of horizontal morphisms in POLYFUN, i.e. $\mathrm{POLYFUN}_{[1]}$ if we view POLYFUN as a cocartesian fibration over $\mathbf{\Delta}^{\mathrm{op}}$.

Applying Proposition A.4.4, we get:

Corollary 2.4.5. *The source-and-target projection* $\mathrm{PolyFun} \to \mathcal{S} \times \mathcal{S}$ *is both cartesian and cocartesian.* □

Remark 2.4.6. For morphisms $f\colon I \to I'$, $g\colon J \to J'$, the cocartesian pushforward of $F \in \mathrm{PolyFun}(I, J)$ along (f, g) is the composite $g_! F f^*$, while the cartesian pullback of $G \in \mathrm{PolyFun}(I', J')$ is $g^* F f_!$. Note that if F corresponds to the diagram

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$

then the pushforward $g_! F f^*$ corresponds to

$$I' \xleftarrow{fs} E \xrightarrow{p} B \xrightarrow{gt} J',$$

while the cartesian pullback is more complicated to describe diagrammatically.

Definition 2.4.7. Let Π denote the category $\bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$, which decomposes as a colimit $(\Delta^1)^{\mathrm{op}} \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \Delta^1$ in Cat_∞. We define Poly to be the subcategory of $\mathrm{Fun}(\Pi, \mathcal{S})$ containing only those morphisms where the middle commuting square is cartesian. In other words, we have a natural equivalence

$$\mathrm{Poly} \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S}).$$

We now define a functor $\Phi\colon \mathrm{Poly} \to \mathrm{PolyFun}$; we do this by defining three functors to PolyFun and then combining them using the horizontal composition in POLYFUN.

Definition 2.4.8. Let $\mathcal{S}^*_{/\_}$ denote the functor $\mathcal{S}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ taking $I \in \mathcal{S}$ to $\mathcal{S}_{/I}$ and $f\colon I \to J$ to $f^*\colon \mathcal{S}_{/J} \to \mathcal{S}_{/I}$. This induces a functor

$$\Phi_1\colon \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Map}(\Delta^\bullet \times (\Delta^1)^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Sq}_{\bullet,1}(\widehat{\mathrm{Cat}}_\infty)^{\text{v=radj,v-op}} \to \mathrm{Sq}^{\mathrm{colax}}(\widehat{\mathrm{CAT}}_\infty)^{\text{v=radj,v-op}},$$

which clearly passes through PolyFun.

Combining $\mathcal{S}^*_{/\_}$ instead with the functor

$$\mathrm{Sq}(\mathrm{Cat}_\infty)^{\text{h=ladj}} \to \mathrm{Sq}^{\mathrm{lax}}(\mathrm{CAT}_\infty)^{\text{h=radj,h-op}}$$

of Proposition A.3.1, which takes mates in the horizontal direction (replacing f^* with f_*), we get a functor

$$\mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathrm{Sq}^{\mathrm{lax}}(\mathrm{CAT}_\infty)_1^{\text{h=radj}}.$$

Restricting to $\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}$, this actually lands in commuting squares, giving

$$\Phi_2\colon \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \to \mathrm{Sq}(\mathrm{CAT}_\infty)_1^{\text{h=radj}},$$

which factors through PolyFun.

Finally, combining $\mathcal{S}^*_{/\_}$ with the functor

$$\mathrm{Sq}(\mathrm{Cat}_\infty)^{\text{h=radj}} \to \mathrm{Sq}^{\mathrm{colax}}(\mathrm{CAT}_\infty)^{\text{h=ladj,h-op}},$$

of Proposition A.3.1, which also takes mates in the horizontal direction (replacing f^* with $f_!$), we get a functor

$$\Phi_3\colon \mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathrm{Sq}^{\mathrm{colax}}(\mathrm{CAT}_\infty)_1^{\text{h=ladj}}.$$

This again factors through PolyFun.

Remark 2.4.9. More explicitly, Φ_1 is given by

$$\begin{array}{ccc} J & \xleftarrow{a} & I \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xleftarrow[b]{} & K \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{S}_{/J} & \xrightarrow{a^*} & \mathcal{S}_{/I} \\ {\scriptstyle g^*}\uparrow & \swarrow\!\!\sim & \uparrow{\scriptstyle f^*} \\ \mathcal{S}_{/L} & \xrightarrow[b^*]{} & \mathcal{S}_{/K}. \end{array}$$

Similarly, the functor Φ_2 is given by

$$\begin{array}{ccc} I & \xrightarrow{a} & J \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ K & \xrightarrow[b]{} & L \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{S}_{/I} & \xrightarrow{a_*} & \mathcal{S}_{/J} \\ {\scriptstyle f^*}\uparrow & \nwarrow\!\!\sim & \uparrow{\scriptstyle g^*} \\ \mathcal{S}_{/K} & \xrightarrow[b_*]{} & \mathcal{S}_{/L}. \end{array}$$

Finally, Φ_3 is given by

$$\begin{array}{ccc} I & \xrightarrow{a} & J \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[b]{} & L \end{array} \quad \mapsto \quad \begin{array}{ccc} \mathcal{S}_{/I} & \xrightarrow{a_!} & \mathcal{S}_{/J} \\ {\scriptstyle f^*}\uparrow & \searrow & \uparrow{\scriptstyle g^*} \\ \mathcal{S}_{/K} & \xrightarrow[b_!]{} & \mathcal{S}_{/L}. \end{array}$$

Definition 2.4.10. The functors Φ_i agree appropriately under restriction to $\mathcal{S}$ to determine a functor

$$\mathrm{Poly} \xrightarrow{\sim} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathrm{PolyFun} \times_{\mathcal{S}} \mathrm{PolyFun} \times_{\mathcal{S}} \mathrm{PolyFun}.$$

Combining this with the horizontal composition in POLYFUN,

$$\mathrm{PolyFun} \times_{\mathcal{S}} \mathrm{PolyFun} \times_{\mathcal{S}} \mathrm{PolyFun} \xleftarrow{\sim} \mathrm{POLYFUN}_{\bullet,3} \to \mathrm{PolyFun},$$

we get the required functor $\Phi\colon \mathrm{Poly} \to \mathrm{PolyFun}$.

Theorem 2.4.11. *The functor* $\Phi\colon \mathrm{Poly} \to \mathrm{PolyFun}$ *is an equivalence.*

The following lemma shows that it is enough to prove this fibrewise:

Lemma 2.4.12. *The projection* $\mathrm{ev}_{0,3}\colon \mathrm{Poly} \to \mathcal{S} \times \mathcal{S}$ *is a cocartesian fibration, and* $\Phi\colon \mathrm{Poly} \to \mathrm{PolyFun}$ *preserves cocartesian morphisms.*

Proof. For a polynomial P given by $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$ and a morphism $(f\colon I \to I', g\colon J \to J')$ out of $\mathrm{ev}_{0,3}(P)$, a cocartesian lift is given by

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & J \\ {\scriptstyle f}\downarrow & & \| & \lrcorner & \| & & \downarrow{\scriptstyle g} \\ I' & \xleftarrow[fs]{} & E & \xrightarrow[p]{} & B & \xrightarrow[gt]{} & J'; \end{array}$$

the cocartesian property is readily checked. Such a diagram is sent by Φ to the colax square

$$\begin{array}{ccc} \mathcal{S}_{/I} & \xrightarrow{P} & \mathcal{S}_{/J} \\ {\scriptstyle f^*}\uparrow & \searrow & \uparrow{\scriptstyle g^*} \\ \mathcal{S}_{/I'} & \xrightarrow[g_! P f^*]{} & \mathcal{S}_{/J'} \end{array}$$

with the natural transformation given by the unit transformation $Pf^* \to g^* g_! P f^*$. But this is precisely the form of the cocartesian edges in PolyFun, as noted in Remark 2.4.6. □

Proposition 2.4.13. *For fixed spaces* I *and* J*, the functor* Φ *gives an equivalence*

$$\mathrm{Poly}(I, J) \to \mathrm{PolyFun}(I, J)$$

when restricted to the fibre over (I, J)*.*

Proof. Both sides are right fibrations over $\mathcal{S}_{/J}$, so it suffices to show that we get an equivalence on fibres over every $B \to J$ in $\mathcal{S}_{/J}$. The fibre of $\mathrm{Poly}(I,J)$ is the ∞-groupoid of spans $I \leftarrow M \to B$, and the fibre of $\mathrm{PolyFun}(I,J)$ is the ∞-groupoid $\mathrm{Map}^{\mathrm{R}}(\mathcal{S}_{/I}, \mathcal{S}_{/B})$. The functor restricts precisely to the functor in Lemma 2.3.8, shown there to be an equivalence. □

Proof of Theorem 2.4.11. By Lemma 2.4.12 the functor Φ is a map between cocartesian fibrations and preserves cocartesian edges. It therefore suffices to show that the induced map on fibres $\mathrm{Poly}(I,J) \to \mathrm{PolyFun}(I,J)$ is an equivalence, which is Proposition 2.4.13. □

2.5. **Colimits of Polynomial Functors.** In this subsection we will give two descriptions of colimits of polynomial functors: First, we will see that colimits in Poly can be computed in $\mathrm{Fun}(\Pi, \mathcal{S})$, i.e. pointwise in the diagram. We will also show that colimits in $\mathrm{PolyFun}(I,J)$ can be computed in $\mathrm{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$.

Proposition 2.5.1. *Let $\mathcal{C}$ be a small ∞-category and let $\mathcal{X}$ be an ∞-topos. The forgetful functor $\mathrm{Fun}(\mathcal{C},\mathcal{X})^{\mathrm{cart}} \to \mathrm{Fun}(\mathcal{C},\mathcal{X})$ preserves and reflects all limits and colimits.*

Proof. We first consider the case of colimits. Given a diagram $\phi\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{C},\mathcal{X})^{\mathrm{cart}}$, let $\bar{\phi}\colon \mathcal{I}^{\triangleright} \to \mathrm{Fun}(\mathcal{C},\mathcal{X})$ be a colimit diagram extending the image of ϕ in $\mathrm{Fun}(\mathcal{C},\mathcal{X})$. We claim that this colimiting cocone in $\mathrm{Fun}(\mathcal{C},\mathcal{X})$ is also a colimiting cocone in the subcategory $\mathrm{Fun}(\mathcal{C},\mathcal{X})^{\mathrm{cart}}$.

To show this we must first prove that the commutative squares

$$\begin{array}{ccc} \phi(i)(c) & \longrightarrow & \phi(i)(c') \\ \downarrow & & \downarrow \\ \bar{\phi}(\infty)(c) & \longrightarrow & \bar{\phi}(\infty)(c') \end{array}$$

are cartesian, for all maps $c \to c'$ in $\mathcal{C}$. Since colimits in functor ∞-categories are computed objectwise, this is true by descent for the ∞-topos $\mathcal{X}$, using [Lur09, Theorem 6.1.3.9(4)].

Second, we must check that for any cocone $\phi'\colon \mathcal{I}^{\triangleright} \to \mathrm{Fun}(\mathcal{C},\mathcal{X})^{\mathrm{cart}}$, the canonical map $\bar{\phi} \to \phi'$ in $\mathrm{Fun}(\mathcal{C},\mathcal{X})$ actually belongs to $\mathrm{Fun}(\mathcal{C},\mathcal{X})^{\mathrm{cart}}$, i.e. it is a cartesian transformation. Since the transformations $\phi(i) \to \phi'(\infty)$ are cartesian, we have pullback squares

$$\begin{array}{ccc} \phi(i)(c) & \longrightarrow & \phi(i)(c') \\ \downarrow & \lrcorner & \downarrow \\ \phi'(\infty)(c) & \longrightarrow & \phi'(\infty)(c'). \end{array}$$

Colimits in $\mathcal{X}$ are universal, so this induces a pullback square of colimits

$$\begin{array}{ccc} \bar{\phi}(\infty)(c) & \longrightarrow & \bar{\phi}(\infty)(c') \\ \downarrow & \lrcorner & \downarrow \\ \phi'(\infty)(c) & \longrightarrow & \phi'(\infty)(c'). \end{array}$$

as required.

The proof for limits is the same, but simpler, using that limits commute and pullbacks preserve limits, which is true in any ∞-category. □

Corollary 2.5.2. *Colimits in* Poly *are constructed in* $\mathrm{Fun}(\Pi, \mathcal{S})$*, and colimits in* $\mathrm{Poly}(I,J)$ *are constructed in* $\mathrm{Fun}(\Delta^1, \mathcal{S})$ *for all spaces I, J. In particular, the ∞-categories* Poly *and* $\mathrm{Poly}(I,J)$ *are cocomplete.*

Proof. By definition, the ∞-category Poly is the fibre product $\mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})$. The projections to $\mathcal{S}$ all preserve colimits (using Proposition 2.5.1 for the middle term), so by [Lur09, Lemma 5.4.5.5] a diagram in Poly is a colimit if and only if its composition with the projections to the terms in this fibre product are colimits. Now Proposition 2.5.1 implies that colimits are computed in

$$\mathrm{Fun}(\Delta^{1,\mathrm{op}}, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S}) \simeq \mathrm{Fun}(\Pi, \mathcal{S}).$$

By the same argument, a diagram in

$$\mathrm{Poly}(I, J) \simeq \mathcal{S}_{/I} \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \times_{\mathcal{S}} \mathcal{S}_{/J}$$

is a colimit if and only if its images in $\mathcal{S}_{/I}$, $\mathrm{Fun}(\Delta^1, \mathcal{S})$, and $\mathcal{S}_{/J}$ are colimits. But colimits in these over-categories are computed in $\mathcal{S}$, so a diagram in $\mathrm{Poly}(I, J)$ is a colimit if and only if its image in $\mathrm{Fun}(\Delta^1, \mathcal{S})$ is a colimit.

Since $\mathrm{Fun}(\Pi, \mathcal{S})$ and $\mathrm{Fun}(\Delta^1, \mathcal{S})$ are cocomplete, it follows that so are the ∞-categories Poly and $\mathrm{Poly}(I, J)$. □

Remark 2.5.3. Note that the corresponding result does not hold in the classical 1-categorical setting of [GK13] (such as in Set), since a 1-topos does not have descent in general. In the 1-categorical setting, only colimits of diagrams of monomorphisms can be computed pointwise, as exemplified by grafting of trees [Koc11], as will be important below (cf. Remark 3.3.4).

Proposition 2.5.4. *The forgetful functor* $\mathrm{PolyFun}(I, J) \to \mathrm{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$ *preserves colimits. In particular, the colimit of a diagram of polynomial functors and cartesian transformations is again a polynomial functor.*

Proof. Consider a diagram $\phi\colon \mathcal{I} \to \mathrm{PolyFun}(I, J)$, where the functor ϕ_x corresponds to the diagram

$$I \xleftarrow{s_x} E(x) \xrightarrow{p_x} B(x) \xrightarrow{t_x} J.$$

By Corollary 2.5.2 and Theorem 2.4.11 the colimit of ϕ in $\mathrm{PolyFun}(I, J)$ corresponds to the diagram

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$

where $E := \mathrm{colim}_{x \in \mathcal{I}} E(x)$ and $B := \mathrm{colim}_{x \in \mathcal{I}} B(x)$. On the other hand, since colimits in functor ∞-categories are computed pointwise, the colimit of the diagram in $\mathrm{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$ is the functor

$$\phi\colon (K \to I) \mapsto \operatorname*{colim}_{x \in \mathcal{I}} \phi_x(K \to I).$$

Let us view ϕ_x as a functor $\mathrm{Fun}(I, \mathcal{S}) \to \mathrm{Fun}(J, \mathcal{S})$; then evaluating at $\alpha\colon I \to \mathcal{S}$ and $j \in J$ we have

$$\phi_x(\alpha)(j) \simeq \operatorname*{colim}_{b \in B(x)_j} \lim_{e \in E(x)_b} \alpha(s_x e).$$

Let $\mathcal{B} \to \mathcal{I}$ be the left fibration corresponding to the functor $B(-)$. This has a map to J, and the fibre $\mathcal{B}_j \to \mathcal{I}$ is the left fibration for the functor $B(-)_j$. Since iterated colimits are colimits over cocartesian fibrations, we get

$$\phi(\alpha)(j) \simeq \operatorname*{colim}_{(b,x) \in \mathcal{B}_j} \lim_{e \in E(x)_b} \alpha(s_x e).$$

Now we observe that the functor $(b, x) \mapsto \lim_{e \in E(x)_b} \alpha(s_x e)$ takes every morphism in $\mathcal{B}_j$ to an equivalence of spaces: Since $\mathcal{B}_j \to \mathcal{I}$ is a left fibration it suffices to consider morphisms of the form $(b, x) \to (B(f)b, x')$ over $f\colon x \to x'$ in $\mathcal{I}$. Then as $\phi(f)$ is a cartesian natural transformation the map $E(f)_b\colon E(x)_b \to E(x')_b$ is an equivalence and we have

$$\lim_{e \in E(x)_b} \alpha(s_x e) \simeq \lim_{e \in E(x)_b} \alpha(s_{x'} E(f) e) \simeq \lim_{e' \in E(x')_b} \alpha(s_{x'} e').$$

Thus this functor from $\mathcal{B}_j$ factors through the space obtained by inverting all morphisms in $\mathcal{B}_j$. This space is precisely $B_j \simeq \mathrm{colim}_{x \in \mathcal{I}} B(x)_j$ by [Lur09, Corollary 3.3.4.6]. Since $\mathcal{B}_j \to B_j$ is cofinal

by [Lur09, Corollary 4.1.2.6], this means we can replace the colimit over $\mathcal{B}_j$ by a colimit over the space B_j. Moreover, since we have pullbacks

$$\begin{array}{ccc} E(x) & \xrightarrow{\epsilon_x} & E \\ {\scriptstyle p_x}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B(x) & \xrightarrow[\beta_x]{} & B \end{array}$$

by [Lur09, Theorem 6.1.3.9], for $b \in B(x)_j$ we can identify $\lim_{e \in E(x)_b} \alpha(s_x e)$ with $\lim_{e \in E_b} \alpha(se)$. Thus we have produced a natural equivalence

$$\phi(x) \simeq \operatorname*{colim}_{b \in B_j} \lim_{e \in E_b} \alpha(se),$$

where the right-hand side is the formula for the polynomial functor corresponding to the diagram

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$

as required. □

2.6. **Slices over Polynomial Functors.** In this subsection we consider slices of PolyFun, i.e. overcategories $\mathrm{PolyFun}_{/P}$. We will show that these ∞-categories are very well-behaved; specifically, we will prove:

Theorem 2.6.1. *For any polynomial functor P, the slice ∞-category $\mathrm{PolyFun}_{/P}$ is an ∞-topos; in particular, this ∞-category is presentable. Furthermore, the full inclusion*

$$\mathrm{PolyFun}_{/P} \simeq \mathrm{Poly}_{/P} \to \mathrm{Fun}(\Pi, \mathcal{S})_{/P}$$

preserves all limits and colimits; it is thus the inverse-image part of a geometric morphism.

Remark 2.6.2. This theorem is also true in the 1-categorical case of Set, although we are not aware of a reference. This is a consequence of the observation that the maps in Poly form a class of standard étale maps in $\mathcal{P}(\Pi)$, in the axiomatic sense of Joyal–Moerdijk [JM94]. The result now follows from their Corollary 2.3.

Lemma 2.6.3. *For any morphism $p\colon E \to B$ in $\mathcal{S}$, the functor*

$$\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_{/p} \to \mathcal{S}_{/B}$$

is an equivalence.

Proof. The 2-of-3 property for pullback squares implies that $\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_{/p}$ can be identified with the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{S})_{/p}$ spanned by cartesian squares. We can thus identify the map to $\mathcal{S}_{/B}$ with a pullback of the forgetful functor from the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S})$ spanned by cartesian squares to the ∞-category of functors from $\Delta^1 \times \Delta^1 \setminus \{(0,0)\}$ to spaces. The latter is an equivalence by [Lur09, Proposition 4.3.2.15], since it is the forgetful functor from squares that are right Kan extended from $\Delta^1 \times \Delta^1 \setminus \{(0,0)\}$. □

The main point of the proof of the theorem is the following general lemma.

Lemma 2.6.4. *For any map of spaces $p\colon E \to B$, the full inclusion*

$$j_p \colon \mathcal{S}_{/B} \simeq \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_{/p} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{S})_{/p}$$

has both a left and a right adjoint. The left adjoint of j_p takes a square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle x}\downarrow & \alpha & \downarrow{\scriptstyle y} \\ E & \xrightarrow[p]{} & B \end{array}$$

to $y \in \mathcal{S}_{/B}$ and the right adjoint of j_p takes it to $p_* x \times_{p_* p^* y} y$.

Proof. The functor $\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathcal{S}$ has as right adjoint the constant diagram functor (which is also given by right Kan extension). By [Lur09, Proposition 5.2.5.1] this induces for $p \colon E \to B$ an adjunction on slice categories

$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{S})_{/p} \rightleftarrows \mathcal{S}_{/B} : g_p,$$

where the right adjoint g_p takes $y \colon Y \to B$ to the pullback

$$\begin{array}{ccc} E \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle y} \\ E & \xrightarrow[p]{} & B \end{array}$$

induced by the unit map. We can thus identify the right adjoint g_p with j_p, so j_p has a left adjoint with the stated description.

It follows from Proposition 2.5.1 that j_p preserves limits and colimits, so since $\mathcal{S}_{/B}$ and $\mathrm{Fun}(\Delta^1, \mathcal{S})_{/p}$ are presentable the adjoint functor theorem implies that j_p also has a right adjoint. To show the right adjoint has the claimed description, for f in $\mathcal{S}_{/B}$ and a square α as above, we must establish the equivalence

$$\mathrm{Map}_{/B}(f, p_* x \times_{p_* p^* y} y) \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{S})_{/p}}(j_p f, \alpha).$$

But the mapping space on the right is the pullback

$$\mathrm{Map}_{/E}(p^* f, x) \underset{\mathrm{Map}_{/B}(p_! p^* f, y)}{\times} \mathrm{Map}_{/B}(f, y)$$

which is naturally equivalent to

$$\mathrm{Map}_{/B}(f, p_* x) \underset{\mathrm{Map}_{/B}(f, p_* p^* y)}{\times} \mathrm{Map}_{/B}(f, y) \simeq \mathrm{Map}_{/B}(f, p_* x \times_{p_* p^* y} y)$$

as required. □

Proof of Theorem 2.6.1. Suppose P is represented by $I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$. By Lemma 2.6.3 we have an equivalence

$$\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_{/p} \simeq \mathcal{S}_{/B}.$$

Using this equivalence, we have (via Theorem 2.4.11 and Definition 2.4.7):

$$\begin{aligned} \mathrm{PolyFun}_{/P} \simeq \mathrm{Poly}_{/P} &\simeq \left(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S}) \underset{\mathcal{S}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}} \underset{\mathcal{S}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{S}) \right)_{/(s,p,t)} \\ &\simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S})_{/s} \underset{\mathcal{S}_{/E}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_{/p} \underset{\mathcal{S}_{/B}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{S})_{/t} \\ &\simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S})_{/s} \underset{\mathcal{S}_{/E}}{\times} \mathcal{S}_{/B} \underset{\mathcal{S}_{/B}}{\times} \mathrm{Fun}(\Delta^1, \mathcal{S})_{/t}. \end{aligned}$$

This is a double pullback of ∞-categories which are ∞-topoi, and the functors involved in the pullbacks are left exact left adjoints. Hence the result is again an ∞-topos by [Lur09, Proposition 6.3.2.2]. In detail, the four functors involved are

$$\begin{array}{ccccc} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{S})_{/s} & & \mathcal{S}_{/B} & & \mathrm{Fun}(\Delta^1, \mathcal{S})_{/t} \\ & {\scriptstyle f_1} \searrow \quad \swarrow {\scriptstyle p^*} & & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle f_4} & \\ & \mathcal{S}_{/E} & & \mathcal{S}_{/B} & \end{array}$$

where f_1 and f_4 are slices of restriction functors along appropriate $\Delta^0 \to \Delta^1$, hence have both adjoints by [Lur09, Proposition 5.2.5.1] and its dual. (The pullbacks are pullbacks in $\widehat{\mathrm{Cat}}_\infty$, or equivalently, pushouts in the ∞-category of ∞-topoi and geometric morphisms, cf. [Lur09, 6.3.1.5].)

The functor $\mathrm{Poly}_{/P} \to \mathrm{Fun}(\Pi, \mathcal{S})_{/P}$ is given by $\mathrm{id} \times_{\mathcal{S}_{/E}} j_p \times_{\mathcal{S}_{/B}} \mathrm{id}$, where $j_p : \mathcal{S}_{/B} \to \mathrm{Fun}(\Delta^1, \mathcal{S})_{/p}$ is from Lemma 2.6.4. We know from Proposition 2.5.1 that this functor preserves all limits and colimits. □

Corollary 2.6.5. *For a fixed polynomial functor P represented by $I \leftarrow E \rightarrow B \rightarrow J$, we have*

$$\mathrm{PolyFun}(I,J)_{/P} \simeq \mathcal{S}_{/B}.$$

In particular, the ∞-category $\mathrm{PolyFun}(I,J)_{/P}$ is an ∞-topos. □

Definition 2.6.6. We define the ∞-category PolyEnd of polynomial endofunctors by the pullback square

$$\begin{array}{ccc} \mathrm{PolyEnd} & \longrightarrow & \mathrm{PolyFun} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{S} \times \mathcal{S}. \end{array}$$

Since PolyFun is cocomplete and the projection to $\mathcal{S} \times \mathcal{S}$ preserves colimits by Corollary 2.5.2, it follows from [Lur09, Lemma 5.4.5.5] that PolyEnd is also cocomplete.

Proposition 2.6.7. *For a fixed polynomial endofunctor P, the ∞-category $\mathrm{PolyEnd}_{/P}$ is an ∞-topos.*

Proof. In the pullback diagram

$$\begin{array}{ccc} \mathrm{PolyEnd}_{/P} & \longrightarrow & \mathrm{PolyFun}_{/P} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_{0,3}} \\ \mathcal{S}_{/I} & \xrightarrow[\Delta]{} & \mathcal{S}_{/I} \times \mathcal{S}_{/I}, \end{array}$$

both Δ and $\mathrm{ev}_{0,3}$ are left exact left adjoints. The former because it is pullback along the codiagonal $I \amalg I \rightarrow I$, the latter because it is the composite $\mathrm{Poly}_{/P} \rightarrow \mathcal{P}(\Pi)_{/P} \rightarrow \mathcal{S}_{/I} \times \mathcal{S}_{/I}$ and here the first functor is a left exact left adjoint by Theorem 2.6.1 and the second is clear. Since the three ∞-categories are ∞-topoi, the pullback is again an ∞-topos by [Lur09, Proposition 6.3.2.2]. □

3. Analytic Functors

3.1. Analytic Functors and κ-Accessible Polynomial Functors.

Definition 3.1.1. A functor $\mathcal{S}_{/I} \rightarrow \mathcal{S}_{/J}$ is *analytic* if it preserves weakly contractible limits and sifted colimits.

Warning 3.1.2. This definition of analytic would not be correct if working over the category of sets instead of the category of spaces. See Remark 3.2.11 for further discussion of this subtle issue.

From this definition it is immediate (using Theorem 2.2.3) that an analytic functor is polynomial. We write AnFun for the full subcategory of PolyFun spanned by the analytic functors, and $\mathrm{AnFun}(I,J)$ for the corresponding subcategory of $\mathrm{PolyFun}(I,J)$. Similarly, we define the ∞-category AnEnd of analytic endofunctors as the pullback

$$\begin{array}{ccc} \mathrm{AnEnd} & \longrightarrow & \mathrm{AnFun} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{S} \times \mathcal{S}. \end{array}$$

We now wish to characterize the analytic functors (and also the κ-accessible polynomial functors) in terms of their representing diagrams.

Definition 3.1.3. Let $\mathcal{C}$ be a cocomplete ∞-category, and let κ be a regular cardinal. Recall that an object x is called *κ-compact* when $\mathrm{Map}_{\mathcal{C}}(x,-)\colon \mathcal{C} \rightarrow \mathcal{S}$ preserves κ-filtered colimits [Lur09, 5.3.4.5], and that it is called *projective* when $\mathrm{Map}_{\mathcal{C}}(x,-)$ preserves geometric realizations [Lur09, 5.5.8.18].

Remark 3.1.4. By [Lur09, Corollary 5.5.8.17] if $\mathcal{C}$ is cocomplete then a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves filtered colimits and geometric realizations if and only if it preserves sifted colimits. In particular, $x \in \mathcal{C}$ is compact and projective if and only if $\mathrm{Map}_{\mathcal{C}}(x,-)\colon \mathcal{C} \to \mathcal{S}$ preserves sifted colimits. If $\mathcal{C}$ is not assumed to be cocomplete, we still say that an object x is *compact projective* if mapping out of it preserves sifted colimits.

Lemma 3.1.5. *Let $\mathcal{C}$ be an ∞-category with finite products. Then $f\colon x \to y$ is a projective or κ-compact object in $\mathcal{C}_{/y}$ if x is a projective or κ-compact object of $\mathcal{C}$. If $\mathcal{C}$ is cartesian closed, then the converse is also true.*

Proof. Consider a diagram $p\colon \mathcal{I} \to \mathcal{C}_{/y}$ that has a colimit. Since this colimit is preserved by the forgetful functor to $\mathcal{C}$, we have a commutative diagram

$$\begin{array}{ccccc} \operatorname{colim} \mathrm{Map}_{/y}(x,p) & \longrightarrow & \mathrm{Map}_{/y}(x, \operatorname{colim} p) & \longrightarrow & \{f\} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{colim} \mathrm{Map}(x,p) & \longrightarrow & \mathrm{Map}(x, \operatorname{colim} p) & \longrightarrow & \mathrm{Map}(x,y). \end{array}$$

Here the right square is clearly cartesian, and the composite square is cartesian since colimits are universal in $\mathcal{S}$. Therefore the left square is also cartesian, so if the lower left horizontal morphism is an equivalence, so is the top left horizontal morphism.

If $\mathcal{C}$ is cartesian closed, then $\mathrm{Map}_{\mathcal{C}}(x, \operatorname{colim} p) \simeq \mathrm{Map}_{/y}(x, y \times \operatorname{colim} p) \simeq \mathrm{Map}_{/y}(x, \operatorname{colim} y \times p)$, which gives the converse. □

Proposition 3.1.6. *Consider an adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

If G and $\mathrm{Map}_{\mathcal{C}}(x,-)$ both preserve $\mathcal{I}$-shaped colimits, then also $\mathrm{Map}_{\mathcal{D}}(Fx,-)$ preserves $\mathcal{I}$-shaped colimits. Conversely, if equivalences in $\mathcal{C}$ are detected by mapping out of a collection of objects x such that the functors $\mathrm{Map}_{\mathcal{C}}(x,-)$ and $\mathrm{Map}_{\mathcal{D}}(Fx,-)$ both preserve $\mathcal{I}$-shaped colimits, then also G preserves $\mathcal{I}$-shaped colimits.

Proof. Consider a diagram $p\colon \mathcal{I} \to \mathcal{D}$ that has a colimit. If G and $\mathrm{Map}_{\mathcal{C}}(x,-)$ both preserve $\mathcal{I}$-shaped colimits, then we have

$$\mathrm{Map}_{\mathcal{D}}(Fx, \operatorname{colim} p) \simeq \mathrm{Map}_{\mathcal{C}}(x, \operatorname{colim} Gp) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(x, Gp) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}(Fx, p),$$

and so $\mathrm{Map}_{\mathcal{D}}(Fx,-)$ also preserves $\mathcal{I}$-shaped colimits. Conversely, for an object x in the collection we have equivalences

$$\mathrm{Map}_{\mathcal{C}}(x, G(\operatorname{colim} p)) \simeq \mathrm{Map}_{\mathcal{D}}(Fx, \operatorname{colim} p) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}(Fx, p) \simeq \mathrm{Map}_{\mathcal{C}}(x, \operatorname{colim} Gp),$$

and thus $\operatorname{colim} Gp \to G(\operatorname{colim} p)$ is an equivalence. □

Three special cases of this result are listed in the following corollary:

Corollary 3.1.7. *Consider an adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

(i) If G preserves κ-filtered colimits, then F preserves κ-compact objects. If equivalences in $\mathcal{C}$ are detected by mapping out of κ-compact objects, then the converse is true.
(ii) If G preserves geometric realizations, then F preserves projective objects. If equivalences in $\mathcal{C}$ are detected by mapping out of projective objects, then the converse is true.
(iii) If G preserves sifted colimits, then F preserves compact projective objects. If equivalences in $\mathcal{C}$ are detected by mapping out of compact projective objects, then the converse is true.

Lemma 3.1.8. *Consider a span of spaces $I \xleftarrow{f} X \xrightarrow{q} *$. The functor $q_* f^*$ preserves κ-filtered colimits if and only if X is κ-compact, and sifted colimits if and only if X is compact projective, i.e. is a finite set.*

Proof. Since equivalences in $\mathcal{S}$ are detected by maps out of $*$, and $\mathrm{Map}(*,-)$ preserves all colimits, Corollary 3.1.7 implies that q_*f^* preserves κ-filtered colimits if and only if $f_!q^*$ preserves κ-compact objects, and sifted colimits if and only if $f_!q^*$ preserves compact projective objects. By Lemma 3.1.5 this is equivalent to q^* preserving κ-compact or compact projective objects, respectively. Once again using that these are detected in $\mathcal{S}$, this is equivalent to $X\times Y$ being κ-compact or compact projective for all κ-compact or compact projective Y. Thus in particular (taking $Y=*$) X is κ-compact or compact projective, but this is enough since if X and Y are κ-compact or compact projective and $p\colon \mathcal{I}\to\mathcal{S}$ is a κ-filtered or sifted diagram, then

$$\mathrm{Map}(X\times Y,\operatorname{colim} p)\simeq \mathrm{Map}(X,\mathrm{Map}(Y,\operatorname{colim} p))\simeq \mathrm{Map}(X,\operatorname{colim}\mathrm{Map}(Y,p))\simeq \operatorname{colim}\mathrm{Map}(X\times Y,p),$$

so $X\times Y$ is also κ-compact or compact projective. □

Proposition 3.1.9. *Suppose $F\colon \mathcal{S}_{/I}\to\mathcal{S}_{/J}$ is a polynomial functor represented by a diagram*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J.$$

(i) F is κ-accessible if and only if the fibres of p are κ-compact spaces.
(ii) F is analytic if and only if the fibres of p are finite sets.

Proof. We first prove (i): Since $t_!$ preserves and reflects colimits, F is κ-accessible if and only if the functor p_*s^* preserves κ-filtered colimits. Using the equivalence $\mathcal{S}_{/B}\simeq\mathrm{Fun}(B,\mathcal{S})$, we see that p_*s^* preserves κ-filtered colimits if and only if the same holds for $b^*p_*s^*$ for every point $b\colon *\to B$. Consider the pullback square

$$\begin{array}{ccc} E_b & \xrightarrow{q} & \{b\} \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ E & \xrightarrow[p]{} & B. \end{array}$$

We have a Beck-Chevalley equivalence $b^*p_*s^*\simeq q_*(si)^*$. By Lemma 3.1.8 this preserves κ-filtered colimits if and only if E_b is κ-compact. The proof of (ii) is the same, using sifted colimits instead of κ-filtered colimits. □

Remark 3.1.10. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{F}$ a collection of morphisms in $\mathcal{X}$ that is stable under pullback. Then $\mathcal{F}$ determines a functor $\mathcal{X}^{\mathrm{op}}\to\widehat{\mathrm{Cat}}_\infty$ taking X to the full subcategory $\mathcal{X}^{\mathcal{F}}_{/X}\subseteq\mathcal{X}_{/X}$ spanned by the morphisms in $\mathcal{F}$, and given on morphisms by taking pullbacks. The class $\mathcal{F}$ is called *local* if this functor preserves limits (i.e. is a sheaf on $\mathcal{X}$); see also [Lur09, Lemma 6.1.3.7] for alternative characterizations. Following [GK17] we say that $\mathcal{F}$ is a *bounded local class* if in addition the ∞-categories $\mathcal{X}^{\mathcal{F}}_{/X}$ are all essentially small. By [Lur09, Proposition 6.1.6.3] these are exactly the classes of morphisms in $\mathcal{X}$ for which there exists a *classifier* $U'_{\mathcal{F}}\to U_{\mathcal{F}}$, meaning a terminal object in the subcategory $\mathcal{O}^{(\mathcal{F})}_{\mathcal{X}}$ of $\mathrm{Fun}(\Delta^1,\mathcal{X})$ with objects the morphisms in $\mathcal{F}$ and cartesian squares as morphisms.

Proposition 3.1.11. *Let $\mathcal{F}$ be a bounded local class of morphisms in $\mathcal{S}$, with classifier $U'_{\mathcal{F}}\to U_{\mathcal{F}}$, and let F be the polynomial functor represented by $*\leftarrow U'_{\mathcal{F}}\to U_{\mathcal{F}}\to *$. Then the forgetful functor*

$$\mathrm{PolyFun}_{/F}\to\mathrm{PolyFun}$$

is fully faithful, and its image is the full subcategory $\mathrm{PolyFun}_{\mathcal{F}}$ spanned by the polynomial functors with "middle map" in $\mathcal{F}$.

Proof. A morphism $P\to F$ in PolyFun is represented by a diagram

$$\begin{array}{ccccccc} I & \longleftarrow & E & \xrightarrow{p} & B & \longrightarrow & J \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ * & \longleftarrow & U'_{\mathcal{F}} & \longrightarrow & U_{\mathcal{F}} & \longrightarrow & *. \end{array}$$

Since $U_{\mathcal{F}}$ is the classifier for maps in the class $\mathcal{F}$, such a morphism exists if and only if p belongs to $\mathcal{F}$, and the morphism is unique if it exists. Thus the forgetful functor from $\mathrm{PolyFun}_{/F}$ to PolyFun is fully faithful, and its image is precisely the full subcategory $\mathrm{PolyFun}_{\mathcal{F}}$. □

Combining this with Theorem 2.6.1, we get:

Corollary 3.1.12. *Let $\mathcal{F}$ be a bounded local class in $\mathcal{S}$. Then the ∞-category $\mathrm{PolyFun}_{\mathcal{F}}$ is an ∞-topos.*

Specializing to κ-accessible and analytic functors, this gives:

Corollary 3.1.13. *Let κ be a regular cardinal.*

(i) Let $U'_\kappa \to U_\kappa$ be the classifying morphism for maps whose fibres are κ-compact spaces, and let P_κ be the polynomial functor represented by

$$* \leftarrow U'_\kappa \to U_\kappa \to *.$$

Then the ∞-category $\mathrm{PolyFun}_\kappa$ of κ-accessible polynomial functors is equivalent to $\mathrm{PolyFun}_{/P_\kappa}$. Moreover, $\mathrm{PolyFun}_\kappa$ is an ∞-topos.

(ii) Let $\mathbf{E}$ be the polynomial functor represented by

$$* \leftarrow \iota\mathrm{Fin}_* \to \iota\mathrm{Fin} \to *,$$

where the middle map is the classifier for morphisms with finite discrete fibres. Then AnFun is equivalent to $\mathrm{PolyFun}_{/\mathbf{E}}$. Moreover, the ∞-category AnFun is an ∞-topos.

Proof. By Proposition 3.1.9, the κ-accessible polynomial functors are those whose "middle map" belong to the bounded local class $\mathcal{F}_\kappa$ of maps with κ-compact fibres. This is equivalent to $\mathrm{PolyFun}_{/P_\kappa}$ by Proposition 3.1.11, and is an ∞-topos by Corollary 3.1.12. This proves (i), and (ii) follows similarly since analytic functors are characterized by having "middle map" in the bounded local class of maps with finite discrete fibres. □

Remark 3.1.14. Note that Corollary 3.1.13 does not have an analogue in ordinary category theory, because of the lack of classifiers.

Remark 3.1.15. The whole ∞-category PolyFun (without cardinal bounds on the middle representing maps) is cocomplete by Corollary 2.5.2, but it is not accessible, since neither is $\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}$. (In particular, PolyFun does not admit a terminal object.) In the κ-bounded case, the minimal generating set for $\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}_\kappa \simeq \mathcal{S}_{/U_\kappa}$ is the set of isomorphism classes of κ-compact spaces. Without the cardinal bound, $\mathrm{Fun}(\Delta^1, \mathcal{S})^{\mathrm{cart}}$ is the union of all these, and a generating set would have to exhaust $\bigcup_\kappa U_\kappa \simeq \iota\mathcal{S}$, which is too big to form a set.

3.2. **Analytic Endofunctors, Symmetric Sequences, and Homotopical Species.** In this subsection we will relate analytic endofunctors to (coloured) symmetric sequences and the homotopical analogue of Joyal's species.

We saw in Corollary 3.1.13 that the ∞-category AnFun of analytic functors is equivalent to the slice $\mathrm{PolyFun}_{/\mathbf{E}}$. Combining this with Corollary 2.6.5, we get:

Corollary 3.2.1. *We have*

$$\mathrm{AnEnd}(*) \simeq \mathcal{S}_{/\iota\mathrm{Fin}} \simeq \mathrm{Fun}(\iota\mathrm{Fin}, \mathcal{S}) \simeq \prod_{n=0}^{\infty} \mathrm{Fun}(B\Sigma_n, \mathcal{S}).$$

Remark 3.2.2. In the corollary, $\prod_{n=0}^{\infty} \mathrm{Fun}(B\Sigma_n, \mathcal{S})$ is the ∞-category of *symmetric sequences* in $\mathcal{S}$. The canonical monoidal structure on $\mathrm{AnEnd}(*)$ given by composition thus carries over to a monoidal structure on the ∞-category of symmetric sequences. Unravelling the formula for composition from Theorem 2.1.8, we see that this is an ∞-categorical version of the substitution product on symmetric sequences, introduced by Kelly [Kel05] to exhibit operads as monoids therein.

Definition 3.2.3. More generally, for a space I, we can consider *I-coloured symmetric sequences* (or *I-collections*): these are by definition presheaves on $\mathbf{E}(I) \times I$. (We shall see a tree interpretation later on in Definition 3.3.7.)

Proposition 3.2.4. *The ∞-category* $\mathrm{AnEnd}(I)$ *of analytic endofunctors of* $\mathcal{S}_{/I}$ *is equivalent to that of I-coloured symmetric sequences.*

Proof. Let $\mathbf{E}_I$ be the cartesian pullback (in the fibration $\mathrm{AnEnd} \to \mathcal{S}$) of $\mathbf{E}$ to an endofunctor of I, i.e. the pullback along (i, i) for $i\colon I \to *$; by Remark 2.4.6 this is the composite $i^*\mathbf{E}i_!$. Then $\mathrm{AnEnd}(I)$ is equivalent to $\mathrm{PolyEnd}(I)_{/\mathbf{E}_I}$. By Corollary 2.6.5 this means that $\mathrm{AnEnd}(I)$ is equivalent to $\mathcal{S}_{/\mathbf{E}_I(\mathrm{id}_I)}$. But here

$$\mathbf{E}_I(\mathrm{id}_I) \simeq i^*\mathbf{E}i_!(\mathrm{id}_I) \simeq i^*\mathbf{E}(I) \simeq I \times \mathbf{E}(I). \qquad \square$$

Lemma 3.2.5. *We have the following explicit formula for evaluation of* $\mathbf{E}$ *on a space X:*

$$\mathbf{E}(X) = \operatorname*{colim}_{k \in \iota\mathrm{Fin}} \mathrm{Map}(k, X) = \coprod_{k \in \mathbb{N}} X^{\times k}_{\mathrm{h}\Sigma_k}. \qquad \square$$

The relationship with $\mathbf{E}$ leads to a useful explicit formula for evaluation of analytic endofunctors:

Proposition 3.2.6. *Suppose P is an analytic endofunctor, represented by the diagram*

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\ {\scriptstyle \jmath}\downarrow & & {\scriptstyle \epsilon}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \jmath} \\ * & \xleftarrow[u]{} & \iota\mathrm{Fin}_* & \xrightarrow[q]{} & \iota\mathrm{Fin} & \longrightarrow & *. \end{array}$$

Then for every map $f\colon X \to I$ there is a natural pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{E}(X) \\ {\scriptstyle p_*s^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{E}(f)} \\ B & \xrightarrow[\bar{s}]{} & \mathbf{E}(I), \end{array}$$

where $\bar{s}\colon B \to q_*u^*I = \mathbf{E}(I)$ corresponds to $u_!q^*B = E \xrightarrow{s} I$ under the adjunction $u_!q^* \dashv q_*u^*$.

Proof. By Lemma 2.1.5 we have a cartesian natural transformation $p_*s^* \to p_*s^*\jmath^*\jmath_! \simeq p_*\epsilon^*u^*\jmath_!$, and using Lemma 2.1.6 we have a Beck-Chevalley equivalence that identifies this with a natural transformation $\eta\colon p_*s^* \to \beta^*q_*u^*\jmath_!$. Consider the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \beta^*q_*u^*X & \longrightarrow & q_*u^*X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \beta^*q_*u^*I & \longrightarrow & q_*u^*I \\ & & \downarrow & & \downarrow \\ & & B & \xrightarrow[\beta]{} & \iota\mathrm{Fin}. \end{array}$$

Here the bottom right square and the composite square in the right column are cartesian by definition of β^*, hence the top right square is also cartesian. The top left square is cartesian since η is a cartesian natural transformation, so the composite square in the top row is cartesian. $\square$

Corollary 3.2.7. *For $P\colon \mathcal{S} \to \mathcal{S}$ an analytic endofunctor as in Proposition 3.2.6, we have*

$$P(X) \simeq \coprod_{n \in \mathbb{N}} B_n \times_{\Sigma_n} X^{\times n}.$$

where B_n is the fibre of $B \to \iota\mathrm{Fin}$ at an n-element set.

Proof. We calculate, using Proposition 3.2.6 and Lemma 3.2.5:

$$P(X) \simeq B \times_{\iota\mathrm{Fin}} \mathbf{E}(X) \simeq B \times_{\iota\mathrm{Fin}} \coprod_{n\in\mathbb{N}} (X^{\times n})_{h\Sigma_n} \simeq \coprod_{n\in\mathbb{N}} (B_n \times X^{\times n})_{h\Sigma_n}. \qquad \square$$

Remark 3.2.8. The formula in Corollary 3.2.7 is the origin of the terminology "analytic": the spaces B_n are the coefficients of the "Taylor expansion" of P. Joyal [Joy86] introduced analytic functors as a categorical analogue of exponential generating functions of species, defining them as left Kan extensions of species (which are functors $\iota\mathrm{Fin} \to \mathrm{Set}$). He characterized analytic endofunctors of the category of sets as those endofunctors that preserve filtered colimits and weakly preserve wide pullbacks. In our approach we have *defined* analytic functors in terms of exactness properties, but can state an ∞-version of Joyal's theorem as follows:

Definition 3.2.9. We call a functor $F\colon \iota\mathrm{Fin} \to \mathcal{S}$ a *homotopical species*. By left Kan extension along the (non-full) inclusion $\iota\mathrm{Fin} \to \mathcal{S}$, it defines an endofunctor $F\colon \mathcal{S} \to \mathcal{S}$, described explicitly by the formula

$$F(X) \simeq \operatorname*{colim}_{n\in\iota\mathrm{Fin}} \operatorname*{colim}_{n\to X} F[n] \simeq \operatorname*{colim}_{n\in\iota\mathrm{Fin}} F[n] \times X^n \simeq \coprod_{n\in\mathbb{N}} (F[n] \times X^n)_{h\Sigma_n}.$$

On the other hand, by unstraightening, it corresponds to a map $B \to \iota\mathrm{Fin}$, and hence to an analytic functor (via Corollary 3.2.1). This analytic endofunctor is canonically identified with $F\colon \mathcal{S} \to \mathcal{S}$, by Corollary 3.2.7, giving:

Proposition 3.2.10 ("Joyal's theorem for homotopical species")**.** *An endofunctor $P\colon \mathcal{S} \to \mathcal{S}$ is analytic (i.e. preserves filtered colimits and weakly contractible limits) if and only if it is the left Kan extension of a "homotopical species" (i.e. a functor $F\colon \iota\mathrm{Fin} \to \mathcal{S}$).* $\square$

Remark 3.2.11. From the viewpoint of species, analytic functors over sets are actually not the optimal notion, since it is not true in general that the exponential generating function of a species agrees with the cardinality of its associated analytic functor. This *is* true over spaces (and in fact already over groupoids, as first observed by Baez and Dolan [BD01] who introduced groupoid-valued species under the name *stuff types*). What goes wrong in the set case is the behaviour of quotients of group actions, which is also responsible for the mere *weak* preservation of connected limits in Joyal's original theorem.

3.3. **Trees and Analytic Endofunctors.** In this subsection we will describe analytic endofunctors in terms of trees. This uses the interpretation of trees as polynomial endofunctors from [Koc11]:

Definition 3.3.1. A *tree* is by definition a polynomial

$$A \xleftarrow{s} M \xrightarrow{p} N \xrightarrow{t} A$$

for which:

(1) The spaces A, M, and N are all finite sets.
(2) The function t is injective.
(3) The function s is injective, with a unique element R (the *root*) in the complement of its image.
(4) Define a successor function $\sigma\colon A \to A$ as follows: First, set $\sigma(R) = R$. For $e \in s(M)$ (which is the complement of R in A), take e' in M with $s(e') = e$ and set $\sigma(e) = t(p(e'))$. Then for every e there exists some $k \in \mathbb{N}$ such that $\sigma^k(e) = R$.

Remark 3.3.2. The intuition behind this notion of "tree" is as follows: we think of A as the set of edges of the tree, N as the set of nodes (our trees do not have nodes at their leaves or root), and M as the set of pairs (v, e) where v is a node and e is an incoming edge of v. The function s is the projection $s(v, e) = e$, the function p is the projection $p(v, e) = v$, and the function t assigns to each node its unique outgoing edge.

Definition 3.3.3. The *elements* of a tree are its edges and nodes, and a tree can be constructed by gluing edges and nodes, as will be formalized below. Let η denote the tree

$$* \leftarrow \emptyset \rightarrow \emptyset \rightarrow *$$

consisting of an edge without nodes; it is called the trivial tree. For $n = 0, 1, \dots$ let C_n denote

$$n+1 \hookleftarrow n \rightarrow * \hookrightarrow n+1,$$

where the first and last morphisms are disjoint inclusions of 1 and n elements in $n+1$; it is the corolla (one-node tree) with n incoming edges. We refer to the trivial tree and the corollas as *elementary trees*.

We define $\mathbf{\Omega}_{\mathrm{el}}$ and $\mathbf{\Omega}_{\mathrm{int}}$ to be the full subcategories of AnEnd spanned by the elementary trees and all the trees, respectively.

Remark 3.3.4. Since trees correspond to diagrams of sets, $\mathbf{\Omega}_{\mathrm{el}}$ and $\mathbf{\Omega}_{\mathrm{int}}$ are ordinary categories, and they are equivalent to those considered by Kock [Koc11] (where they are denoted elTr and tEmb, respectively). It is a consequence of the tree axioms (see [Koc11, Proposition 1.1.3]) that the morphisms in $\mathbf{\Omega}_{\mathrm{int}}$ are *tree embeddings*, meaning injective on nodes and edges. The subscript "int" stands for *inert*; in §5.3 we will embed $\mathbf{\Omega}_{\mathrm{int}}$ into a bigger category of trees $\mathbf{\Omega}$, where the inert morphisms become the right class of an (active, inert) factorization system.

The category $\mathbf{\Omega}_{\mathrm{int}}$ admits certain pushouts (and colimits built from them), namely ones corresponding to grafting of trees: if $\eta \rightarrow S$ picks out the root and $\eta \rightarrow R$ picks out a leaf, then the pushout $S \amalg_\eta R$ calculated in AnEnd (where it exists since colimits in AnEnd can be calculated in $\mathrm{Fun}(\amalg, \mathcal{S})$) is again a tree T, in which R and S are naturally subtrees — T is "S grafted onto R". Hence the pushout is also a pushout in $\mathbf{\Omega}_{\mathrm{int}}$. Furthermore, since the spaces involved in the colimit are just sets and since the maps are injections, the colimit can actually be calculated in Set. The details can be found in [Koc11].

For a tree $T \in \mathbf{\Omega}_{\mathrm{int}}$, we write $\mathrm{el}(T) = \mathbf{\Omega}_{\mathrm{el}/T}$ for the category $\mathbf{\Omega}_{\mathrm{el}} \times_{\mathbf{\Omega}_{\mathrm{int}}} (\mathbf{\Omega}_{\mathrm{int}})_{/T}$, and call it the *category of elements* of T. (Seeing T as a presheaf on $\mathbf{\Omega}_{\mathrm{el}}$ given by $E \mapsto \mathrm{Map}_{\mathbf{\Omega}_{\mathrm{int}}}(E, T)$, this really is its category of elements.)

The grafting construction can readily be iterated to establish the following result, which is intuitively clear:

Lemma 3.3.5. *Every tree T is canonically the colimit in $\mathbf{\Omega}_{\mathrm{int}}$, and in* AnEnd*, of its elementary subtrees:*

$$T \simeq \operatorname*{colim}_{E \in \mathrm{el}(T)} E.$$

Proof. This is a reformulation of [Koc11, Corollary 1.1.24]. □

Lemma 3.3.6. *Given an analytic endofunctor P represented by a diagram*

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\ \downarrow & & \downarrow{\scriptstyle \epsilon} & \lrcorner & \downarrow{\scriptstyle \beta} & & \downarrow \\ * & \longleftarrow & \iota\mathrm{Fin}_* & \longrightarrow & \iota\mathrm{Fin} & \longrightarrow & *, \end{array}$$

there are natural equivalences

$$\mathrm{Map}(\eta, P) \simeq I, \qquad \mathrm{Map}(C_n, P) \simeq B_n,$$

where B_n is the fibre of β at an n-element set.

Proof. It is clear that a map $\eta \rightarrow P$ is uniquely determined by the map $* \rightarrow I$, so $\mathrm{Map}(\eta, P) \simeq I$. For C_n, observe that since $n+1$ is the disjoint union of the images of $*$ and n, the space of maps

$C_n \to P$ is equivalent to the space of cartesian squares

$$\begin{array}{ccc} n & \xrightarrow{u} & * \\ \downarrow & \lrcorner & \downarrow \\ E & \xrightarrow[p]{} & B. \end{array}$$

More formally, this space is described as the pullback

$$\begin{array}{ccc} \mathrm{Map}(C_n, P) \simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{cart}}(\Delta^1, \mathcal{S})}(u,p) & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ \iota\mathrm{Fin}_{n/} & \xrightarrow[\mathrm{codom}]{} & \iota\mathrm{Fin}. \end{array}$$

But $\iota\mathrm{Fin}_{n/}$ is contractible, so the pullback is B_n, as asserted. □

Definition 3.3.7. A *coloured collection* or *coloured symmetric sequence* is a presheaf on $\mathbf{\Omega}_{\mathrm{el}}$.

Remark 3.3.8. The intuition is that the inclusion $\{\eta\} \to \mathbf{\Omega}_{\mathrm{el}}$ defines a projection $\mathcal{P}(\mathbf{\Omega}_{\mathrm{el}}) \to \mathcal{S}$, which can be interpreted as assigning to a coloured collection its space of colours, and that the value of a presheaf on the corolla C_n is the space of n-ary operations of the coloured collection. The $n+1$ different maps of trees $\eta \to C_n$ then extract from an n-ary operation its n input colours and its output colour.

Definition 3.3.9. The inclusion $i\colon \mathbf{\Omega}_{\mathrm{el}} \to \mathrm{AnEnd}$ extends to a unique colimit-preserving functor $i_!\colon \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}}) \to \mathrm{AnEnd}$ with right adjoint $i^*\colon \mathrm{AnEnd} \to \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}})$ given by the restricted Yoneda functor, i.e.

$$P \mapsto \mathrm{Map}_{\mathrm{AnEnd}}(i(-), P).$$

Proposition 3.3.10. *The functor $i^*\colon \mathrm{AnEnd} \to \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}})$ is an equivalence.*

To prove this, we shall use the following general criterion.

Lemma 3.3.11. *Suppose $\mathcal{C}$ is a cocomplete and locally small ∞-category and $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is the inclusion of an essentially small full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that*

(i) the objects of $\mathcal{C}_0$ are completely compact*, i.e. for $C \in \mathcal{C}_0$ the functor $\mathrm{Map}_{\mathcal{C}}(C, -)$ preserves colimits,*

(ii) the functors $\mathrm{Map}_{\mathcal{C}}(C, -)$ for $C \in \mathcal{C}_0$ are jointly conservative*, i.e. if a map $f\colon X \to Y$ in $\mathcal{C}$ is such that $f_*\colon \mathrm{Map}_{\mathcal{C}}(C, X) \to \mathrm{Map}_{\mathcal{C}}(C, Y)$ is an equivalence for all $C \in \mathcal{C}_0$, then f is an equivalence.*

Then the adjunction

$$i_! : \mathcal{P}(\mathcal{C}_0) \rightleftarrows \mathcal{C} : i^*$$

is an adjoint equivalence.

Proof. The functor i^* preserves colimits since the objects of $\mathcal{C}_0$ are completely compact, and detects equivalences since they are jointly conservative. The composite $i^* i_!\colon \mathcal{P}(\mathcal{C}_0) \to \mathcal{P}(\mathcal{C}_0)$ is thus a colimit-preserving functor that restricts to the Yoneda embedding on $\mathcal{C}_0$; it must therefore be the identity, and so the unit transformation $\mathrm{id} \to i^* i_!$ is an equivalence. To see that the counit transformation $i_! i^* \to \mathrm{id}$ is also an equivalence, it suffices to show that it is an equivalence after applying the conservative functor i^*, which now follows from the invertibility of the unit and one of the adjunction identities. □

Proof of Proposition 3.3.10. By Lemma 3.3.11 it suffices to check that the objects in $\mathbf{\Omega}_{\mathrm{el}}$ jointly detect equivalences and are completely compact. A morphism

$$\begin{array}{ccccccc} I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & I \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ I' & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & I' \end{array}$$

in AnEnd is an equivalence if and only if the maps $I \to I'$ and $B \to B'$ are equivalences. The latter map is an equivalence if and only if for every n the map on fibres $B_n \to B'_n$ is an equivalence. It thus follows from Lemma 3.3.6 that the objects in $\mathbf{\Omega}_{\mathrm{el}}$ detect equivalences. Similarly, mapping out of them preserves colimits since these are computed levelwise by Corollary 2.5.2 and pullbacks preserve colimits. □

Having described analytic functors in terms of elementary trees, we now describe them in terms of general trees.

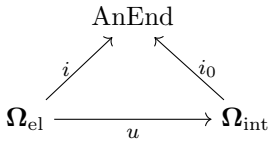
Definition 3.3.12. The inclusion $u\colon \mathbf{\Omega}_{\mathrm{el}} \to \mathbf{\Omega}_{\mathrm{int}}$ induces a geometric morphism $u_*\colon \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}}) \to \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}})$, fully faithful since u is, hence identifying $\mathcal{P}(\mathbf{\Omega}_{\mathrm{el}})$ as a left exact localization of $\mathcal{P}(\mathbf{\Omega}_{\mathrm{int}})$. We denote the image by $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}})$ and call its objects *Segal presheaves*:

$$\mathcal{P}(\mathbf{\Omega}_{\mathrm{el}}) \xrightarrow{\sim} \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}}) \subset \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}})$$

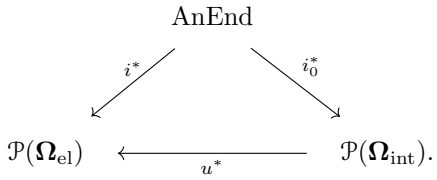
A presheaf $\Phi \in \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}})$ is thus a Segal presheaf if it is a right Kan extension of its restriction to $\mathbf{\Omega}_{\mathrm{el}}$. This right Kan extension is calculated in the standard way using limits: A presheaf Φ is Segal when the natural map $\Phi(T) \to \lim_{E \in \mathrm{el}(T)^{\mathrm{op}}} \Phi(E)$ is an equivalence. (Recall that $\mathrm{el}(T) = \mathbf{\Omega}_{\mathrm{el}/T}$ is the category of elements of T.)

Definition 3.3.13. Let i_0 denote the inclusion $\mathbf{\Omega}_{\mathrm{int}} \hookrightarrow \mathrm{AnEnd}$. This extends to a unique colimit-preserving functor $i_{0,!}\colon \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}}) \to \mathrm{AnEnd}$ with right adjoint i_0^* given by the restricted Yoneda embedding.

The commutative triangle of inclusion functors



induces a commutative diagram of right adjoint functors



The functor u^* given by composition with u also has a right adjoint u_*, given by right Kan extension along u^{op}.

Lemma 3.3.14. *The natural transformation*

$$i_0^* \to u_* u^* i_0^* \simeq u_* i^*,$$

induced by the unit for the adjunction $u^* \dashv u_*$, is an equivalence.

Proof. For $P \in \mathrm{AnEnd}$ and $T \in \mathbf{\Omega}_{\mathrm{int}}$, we have

$$(i_0^* P)(T) \simeq \mathrm{Map}(i_0 T, P) \simeq \mathrm{Map}\left(i_0\left(\operatorname*{colim}_{E \in \mathrm{el}(T)} uE\right), P\right) \simeq \mathrm{Map}\left(\operatorname*{colim}_{E \in \mathrm{el}(T)} i_0 uE, P\right) \simeq \lim_{E \in \mathrm{el}(T)^{\mathrm{op}}} \mathrm{Map}(iE, P),$$

since T is the colimit of its elementary subtrees in $\mathbf{\Omega}_{\mathrm{int}}$ and this colimit is preserved by the inclusion i_0 by Lemma 3.3.5. But this gives precisely the limit formula for the right Kan extension $u_* i^* P$. □

Proposition 3.3.15. *The functor $i_0^*\colon \mathrm{AnEnd} \to \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}})$ is fully faithful with image the Segal presheaves. In other words, it induces an equivalence*

$$\mathrm{AnEnd} \xrightarrow{\sim} \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}}).$$

Proof. By Lemma 3.3.14, i_0^* factors as i^* followed by u_*. But i^* is an equivalence by Proposition 3.3.10, so i_0^* is fully faithful because u_* is, and has the same image as u_*, which is $\mathcal{P}_{\mathrm{Seg}}(\boldsymbol{\Omega}_{\mathrm{int}})$ by definition. □

4. Initial Algebras and Free Monads

4.1. **Initial Lambek Algebras.** In this subsection we prove an ∞-categorical version of the existence theorem for initial Lambek algebras. In ordinary category theory, the study of initial algebras for endofunctors goes back to Lambek [Lam68], while the existence result is due to Adámek [Adá74]. In the present account, we establish the initial-algebra theorem via a lightweight version of bar-cobar duality.

Definition 4.1.1. Let $P : \mathcal{C} \to \mathcal{C}$ be any endofunctor. Recall that a *Lambek P-algebra* is a pair (A, a) where A is an object of $\mathcal{C}$ and $a\colon PA \to A$ is a morphism of $\mathcal{C}$. Dually, a *Lambek P-coalgebra* is a pair (C, c) where C is an object and $c\colon C \to PC$ is a morphism. (We shall omit the attribute 'Lambek' for the rest of this subsection.) Formally, the ∞-categories of P-algebras and P-coalgebras are defined as pullbacks

$$\begin{array}{ccc} \mathrm{alg}_P(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow[(P,\mathrm{id})]{} & \mathcal{C}\times\mathcal{C}, \end{array} \qquad \begin{array}{ccc} \mathrm{coalg}_P(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow[(\mathrm{id},P)]{} & \mathcal{C}\times\mathcal{C}. \end{array}$$

Definition 4.1.2. If (A, a) is a P-algebra and (C, c) is a P-coalgebra, then a *P-twisting morphism* is a morphism $f\colon C \to A$ in $\mathcal{C}$ together with a commutative square

$$\begin{array}{ccc} C & \xrightarrow{c} & PC \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Pf} \\ A & \xrightarrow{a} & PA. \end{array}$$

We define the space $\mathrm{Tw}_P(C, A)$ of P-twisting morphisms from C to A as the equalizer

$$\mathrm{Tw}_P(C, A) \to \mathrm{Map}(C, A) \rightrightarrows \mathrm{Map}(C, A),$$

where the two maps $\mathrm{Map}(C, A) \to \mathrm{Map}(C, A)$ are the identity and $f \mapsto a \circ Pf \circ c$. (The equation $f \simeq a \circ Pf \circ f$ may be viewed as the analogue of the *Maurer–Cartan equation* in this context.)

It will be useful to note that a P-twisting morphism may also be seen as a $\mathrm{Tw}(P)$-algebra in the twisted arrow ∞-category $\mathrm{Tw}(\mathcal{C})$, as we proceed to establish.

Definition 4.1.3. Recall that, for $\mathcal{C}$ an ∞-category, the *twisted arrow ∞-category* $\mathrm{Tw}(\mathcal{C})$ has as objects the morphisms in $\mathcal{C}$, and a morphism in $\mathrm{Tw}(\mathcal{C})$ from $f'\colon X' \to Y'$ to $f\colon X \to Y$ is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle f} & & {\scriptstyle f'}\downarrow \\ Y & \longrightarrow & Y'. \end{array}$$

See [Bar17] or [Lur17, §5.2.1] for a more formal definition. Note that in our convention it is the *codomain* component that determines the direction of the morphism. (Lurie [Lur17, §5.2.1] uses the opposite convention.)

There is a canonical left fibration

$$(\mathrm{dom}, \mathrm{codom})\colon \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C},$$

corresponding to the functor $\mathrm{Map}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$.

Proposition 4.1.4. *The endofunctor $P\colon \mathcal{C} \to \mathcal{C}$ induces an endofunctor $\mathrm{Tw}(P)\colon \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{C})$ and a morphism of endofunctors*

$$\begin{array}{ccc} \mathrm{Tw}(\mathcal{C}) & \xrightarrow{\mathrm{Tw}(P)} & \mathrm{Tw}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow[P^{\mathrm{op}} \times P]{} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C}. \end{array}$$

This induces a functor

$$\mathrm{alg}_{\mathrm{Tw}(P)}(\mathrm{Tw}(\mathcal{C})) \longrightarrow \mathrm{alg}_{P^{\mathrm{op}} \times P}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \simeq \mathrm{alg}_{P^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{alg}_P(\mathcal{C}) \simeq \mathrm{coalg}_P(\mathcal{C})^{\mathrm{op}} \times \mathrm{alg}_P(\mathcal{C})$$

which is a left fibration such that the fibre over $(C, A) \in \mathrm{coalg}_P(\mathcal{C})^{\mathrm{op}} \times \mathrm{alg}_P(\mathcal{C})$ is the space $\mathrm{Tw}_P(C, A)$ of P-twisting morphism from C to A.

Before we prove this, we make some simple observations:

Lemma 4.1.5. *Consider a commutative diagram of ∞-categories*

$$\begin{array}{ccccc} \mathcal{E}' & \longrightarrow & \mathcal{E} & \longleftarrow & \mathcal{E}'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}' & \longrightarrow & \mathcal{B} & \longleftarrow & \mathcal{B}'' \end{array}$$

where the vertical maps are cocartesian fibrations and the upper horizontal maps preserve cocartesian morphisms. Then the induced functor

$$\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}'' \to \mathcal{B}' \times_{\mathcal{B}} \mathcal{B}''$$

is again a cocartesian fibration, and the canonical functors to $\mathcal{E}'$, $\mathcal{E}$, and $\mathcal{E}''$ all preserve cocartesian morphisms. Moreover, if the vertical maps are actually left fibrations, then so is this new map.

Proof. Given a morphism f in $\mathcal{B}' \times_{\mathcal{B}} \mathcal{B}''$ it is easy to see that the morphism in $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}''$ corresponding to a compatible choice of cocartesian morphisms over the images of f in $\mathcal{B}'$, $\mathcal{B}$, and $\mathcal{B}''$ is cocartesian. □

Lemma 4.1.6. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories equipped with endofunctors $P\colon \mathcal{C} \to \mathcal{C}$ and $Q\colon \mathcal{D} \to \mathcal{D}$, and $F\colon \mathcal{D} \to \mathcal{C}$ is a cocartesian fibration which is compatible with P and Q in the sense that there is a commutative square*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{Q} & \mathcal{D} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow[P]{} & \mathcal{C}, \end{array}$$

and Q preserves F-cocartesian morphisms. Then the resulting functor $\mathrm{alg}_Q(\mathcal{D}) \to \mathrm{alg}_P(\mathcal{C})$ is a cocartesian fibration. Furthermore, if F is actually a left fibration, then $\mathrm{alg}_Q(\mathcal{D}) \to \mathrm{alg}_P(\mathcal{C})$ is itself a left fibration.

Proof. The functor F induces morphisms of cocartesian (respectively, left) fibrations

$$\begin{array}{ccc} \mathcal{D}^{\Delta^1} & \longrightarrow & \mathcal{D}^{\partial\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\Delta^1} & \longrightarrow & \mathcal{C}^{\partial\Delta^1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{D} & \xrightarrow{(Q,\mathrm{id})} & \mathcal{D} \times \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[(P,\mathrm{id})]{} & \mathcal{C} \times \mathcal{C} \end{array}$$

(note that this second square commutes by virtue of our assumption on F). Taking pullbacks, we obtain the natural map $\mathrm{alg}_Q(\mathcal{D}) \to \mathrm{alg}_P(\mathcal{C})$ which is therefore a cocartesian (respectively, left) fibration by Lemma 4.1.5. □

Proof of Proposition 4.1.4. It follows from Lemma 4.1.6 applied to the left fibration $F\colon \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ that the induced map

$$\mathrm{alg}_{\mathrm{Tw}(P)}(\mathrm{Tw}(\mathcal{C})) \to \mathrm{coalg}_P(\mathcal{C})^{\mathrm{op}} \times \mathrm{alg}_P(\mathcal{C})$$

is a left fibration. It remains to see that the fibre over $(C, A) \in \mathrm{coalg}_P(\mathcal{C})^{\mathrm{op}} \times \mathrm{alg}_P(\mathcal{C})$ is the space of P-twisting morphisms. By construction, this fibre is obtained as the pullback of fibres of the induced left fibrations. The fibre of F over the object (C, A) is the space $\mathrm{Map}(C, A)$, and the fibre of $F^{\Delta^1}\colon \mathrm{Tw}(\mathcal{C})^{\Delta^1} \to (\mathcal{C}^{\mathrm{op}})^{\Delta^1} \times \mathcal{C}^{\Delta^1}$ is computed as $\mathrm{Map}(C, A)^{\Delta^1} \simeq \mathrm{Map}(C, A)$. Hence the pullback M of the fibres fits into the commutative square

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Map}(C, A) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{diag.}} \\ \mathrm{Map}(C, A) & \xrightarrow[f \mapsto (a \circ Pf \circ c, f)]{} & \mathrm{Map}(C, A) \times \mathrm{Map}(C, A). \end{array}$$

But this M is just a pullback reformulation of the equalizer definition of $\mathrm{Tw}_P(C, A)$. □

Lemma 4.1.7. *A P-coalgebra morphism $(C, c) \to (C', c')$ induces a map $\mathrm{Tw}_P(C', A) \to \mathrm{Tw}_P(C, A)$ by pre-composition. Similarly, a P-algebra morphism $(A', a') \to (A, a)$ induces a map $\mathrm{Tw}_P(C, A') \to \mathrm{Tw}_P(C, A)$ by post-composition.*

Proof. This is immediate from the description of these spaces as fibres of a left fibration. □

If (C, c) is a P-coalgebra, then (PC, Pc) is a P-coalgebra, and $c\colon (C, c) \to (PC, Pc)$ is a P-coalgebra morphism. The following is the key property of twisting morphisms:

Lemma 4.1.8. *For a P-coalgebra (C, c) and a P-algebra (A, a), the map $\mathrm{Tw}_P(PC, A) \to \mathrm{Tw}_P(C, A)$ which sends g to $g \circ c$, is an equivalence, with inverse the map $\mathrm{Tw}_P(C, A) \to \mathrm{Tw}_P(PC, A)$ which sends f to $a \circ Pf$.*

Proof. We first detail the inverse. If $f\colon C \to A$ is a twisting morphism with square

$$\begin{array}{ccc} PC & \xleftarrow{c} & C \\ {\scriptstyle Pf}\downarrow & & \downarrow{\scriptstyle f} \\ PA & \xrightarrow[a]{} & A, \end{array}$$

apply P and paste with a trivial square like this:

$$\begin{array}{ccc} PPC & \xleftarrow{Pc} & PC \\ {\scriptstyle PPf}\downarrow & & \downarrow{\scriptstyle Pf} \\ PPA & \xrightarrow[Pa]{} & PA \\ {\scriptstyle Pa}\downarrow & & \downarrow{\scriptstyle a} \\ PA & \xrightarrow[a]{} & A. \end{array}$$

The left vertical composition is $Pa \circ PPf \simeq P(a \circ Pf)$, so the composite square exhibits $a \circ Pf$ as a twisting morphism, as required.

To see that the two constructions are inverse, we check that the respective composites are naturally equivalent to the respective identity functors. Starting with the square for $f\colon C \to A$, going

left and then back right gives

$$\begin{array}{ccc} PC & \xleftarrow{c} & C \\ {\scriptstyle Pc}\downarrow & & \downarrow{\scriptstyle c} \\ PPC & \xleftarrow{Pc} & PC \\ {\scriptstyle PPf}\downarrow & & \downarrow{\scriptstyle Pf} \\ PPA & \xrightarrow[Pa]{} & PA \\ {\scriptstyle Pa}\downarrow & & \downarrow{\scriptstyle a} \\ PA & \xrightarrow[a]{} & A, \end{array}$$

but this is homotopic to the original square for f since f is twisting. On the other hand, starting with the square for $g\colon PC \to A$, going right and then back left gives

$$\begin{array}{ccc} PPC & \xleftarrow{Pc} & PC \\ {\scriptstyle PPc}\downarrow & & \downarrow{\scriptstyle Pc} \\ PPPC & \xleftarrow{PPc} & PPC \\ {\scriptstyle PPg}\downarrow & & \downarrow{\scriptstyle Pg} \\ PPA & \xrightarrow[Pa]{} & PA \\ {\scriptstyle Pa}\downarrow & & \downarrow{\scriptstyle a} \\ PA & \xrightarrow[a]{} & A \end{array}$$

which is also homotopic to the original square for g since g is twisting. □

Definition 4.1.9. Assume $\mathcal{C}$ has filtered colimits and that $P\colon \mathcal{C} \to \mathcal{C}$ preserves them. Then for a P-coalgebra C we have a diagram

$$C \xrightarrow{c} PC \xrightarrow{Pc} P^2C \to \cdots.$$

Let $I_C := \operatorname{colim}_{n\to\infty} P^nC$ be the colimit of this sequence. Then there is a canonical map

$$I_C \simeq \operatorname*{colim}_n P^{n+1}C \to P(\operatorname*{colim}_n P^n\mathcal{C}) \simeq PI_C.$$

Since P preserves filtered colimits, this map is an equivalence. If u denotes its inverse, the pair (I_C, u) is a P-algebra. We denote this P-algebra ΩC and refer to it as the *cobar construction* of C.

We will now establish a universal property of the cobar construction, which in particular implies that it determines a functor $\Omega\colon \operatorname{coalg}_P(\mathcal{C}) \to \operatorname{alg}_P(\mathcal{C})$. Under further assumptions, we will see that it is left adjoint to a dual *bar* construction, which gives a P-coalgebra from a P-algebra.

Lemma 4.1.10. *If (U, u) is a P-coalgebra for which u is an equivalence, with inverse v giving a P-algebra (U, v), then for any P-algebra (A, a) we have*

$$\operatorname{Map}_{\operatorname{alg}_P}(U, A) \simeq \operatorname{Tw}_P(U, A).$$

Proof. Consider the diagram

$$\begin{array}{ccc} \operatorname{Map}_{\operatorname{alg}_P(\mathcal{C})}((U,v),(A,a)) & \longrightarrow & \operatorname{Map}_{\mathcal{C}^{\Delta^1}}(v,a) \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Map}_{\mathcal{C}}(U,A) & \xrightarrow{f\mapsto(Pf,f)} & \operatorname{Map}_{\mathcal{C}}(PU,PA)\times\operatorname{Map}_{\mathcal{C}}(U,A) \\ {\scriptstyle f\mapsto a\circ Pf\circ u}\downarrow\downarrow{\scriptstyle f\mapsto f} & & {\scriptstyle g\mapsto a\circ g}\downarrow\downarrow{\scriptstyle f\mapsto f\circ v} \\ \operatorname{Map}_{\mathcal{C}}(U,A) & \xrightarrow[f\mapsto f\circ v]{} & \operatorname{Map}_{\mathcal{C}}(PU,A) \end{array}$$

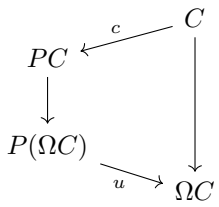
The top square is a pullback by definition of $\operatorname{alg}_P(\mathcal{C})$ as a pullback. The bottom map is an equivalence since v is an equivalence. Since the right fork is an equalizer, it follows (by a standard argument,

for example by expressing equalizers as pullbacks) that also the left fork is an equalizer, hence $\mathrm{Map}_{\mathrm{alg}_P}(U, A) \simeq \mathrm{Tw}_P(U, A)$ as required. □

Definition 4.1.11. Assume that $\mathcal{C}$ has filtered colimits and that $P\colon \mathcal{C} \to \mathcal{C}$ preserves them. Given a P-coalgebra (C, c), the *universal P-twisting morphism* $(C, c) \to (\Omega C, u)$ is the canonical map

$$C \to \operatorname*{colim}_{n\to\infty} P^n C,$$

which is P-twisting by virtue of the diagram



(where all the morphisms are the canonical ones from the colimit diagram defining ΩC).

Proposition 4.1.12. *For a P-coalgebra (C, c) and a P-algebra (A, a) there is a canonical equivalence*

$$\mathrm{Map}_{\mathrm{alg}_P}(\Omega C, A) \simeq \mathrm{Tw}_P(C, A)$$

given by precomposing with the universal P-twisting morphism.

Proof. By Lemma 4.1.10, we have

$$\mathrm{Map}_{\mathrm{alg}_P}(\Omega C, A) \simeq \mathrm{Tw}_P(\operatorname*{colim}_n P^n C, A),$$

and the latter space is described as an equalizer

$$\mathrm{Tw}_P(\operatorname*{colim}_n P^n C, A) \to \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_n P^n C, A) \rightrightarrows \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_n P^n C, A).$$

The mapping spaces are in turn limits. Altogether we can write down a big commutative diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Map}(P^2C, A) & \underset{a\circ P(-)\circ P^2(c)}{\overset{\mathrm{id}}{\rightrightarrows}} & \mathrm{Map}(P^2C, A) \\
\downarrow & & \downarrow \\
\mathrm{Map}(PC, A) & \underset{a\circ P(-)\circ P(c)}{\overset{\mathrm{id}}{\rightrightarrows}} & \mathrm{Map}(PC, A) \\
\downarrow & & \downarrow \\
\mathrm{Map}(C, A) & \underset{a\circ P(-)\circ c}{\overset{\mathrm{id}}{\rightrightarrows}} & \mathrm{Map}(C, A)
\end{array}$$

(It is clear that it commutes, both for the identity maps and for the other horizontal maps.)

We calculate the limit of this diagram in two ways. First we calculate the limit of each column, yielding the parallel pair of maps

$$\mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_n P^n C, A) \rightrightarrows \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_n P^n C, A),$$

and then we take the equalizer of this to obtain $\mathrm{Tw}_P(\mathrm{colim}_n P^n C, A)$. On the other hand, we can calculate the limit by first taking the equalizer of each row. That gives in each row the space $\mathrm{Tw}_P(P^n C, A)$, and then we can calculate the sequential limit of this new column. Now note that all the maps in the new column are equivalences: this follows from Lemma 4.1.8. So the limit is equivalent to just the zeroth space $\mathrm{Tw}_P(C, A)$ as claimed. □

Proposition 4.1.13. *If $\mathcal{C}$ is an ∞-category with filtered colimits and an initial object $\emptyset$, and $P\colon \mathcal{C} \to \mathcal{C}$ is a filtered-colimit-preserving endofunctor, then the ∞-category $\mathrm{alg}_P(\mathcal{C})$ has an initial object, given by $\Omega\emptyset$.*

Proof. $\emptyset$ has a unique coalgebra structure, and Proposition 4.1.12 gives, for any P-algebra A,

$$\mathrm{Map}_{\mathrm{alg}_P}(\Omega\emptyset, A) \simeq \mathrm{Tw}_P(\emptyset, A).$$

Since the latter space is clearly contractible, it follows that $\Omega\emptyset$ is an initial P-algebra. □

Remark 4.1.14. The constructions, results and proofs go through more generally when $\mathcal{C}$ has κ-filtered colimits and $P\colon \mathcal{C} \to \mathcal{C}$ preserves them. The important point is that even if P does not preserve ω-filtered colimits, there is still a transition map $I \to PI$ at each colimit step, so that I is again a P-coalgebra. P can now be applied iteratively again, the next colimit can be taken, and so on, until the resulting chain is longer than κ, and P will preserve the colimit to yield an invertible structure map for the resulting P-coalgebra, and hence a P-algebra. Accepting notation such as $\mathrm{colim}_{n\leq\kappa} P^n C$ for transfinite application of P alternated with taking colimits, all the subsequent constructions go through.

Remark 4.1.15. All the constructions, results and proofs can be dualized: assume that $\mathcal{C}$ has cofiltered limits, and that P preserves them. Then there is a functor

$$B\colon \mathrm{alg}_P \to \mathrm{coalg}_P$$

taking a P-algebra (A, a) to the limit of the chain $A \xleftarrow{a} PA \xleftarrow{Pa} P^2A \leftarrow \cdots$. (This is called the *bar construction*.)

The notion of P-twisting morphism is still the same, but now the results are about applying P to A instead of C. Lemma 4.1.8 becomes the statement that the following maps are inverse homotopy equivalences:

$$\begin{array}{rcl} \mathrm{Tw}_P(C, PA) & \longrightarrow & \mathrm{Tw}_P(C, A) \\ g & \longmapsto & a \circ g \\ Pf \circ c & \longleftarrow\!\shortmid & f \end{array}$$

Assuming that $\mathcal{C}$ has cofiltered limits and P preserves them, we get

$$\mathrm{Map}_{\mathrm{alg}_P}(C, BA) \simeq \mathrm{Tw}_P(C, A).$$

Putting together the two sides of duality, we get:

Theorem 4.1.16. *If $\mathcal{C}$ has filtered colimits and cofiltered limits, and if $P\colon \mathcal{C} \to \mathcal{C}$ preserves them, then Ω is left adjoint to B. Altogether*

$$\mathrm{Map}_{\mathrm{alg}_P}(\Omega C, A) \simeq \mathrm{Tw}_P(C, A) \simeq \mathrm{Map}_{\mathrm{coalg}_P}(C, BA).$$

In the case of interest here, $\mathcal{C}$ will be presentable, and P will be analytic. In particular P then preserves filtered colimits, and also preserves cofiltered limits (since these are weakly contractible), so the theorem applies.

4.2. **Free Monads.**

Definition 4.2.1. Let $\mathcal{C}$ be an ∞-category with binary coproducts, let $P\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor, and let X be an object of $\mathcal{C}$. Define a new endofunctor $P_X\colon \mathcal{C}_{X/} \to \mathcal{C}_{X/}$ as the composite

$$\mathcal{C}_{X/} \xrightarrow{u_X} \mathcal{C} \xrightarrow{P} \mathcal{C} \xrightarrow{a_X} \mathcal{C}_{X/},$$

where u_X is the forgetful functor, with left adjoint $a_X = X \amalg (\ )$.

Lemma 4.2.2. *In the situation of the previous definition, there is a canonical equivalence*

$$\mathrm{alg}_{P_X}(\mathcal{C}_{X/}) \simeq \mathrm{alg}_P(\mathcal{C})_{X/},$$

where $\mathrm{alg}_P(\mathcal{C})_{X/} := \mathrm{alg}_P(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/X}$.

Proof. Both ∞-categories are defined as pullbacks:

$$\begin{array}{ccc} \mathrm{alg}_{P_X}(\mathcal{C}_{X/}) & \longrightarrow & (\mathcal{C}_{X/})^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C}_{X/} & \xrightarrow[(P_X,\mathrm{id})]{} & \mathcal{C}_{X/} \times \mathcal{C}_{X/} \end{array} \qquad \begin{array}{ccccc} \mathrm{alg}_P(\mathcal{C})_{X/} & \longrightarrow & \mathrm{alg}_P(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C}_{X/} & \xrightarrow[u_X]{} & \mathcal{C} & \xrightarrow[(P,\mathrm{id})]{} & \mathcal{C} \times \mathcal{C}. \end{array}$$

The bottom functors can be factored into three steps, respectively:

$$\mathcal{C}_{X/} \xrightarrow[(u_X,\mathrm{id})]{} \mathcal{C} \times \mathcal{C}_{X/} \xrightarrow[P\times\mathrm{id}]{} \mathcal{C} \times \mathcal{C}_{X/} \xrightarrow[a_X\times\mathrm{id}]{} \mathcal{C}_{X/} \times \mathcal{C}_{X/},$$

$$\mathcal{C}_{X/} \xrightarrow[(u_X,\mathrm{id})]{} \mathcal{C} \times \mathcal{C}_{X/} \xrightarrow[P\times\mathrm{id}]{} \mathcal{C} \times \mathcal{C}_{X/} \xrightarrow[\mathrm{id}\times u_X]{} \mathcal{C} \times \mathcal{C}.$$

The first two steps are the same, and for the last step we have an equivalence of pullbacks

$$\begin{array}{ccccc} (\mathcal{C}_{X/})^{\Delta^1} & \longleftarrow & \cdot & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_{X/} \times \mathcal{C}_{X/} & \xleftarrow[a_X\times\mathrm{id}]{} & \mathcal{C} \times \mathcal{C}_{X/} & \xrightarrow[\mathrm{id}\times u_X]{} & \mathcal{C} \times \mathcal{C} \end{array}$$

since a_X is left adjoint to u_X. □

Lemma 4.2.3. *Suppose $\mathcal{C}$ has colimits of shape K and $P\colon \mathcal{C} \to \mathcal{C}$ preserves them. Then $\mathrm{alg}_P(\mathcal{C})$ has colimits of shape K and the forgetful functor $U\colon \mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}$ preserves and reflects them.*

Proof. The ∞-category $\mathrm{alg}_P(\mathcal{C})$ is defined as a pullback

$$\begin{array}{ccc} \mathrm{alg}_P(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C} & \xrightarrow[(P,\mathrm{id})]{} & \mathcal{C} \times \mathcal{C}. \end{array}$$

Here the ∞-categories $\mathcal{C}^{\Delta^1}$, $\mathcal{C}$, and $\mathcal{C} \times \mathcal{C}$ have colimits of shape K, and the functors $(\mathrm{ev}_0, \mathrm{ev}_1)$ and (P, id) preserve them. It therefore follows from [Lur09, Lemma 5.4.5.5] that $\mathrm{alg}_P(\mathcal{C})$ has colimits of shape K, and that a diagram $K^{\triangleright} \to \mathrm{alg}_P(\mathcal{C})$ is a colimit if and only if its images in $\mathcal{C}$ and $\mathcal{C}^{\Delta^1}$ are colimits. Since the functor $(\mathrm{ev}_0, \mathrm{ev}_1)$ preserves and reflects colimits, this is equivalent to the image under U being a colimit. □

Proposition 4.2.4. *Suppose $\mathcal{C}$ is an ∞-category with filtered colimits and binary coproducts, and $P\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor that preserves filtered colimits. Then the forgetful functor $U\colon \mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}$ has a left adjoint, and the resulting adjunction is monadic.*

Proof. To see that U has a left adjoint, it suffices by [GK17, Corollary 2.3] to show that for every $X \in \mathcal{C}$ the ∞-category

$$\mathrm{alg}_P(\mathcal{C})_{X/} := \mathrm{alg}_P(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{X/}$$

has an initial object. But $\mathrm{alg}_P(\mathcal{C})_{X/}$ can be identified with $\mathrm{alg}_{P_X}(\mathcal{C}_{X/})$ by Lemma 4.2.2, where $P_X = a_X \circ P \circ u_X$ as in Definition 4.2.1. Moreover, the functor P_X preserves filtered colimits (indeed u_X preserves filtered colimits by the dual of Lemma 2.2.7, P preserves filtered colimits by assumption, and a_X is a left adjoint). Therefore $\mathrm{alg}_{P_X}(\mathcal{C}_{X/})$ has an initial object by Proposition 4.1.13 since $\mathcal{C}_{X/}$ obviously has an initial object, and has filtered colimits by the dual of Lemma 2.2.7.

To show that the resulting adjunction is monadic, we apply the Lurie–Barr–Beck monadicity theorem [Lur17, Theorem 4.7.3.5]. For this we must show that U detects equivalences, which is clear, and that $\mathrm{alg}_P(\mathcal{C})$ has colimits of U-split simplicial diagrams, and U preserves these. Consider a U-split simplicial diagram $A_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{alg}_P(\mathcal{C})$. By (the proof of) [Lur09, Lemma 5.4.5.5] it is enough to show that the images of $A_\bullet$ in $\mathcal{C}$ and $\mathcal{C}^{\Delta^1}$ have colimits, and these are preserved by

the functors $(P, \mathrm{id})\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathcal{C}^{\Delta^1} \to \mathcal{C} \times \mathcal{C}$. Since $A_\bullet$ is U-split, it follows from [Lur17, Remark 4.7.2.3] that $U(A_\bullet)$ has a colimit $C \in \mathcal{C}$, and this is preserved by any functor, in particular by P. Thus the map $PC \simeq \operatorname{colim} PU(A_\bullet) \to \operatorname{colim} U(A_\bullet) \simeq C$ induced by the algebra structure maps in $A_\bullet$ is a colimit in $\mathcal{C}^{\Delta^1}$. Thus $A_\bullet$ has a colimit in $\mathrm{alg}_P(\mathcal{C})$ and U preserves it. □

Notation 4.2.5. For any ∞-category $\mathcal{C}$ we write $\mathrm{Mnd}(\mathcal{C})$ for the ∞-category of monads on $\mathcal{C}$, defined as the ∞-category $\mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ of associative algebras in $\mathrm{End}(\mathcal{C})$ with respect to the monoidal structure given by composition, as in [Lur17, §4.7.1]. If $T \in \mathrm{Mnd}(\mathcal{C})$ is a monad on $\mathcal{C}$, we write $\mathrm{Alg}_T(\mathcal{C})$ for the ∞-category of T-algebras in $\mathcal{C}$ (which can be defined as the ∞-category $\mathrm{LMod}_T(\mathcal{C})$ of left T-modules via the action of $\mathrm{End}(\mathcal{C})$ on $\mathcal{C}$). Note that we write lowercase alg for Lambek algebras for an endofunctor and uppercase Alg for algebras for a monad.

Definition 4.2.6. If $\mathcal{C}$ is an ∞-category with filtered colimits and binary coproducts, and $P \in \mathrm{End}(\mathcal{C})$ is a filtered-colimit-preserving endofunctor on $\mathcal{C}$, we write $\overline{P} \in \mathrm{Mnd}(\mathcal{C})$ for the monad associated to the monadic adjunction

$$\mathcal{C} \rightleftarrows \mathrm{alg}_P(\mathcal{C})$$

of Proposition 4.2.4; this exists by [Lur17, Proposition 4.7.3.3].

With this notation, we have:

Corollary 4.2.7. *There is a canonical equivalence*

$$\mathrm{alg}_P(\mathcal{C}) \simeq \mathrm{Alg}_{\overline{P}}(\mathcal{C})$$

over $\mathcal{C}$. □

The following result shows that $\overline{P}$ is the *free* monad on P:

Proposition 4.2.8. *Suppose $\mathcal{C}$ is an ∞-category with filtered colimits and binary coproducts, and $P\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor that preserves filtered colimits. Then for every monad T on $\mathcal{C}$ the morphism*

$$\mathrm{Map}_{\mathrm{Mnd}(\mathcal{C})}(\overline{P}, T) \to \mathrm{Map}_{\mathrm{End}(\mathcal{C})}(P, T)$$

induced by the natural transformation $P \to \overline{P}$ is an equivalence.

The final ingredient needed for the proof of Proposition 4.2.8 is the following observation:

Proposition 4.2.9. *For any endofunctor $P\colon \mathcal{C} \to \mathcal{C}$ and any adjunction*

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

there is a natural equivalence

$$\mathrm{Map}_{/\mathcal{C}}(\mathcal{D}, \mathrm{alg}_P(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{End}(\mathcal{C})}(P, RL).$$

Proof. It is enough to establish

$$\mathrm{Map}_{/\mathcal{C}}(\mathcal{D}, \mathrm{alg}_P(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C})}(PR, R),$$

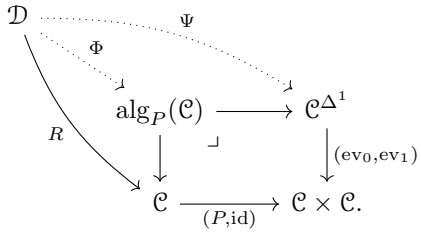
because the latter space is equivalent to $\mathrm{Map}_{\mathrm{End}(\mathcal{C})}(P, RL)$ by adjunction: precomposing with the adjunction $L \dashv R$ we get for any ∞-category $\mathcal{X}$ an adjunction

$$R^* : \mathrm{Fun}(\mathcal{D}, \mathcal{X}) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{X}) : L^*,$$

with R^* left adjoint to L^*, and hence a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{X})}(FR, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{X})}(F, GL).$$

Consider the diagram



Since $\mathrm{alg}_P(\mathcal{C})$ is defined as a pullback, we see that giving $\Phi \in \mathrm{Map}_{/\mathcal{C}}(\mathcal{D}, \mathrm{alg}_P(\mathcal{C}))$ is equivalent to giving Ψ, which amounts precisely to giving a natural transformation from PR to R, as required. □

Proof of Proposition 4.2.8. Combine the equivalences of Proposition 4.2.9, Corollary 4.2.7, and Corollary A.5.11. □

Definition 4.2.10. Let $\mathcal{C}$ be an ∞-category with filtered colimits. We write $\mathrm{End}^{\omega}(\mathcal{C})$ for the full subcategory of $\mathrm{End}(\mathcal{C})$ spanned by the endofunctors that preserve filtered colimits. These are closed under composition, and so we get an ∞-category $\mathrm{Mnd}^{\omega}(\mathcal{C}) := \mathrm{Alg}(\mathrm{End}^{\omega}(\mathcal{C}))$, the full subcategory of $\mathrm{Mnd}(\mathcal{C})$ spanned by the monads that preserve filtered colimits.

Corollary 4.2.11. *Suppose $\mathcal{C}$ is an ∞-category with filtered colimits and binary coproducts. Then the forgetful functor $\mathrm{Mnd}^{\omega}(\mathcal{C}) \to \mathrm{End}^{\omega}(\mathcal{C})$ has a left adjoint.*

Proof. By Proposition 4.2.8, for each $P \in \mathrm{End}^{\omega}(\mathcal{C})$ the ∞-category $\mathrm{Mnd}^{\omega}(\mathcal{C})_{P/}$ has an initial object, namely the free monad $\overline{P}$ on P. This implies that the forgetful functor has a left adjoint, which assigns to every endofunctor P its free monad $\overline{P}$. □

Our next goal is to prove that this free monad adjunction is itself monadic, at least if we impose further restrictions on the monads:

Definition 4.2.12. Suppose $\mathcal{C}$ is an ∞-category with sifted colimits. We write $\mathrm{End}^{\sigma}(\mathcal{C})$ for the full subcategory of $\mathrm{End}(\mathcal{C})$ spanned by the endofunctors that preserve sifted colimits, and let $\mathrm{Mnd}^{\sigma}(\mathcal{C})$ denote the full subcategory of $\mathrm{Mnd}(\mathcal{C})$ of monads whose underlying endofunctors preserve sifted colimits.

Lemma 4.2.13. *Suppose $\mathcal{C}$ is an ∞-category with sifted colimits and binary coproducts. If $P\colon \mathcal{C} \to \mathcal{C}$ preserves sifted colimits, then the underlying endofunctor of the free monad $\overline{P}$ on P also preserves sifted colimits.*

Proof. It suffices to show that the forgetful functor $U\colon \mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}$ preserves sifted colimits, but this is a special case of Lemma 4.2.3. □

Thus if $\mathcal{C}$ is an ∞-category with sifted colimits and binary coproducts, then the free monad functor restricts to give an adjunction

$$F : \mathrm{End}^{\sigma}(\mathcal{C}) \rightleftarrows \mathrm{Mnd}^{\sigma}(\mathcal{C}) : U.$$

Proposition 4.2.14. *Suppose $\mathcal{C}$ is an ∞-category with sifted colimits. Then $\mathrm{Mnd}^{\sigma}(\mathcal{C})$ has sifted colimits, and these are preserved by U.*

Proof. $\mathrm{Mnd}^{\sigma}(\mathcal{C})$ is the ∞-category of associative algebras in the monoidal ∞-category $\mathrm{End}^{\sigma}(\mathcal{C})$, where the tensor product, i.e. composition, commutes with sifted colimits in each variable (since we are considering endofunctors that preserve these colimits). The result is therefore a special case of [Lur17, Proposition 3.2.3.1]. □

Corollary 4.2.15. *Suppose $\mathcal{C}$ is an ∞-category with sifted colimits and binary coproducts. Then the adjunction*

$$F : \mathrm{End}^{\sigma}(\mathcal{C}) \rightleftarrows \mathrm{Mnd}^{\sigma}(\mathcal{C}) : U.$$

is monadic.

Proof. We already know from Proposition 4.2.14 that $\mathrm{Mnd}^\sigma(\mathcal{C})$ has all sifted colimits and that U preserves these. It therefore suffices by [Lur17, Theorem 4.7.3.5] to show that U detects equivalences, which follows from [Lur17, Lemma 3.2.2.6]. □

We end this subsection by noting that, under rather restrictive hypotheses on $\mathcal{C}$, this implies that $\mathrm{Mnd}^\sigma(\mathcal{C})$ is presentable:

Definition 4.2.16. Let us say that an ∞-category $\mathcal{C}$ is *compact projectively generated* if it is of the form $\mathcal{P}_\Sigma(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$ with finite coproducts, using the notation of [Lur09, §5.5.8].

Remark 4.2.17. The only reason for introducing this notion is that it implies that $\mathrm{End}^\sigma(\mathcal{C})$ is presentable. We believe this should be true for any presentable ∞-category $\mathcal{C}$, but we will not attempt to prove this as it is not needed for our purposes.

Corollary 4.2.18. *Suppose $\mathcal{C}$ is a compact projectively generated ∞-category. Then $\mathrm{Mnd}^\sigma(\mathcal{C})$ is a presentable ∞-category.*

Proof. Since $\mathcal{C}$ is compact projectively generated, the ∞-category $\mathrm{End}^\sigma(\mathcal{C})$ is equivalent to $\mathrm{Fun}(\mathcal{C}_0, \mathcal{C})$ where $\mathcal{C}_0$ is a small ∞-category, and so this ∞-category is presentable. Moreover, the ∞-category $\mathrm{Mnd}^\sigma(\mathcal{C})$ has sifted colimits by Proposition 4.2.14 and these are preserved by the forgetful functor to $\mathrm{End}^\sigma(\mathcal{C})$. Applying [GH15, Lemma A.5.8, Proposition A.5.9] to the adjunction

$$F : \mathrm{End}^\sigma(\mathcal{C}) \rightleftarrows \mathrm{Mnd}^\sigma(\mathcal{C}) : U,$$

which is monadic by Corollary 4.2.15, it follows that $\mathrm{Mnd}^\sigma(\mathcal{C})$ is presentable. □

4.3. **An Explicit Description of the Free Monad.** We will now give a more explicit description of the free monad $\overline{P}$ as the colimit of a sequence of functors.

Definition 4.3.1. For $\mathcal{C}$ an ∞-category with filtered colimits and binary coproducts, and $P\colon \mathcal{C} \to \mathcal{C}$ an endofunctor that preserves filtered colimits, we will recursively define endofunctors P_n and natural transformations

$$P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} P_2 \xrightarrow{f_3} \cdots .$$

Here $P_0 := \mathrm{id}_\mathcal{C}$, and recursively $P_{n+1} := \mathrm{id}_\mathcal{C} \amalg (P \circ P_n)$. For the natural transformations, f_1 is the coproduct inclusion, and $f_{n+1} := \mathrm{id}_\mathcal{C} \amalg P(f_n)$.

Proposition 4.3.2. *With notation as above, we have a natural equivalence*

$$\operatorname*{colim}_{n\to\infty} P_n \xrightarrow{\sim} \overline{P}.$$

Lemma 4.3.3. *Let $F\colon \mathcal{C} \to \mathrm{alg}_P(\mathcal{C})$ be the left adjoint to the forgetful functor $U\colon \mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}$. The composite of F with the forgetful functor $\mathrm{alg}_P(\mathcal{C}) \to \mathcal{C}^{\Delta^1}$ corresponds to a natural transformation $\phi\colon P\overline{P} \to \overline{P}$. The induced transformation*

$$\mathrm{id}_\mathcal{C} \amalg P\overline{P} \xrightarrow{\eta|\phi} \overline{P},$$

where η is the unit for the adjunction $F \dashv U$ and $\eta|\phi$ means η on the first summand and ϕ on the second summand, is an equivalence.

Proof. Evaluating at $X \in \mathcal{C}$ the map $X \amalg P\overline{P}(X) \to \overline{P}(X)$ is the structure map $P_X\overline{P}X \to \overline{P}X$ exhibiting $\overline{P}X$ as the initial P_X-algebra, which we know is an equivalence. □

Proof of Proposition 4.3.2. To define a natural transformation $\mathrm{colim}_{n\to\infty} P_n \to \overline{P}$ we recursively define natural transformations $\pi_n\colon P_n \to \overline{P}$ and equivalences $\pi_n \circ f_n \simeq \pi_{n-1}$.

We start by setting $\pi_0 := \eta\colon \mathrm{id}_\mathcal{C} \to \overline{P}$, and then given π_n we define π_{n+1} as the composite

$$P_{n+1} = \mathrm{id} \amalg PP_n \xrightarrow{\mathrm{id}\amalg P\pi_n} \mathrm{id} \amalg P\overline{P} \xrightarrow{\eta|\phi} \overline{P}.$$

We then have a commutative diagram

$$\begin{array}{ccccc} \mathrm{id} & & \xrightarrow{\quad\eta\quad} & & \\ {\scriptstyle f_1}\downarrow & & & \searrow & \\ \mathrm{id} \amalg P & \xrightarrow[\mathrm{id}\amalg P\eta]{} & \mathrm{id} \amalg P\overline{P} & \xrightarrow[\eta|\phi]{} & \overline{P}, \end{array}$$

which gives an equivalence $\pi_1 \circ f_1 \simeq \pi_0$. For $n > 0$, the composite $\pi_{n+1} \circ f_{n+1}$ is equivalent to the composite

$$\mathrm{id} \amalg PP_{n-1} \xrightarrow{\mathrm{id}\amalg P f_{n-1}} \mathrm{id} \amalg PP_n \xrightarrow{\mathrm{id}\amalg P\pi_n} \mathrm{id} \amalg P\overline{P} \xrightarrow{\eta\amalg\phi} \overline{P}.$$

We can rewrite this as

$$\mathrm{id} \amalg PP_{n-1} \xrightarrow{\eta|\phi\circ P(\pi_n f_n)} \overline{P},$$

which is equivalent to $\eta|\phi\pi_{n-1}$ — this is the same as π_n by definition, giving the required equivalence $\pi_{n+1} \circ f_{n+1} \simeq \pi_n$.

It remains to show that the induced map $\mathrm{colim}_n P_n X \to \overline{P}X$ is an equivalence for all $X \in \mathcal{C}$. By definition, $\overline{P}(X)$ is the underlying object in $\mathcal{C}$ of the initial P_X-algebra (notation as in Definition 4.2.1), in turn described in Proposition 4.1.13 as the colimit of $P_X^n(\mathrm{id}_X)$ as $n \to \infty$. But on underlying objects we clearly have $P_n X \simeq P_X^n(\mathrm{id})$, and under this identification, the transition maps $f_n \colon P_{n-1}X \to P_n X$ are the iterated P_X-coalgebra structure maps $c_n \colon P_X^{n-1}(\mathrm{id}) \to P_X^n(\mathrm{id})$, as in Definition 4.1.9. Hence $\overline{P} \simeq \mathrm{colim}_{n\to\infty} P_n$ as required. □

Lemma 4.3.4. *For (A, a) a P-algebra, the underlying map of the counit, $U\epsilon_A \colon UFUA \to UA$ is naturally identified with the colimit of the sequence of maps $e_n \colon P_n A \to A$, defined recursively with $e_0 \colon A \to A$ the identity, and e_{n+1} defined as the composite*

$$P_{n+1}A = A \amalg PP_n A \xrightarrow{\mathrm{id}\amalg P(e_n)} A \amalg PA \xrightarrow{\mathrm{id}|a} A.$$

Proof. We know that FUA is the underlying object in $\mathrm{alg}_P(\mathcal{C})$ of $\mathrm{colim}_{n\to\infty} P_A^n(\mathrm{id}_A)$, the initial P_A-algebra. Since (A, a) is a P-algebra, the morphism id_A becomes naturally a P_A-algebra, hence there is a unique homomorphism of P_A-algebras $\mathrm{colim}_{n\to\infty} P_A^n(\mathrm{id}_A) \to \mathrm{id}_A$. The image under the forgetful functor $\mathrm{alg}_{P_A}(\mathcal{C}_{A/}) \to \mathrm{alg}_P(\mathcal{C})$ is the counit ϵ_A. By Proposition 4.1.12, this corresponds to the unique P_A-twisting morphism $\mathrm{id}_A \to \mathrm{id}_A$. By (the proof of) Lemma 4.1.8, the counit $\epsilon_A \colon \mathrm{colim}_{n\to\infty} P_A^n(\mathrm{id}_A) \to \mathrm{id}_A$ is induced by the sequence of twisting morphisms

$$e_n \colon P_A^n(\mathrm{id}_A) \to \mathrm{id}_A,$$

where $e_0 \colon \mathrm{id}_A \to \mathrm{id}_A$ is the identity map, and recursively e_{n+1} is defined as the composite

$$(A \amalg P)^{n+1}(\mathrm{id}_A) \simeq (A \amalg P)(A \amalg P)^n(\mathrm{id}_A) \xrightarrow{(A\amalg P)(e_n)} (A \amalg P)(A) \xrightarrow{\mathrm{id}|a} A.$$

The forgetful functor $U \colon \mathrm{alg}_{P_A}(\mathcal{C}_{A/}) \to \mathcal{C}$ preserves filtered colimits by Lemma 4.2.3, so under the identifications $P_n A \simeq P_A^n(\mathrm{id}_A)$, valid in $\mathcal{C}$, this is precisely the sequence of maps of the statement. □

Construction 4.3.5. We define natural transformations, for $m, n \geq 0$

$$\mu_{m,n} \colon P_n \circ P_m \to P_{m+n}$$

recursively as follows. For the base case $n = 0$ (all $m \geq 0$) we take $\mu_{m,0} \colon \mathrm{id}_{\mathcal{C}} \circ P_m \to P_m$ to be the identity natural transformation. Assuming $\mu_{m,n} \colon P_n \circ P_m \to P_{m+n}$ defined, define $\mu_{m,n+1}$ to be the composite

$$P_{n+1} \circ P_m \simeq P_m \amalg (P \circ P_n \circ P_m) \xrightarrow{\mathrm{id}\amalg P(\mu_{m,n})} P_m \amalg P \circ P_{m+n} \to P_m \amalg P_{m+n+1} \to P_{m+n+1}.$$

Here the second arrow is the sum inclusion $P \circ P_{m+n} \to \mathrm{id}_{\mathcal{C}} \amalg (P \circ P_{m+n}) \simeq P_{m+n+1}$ and the third adds the natural transformation $P_m \to P_{m+n+1}$ which is a composite of f_k in the defining chain.

Lemma 4.3.6. *The natural transformations $\mu_{m,n}$ are compatible with the transition maps f_k in both variables. More precisely, we have commutative diagrams for all $m, n \geq 0$*

$$\begin{array}{ccc} P_n \circ P_m & \xrightarrow{\mu_{m,n}} & P_{m+n} \\ {\scriptstyle f_{n+1}P_m}\downarrow & (1) & \downarrow{\scriptstyle f_{m+n+1}} \\ P_{n+1} \circ P_m & \xrightarrow[\mu_{m,n+1}]{} & P_{m+n+1}, \end{array} \qquad \begin{array}{ccc} P_n \circ P_m & \xrightarrow{\mu_{m,n}} & P_{m+n} \\ {\scriptstyle P_n(f_{m+1})}\downarrow & (2) & \downarrow{\scriptstyle f_{m+1+n}} \\ P_n \circ P_{m+1} & \xrightarrow[\mu_{m+1,n}]{} & P_{m+1+n}. \end{array}$$

Proof. For both statements, the proof is by induction on n, the $n = 0$ cases being trivial. Assuming we have the square (1), we also have

$$\begin{array}{ccccccc} P_{n+1} \circ P_m & = & P_m \amalg P(P_n P_m) & \xrightarrow{\mathrm{id}\amalg P(\mu_{m,n})} & P_m \amalg PP_{m+n} & \longrightarrow & P_m \amalg P_{m+n+1} \longrightarrow P_{m+n+1} \\ {\scriptstyle f_{n+2}P_m}\downarrow & & {\scriptstyle \mathrm{id}\amalg P(f_{n+1}P_m)}\downarrow & (3) & \downarrow{\scriptstyle \mathrm{id}\amalg P(f_{m+n+1})} & & \downarrow \qquad\qquad \downarrow{\scriptstyle f_{m+n+2}} \\ P_{n+2} \circ P_m & = & P_m \amalg P(P_{n+1}P_m) & \xrightarrow[\mathrm{id}\amalg P(\mu_{m,n+1})]{} & P_m \amalg PP_{m+n+1} & \longrightarrow & P_m \amalg P_{m+n+2} \longrightarrow P_{m+n+2} \end{array}$$

Indeed, square (3) commutes by induction (the right summand is P applied to the square (1) of the induction hypothesis) and the two following squares obviously commute. The horizontal composites are precisely $\mu_{m,n+1}$ and $\mu_{m,n+2}$.

Assuming that square (2) commutes, we also have

$$\begin{array}{ccccccc} P_{n+1} \circ P_m & = & P_m \amalg P(P_n P_m) & \xrightarrow{\mathrm{id}\amalg P(\mu_{m,n})} & P_m \amalg PP_{m+n} & \longrightarrow & P_m \amalg P_{m+n+1} \longrightarrow P_{m+n+1} \\ {\scriptstyle P_n(f_{m+1})}\downarrow & & {\scriptstyle f_{m+1}\amalg PP_n(f_{m+1})}\downarrow & (4) & \downarrow{\scriptstyle f_{m+1}\amalg P(f_{m+1+n})} & & \downarrow \qquad\qquad \downarrow{\scriptstyle f_{m+n+2}} \\ P_{n+1} \circ P_{m+1} & = & P_{m+1} \amalg P(P_n P_{m+1}) & \xrightarrow[\mathrm{id}\amalg P(\mu_{m+1,n})]{} & P_{m+1} \amalg PP_{m+1+n} & \longrightarrow & P_{m+1} \amalg P_{m+n+2} \longrightarrow P_{m+n+2} \end{array}$$

Indeed, the square (4) commutes by induction (the right summand is P applied to the square (2) of the induction hypothesis) and the two following squares obviously commute. The horizontal composites are precisely $\mu_{m,n+1}$ and $\mu_{m+1,n+1}$. □

Lemma 4.3.7. *The colimit of the sequence of natural transformations*

$$\mu_{m,n} \colon P_n \circ P_m \to P_{m+n}$$

for $m \to \infty$ is naturally identified with the maps

$$e_n \colon P_n \circ \overline{P} \to \overline{P}$$

of Lemma 4.3.4.

Proof. Induction on n. The case $n = 0$ is clear, as both maps are the identity. Suppose $\operatorname*{colim}_{m\to\infty} \mu_{m,n} \simeq e_n$. Write down $\mu_{m,n+1}$ according to the recursive definition:

$$P_{n+1}P_m \simeq P_m \amalg (PP_nP_m) \xrightarrow{\mathrm{id}\amalg P(\mu_{m,n})} P_m \amalg PP_{m+n} \to P_m \amalg P_{m+n+1} \to P_{m+n+1}.$$

Take the colimit as $m \to \infty$ to find

$$P_{n+1}\overline{P} \simeq \overline{P} \amalg (PP_n\overline{P}) \xrightarrow{\mathrm{id}\amalg P(e_n)} \overline{P} \amalg P \circ \overline{P} \to \overline{P} \amalg \overline{P} \to \overline{P},$$

by induction, using that all the functors commute with filtered colimits. But this is precisely the recursive description of e_{n+1}, given in 4.3.4. □

Proposition 4.3.8. *The multiplication $\mu \colon \overline{P} \circ \overline{P} \to \overline{P}$ is the colimit, for $n \to \infty$, of the natural transformations $\mu_n \colon P_n \circ P_n \to P_{2n}$ of Construction 4.3.5. The unit $\eta \colon \mathrm{id}_{\mathcal{C}} \to \overline{P}$ is the colimit of the sequence of natural transformations $\eta_n \colon \mathrm{id}_{\mathcal{C}} \to P_n$.*

Proof. Thanks to the compatibilities with the transition maps of Lemma 4.3.6, we can compute the colimit first by holding n fixed. Lemma 4.3.7 tells us that for each n fixed, the $m \to \infty$ colimit is the map $e_n \colon P_n\overline{P} \to \overline{P}$, and Lemma 4.3.4 then tells us that the $n \to \infty$ colimit of those is the monad multiplication. □

4.4. **Free Monads in Families.** In this section we will extend our results on free monads to the setting of monads and endofunctors on varying base ∞-categories. In §A.5, we review results of [Hau20] that lead to a commutative triangle

$$\begin{array}{ccc} \mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}} & \longrightarrow & \mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \\ & \searrow \quad \swarrow & \\ & \mathrm{Cat}_\infty, & \end{array}$$

where $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ is an ∞-category of monads and lax morphisms, $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ is an ∞-category of endofunctors and lax morphisms, and the functors to Cat_∞ send monads and endofunctors to the ∞-category they are defined on. This is given fibrewise by the forgetful functor $\mathrm{Alg}(\mathrm{End}(\mathcal{C}))^{\mathrm{op}} \to \mathrm{End}(\mathcal{C})^{\mathrm{op}}$. We then show in Corollary A.5.14 that if we restrict to the subcategory $\mathrm{Cat}_\infty^{\mathrm{radj}}$ where the morphisms are right adjoint functors, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}^{\mathrm{radj}} & \longrightarrow & \mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}^{\mathrm{radj}} \\ & \searrow \quad \swarrow & \\ & \mathrm{Cat}_\infty^{\mathrm{radj}}, & \end{array}$$

where the two downward functors are cocartesian fibrations, the horizontal functor preserves cocartesian morphisms, and the right-hand functor is also a cartesian fibration. We need to introduce notation for a restricted version of these ∞-categories:

Definition 4.4.1. Let $\widehat{\mathrm{Cat}}_\infty^{\sigma,\mathrm{radj}}$ be the ∞-category of compact projectively generated ∞-categories (in the sense of Definition 4.2.16 — but see Remark 4.2.17), with morphisms the functors that are right adjoints and preserve sifted colimits. Then we define $\widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}}$ to be the full subcategory of the pullback of $\widehat{\mathrm{End}}(\widehat{\mathrm{CAT}}_\infty)_{\mathrm{lax}} \to \widehat{\mathrm{Cat}}_\infty$ to $\widehat{\mathrm{Cat}}_\infty^{\sigma,\mathrm{radj}}$ spanned by the endofunctors that preserve sifted colimits; we also define $\widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}}$ similarly.

Proposition 4.4.2. *There is a commuting diagram*

$$\begin{array}{ccc} \widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} & \longrightarrow & \widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \\ & \searrow \quad \swarrow & \\ & \widehat{\mathrm{Cat}}_\infty^{\sigma,\mathrm{radj}}, & \end{array}$$

where the two downward functors are cocartesian fibrations and the horizontal functor preserves cocartesian morphisms. Moreover, both the downward functors are also cartesian fibrations.

Proof. It is immediate from Corollary A.5.14 that the downward functors are cocartesian fibrations and the horizontal functor preserves cocartesian morphisms: from the description of the cocartesian morphisms there it follows that these full subcategories contain the cocartesian morphisms whose sources lie in the subcategories. Similarly, the right-hand functor is a cartesian fibration.

It remains to prove that the functor $\widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \to \widehat{\mathrm{Cat}}_\infty^{\sigma,\mathrm{radj}}$ is a cartesian fibration. Since we know it is a cocartesian fibration, this is equivalent to showing that the functor $\phi_!^{\mathrm{op}} \colon \mathrm{Mnd}^\sigma(\mathcal{C}) \to \mathrm{Mnd}^\sigma(\mathcal{D})$ corresponding to the cocartesian pushforward along a map $\phi \colon \mathcal{C} \to \mathcal{D}$ has a left adjoint. Since the forgetful functor $\widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \to \widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}}$ preserves cocartesian morphisms we have a commutative

square

$$\begin{array}{ccc} \mathrm{Mnd}^{\sigma}(\mathcal{C}) & \xrightarrow{\phi_!^{\mathrm{op}}} & \mathrm{Mnd}^{\sigma}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{End}^{\sigma}(\mathcal{C}) & \xrightarrow[\phi_!^{\mathrm{op}}]{} & \mathrm{End}^{\sigma}(\mathcal{D}). \end{array}$$

Here we know that $\mathrm{Mnd}^{\sigma}(\mathcal{C})$ and $\mathrm{Mnd}^{\sigma}(\mathcal{D})$ are presentable ∞-categories by Corollary 4.2.18. By the adjoint functor theorem it therefore suffices to show that the functor $\phi_!^{\mathrm{op}}$ is accessible and preserves limits. But the forgetful functor $\mathrm{Mnd}^{\sigma}(\mathcal{D}) \to \mathrm{End}^{\sigma}(\mathcal{D})$ is a monadic right adjoint by Corollary 4.2.15 and preserves sifted colimits by Proposition 4.2.14. Thus limits and sifted colimits are computed in $\mathrm{End}^{\sigma}(\mathcal{D})$, so it suffices to show that the composite $\mathrm{Mnd}^{\sigma}(\mathcal{C}) \to \mathrm{End}^{\sigma}(\mathcal{D})$ preserves limits and is accessible. The same observations apply to the forgetful functor $\mathrm{Mnd}^{\sigma}(\mathcal{C}) \to \mathrm{End}^{\sigma}(\mathcal{C})$, so in the end it is enough to prove that $\phi_!^{\mathrm{op}} \colon \mathrm{End}^{\sigma}(\mathcal{C}) \to \mathrm{End}^{\sigma}(\mathcal{D})$ preserves limits and is accessible, or equivalently that this is a right adjoint. But this follows from the projection $\widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \to \widehat{\mathrm{Cat}}_{\infty}^{\sigma,\mathrm{radj}}$ being a cartesian and cocartesian fibration (or from the explicit description of the cocartesian morphisms). □

Proposition 4.4.3. *The forgetful functor $\widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \to \widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}}$ has a right adjoint that commutes with the projections to $\widehat{\mathrm{Cat}}_{\infty}^{\sigma,\mathrm{radj}}$, which takes an endofunctor to its free monad.*

Proof. By (the dual of) [Lur17, Proposition 7.3.2.6] it suffices to show that the functor on fibres over each $\mathcal{C} \in \widehat{\mathrm{Cat}}_{\infty}^{\sigma,\mathrm{radj}}$ has a right adjoint. But this functor can be identified with the forgetful functor $\mathrm{Mnd}^{\sigma}(\mathcal{C})^{\mathrm{op}} \to \mathrm{End}^{\sigma}(\mathcal{C})^{\mathrm{op}}$, so this follows from Corollary 4.2.11. □

We now wish to prove that this free monad adjunction is in fact monadic (which we saw fibrewise in Corollary 4.2.15):

Theorem 4.4.4. *The forgetful functor $\widehat{\mathrm{Mnd}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}} \to \widehat{\mathrm{End}}_{\mathrm{lax}}^{\sigma,\mathrm{radj}}$ has a right adjoint that commutes with the projections to $\widehat{\mathrm{Cat}}_{\infty}^{\sigma,\mathrm{radj}}$, and the resulting adjunction is comonadic.*

To prove this we will use the following general observation:

Proposition 4.4.5. *Suppose we have a diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{U} & \mathcal{D} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{B} & \end{array}$$

where

(1) p and q are cocartesian fibrations,
(2) for $b \in \mathcal{B}$ the ∞-category $\mathcal{C}_b$ has geometric realizations,
(3) for $f \colon b \to b'$ in $\mathcal{B}$, the cocartesian pushforward functor $f_! \colon \mathcal{C}_b \to \mathcal{C}_{b'}$ preserves geometric realizations,
(4) U has a left adjoint $F \colon \mathcal{D} \to \mathcal{C}$ such that $pF \simeq q$,
(5) the adjunction $F \dashv U$ restricts in each fibre to an adjunction $F_b \dashv U_b$,
(6) $U_b \colon \mathcal{C}_b \to \mathcal{D}_b$ preserves geometric realizations,
(7) U_b detects equivalences for every $b \in \mathcal{B}$.

Then the adjunction $F \dashv U$ is monadic.

Remark 4.4.6. It follows from these assumptions that each adjunction $F_b \dashv U_b$ is monadic.

Remark 4.4.7. Our proof of this result follows the argument used to prove monadicity for enriched categories in [Wol74].

Lemma 4.4.8. *Let $\mathcal{C}$ be an ∞-category with all small colimits, and suppose $F\colon (\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\to\mathcal{C}$ is a diagram such that for every $[n]\in\mathbf{\Delta}$ the diagrams $F|_{\{[n]\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ and $F|_{(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times\{[n]\}}$ are colimit diagrams. Then the following are equivalent:*

(i) the restriction $F_{\{\infty\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ is a colimit diagram,
(ii) the restriction $F_{(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times\{\infty\}}$ is a colimit diagram,
(iii) the commutative square

$$\begin{array}{ccc} F([0],[0]) & \longrightarrow & F(\infty,[0]) \\ \downarrow & & \downarrow \\ F([0],\infty) & \longrightarrow & F(\infty,\infty) \end{array}$$

is a pushout,
(iv) F is the left Kan extension of its restriction to $\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}$.

Proof. Functoriality of left Kan extensions and some easy cofinality arguments. □

Proof of Proposition 4.4.5. By [Lur17, Theorem 4.7.3.5] it remains to show that $\mathcal{C}$ has colimits of U-split simplicial diagrams, these colimits are preserved by U, and U detects equivalences.

Let us first check that U detects equivalences. Suppose therefore that $f\colon c\to c'$ is a morphism in $\mathcal{C}$ such that Uf is an equivalence in $\mathcal{D}$. Then $g := qUf\simeq pf$ is an equivalence in $\mathcal{B}$. We can factor f as $c\xrightarrow{\phi} g_!c\xrightarrow{f'}c'$ where ϕ is a p-cocartesian morphism over g and f' is a morphism in the fibre $\mathcal{C}_{p(c')}$. But then ϕ is an equivalence since it is cocartesian over the equivalence g, and f' is an equivalence since by assumption (7) U detects equivalences fibrewise over $\mathcal{B}$.

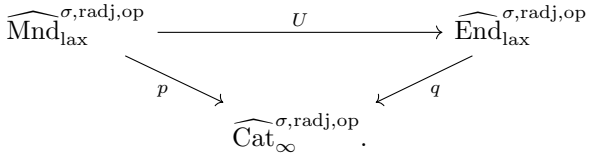
Using assumptions (1), (2), and (3) we see by [Lur09, Corollary 4.3.1.11] that p-colimits of simplicial diagrams exist in $\mathcal{C}$. Moreover, by [Lur09, Proposition 4.3.1.5] a p-colimit diagram whose underlying diagram in $\mathcal{B}$ is a colimit is a colimit diagram in $\mathcal{C}$. Thus $\mathcal{C}$ has colimits for simplicial diagrams whose underlying diagrams in $\mathcal{B}$ have colimits — in particular, this holds for U-split simplicial diagrams, since by definition their underlying diagrams in $\mathcal{B}$ can be extended to split simplicial diagrams, which are colimit diagrams by [Lur09, Lemma 6.1.3.16].

Moreover, since $\mathbf{\Delta}^{\mathrm{op}}$ is weakly contractible, it follows from [Lur09, Proposition 4.3.1.10] that if $\phi\colon\mathbf{\Delta}^{\mathrm{op}}\to\mathcal{C}$ is a diagram in $\mathcal{C}_b$ for some b, then its colimit in $\mathcal{C}_b$ is also a colimit in $\mathcal{C}$.

Suppose then that $\phi\colon\mathbf{\Delta}^{\mathrm{op}}\to\mathcal{C}$ is a U-split diagram. Using the monad $T := UF$ we can extend this to a diagram $\Phi\colon\mathbf{\Delta}^{\mathrm{op}}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\to\mathcal{C}$, where $\Phi|_{\mathbf{\Delta}^{\mathrm{op}}\times\{\infty\}}\simeq\phi$ and $\Phi|_{\mathbf{\Delta}^{\mathrm{op}}\times\{[n]\}}\simeq FT^nU\phi$. The underlying diagram in $\mathcal{B}$ of each row $\Phi|_{\mathbf{\Delta}^{\mathrm{op}}\times\{[n]\}}$ is split, hence the rows all have colimits in $\mathcal{C}$. Let $\overline{\Phi}\colon(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\to\mathcal{C}$ denote the left Kan extension of Φ. Observe that the column $\Phi|_{\{[n]\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ is a free resolution of $\phi([n])$ in the fibre $\mathcal{C}_b$. It is therefore a colimit diagram in $\mathcal{C}_b$, and hence in $\mathcal{C}$. Thus by Lemma 4.4.8 the last column $\Phi|_{\{\infty\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ is also a colimit diagram.

Now consider $U\overline{\Phi}$. The rows $(U\overline{\Phi})|_{(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times\{[n]\}}$ can all be extended to split simplicial diagrams, and are therefore colimits in $\mathcal{D}$. The columns $(U\overline{\Phi})|_{\{[n]\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ can similarly be extended to split simplicial diagrams (in a single fibre) so they are also colimit diagrams. Finally, the underlying diagram in $\mathcal{B}$ is constant, so the last column $(U\overline{\Phi})|_{\{\infty\}\times(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}}$ lies in a single fibre; it is therefore a colimit in $\mathcal{D}$ since U preserves geometric realizations in each fibre. Applying Lemma 4.4.8 again we conclude that the last row $(U\overline{\Phi})|_{(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}\times\{\infty\}}$ is also a colimit, i.e. the colimit of ϕ is preserved by U. □

Proof of Theorem 4.4.4. We will apply Proposition 4.4.5 to the commutative triangle



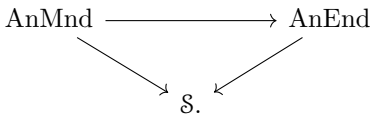
From our previous results the required conditions are satisfied here:

(1) p and q are cocartesian fibrations by Proposition 4.4.2.
(2) The cocartesian pushforward functors are left adjoints, since p and q are also cartesian fibrations, and so preserve all colimits.
(3) U has a left adjoint F such that $pF \simeq q$ by Proposition 4.4.3.
(4) This adjunction restricts to an adjunction in each fibre by construction.
(5) The fibrewise right adjoints preserve sifted colimits by Proposition 4.2.14.
(6) The fibrewise right adjoints detect equivalences since they are monadic by Corollary 4.2.15.

□

5. Analytic Monads and ∞-Operads

5.1. **Analytic Monads.** An *analytic monad* is a monad on $\mathcal{S}_{/I}$ whose underlying endofunctor is analytic, and whose unit and multiplication transformations are cartesian. In other words, it is an associative algebra in $\mathrm{AnEnd}(I)$ under composition. We write $\mathrm{AnMnd}(I)$ for the ∞-category of analytic monads on $\mathcal{S}_{/I}$, defined as the subcategory of $\mathrm{Mnd}(\mathcal{S}_{/I})$ with analytic monads as objects and the morphisms of monads whose underlying maps in $\mathrm{End}(\mathcal{S}_{/I})$ are cartesian transformations as morphisms. Similarly, we define an ∞-category AnMnd over $\mathcal{S}$ of analytic monads over varying base spaces as a subcategory of the pullback of $\widehat{\mathrm{Mnd}}^{\mathrm{op}}_{\mathrm{lax}} \to \widehat{\mathrm{Cat}}^{\mathrm{op}}_{\infty}$ along $\mathcal{S}^*_{/\_} \colon \mathcal{S} \to \widehat{\mathrm{Cat}}^{\mathrm{op}}_{\infty}$. We then get a commutative diagram



We will now use our results on free monads to show that the forgetful functor $\mathrm{AnMnd} \to \mathrm{AnEnd}$ has a left adjoint, and the resulting adjunction is monadic.

To prove this, we will first show that the free monad on an analytic endofunctor is analytic:

Proposition 5.1.1. *The free monad $\overline{P}$ on an analytic endofunctor P is again an analytic endofunctor, and its structure maps $\mu \colon \overline{P} \circ \overline{P} \to \overline{P}$ and $\eta \colon \mathrm{id} \to \overline{P}$ are cartesian natural transformations.*

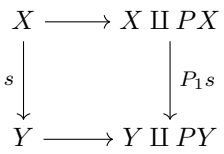
We will prove this using the colimit description of the free monad

$$\overline{P} \simeq \operatorname*{colim}_n P_n$$

from Proposition 4.3.2. The key observation is the following:

Lemma 5.1.2. *Each endofunctor P_n is analytic, and the transition maps $f_{n+1} \colon P_n \to P_{n+1}$ are cartesian.*

Proof. The case $n = 0$ is clear since $P_0 \simeq \mathrm{id}$ is certainly analytic, and $f_1 \colon \mathrm{id} \to \mathrm{id} \amalg P$ is cartesian since for any map $s \colon X \to Y$ over I, the square



is cartesian, as coproducts of cartesian squares are cartesian in an ∞-topos. If P_n is analytic, then $P_{n+1} \simeq \mathrm{id} \amalg (P \circ P_n)$ is analytic, as composites and colimits of analytic functors are analytic, and if f_n is cartesian, then $f_{n+1} \simeq \mathrm{id} \amalg P(f_n)$ is cartesian: the squares

$$\begin{array}{ccc} X \amalg P(P_{n-1}X) & \longrightarrow & X \amalg P(P_n X) \\ \downarrow & & \downarrow \\ Y \amalg P(P_{n-1}Y) & \longrightarrow & Y \amalg P(P_n Y) \end{array}$$

are cartesian since P preserves pullbacks, and coproducts of cartesian squares are cartesian in an ∞-topos. This implies the required result by induction. □

Proof of Proposition 5.1.1. We have $\overline{P} \simeq \mathrm{colim}_n P_n$. By Proposition 2.5.4, the colimit of any diagram of polynomial functors and cartesian transformations is again a polynomial functor (corresponding to the colimit of the associated polynomials). Since analytic endofunctors are a slice of polynomial endofunctors by Corollary 3.1.13 (or since we know from Lemma 4.2.13 that $\overline{P}$ preserves sifted colimits), the endofunctor $\overline{P}$ is therefore analytic.

According to Proposition 4.3.8, the multiplication $\mu\colon \overline{P} \circ \overline{P} \to \overline{P}$ is the colimit of the natural transformations $\mu_h\colon P_h \circ P_h \to P_{2h}$ of Construction 4.3.5. Tracing through the definitions, these are constructed from sum inclusions (which are cartesian since in slices $\mathcal{S}_{/I}$ sums are disjoint), applying P (which preserves cartesianness since it is itself cartesian), and sums of cartesian natural transformations, which are again cartesian (since $\mathcal{S}_{/I}$ is locally cartesian closed). So all the natural transformations $\mu_h\colon P_h \circ P_h \to P_{2h}$ are cartesian. Finally, a filtered colimit of cartesian natural transformations is again cartesian by Proposition 2.5.1, so $\mu \simeq \mathrm{colim}_h \mu_h$ is also cartesian. As to the unit $\eta\colon \mathrm{id} \to \overline{P}$, by Proposition 4.3.8 it is the filtered colimit of the natural transformations $\eta_h\colon \mathrm{id} \to P_h$, each being just a sum inclusion and hence cartesian. Thus η is again cartesian. □

Lemma 5.1.3. *If $u\colon R' \to R$ is a cartesian natural transformation between polynomial endofunctors on $\mathcal{S}_{/I}$, then the induced natural transformation $\overline{u}\colon \overline{R'} \to \overline{R}$ of free monads is again cartesian.*

Proof. The natural transformation $\overline{u}$ is the colimit of the sequence of natural transformations $u_h : R'_h \to R_h$ with $u_0 := \mathrm{id}_{\mathrm{id}}$ and $u_{h+1} : R'_{h+1} \to R_{h+1}$ defined as the composite

$$\mathrm{id} \amalg (R' \circ R'_h) \xrightarrow{\mathrm{id} \amalg R'(u_h)} \mathrm{id} \amalg (R' \circ R_h) \xrightarrow{\mathrm{id} \amalg u} \mathrm{id} \amalg (R \circ R_h).$$

Each u_h is a cartesian natural transformation. Indeed, $u_0 \simeq \mathrm{id}$ clearly is, and if u_h is then so is u_{h+1} since R' preserves pullbacks and u is cartesian. Finally, since $\overline{u}$ is the filtered colimit of cartesian natural transformations, it is again cartesian (since pullbacks distribute over filtered colimits). □

Lemma 5.1.4.

(i) If P is an analytic endofunctor, then the unit map $P \to \overline{P}$ is cartesian.
(ii) If (T, μ, η) is an analytic monad, then the counit map $\overline{T} \to T$ is cartesian.

Proof. The unit map $P \to \overline{P}$ is the sum inclusion $P \to \mathrm{id} \amalg P = P_1$ followed by the colimit map $P_1 \to \overline{P}$. Sum inclusions are cartesian by disjointness of sums, and the colimit map is cartesian by Proposition 2.5.1, since all the transition maps are cartesian by Lemma 5.1.2.

The counit map is the map of monads corresponding to the forgetful functor $\phi\colon \mathrm{Alg}_T(\mathcal{S}_{/I}) \to \mathrm{alg}_T(\mathcal{S}_{/I})$. If we denote these two monadic adjunctions by

$$F : \mathcal{C} \rightleftarrows \mathrm{Alg}_T(\mathcal{S}_{/I}) : U \qquad f : \mathcal{C} \rightleftarrows \mathrm{alg}_T(\mathcal{S}_{/I}) : u,$$

then this natural transformation $\overline{T} \simeq uf \to UF \simeq T$ is given as the composite

$$uf \to ufUF \simeq uf u \phi F \to u \phi F \simeq UF,$$

where the first map comes from the unit for $F \dashv U$ and the second map from the counit for $f \dashv u$. Unwinding our description of the counit $ufu \to u$ from Lemma 4.3.4, we see that this

is the colimit of a sequence of natural transformations $\varepsilon_h : T_h \to T$ defined recursively as follows: $\varepsilon_0 : T_0 = \mathrm{id} \to T$ is the unit of the monad, cartesian by assumption. Assuming we have a cartesian natural transformation $\varepsilon_h : T_h \to T$, the next map $\varepsilon_{h+1} : \mathrm{id} \amalg (T \circ T_h) \to T$ is defined as the sum of the unit $\mathrm{id} \to T$ and the composite

$$T \circ T_h \xrightarrow{T(\varepsilon_h)} T \circ T \xrightarrow{\mu} T.$$

But ε_h is cartesian by induction, and T preserves cartesian maps since it is analytic, and μ is cartesian by assumption. The colimit of all the ε_h is the natural transformation $\overline{T} \to T$, which is then cartesian by descent (Proposition 2.5.1 again). □

Corollary 5.1.5. *The forgetful functor* $\mathrm{AnMnd}(I) \to \mathrm{AnEnd}(I)$ *has a left adjoint, taking an analytic endofunctor to its free monad, and the resulting adjunction is monadic.*

Proof. By Corollary 4.2.15 we have a monadic adjunction

$$F : \mathrm{End}^{\sigma}(\mathcal{S}_{/I}) \rightleftarrows \mathrm{Mnd}^{\sigma}(\mathcal{S}_{/I}) : U.$$

From Proposition 5.1.1 and Lemma 5.1.3 we know that the composite

$$\mathrm{AnEnd}(I) \to \mathrm{End}^{\sigma}(\mathcal{S}_{/I}) \xrightarrow{F} \mathrm{Mnd}^{\sigma}(\mathcal{S}_{/I})$$

lands in the subcategory $\mathrm{AnMnd}(I)$. Moreover, by Lemma 5.1.4 the unit and counit transformations for $F \dashv U$ restrict to natural transformations valued in $\mathrm{AnEnd}(I)$ and $\mathrm{AnMnd}(I)$. Since these restrictions still satisfy the adjunction identities, the adjunction restricts to an adjunction between $\mathrm{AnEnd}(I)$ and $\mathrm{AnMnd}(I)$. Identifying $\mathrm{AnMnd}(I)$ with $\mathrm{Alg}(\mathrm{AnEnd}(I))$, we see by the same proofs as for Proposition 4.2.14 and Corollary 4.2.15 that $\mathrm{AnMnd}(I) \to \mathrm{AnEnd}(I)$ is a monadic right adjoint. □

Now, by the exact same argument as in the proof of Theorem 4.4.4, we get:

Corollary 5.1.6. *The forgetful functor* $\mathrm{AnMnd} \to \mathrm{AnEnd}$ *has a left adjoint, compatible with the projections to* $\mathcal{S}$*, and the resulting adjunction is monadic.*

5.2. **Free Analytic Monads in Terms of Trees.** In this section we will obtain an explicit description of the free monad on an analytic endofunctor in terms of trees, thus extending the description of free analytic monads from [Koc11] to the ∞-categorical setting.

Definition 5.2.1. We shall need various groupoids derived from $\boldsymbol{\Omega}_{\mathrm{int}}$. First of all let

$$\mathrm{tr} := \iota\boldsymbol{\Omega}_{\mathrm{int}}$$

denote the groupoid of all trees, and let $\mathrm{cor} \subseteq \mathrm{tr}$ denote the subgroupoid of corollas (which is equivalent to $\coprod_{n \geq 0} B\Sigma_n$).

Definition 5.2.2. If T is a tree

$$A \xleftarrow{s} M \xrightarrow{p} N \xrightarrow{t} A,$$

then the *leaves* of T are the elements of A that are not in the image of t. Morphisms of trees do not necessarily preserve leaves, but isomorphisms do, yielding a functor $\mathrm{leaves} : \mathrm{tr} \to \iota\mathrm{Fin}$.

Definition 5.2.3. Suppose P is an analytic endofunctor represented by the diagram

$$I \leftarrow E \xrightarrow{p} B \to I.$$

Define spaces tr', $\mathrm{tr}(P)$ and $\mathrm{tr}'(P)$ by pullback as follows:

$$\begin{array}{ccccc}
\mathrm{tr}'(P) & \longrightarrow & \mathrm{tr}(P) & \longrightarrow & \mathrm{AnEnd}_{/P} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{tr}' & \longrightarrow & \mathrm{tr} & \longrightarrow & \mathrm{AnEnd} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{leaves}} & & \\
\iota\mathrm{Fin}_* & \longrightarrow & \iota\mathrm{Fin}. & &
\end{array}$$

The objects of $\mathrm{tr}(P)$ are *P-trees*, defined as diagrams

$$\begin{array}{ccccccc} A & \longrightarrow & M & \longrightarrow & N & \longrightarrow & A \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & I, \end{array}$$

where the first row is a tree. The objects of $\mathrm{tr}'(P)$ are P-trees with a marked leaf, which amount to diagrams

$$\begin{array}{ccccccc} * & \longrightarrow & \emptyset & \longrightarrow & \emptyset & \longrightarrow & * \\ \downarrow & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & M & \longrightarrow & N & \longrightarrow & A \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & I. \end{array}$$

Here the upper right square being a pullback expresses that the edge is a leaf (cf. [Koc11]). (Note that while tr and tr' are 1-truncated (i.e. ordinary groupoids), $\mathrm{tr}(P)$ and $\mathrm{tr}'(P)$ are not in general so. For example, $\mathrm{tr}(P)$ contains the space B (cf. 3.3.6).) The map $\mathrm{tr}(P) \to \mathrm{tr}$ corresponds to the functor $\mathrm{Map}(-, P)\colon \mathrm{tr} \to \mathcal{S}$, and we have the explicit formula

$$\mathrm{tr}(P) \simeq \operatorname*{colim}_{T \in \mathrm{tr}} \mathrm{Map}_{\mathrm{AnEnd}}(T, P) \simeq \coprod_{T \in \mathrm{tr}} \mathrm{Map}_{\mathrm{AnEnd}}(T, P)_{\mathrm{hAut}(T)},$$

where we are implicitly identifying the groupoid tr with its opposite.

The vertical composite $\mathrm{tr}(P) \longrightarrow \mathrm{tr} \xrightarrow{\text{leaves}} \iota\mathrm{Fin}$ factors also through "P-coloured finite sets", which could be denoted $\iota\mathrm{Fin}(P)$ or $\mathbf{E}(I)$:

$$\mathrm{tr}(P) \xrightarrow{\text{leaves}} \mathbf{E}(I) \to \iota\mathrm{Fin}.$$

There is a canonical map $\rho\colon \mathrm{tr}(P) \to I$ which to a P-tree assigns the colour of its root edge. Formally, for each tree T consider the inclusion of the root edge $\eta \to T$. The associated maps $\mathrm{Map}(T, P) \to \mathrm{Map}(\eta, P) \simeq I$ assemble into $\mathrm{tr}(P) \simeq \mathrm{colim}_{T \in \mathrm{tr}} \mathrm{Map}(T, P) \to \mathrm{Map}(\eta, P) \simeq I$. Similarly, there is a canonical map $\lambda\colon \mathrm{tr}'(P) \to I$ which returns the colour of the marked leaf. Formally, this is the composite $\mathrm{tr}'(P) \simeq \mathrm{colim}_{T \in \mathrm{tr}'} \mathrm{Map}(T, P) \to \mathrm{Map}(\eta, P) \simeq I$, where this time $\eta \to T$ picks out the marked leaf.

Theorem 5.2.4. *If P is an analytic endofunctor, then $\overline{P}$, the (underlying endofunctor of the) free monad on P, is represented by the polynomial*

$$I \xleftarrow{\lambda} \mathrm{tr}'(P) \longrightarrow \mathrm{tr}(P) \xrightarrow{\rho} I.$$

Remark 5.2.5. See [Koc11] for the analogous result in the case of sets, and [Koc17] for the groupoid case.

To prove this, we use the description of $\overline{P}$ given in Definition 4.3.1 and Proposition 4.3.2, as the colimit of the sequence of functors defined by $P_0 \simeq \mathrm{id}$ and $P_{h+1} \simeq \mathrm{id} \amalg (P \circ P_h)$, which we shall describe in terms of trees of bounded height. For this we need some notation.

Definition 5.2.6. For $A \leftarrow M \to N \to A$ a tree, the *height* of $e \in A$ is the minimal $k \in \mathbb{N}$ such that $\sigma^k(e)$ is the root edge. Here σ is the 'successor' function (or walk-to-the-root function) from the definition of tree (3.3.1). The *height* of the tree T is the maximal height of its edges. Hence the trivial tree has height 0 and any corolla has height 1.

Let $\mathrm{tr}_{\leq h}$ denote the subgroupoid of tr containing only the trees of height $\leq h$. For P an analytic endofunctor, and for each $h \in \mathbb{N}$, define groupoids $\mathrm{tr}'_{\leq h}$, $\mathrm{tr}_{\leq h}(P)$ and $\mathrm{tr}'_{\leq h}(P)$ by pullbacks

$$\begin{array}{ccccc}
\mathrm{tr}'_{\leq h}(P) & \xrightarrow{p_h} & \mathrm{tr}_{\leq h}(P) & \longrightarrow & \mathrm{AnEnd}_{/P} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{tr}'_{\leq h} & \longrightarrow & \mathrm{tr}_{\leq h} & \longrightarrow & \mathrm{AnEnd} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{leaves}} & & \\
\iota\mathrm{Fin}_* & \longrightarrow & \iota\mathrm{Fin} & &
\end{array}$$

We have

$$\mathrm{tr}_{\leq h}(P) := \operatorname*{colim}_{T \in \mathrm{tr}_{\leq h}} \mathrm{Map}_{\mathrm{AnEnd}}(T, P).$$

Let $\mathrm{tr}_{\bullet,\leq h}$ denote the subgroupoid containing the trees of height $\leq h$ that have a root node (i.e. we exclude η). We have a forgetful functor $\mathrm{bot}\colon \mathrm{tr}_{\bullet,\leq h} \to \mathrm{tr}_{\bullet,\leq 1} \simeq \mathrm{cor} \simeq \iota\mathrm{Fin}$ that takes a tree to its root corolla.

Lemma 5.2.7. *For each fixed $k \in \iota\mathrm{Fin}$, we have a pullback square*

$$\begin{array}{ccc}
(\mathrm{tr}_{\leq h})^{\times k} & \longrightarrow & \mathrm{tr}_{\bullet,\leq h+1} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{bot}} \\
* & \xrightarrow[k]{} & \iota\mathrm{Fin}
\end{array}$$

Proof. The map $(\mathrm{tr}_{\leq h})^{\times k} \to \mathrm{tr}_{\bullet,\leq h+1}$ takes a k-tuple of height-h trees and grafts them onto the corolla k. It is clear that the fibre of this map is the set of automorphisms of k, just as the fibre of $k\colon * \to \iota\mathrm{Fin}$. □

Let P be an analytic endofunctor. Recall from Definition 4.3.1 the sequence of endofunctors P_h defined by $P_0 := \mathrm{id}$, and $P_{h+1} := \mathrm{id} \amalg (P \circ P_h)$. By Lemma 5.1.2, each P_h is analytic.

Proposition 5.2.8. *If P is represented by the polynomial*

$$I \leftarrow E \to B \to I,$$

then P_h is represented by the polynomial

$$I \xleftarrow{s_h} \mathrm{tr}'_{\leq h}(P) \xrightarrow{p_h} \mathrm{tr}_{\leq h}(P) \xrightarrow{t_h} I.$$

Here s_h assigns to a leaf-marked tree the colour of the marked leaf, and t_h assigns the colour of the root edge.

The proof requires a couple of auxiliary results, exploiting that the analytic functor P lives over $\mathbf{E}$. With notation as in Proposition 3.2.6, we have the diagram

$$\begin{array}{ccccccc}
I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\
{\scriptstyle \jmath}\downarrow & & \downarrow{\scriptstyle \epsilon} & \lrcorner & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \jmath} \\
* & \xleftarrow[u]{} & \iota\mathrm{Fin}_* & \xrightarrow[q]{} & \iota\mathrm{Fin} & \longrightarrow & *.
\end{array}$$

Lemma 5.2.9. *With notation as above, there is a natural pullback square*

$$\begin{array}{ccc}
\mathrm{tr}_{\bullet,\leq h+1}(P) & \longrightarrow & \mathbf{E}(\mathrm{tr}_{\leq h}(P)) \\
{\scriptstyle \mathrm{bot}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{E}(t_h)} \\
\mathrm{cor}(P) \simeq B & \xrightarrow[\mathrm{leaves}=\bar{s}]{} & \mathbf{E}(I),
\end{array}$$

where $\bar{s}\colon B \to q_* u^* I = \mathbf{E}(I)$ corresponds to $u_! q^* B = E \xrightarrow{s} I$ under the adjunction $u_! q^* \dashv q_* u^*$.

Unravelling the definitions, this says that giving a P-tree with a bottom node and of height $\leq h+1$ is the same thing as giving the bottom node (a P-corolla) and a P-forest of trees of height $\leq h$ whose root edges match the leaves of the bottom node.

Proof. Expanded in colimits, the asserted square reads as follows.

$$\begin{array}{ccc} \operatorname*{colim}_{T\in \mathrm{tr}_{\bullet,\leq h+1}} \mathrm{Map}(T,P) & \longrightarrow & \operatorname*{colim}_{k\in \iota\mathrm{Fin}} \mathrm{tr}_{\leq h}(P)^{\times k} \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{k\in \iota\mathrm{Fin}} \mathrm{Map}(C_k,P) & \longrightarrow & \operatorname*{colim}_{k\in \iota\mathrm{Fin}} \mathrm{Map}(\eta,P)^{\times k}. \end{array}$$

By Lemma 5.2.7, we can rewrite $\mathrm{colim}_{T\in \mathrm{tr}_{\bullet,\leq h+1}} \mathrm{Map}(T,P)$ as an iterated colimit

$$\operatorname*{colim}_{k\in \iota\mathrm{Fin}} \operatorname*{colim}_{(T_i)\in(\mathrm{tr}_{\leq h})^{\times k}} \mathrm{Map}(T,P).$$

Since we have natural pushouts $T \simeq C_k \amalg_{\coprod_{i=1}^k \eta} \coprod_{i=1}^k T_i$ this is equivalent to

$$\operatorname*{colim}_{k\in \iota\mathrm{Fin}} \operatorname*{colim}_{(T_i)\in(\mathrm{tr}_{\leq h})^{\times k}} \mathrm{Map}(C_k,P) \underset{\prod_{i=1}^k \mathrm{Map}(\eta,P)}{\times} \prod_{i=1}^k \mathrm{Map}(T_i,P).$$

Since colimits in $\mathcal{S}$ are universal and products commute with colimits, we can rewrite this as

$$\operatorname*{colim}_{k\in \iota\mathrm{Fin}} \left(\mathrm{Map}(C_k,P) \underset{\prod_{i=1}^k \mathrm{Map}(\eta,P)}{\times} \prod_{i=1}^k \operatorname*{colim}_{T_i\in \mathrm{tr}_{\leq h}} \mathrm{Map}(T_i,P) \right) \simeq \operatorname*{colim}_{k\in \iota\mathrm{Fin}} \left(\mathrm{Map}(C_k,P) \underset{\mathrm{Map}(\eta,P)^{\times k}}{\times} \mathrm{tr}_{\leq h}(P)^{\times k} \right).$$

But colimits over ∞-groupoids commute with weakly contractible limits by Lemma 2.2.8, so this is equivalent to

$$(\operatorname*{colim}_{k\in \iota\mathrm{Fin}} \mathrm{Map}(C_k,P)) \times_{\mathrm{colim}_{k\in \iota\mathrm{Fin}} \mathrm{Map}(\eta,P)^{\times k}} (\operatorname*{colim}_{k\in \iota\mathrm{Fin}} \mathrm{tr}_{\leq h}(P)^{\times k}). \qquad \square$$

Corollary 5.2.10. *P evaluated on the map $t_h\colon \mathrm{tr}_{\leq h}(P) \to I$ yields $t_{h+1}\colon \mathrm{tr}_{\bullet,\leq h+1}(P) \to I$.*

Proof. Proposition 3.2.6 describes $p_* s^*(t_h)$ as the vertical left map in the pullback

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{E}(\mathrm{tr}_{\leq h}(P)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{E}(t_h)} \\ \mathrm{cor}(P) \simeq B & \xrightarrow[\bar{s}]{} & \mathbf{E}(I), \end{array}$$

and Lemma 5.2.9 identifies this map as $\mathrm{bot}\colon \mathrm{tr}_{\bullet,\leq h+1}(P) \to B$. Applying finally $t_!$ is to compose with t, yielding t_{h+1} as required. $\square$

Proof of Proposition 5.2.8. This goes by induction on h. First of all, $P_0 := \mathrm{id}$ is represented by $I \leftarrow I \to I \to I$, but I is also the space of P-trees of height ≤ 0. This establishes the base of the induction. Assuming we have already established that P_h is represented by trees of height $\leq h$, we need to identify the composite $P \circ P_h$, using the explicit description given in Theorem 2.1.8. We have already computed the space in the upper right corner (denoted D in the big diagram in 2.1.8): by Corollary 5.2.10, it is the space $\mathrm{tr}_{\bullet,\leq h+1}(P)$ of trees of height $\leq h+1$ and with a bottom node. To compute the space in the upper left corner (denoted G in the big diagram in 2.1.8), we need first to pull back along $p : E \to B$: this gives the same space of trees but with a marked incoming edge of the bottom node. This space comes with a canonical projection to $\mathrm{tr}_{\leq h}(P)$ given by returning the tree sitting over that marked edge. Finally we need to pull back along p_h, which amounts to marking a leaf of that marked subtree. Together the two pullbacks amount to marking any leaf, giving thus the space $\mathrm{tr}'_{\bullet,\leq h+1}(P)$. Finally, the formula for P_{h+1} adds in the trivial tree by means of the summand id. This compensates precisely for the requirement of having a bottom node. $\square$

Proof of Theorem 5.2.4. We know from Proposition 4.3.2 that the free monad $\overline{P}$ is the colimit of the sequence P_h, where P_h is represented by

$$I \longleftarrow \mathrm{tr}'_{\leq h}(P) \longrightarrow \mathrm{tr}_{\leq h}(P) \longrightarrow I$$

by Proposition 5.2.8. It then follows from Proposition 2.5.4, Theorem 2.4.11, and Corollary 2.5.2 that the colimit $\overline{P}$ is the polynomial functor corresponding to the pointwise colimit of these diagrams, which is clearly

$$I \longleftarrow \mathrm{tr}'(P) \longrightarrow \mathrm{tr}(P) \longrightarrow I$$

as asserted. □

Remark 5.2.11. The monad structure on $\overline{P}$ is also pleasantly described in terms of trees. The space of operations of $\overline{P} \circ \overline{P}$ is $\overline{P}(\mathrm{tr}(P))$, the space of P-trees whose leaves are decorated by P-trees in a compatible way. More precisely, the objects of $\overline{P}(\mathrm{tr}(P))$ are tuples

$$\Big(R, \begin{array}{ccc} \mathrm{leaves}(R) & \xrightarrow{f} & \mathrm{tr}(P) \\ & \searrow & \swarrow_{\rho} \\ & I & \end{array}\Big)$$

where R is a P-tree and f assigns to each leaf of R a P-tree whose root edge has the same colour. The monad multiplication $\overline{P} \circ \overline{P} \to \overline{P}$ now simply takes this configuration and glues those trees onto the leaves of R. Clearly this construction is just the colimit of the same construction with trees of height m and n, which is the tree interpretation of the natural transformations $P_n \circ P_m \to P_{m+n}$ from 4.3.5.

Cartesianness of μ can also be established along these lines: the arity of an operation (R, f) as above is the disjoint union of all the leaves of all the upper trees. Clearly this is the same as the set of all leaves of the resulting total tree.

Proposition 5.2.12. *Let P be an analytic endofunctor on $\mathcal{S}_{/I}$, and $\overline{P}$ the free monad on P. Under the equivalence $i^*\colon \mathrm{AnEnd} \xrightarrow{\sim} \mathcal{P}(\mathbf{\Omega}_{\mathrm{el}})$ of Proposition 3.3.10, the underlying endofunctor of $\overline{P}$ is identified with the presheaf*

$$C_n \longmapsto \operatorname*{colim}_{T \in n\text{-tr}} \mathrm{Map}(T, P).$$

(and $\eta \mapsto I$). Here n-tr is the homotopy fibre over n of the map $\mathrm{tr} \to \iota\mathrm{Fin}$ that sends a tree to its set of leaves.

Proof. The equivalence i^* sends $\overline{P}$ to the presheaf $C_n \mapsto \mathrm{Map}(C_n, \overline{P})$. But we know from Theorem 5.2.4 that $\overline{P}$ is represented by $I \leftarrow \mathrm{tr}'(P) \to \mathrm{tr}(P) \to I$. Now by 3.3.6, we have $\mathrm{Map}(C_n, \overline{P}) \simeq n\text{-tr}(P)$, the latter defined as the left composite pullback

$$\begin{array}{ccccc} n\text{-tr}(P) & \longrightarrow & \mathrm{tr}(P) & \longrightarrow & \mathrm{AnEnd}_{/P} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ n\text{-tr} & \longrightarrow & \mathrm{tr} & \longrightarrow & \mathrm{AnEnd} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{leaves}} & & \\ * & \xrightarrow[n]{} & \iota\mathrm{Fin}. & & \end{array}$$

From the top composite pullback, we get (in analogy with 5.2.3) $n\text{-tr}(P) \simeq \mathrm{colim}_{T \in n\text{-tr}} \mathrm{Map}(T, P)$, as claimed. In summary:

$$i^*\overline{P}(C_n) \simeq \mathrm{Map}(C_n, \overline{P}) \simeq n\text{-tr}(P) \simeq \operatorname*{colim}_{T \in n\text{-tr}} \mathrm{Map}(T, P).$$

□

5.3. **Analytic Monads versus Dendroidal Segal Spaces.** In this subsection we will prove the main result of the paper, that analytic monads are equivalent to ∞-operads. First, we need to recall the model of ∞-operads we will use for the comparison, namely the dendroidal Segal spaces of Cisinski and Moerdijk [CM13a].

Definition 5.3.1. The *dendroidal category* $\mathbf{\Omega}$ is the full subcategory of AnMnd spanned by the image of $\mathbf{\Omega}_{\mathrm{int}}$, i.e. the free monads on the trees.

Remark 5.3.2. Since trees themselves are polynomials in Set, and since the free monad on a set polynomial is again a set polynomial, the definition given here agrees with that of [Koc11], which in turn is just a polynomial reformulation of the original definition of [MW07]. Recall that $\mathbf{\Omega}$ has as morphisms the monad maps between free monads on the trees, and that $\mathbf{\Omega}$ has an active–inert factorization system (also called the generic–free factorization system [BMW12]): The inert maps are the tree inclusions, defined formally as the morphisms of polynomial functors between trees, forming the category $\mathbf{\Omega}_{\mathrm{int}}$ studied so far. The active maps are given by node refinements, characterized also as the monad maps that preserve leaves and root. This includes the codegeneracy case where a unary node is "refined" into a nodeless tree. To specify an active map out of a corolla with set of leaves L amounts to giving a tree with L as set of leaves. The only active map out of the trivial tree η is the identity. A general active map $T \to T'$ is specified by giving an active map out of each node corolla, and then gluing together the resulting trees along roots and leaves, according to the same recipe that gave T as the colimit of its elementary trees.

Definition 5.3.3. A presheaf F on $\mathbf{\Omega}$ is called a *Segal presheaf* if its restriction j^*F along the inclusion $j\colon \mathbf{\Omega}_{\mathrm{int}} \to \mathbf{\Omega}$ is a Segal presheaf on $\mathbf{\Omega}_{\mathrm{int}}$, as in 3.3.12. We define the ∞-category $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega})$ of Segal presheaves to be the pullback

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}) & \longrightarrow & \mathcal{P}(\mathbf{\Omega}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle j^*} \\ \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}}) & \longrightarrow & \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}}). \end{array}$$

Theorem 5.3.4. *The restricted Yoneda embedding $N\colon \mathrm{AnMnd} \to \mathcal{P}(\mathbf{\Omega})$ is fully faithful, and its essential image is $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega})$. We thus have an equivalence of ∞-categories*

$$\mathrm{AnMnd} \simeq \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}).$$

The proof will be based on the following general observation, which is an ∞-categorical version of a result of Berger, Melliès and Weber [BMW12]; they use it to give a proof of the "nerve theorem" for monads (originally due to Weber [Web07]).

Proposition 5.3.5. *Suppose given a commutative square of ∞-categories*

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\bar{\phi}} & \mathcal{E}_2 \\ {\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle U_2} \\ \mathcal{B}_1 & \xrightarrow[\phi]{} & \mathcal{B}_2 \end{array}$$

such that

(1) the functor U_i has a left adjoint F_i for $i = 1, 2$,
(2) the adjunction $F_i \dashv U_i$ is monadic for $i = 1, 2$,
(3) the functor ϕ is fully faithful,
(4) the mate transformation $F_2\phi \to \bar{\phi}F_1$ is a natural equivalence,

Then the functor $\bar{\phi}$ is also fully faithful, and its essential image consists of those $A \in \mathcal{E}_2$ such that U_2A is in the image of ϕ. (In other words, the commutative square above is cartesian.)

Proof. We first prove that $\bar{\phi}$ is fully faithful, i.e. that for all $A, B \in \mathcal{E}_1$ the map

$$\mathrm{Map}_{\mathcal{E}_1}(A,B) \to \mathrm{Map}_{\mathcal{E}_2}(\bar{\phi}A, \bar{\phi}B)$$

is an equivalence.

First suppose A is free, i.e. of the form F_1X for some $X \in \mathcal{B}_1$. Then we have natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{E}_1}(F_1X,B) \simeq \mathrm{Map}_{\mathcal{B}_1}(X,U_1B) \simeq \mathrm{Map}_{\mathcal{B}_2}(\phi X, \phi U_1 B) \simeq \mathrm{Map}_{\mathcal{B}_2}(\phi X, U_2\bar{\phi}B)\\ \simeq \mathrm{Map}_{\mathcal{E}_2}(F_2\phi X, \bar{\phi}B) \simeq \mathrm{Map}_{\mathcal{E}_2}(\bar{\phi}F_1X, \bar{\phi}B).\end{aligned}$$

For a general A, we can choose a U_1-split simplicial free resolution $A_\bullet$. Since each A_n is free, we have natural equivalences

$$\mathrm{Map}_{\mathcal{E}_1}(A,B) \simeq \lim \mathrm{Map}_{\mathcal{E}_1}(A_\bullet, B) \simeq \lim \mathrm{Map}_{\mathcal{E}_2}(\bar{\phi}A_\bullet, \bar{\phi}B).$$

Since $A_\bullet$ is U_1-split and $U_2\bar{\phi} \simeq \phi U_1$, the simplicial diagram $\bar{\phi}A_\bullet$ is U_2-split. Since $F_2 \dashv U_2$ is monadic, this implies that the colimit $|\bar{\phi}A_\bullet|$ exists and is preserved by U_2. There is a canonical map $|\bar{\phi}A_\bullet| \to \bar{\phi}(A)$ and to see that it is an equivalence it suffices to show that it is one after applying U_2. But since the diagram $U_1A_\bullet$ is split, so is $\phi U_1 A_\bullet \simeq U_2\bar{\phi}A_\bullet$ and therefore its colimit is $\phi U_1 A \simeq U_2\bar{\phi}A$, as required. We thus have a natural equivalence

$$\mathrm{Map}_{\mathcal{E}_1}(A,B) \simeq \lim \mathrm{Map}_{\mathcal{E}_2}(\bar{\phi}A_\bullet, \bar{\phi}B) \simeq \mathrm{Map}_{\mathcal{E}_2}(|\bar{\phi}A_\bullet|, \bar{\phi}B) \simeq \mathrm{Map}_{\mathcal{E}_2}(\bar{\phi}A, \bar{\phi}B).$$

This shows that $\bar{\phi}$ is fully faithful; it remains to prove that if $A \in \mathcal{E}_2$ satisfies $U_2A \simeq \phi X$ for some $X \in \mathcal{B}_1$, then A is in the image of $\bar{\phi}$. We can view A as the geometric realization of its canonical free resolution $A_\bullet := F_2(U_2F_2)^\bullet U_2A$. We have $U_2F_2\phi \simeq U_2\bar{\phi}F_1 \simeq \phi U_1F_1$, so $F_2(U_2F_2)^nU_2A \simeq \bar{\phi}F_1(U_1F_1)^nX$. Since $\bar{\phi}$ is fully faithful, the diagram $A_\bullet$ factors through $\mathcal{E}_1$, i.e. we have a simplicial diagram $A'_\bullet$ in $\mathcal{E}_1$ such that $\bar{\phi}A'_\bullet \simeq A_\bullet$. The diagram $A_\bullet$ is also U_2-split, and since $U_2A \simeq \phi X$ the extension of $U_2A_\bullet$ to a split simplicial diagram factors through $\mathcal{B}_1$ as ϕ is fully faithful. Thus $A'_\bullet$ is U_1-split. Since the adjunction $F_1 \dashv U_1$ is monadic, this implies that $A'_\bullet$ has a colimit A' in $\mathcal{E}_1$, and this colimit is preserved by U_1. There is then a canonical map $A \to \bar{\phi}A'$, and this is an equivalence since U_2 detects equivalences. This proves that A is in the essential image of $\bar{\phi}$, as required. □

We are going to apply Proposition 5.3.5 to the commutative diagram

$$\begin{array}{ccc}\mathrm{AnMnd} & \xrightarrow{N} & \mathcal{P}(\boldsymbol{\Omega}) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle j^*} \\ \mathrm{AnEnd} & \xrightarrow[N_{\mathrm{int}}]{} & \mathcal{P}(\boldsymbol{\Omega}_{\mathrm{int}}).\end{array}$$

The vertical functors have left adjoints F and $j_!$, respectively. A key step (which will be Proposition 5.3.16 below) is to show that the mate commutes

$$\begin{array}{ccc}\mathrm{AnMnd} & \xrightarrow{N} & \mathcal{P}(\boldsymbol{\Omega}) \\ \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle j_!} \\ \mathrm{AnEnd} & \xrightarrow[N_{\mathrm{int}}]{} & \mathcal{P}(\boldsymbol{\Omega}_{\mathrm{int}}).\end{array}$$

To establish this, we will need:

- a simplified formula for $j^*j_!$ in terms of active maps (Lemma 5.3.10 below),
- some results regarding compatibility of Segal presheaves with active maps and colimits of subtrees (Corollary 5.3.12 and Lemma 5.3.14),
- the formula for the free monad on an analytic endofunctor in terms of trees, already established in Proposition 5.2.12, which allows for reduction to the case of elementary trees (Lemma 5.3.15).

Lemma 5.3.6. *Let $\beta\colon T \to T'$ be an active map.*

(i) There is an induced functor

$$\beta_! \colon \mathbf{\Omega}_{\mathrm{el}/T} \to \mathbf{\Omega}_{\mathrm{int}/T'}$$

which takes an elementary tree $E \to T$ to the inert part of the active-inert factorization of the composite $E \to T \to T'$, as in

$$\begin{array}{ccc} T & \xrightarrow{\beta} & T' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ E & \xrightarrow[\alpha]{} & \beta_! E. \end{array}$$

(ii) There is an induced colimit decomposition of T' into subtrees $\beta_!(E) \subset T'$:

$$T' \simeq \operatorname*{colim}_{E \in \mathrm{el}(T)} \beta_!(E).$$

Proof. (i) is clear. For (ii), note that active maps preserve leaves and root. The colimit $T \simeq \mathrm{colim}_{E\in\mathrm{el}(T)} E$ is an iterated grafting, i.e. an iterated pushout over trivial trees, each included into one tree as the root and into another tree as a leaf. Since the only active map out of η is the identity, the colimit asserted in (ii) is again an iterated pushout over trivial trees, and since for each $E \in \mathrm{el}(T)$ the active map $E \to \beta_! E$ preserves leaves and root, the colimit asserted in (ii) is again an iterated grafting. Finally, since all the trees $\beta_! E$ are subtrees of T', and are disjoint on nodes, the colimit defines a subtree of T'. Since each node in T' appears in precisely one of these subtrees, the colimit must actually be all of T'. (In fact, the whole map β is the colimit of the maps $E \to \beta_!(E)$, cf. [Koc11, 1.3.4 and 1.3.16].) □

Definition 5.3.7. Let $\mathrm{Fun}^{\mathrm{act}}(\Delta^1, \mathbf{\Omega}) \subset \mathrm{Fun}(\Delta^1, \mathbf{\Omega})$ denote the full subcategory of the arrow category of $\mathbf{\Omega}$ spanned by the active maps. Thanks to the active–inert factorization system in $\mathbf{\Omega}$, the domain projection $\mathrm{Fun}^{\mathrm{act}}(\Delta^1, \mathbf{\Omega}) \to \mathbf{\Omega}$ is a cartesian fibration: the cartesian arrows are the squares with codomain arrow inert (see [GCKT18, Lemma 1.3]). The associated right fibration we pull back to $\mathbf{\Omega}_{\mathrm{int}}$ and straighten to get a presheaf $\mathrm{Act} \colon (\mathbf{\Omega}_{\mathrm{int}})^{\mathrm{op}} \to \mathcal{S}$. Thus $\mathrm{Act}(T)$ is the ∞-groupoid of active maps $T \to T'$ in $\mathbf{\Omega}$ (actually just a 1-groupoid); for example, $\mathrm{Act}(C_n)$ is the groupoid n-tr of trees with n leaves. Note also that $\mathrm{Act}(\eta) \simeq *$, the only active map out of η being the identity.

Remark 5.3.8. As $\mathrm{Act} \colon (\mathbf{\Omega}_{\mathrm{int}})^{\mathrm{op}} \to \mathcal{S}$ factors through the full subcategory $\mathrm{Gpd} \subset \mathcal{S}$ of groupoids (equivalently, 1-truncated spaces), we are really just applying a 1-categorical straightening result here. One could also directly define Act as an explicit (pseudo)functor, but this would involve making arbitrary choices, since inert-active factorizations are only defined up to unique isomorphism.

Lemma 5.3.9. *The presheaf $\mathrm{Act} \colon (\mathbf{\Omega}_{\mathrm{int}})^{\mathrm{op}} \to \mathcal{S}$ satisfies the Segal condition. More precisely, for any tree T, we have*

$$\mathrm{Act}(T) \simeq \lim_{E \in \mathrm{el}(T)} \mathrm{Act}(E) \simeq \prod_{C \in \mathrm{cor}(T)} \mathrm{Act}(C).$$

Proof. The second equivalence follows from $\mathrm{Act}(\eta) \simeq *$. The map $\mathrm{Act}(T) \to \prod_{C\in\mathrm{cor}(T)} \mathrm{Act}(C)$ sends an active map $\beta \colon T \to T'$ to the collection of active maps $\alpha \colon C \to S'$ as in Lemma 5.3.6(i). A map in the other direction is given by gluing together all the subtrees S' according to the same recipe as the corollas C glue together to give T, as in Lemma 5.3.6(ii). This constitutes a bijection at the level of isomorphism classes by [Koc11, 1.3.16]. Since the spaces involved are just 1-groupoids it thus remains to check that the automorphism groups match up. But an automorphism of an active map $\beta \colon T \to T'$ is the same as an automorphism of T' that fixes the edges from T, and this amounts to giving for each $C \in \mathrm{cor}\, T$ an automorphism of the corresponding tree S' that fixes all leaves, which in turn is precisely to give an automorphism of the active map $\alpha \colon C \to S'$. So $\mathrm{Aut}(\beta) = \prod \mathrm{Aut}(\alpha)$ as required. □

Lemma 5.3.10. *The active-inert factorization system on $\mathbf{\Omega}$ induces an equivalence*

$$(j^* j_! \Phi)(T) \simeq \operatorname*{colim}_{T \to T' \in \mathrm{Act}(T)} \Phi(T')$$

for each Segal presheaf Φ and each tree T.

Proof. Since $j\colon \mathbf{\Omega}_{\mathrm{int}} \to \mathbf{\Omega}$ is the identity on objects, $(j^* j_! \Phi)(T) \simeq (j_!\Phi)(jT) \simeq (j_!\Phi)(T)$. By the usual formula for the left Kan extension, we have that

$$(j_!\Phi)(T) \simeq \operatorname*{colim}_{T \to T' \in ((\mathbf{\Omega}_{\mathrm{int}})_{T/})^{\mathrm{op}}} \Phi(T'),$$

so it suffices to show that the functor $\mathrm{Act}(T)^{\mathrm{op}} \to ((\mathbf{\Omega}_{\mathrm{int}})_{T/})^{\mathrm{op}}$ is cofinal. Invoking [Lur09, 4.1.3.1], it suffices to show that for all $f\colon T \to T' \in ((\mathbf{\Omega}_{\mathrm{int}})_{T/})$, the pullback

$$\mathrm{Act}(T) \underset{(\mathbf{\Omega}_{\mathrm{int}})_{T/}}{\times} (\mathbf{\Omega}_{\mathrm{int}})_{T//f}$$

is a weakly contractible ∞-category. But this is precisely the ∞-category of active-inert factorizations of $f\colon T \to T'$, which is contractible by [Lur09, Proposition 5.2.8.17]. □

Proposition 5.3.11. *Let $\Phi\colon \mathbf{\Omega}_{\mathrm{int}}^{\mathrm{op}} \to \mathcal{S}$ be a Segal presheaf. Let $T \simeq S \amalg_\eta R$ be the tree obtained by grafting a tree S onto a leaf of another tree R. Then the canonical map*

$$\Phi(T) \to \Phi(S) \times_{\Phi(\eta)} \Phi(R)$$

is an equivalence.

Proof. Since Φ is Segal, we have $\Phi(T) \xrightarrow{\sim} \lim_{E \in \mathrm{el}(T)} \Phi(E)$. On the other hand, we have $\mathrm{el}(T) \simeq \mathrm{el}(S) \amalg_{\mathrm{el}(\eta)} \mathrm{el}(R)$. It follows that the limit can be computed in steps:

$$\lim_{E \in \mathrm{el}(T)} \Phi(E) \simeq \lim_{E \in \mathrm{el}(S)} \Phi(E) \times_{\Phi(\eta)} \lim_{E \in \mathrm{el}(R)} \Phi(E) \simeq \Phi(S) \times_{\Phi(\eta)} \Phi(R).$$

□

Corollary 5.3.12. *If $\Phi\colon \mathbf{\Omega}_{\mathrm{int}}^{\mathrm{op}} \to \mathcal{S}$ is a Segal presheaf, and if $T \simeq \mathrm{colim}_{\mathcal{I}} R$ is a colimit of certain subtrees grafted to each other, then the canonical map*

$$\Phi(T) \to \lim_{\mathcal{I}} \Phi(R)$$

is an equivalence.

Proof. This follows by iterated application of Proposition 5.3.11. □

Corollary 5.3.13. *For $\Phi\colon \mathbf{\Omega}_{\mathrm{int}}^{\mathrm{op}} \to \mathcal{S}$ a Segal presheaf and $\beta\colon T \to T'$ an active map, the canonical map*

$$\Phi(T') \longrightarrow \lim_{E \in \mathrm{el}(T)} \Phi(\beta_!(E))$$

is an equivalence.

Proof. By Lemma 5.3.6(ii) we have $T' \simeq \mathrm{colim}_{E \in \mathrm{el}(T)} \beta_!(E)$, and the result follows from Corollary 5.3.12. □

Lemma 5.3.14. *For $\Phi\colon \mathbf{\Omega}_{\mathrm{int}}^{\mathrm{op}} \to \mathcal{S}$ a Segal presheaf and T a tree, there is a natural equivalence*

$$\operatorname*{colim}_{T \to T' \in \mathrm{Act}(T)} \Phi(T') \xrightarrow{\sim} \lim_{E \in \mathrm{el}(T)} \operatorname*{colim}_{E \to S' \in \mathrm{Act}(E)} \Phi(S').$$

Proof. Given an active map $\beta\colon T \to T'$ and an elementary subtree $f\colon E \to T$, we can active-inert factor the composite as in Lemma 5.3.6:

$$\begin{array}{ccc} T & \xrightarrow{\beta} & T' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ E & \xrightarrow[\alpha]{} & \beta_! E. \end{array}$$

We now have the map

$$\Phi(T') \xrightarrow{f'^*} \Phi(\beta_! E) \longrightarrow \operatorname*{colim}_{E \to S' \in \mathrm{Act}(E)} \Phi(S'),$$

and letting β and f vary, we get altogether the map of the statement. By construction we have a commutative square of ∞-groupoids

$$\begin{array}{ccc} \operatorname*{colim}_{T\to T'\in \mathrm{Act}(T)} \Phi(T') & \longrightarrow & \lim_{E\in \mathrm{el}(T)} \operatorname*{colim}_{E\to S'\in \mathrm{Act}(E)} \Phi(S') \\ \downarrow & & \downarrow \\ \mathrm{Act}(T) & \xrightarrow{\ \sim\ } & \lim_{E\in \mathrm{el}(T)} \mathrm{Act}(E). \end{array}$$

Since the bottom horizontal map is an equivalence by Lemma 5.3.9, to conclude that the top horizontal map is an equivalence, it suffices to show that, for any given basepoint $\beta\colon T \to T'$ in $\mathrm{Act}(T)$, the map on fibres

$$\Phi(T') \longrightarrow \lim_{E\in \mathrm{el}(T)} \Phi(S')$$

is an equivalence. But this is Corollary 5.3.13, since $S' \simeq \beta_!(E)$. □

Lemma 5.3.15. *For P an analytic endofunctor, the natural transformation*

$$j^* j_! N_{\mathrm{int}} P \to N_{\mathrm{int}} UFP$$

is an equivalence on elementary trees.

Proof. The statement is easily seen to be true for the trivial tree η. Consider now a corolla C. On the left side we compute (using the colimit formula for $j_!$ of Lemma 5.3.10)

$$(j^* j_! N_{\mathrm{int}} P)(C) \simeq \operatorname*{colim}_{C\to T'\in \mathrm{Act}(C)} N_{\mathrm{int}} P(T') \simeq \operatorname*{colim}_{C\to T'\in \mathrm{Act}(C)} \mathrm{Map}(T', P).$$

On the right we compute (using the colimit formula for the free-monad monad of Proposition 5.2.12)

$$(N_{\mathrm{int}} UFP)(C) \simeq \mathrm{Map}(C, UFP) \simeq (UFP)(C) \simeq \operatorname*{colim}_{C\to T'\in \mathrm{Act}(C)} P(T') \simeq \operatorname*{colim}_{C\to T'\in \mathrm{Act}(C)} \mathrm{Map}(T', P). \quad \square$$

Proposition 5.3.16. *The mate square*

$$\begin{array}{ccc} \mathrm{AnMnd} & \xrightarrow{N} & \mathcal{P}(\boldsymbol{\Omega}) \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle j_!} \\ \mathrm{AnEnd} & \xrightarrow[N_{\mathrm{int}}]{} & \mathcal{P}(\boldsymbol{\Omega}_{\mathrm{int}}) \end{array}$$

commutes. In other words, the natural transformation

$$j_! \circ N_{\mathrm{int}} \to N \circ F$$

is an equivalence.

Proof. Since j^* is conservative, it is enough to check that $j^* j_! N_{\mathrm{int}} \to j^* NF \simeq N_{\mathrm{int}} UF$ is an equivalence. Let P be an analytic endofunctor, and put $\Phi := N_{\mathrm{int}} P$ and $\overline{P} := UFP$. For T a tree, we have:

$$\begin{aligned} j^* j_! \Phi(T) &\simeq \operatorname*{colim}_{T\to T'\in \mathrm{Act}(T)} \Phi(T') && \text{by Lemma 5.3.10} \\ &\simeq \lim_{E\in \mathrm{el}(T)} \operatorname*{colim}_{E\to S'\in \mathrm{Act}(E)} \Phi(S') && \text{by Lemma 5.3.14} \\ &\simeq \lim_{E\in \mathrm{el}(T)} j^* j_! \Phi(E) && \text{by Lemma 5.3.10} \\ &\simeq \lim_{E\in \mathrm{el}(T)} N_{\mathrm{int}} \overline{P}(E) && \text{by Lemma 5.3.15} \\ &\simeq N_{\mathrm{int}} \overline{P}(T) && \text{since } N_{\mathrm{int}} \text{ of anything is Segal} \end{aligned}$$

as required. □

Proof of Theorem 5.3.4. Proposition 5.3.16 tells us that the square of left adjoints commutes. Since the adjunction $F \dashv U$ is monadic by Corollary 5.1.6 and N_{int} is fully faithful (by Proposition 3.3.15), we are in position to apply Proposition 5.3.5, which now tells us that the square of right adjoints is a pullback. In particular, the nerve functor $N\colon \mathrm{AnMnd} \to \mathcal{P}(\mathbf{\Omega})$ is fully faithful. Furthermore, N factors through $\mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega})$, since this was defined as a pullback (5.3.3), as in this diagram:

$$\begin{array}{ccccc} \mathrm{AnMnd} & \xrightarrow{N} & \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}) & \longrightarrow & \mathcal{P}(\mathbf{\Omega}) \\ \downarrow{\scriptstyle U} & & \downarrow & & \downarrow{\scriptstyle j^*} \\ \mathrm{AnEnd} & \xrightarrow[N_{\mathrm{int}}]{\sim} & \mathcal{P}_{\mathrm{Seg}}(\mathbf{\Omega}_{\mathrm{int}}) & \longrightarrow & \mathcal{P}(\mathbf{\Omega}_{\mathrm{int}}). \end{array}$$

Since the composite square is a pullback, and the right square is a pullback, also the left square is a pullback, whence the result. □

Appendix A. Mates and Monads

In this appendix we discuss some $(\infty,2)$-categorical results needed in order to set up the ∞-categories of polynomial functors and polynomial monads. For most of these the proofs can be found in the companion paper [Hau20].

A.1. **$(\infty,2)$-Categories and Lax Transformations.** We write $\mathrm{Cat}_{(\infty,2)}$ for the ∞-category of $(\infty,2)$-categories. We will not need to use any specific model for these objects, but we will need to make use of the *lax Gray tensor product* of $(\infty,2)$-categories; several versions of this have recently been constructed [GHL20, Mae20, ORV20] in different models. On the ∞-categorical level all produce functors

$$\otimes^{\mathrm{lax}}\colon \mathrm{Cat}_{(\infty,2)} \times \mathrm{Cat}_{(\infty,2)} \to \mathrm{Cat}_{(\infty,2)}$$

that preserve colimits in each variable.

Remark A.1.1. Let $\mathbf{\Theta}_2 \subseteq \mathrm{Cat}_{(\infty,2)}$ denote Joyal's category of 2-dimensional pasting diagrams. Rezk's presentation [Rez10] of $(\infty,2)$-categories as complete Segal $\mathbf{\Theta}_2$-spaces implies that the Gray tensor product is uniquely determined by its restriction to a functor

$$\mathbf{\Theta}_2 \times \mathbf{\Theta}_2 \to \mathrm{Cat}_{(\infty,2)}.$$

This is given by the classical Gray tensor products of pasting diagrams in all the models, and hence they all produce the same functor of ∞-categories.[5]

Our convention is that a *lax natural transformation* η between functors $F, G\colon \mathcal{X} \to \mathcal{Y}$ assigns to every morphism $f\colon X \to X'$ in $\mathcal{X}$ a lax square

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & \swarrow & \downarrow{\scriptstyle G(f)} \\ F(X') & \xrightarrow[\eta_{X'}]{} & G(X'), \end{array}$$

and that this is given by a functor $\mathcal{X} \otimes^{\mathrm{lax}} \Delta^1 \to \mathcal{Y}$. Similarly, a *colax natural transformation* between the same functors assigns to every morphism $f\colon X \to X'$ in $\mathcal{X}$ a colax square

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & \nearrow & \downarrow{\scriptstyle G(f)} \\ F(X') & \xrightarrow[\eta_{X'}]{} & G(X'), \end{array}$$

[5] At one point in [Hau20] we also need to use the further assumption that this Gray tensor product restricts to the classical one when applied to ordinary 2-categories, or at least for *gaunt* 2-categories, which are those with no non-identity invertible morphisms and 2-morphisms.

and this is given by a functor $\mathcal{X} \otimes^{\mathrm{colax}} \Delta^1 \to \mathcal{Y}$, where

$$\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} := \mathcal{Y} \otimes^{\mathrm{lax}} \mathcal{X}.$$

Since $\otimes^{\mathrm{lax}}$ preserves colimits in each variable, we get by adjunction natural $(\infty,2)$-categories $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ determined by natural equivalences

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{W}, \mathrm{FUN}(\mathcal{X},\mathcal{Y})_{\mathrm{lax}}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X} \otimes^{\mathrm{lax}} \mathcal{W}, \mathcal{Y}),$$

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{W}, \mathrm{FUN}(\mathcal{X},\mathcal{Y})_{\mathrm{colax}}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{W}, \mathcal{Y}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{W} \otimes^{\mathrm{lax}} \mathcal{X}, \mathcal{Y}).$$

We write $\mathrm{Fun}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ for the underlying ∞-category of $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$; this has functors $\mathcal{X} \to \mathcal{Y}$ as objects and (co)lax natural transformations between them as morphisms.

We also write $\mathrm{FUN}(\mathcal{X},\mathcal{Y})$ for the ordinary internal hom in $\mathrm{Cat}_{(\infty,2)}$ (adjoint to the cartesian product) and $\mathrm{Fun}(\mathcal{X},\mathcal{Y})$ for its underlying ∞-category. By [Hau20, Corollary 3.15], we can identify this with the wide sub-$(\infty,2)$-category of $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ with morphisms those (co)lax transformations whose (co)lax naturality squares actually commute.

A.2. **Double ∞-Categories of Squares.** We think of double ∞-categories as simplicial ∞-categories that satisfy the Segal condition, or (essentially equivalently, using the description of ∞-categories as complete Segal spaces) bisimplicial spaces that satisfy the Segal condition in each variable. If $\mathcal{K}_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a double ∞-category, we think of

- the objects of $\mathcal{K}_0$ as the objects of the double ∞-category,
- the morphisms of $\mathcal{K}_0$ as the vertical morphisms,
- the objects of $\mathcal{K}_1$ as the horizontal morphisms,
- the morphisms of $\mathcal{K}_1$ as the squares in the double ∞-category.

The corresponding bisimplicial space is given by

$$\mathcal{K}_{n,m} := \mathrm{Map}(\Delta^m, \mathcal{K}_n),$$

so that

- the space of objects of $\mathcal{K}$ is $\mathcal{K}_{0,0}$,
- the space of vertical morphisms is $\mathcal{K}_{0,1}$,
- the space of horizontal morphisms is $\mathcal{K}_{1,0}$,
- the space of squares is $\mathcal{K}_{1,1}$.

Definition A.2.1. For any $(\infty,2)$-category $\mathcal{X}$ we will define double ∞-categories $\mathrm{Sq}^{\mathrm{lax}}(\mathcal{X})$, $\mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})$ and $\mathrm{Sq}(\mathcal{X})$ where

- the objects are the objects of $\mathcal{X}$,
- both the horizontal and vertical morphisms are the morphisms of $\mathcal{X}$,
- the squares are, respectively, lax squares, colax squares, and commuting squares in $\mathcal{X}$.

These can be defined using the (co)lax Gray tensor product and the cartesian product as the simplicial ∞-categories

$$\begin{aligned} \mathrm{Sq}^{\mathrm{lax}}(\mathcal{X})_\bullet &:= \mathrm{Fun}(\Delta^\bullet, \mathcal{X})_{\mathrm{colax}} \text{ [6]} \\ \mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})_\bullet &:= \mathrm{Fun}(\Delta^\bullet, \mathcal{X})_{\mathrm{lax}} \\ \mathrm{Sq}(\mathcal{X})_\bullet &:= \mathrm{Fun}(\Delta^\bullet, \mathcal{X}). \end{aligned}$$

The Segal condition follows immediately from the assumption that the Gray tensor product preserves colimits in each variable.

Remark A.2.2. Equivalently, these double ∞-categories are given by the bisimplicial spaces

(1) $\mathrm{Sq}^{\mathrm{lax}}(\mathcal{X})_{n,m} := \mathrm{Map}(\Delta^m, \mathrm{Fun}(\Delta^n, \mathcal{X})_{\mathrm{colax}}) \simeq \mathrm{Map}(\Delta^n \otimes^{\mathrm{colax}} \Delta^m, \mathcal{X}) \simeq \mathrm{Map}(\Delta^m \otimes^{\mathrm{lax}} \Delta^n, \mathcal{X})$,
(2) $\mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})_{n,m} := \mathrm{Map}(\Delta^m, \mathrm{Fun}(\Delta^n, \mathcal{X})_{\mathrm{lax}}) \simeq \mathrm{Map}(\Delta^n \otimes^{\mathrm{lax}} \Delta^m, \mathcal{X}) \simeq \mathrm{Map}(\Delta^m \otimes^{\mathrm{colax}} \Delta^n, \mathcal{X})$,
(3) $\mathrm{Sq}(\mathcal{X})_{n,m} := \mathrm{Map}(\Delta^m, \mathrm{Fun}(\Delta^n, \mathcal{X})) \simeq \mathrm{Map}(\Delta^n \times \Delta^m, \mathcal{X}) \simeq \mathrm{Map}(\Delta^m \times \Delta^n, \mathcal{X})$.

Notation A.2.3. If $\mathcal{K}$ is a double ∞-category, regarded as a bisimplicial space, we write $\mathcal{K}^{\text{h-op}}$ for the double ∞-category obtained by reversing direction in the first coordinate, and $\mathcal{K}^{\text{v-op}}$ for that obtained by reversing direction in the second coordinate. We also write $\mathcal{K}^{\text{rev}}$ for the double ∞-category obtained by reversing the order of the coordinates.

Remark A.2.4. Since there is by definition a natural equivalence $\mathcal{X} \otimes^{\text{lax}} \mathcal{Y} \simeq \mathcal{Y} \otimes^{\text{colax}} \mathcal{X}$, we have a natural equivalence

$$\mathrm{Sq}^{\text{lax}}(\mathcal{X})^{\text{rev}} \simeq \mathrm{Sq}^{\text{colax}}(\mathcal{X}).$$

There is also obviously an equivalence $\mathrm{Sq}(\mathcal{X})^{\text{rev}} \simeq \mathrm{Sq}(\mathcal{X})$.

Definition A.2.5. Let $\mathrm{Sq}^{\text{lax}}(\mathcal{X})^{\text{v=ladj}}$ denote the sub-double ∞-category of $\mathrm{Sq}^{\text{lax}}(\mathcal{X})$ containing only the squares where the vertical maps are left adjoints. (We will apply similar notations with right adjoints and horizontal morphisms, and other types of squares, without further comment.)

A.3. **Naturality of Mates.** Given a diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{R} & \mathcal{D} \\ {\scriptstyle\gamma}\downarrow & \swarrow{\scriptstyle\alpha} & \downarrow{\scriptstyle\delta} \\ \mathcal{C}' & \xrightarrow[R']{} & \mathcal{D}', \end{array}$$

where α is a natural transformation $\delta R \to R'\gamma$, and the functors R and R' have left adjoints L and L', respectively, then the *mate* of α is the natural transformation

$$L'\delta \to L'\delta RL \to L'R'\gamma L \to \gamma L,$$

which we can depict as

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{L} & \mathcal{D} \\ {\scriptstyle\gamma}\downarrow & \nwarrow & \downarrow{\scriptstyle\delta} \\ \mathcal{C}' & \xleftarrow[L']{} & \mathcal{D}' \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{L} & \mathcal{C} \\ {\scriptstyle\delta}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ \mathcal{D}' & \xrightarrow[L']{} & \mathcal{C}'. \end{array}$$

We thus pass from a lax square where the horizontal morphisms are right adjoints to a colax square where they are left adjoints. A dual version of this construction takes a colax square where the horizontal morphisms are left adjoints to a lax square, and doing both gives back the original square.

We would like to know that the process of taking mates is natural. The most general form of this statement would be that for any $(\infty, 2)$-category $\mathcal{X}$, taking mates gives a natural equivalence of double ∞-categories

$$\mathrm{Sq}^{\text{lax}}(\mathcal{X})^{\text{h=radj}} \xrightarrow{\sim} \mathrm{Sq}^{\text{colax}}(\mathcal{X})^{\text{h=ladj,h-op}}.$$

We will not establish such an equivalence here; instead, we will observe that the following weaker statement, where the squares in the source are required to commute, follows from the results of [Hau20]:

Proposition A.3.1. *There are morphisms of double ∞-categories*

$$\mathrm{Sq}(\mathcal{X})^{\text{h=ladj}} \to \mathrm{Sq}^{\text{lax}}(\mathcal{X})^{\text{h=radj,h-op}},$$

$$\mathrm{Sq}(\mathcal{X})^{\text{h=radj}} \to \mathrm{Sq}^{\text{colax}}(\mathcal{X})^{\text{h=ladj,h-op}},$$

given by taking mates in the horizontal direction.

Remark A.3.2. Using the equivalence of Remark A.2.4, we can also interpret these as maps

$$\mathrm{Sq}(\mathcal{X})^{\text{v=ladj}} \to \mathrm{Sq}^{\text{colax}}(\mathcal{X})^{\text{v=radj,v-op}},$$

$$\mathrm{Sq}(\mathcal{X})^{\text{v=radj}} \to \mathrm{Sq}^{\text{lax}}(\mathcal{X})^{\text{v=ladj,v-op}},$$

given by taking mates in the vertical direction.

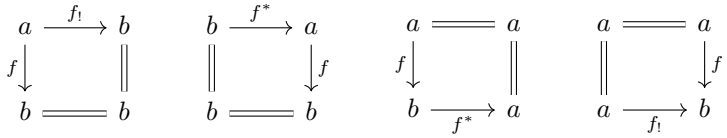
Proof. By [Hau20, Remark 4.11] taking mates gives a natural functor

$$\mathrm{FUN}(\Delta^\bullet, \mathcal{X})_{\mathrm{radj}} \to \mathrm{FUN}((\Delta^\bullet)^{\mathrm{op}}, \mathcal{X})_{\mathrm{ladj,lax}}.$$

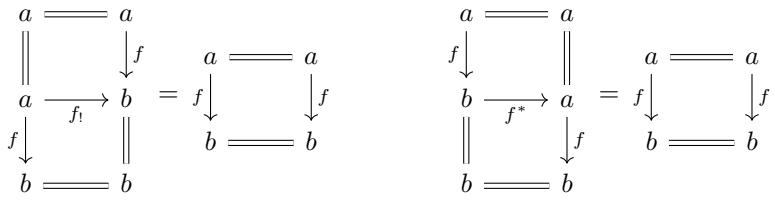
The underlying functor of simplicial ∞-categories gives precisely a functor of double ∞-categories $\mathrm{Sq}(\mathcal{X})^{\mathrm{h=radj}} \to \mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})^{\mathrm{h=ladj,h\text{-}op}}$. The other functor is defined in the same way by reversing 2-morphisms (which swaps lax and colax transformations and left and right adjoints). □

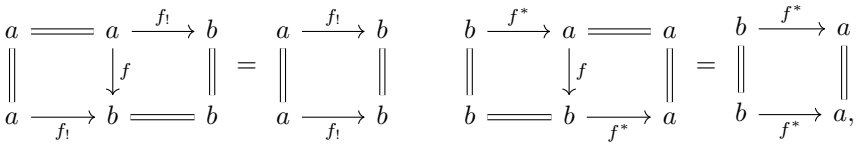
A.4. **Framed Double ∞-Categories.** We will need to know that the source-and-target projection for the double ∞-category $\mathrm{Sq}^{\mathrm{colax}}(\mathrm{CAT}_\infty)^{\mathrm{v=radj}}$ is a cartesian and cocartesian fibration. In order to show this, we will now prove an ∞-categorical version of a result of Shulman [Shu08] on double categories. To state this we first introduce some terminology:

Definition A.4.1 (Shulman [Shu08])**.** A double category is *framed* if for every vertical edge $f\colon a \to b$, there exist horizontal edges $f_!\colon a \to b$ and $f^*\colon b \to a$ together with four squares (2-cells)



such that the following four equations hold:





where on the right-hand side we have the horizontal and vertical identity squares for f, f^*, and $f_!$.

Remark A.4.2. In [Shu08], this structure is called a *framed bicategory* rather than a framed double category.

Definition A.4.3. We say a double ∞-category is *framed* if its homotopy double category is framed.

We have the following ∞-categorical version of [Shu08, Thm.4.1]:

Proposition A.4.4. *Let $\mathcal{X}$ be a double ∞-category, viewed as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfying the Segal condition*

$$\mathcal{X}_n \xrightarrow{\sim} \mathcal{X}_1 \times_{\mathcal{X}_0} \cdots \times_{\mathcal{X}_0} \mathcal{X}_1.$$

Put $\pi := (d_1, d_0)\colon \mathcal{X}_1 \to \mathcal{X}_0 \times \mathcal{X}_0$. Then the following are equivalent:

(i) The double ∞-category $\mathcal{X}$ is framed.
(ii) The functor π is a cocartesian fibration.
(iii) The functor π is a cartesian fibration.

Proof. The proof that (ii) and (iii) imply (i) is exactly as in the case of ordinary double categories, since cartesian (or cocartesian) fibrations induce Grothendieck (op)fibrations on the level of homotopy categories, and condition (i) is a statement about the homotopy double category. The more interesting direction (which is the one we are going to need) is that (i) implies (iii). So assume given (for each vertical edge) the four squares, and assume given homotopy equivalences representing the

four equations. Given an object in $\mathcal{X}_1$, that is, a horizontal edge $M : b \to d$, and an arrow downstairs in $\mathcal{X}_0 \times \mathcal{X}_0$ with codomain (b, d), that is altogether a configuration

$$\begin{array}{ccc} a & & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[M]{} & d, \end{array}$$

we claim that

$$\alpha : \quad \begin{array}{ccccccc} a & \xrightarrow{f_!} & b & \xrightarrow{M} & d & \xrightarrow{g^*} & c \\ {\scriptstyle f}\downarrow & & \| & & \| & & \downarrow{\scriptstyle g} \\ b & = & b & \xrightarrow[M]{} & d & = & d \end{array}$$

is a cartesian lift. Given vertical edges $\begin{array}{c} x \\ \downarrow{\scriptstyle u} \\ a \end{array}$ and $\begin{array}{c} y \\ \downarrow{\scriptstyle v} \\ c \end{array}$, and a horizontal edge $N : x \to y$, the claim is that the natural map

$$\mathrm{Map}_{\mathcal{X}_1}(N, g^* M f_!)_{(u,v)} \longrightarrow \mathrm{Map}_{\mathcal{X}_1}(N, M)_{(fu, gv)}$$

given by pasting the square α to the bottom edge is a homotopy equivalence. But we can construct a homotopy inverse by sending a square

$$\begin{array}{ccc} x & \xrightarrow{N} & y \\ {\scriptstyle uf}\downarrow & & \downarrow{\scriptstyle vg} \\ b & \xrightarrow[M]{} & d \end{array}$$

to the pasting

$$\begin{array}{ccccccc} x & = & x & \xrightarrow{N} & y & = & y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ a & = & a & & c & = & c \\ \| & & \downarrow{\scriptstyle f} & & {\scriptstyle g}\downarrow & & \| \\ a & \xrightarrow[f_!]{} & b & \xrightarrow[M]{} & d & \xrightarrow[g^*]{} & c \end{array}$$

These two assignments are homotopy inverses: explicit homotopies are easily constructed from the homotopy equivalences stipulated in (i).

The proof that (i) implies (ii) is similar. For reference, we note that the cocartesian lifts (of (f, g) to N) can be taken to be of the form

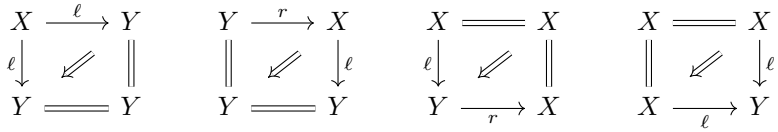
$$\begin{array}{ccccccc} a & = & a & \xrightarrow{N} & c & = & c \\ {\scriptstyle f}\downarrow & & \| & & \| & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[f^*]{} & a & \xrightarrow[N]{} & c & \xrightarrow[g_!]{} & d. \end{array}$$

□

Proposition A.4.5. *The double ∞-categories $\mathrm{Sq}^{\mathrm{lax}}(\mathrm{CAT}_\infty)^{\mathrm{v=ladj}}$ and $\mathrm{Sq}^{\mathrm{colax}}(\mathrm{CAT}_\infty)^{\mathrm{v=radj}}$ are framed.*

Proof. We give the proof for $\mathrm{Sq}^{\mathrm{lax}}(\mathrm{CAT}_\infty)^{\mathrm{v=ladj}}$, the other case is essentially the same. For each vertical edge, that is a left adjoint functor $\ell \colon X \to Y$, with right adjoint $r \colon Y \to X$, we have the

lax squares



where the second square is the counit ϵ_Y, the third square is the unit η_X, and the two other squares are trivial. The four equations required are two trivial ones, and the triangle laws for adjunctions. □

Combining Proposition A.4.5 with Proposition A.4.4, we get:

Corollary A.4.6. *The source-and-target projections*

$$\mathrm{Sq}^{\mathrm{lax}}(\mathrm{CAT}_\infty)_1^{\mathrm{v=ladj}} \to (\mathrm{Cat}_\infty^{\mathrm{ladj}})^{\times 2}, \qquad \mathrm{Sq}^{\mathrm{colax}}(\mathrm{CAT}_\infty)_1^{\mathrm{v=radj}} \to (\mathrm{Cat}_\infty^{\mathrm{radj}})^{\times 2},$$

are cartesian and cocartesian fibrations.

A.5. **Monads.** To define the ∞-category of polynomial monads we need to have a suitable ∞-category of monads on varying base ∞-categories. We can define this in terms of lax natural transformations:

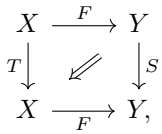
Definition A.5.1. Let $\mathfrak{mnd}$ denote the universal 2-category containing a monad.[7] A *monad* in an $(\infty,2)$-category $\mathcal{X}$ is a functor $\mathfrak{mnd} \to \mathcal{X}$. A *lax morphism of monads* (or *monad functor* in the terminology of [Str72]) is a lax natural transformation of monads, i.e. a functor $\mathfrak{mnd} \otimes^{\mathrm{lax}} \Delta^1 \to \mathcal{X}$. Similarly, a *colax morphism of monads* (or *monad opfunctor*) is a colax natural transformation $\mathfrak{mnd} \otimes^{\mathrm{colax}} \Delta^1 \to \mathcal{X}$. We then have $(\infty,2)$-categories of monads and (co)lax morphisms defined as

$$\mathrm{MND}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}} := \mathrm{FUN}(\mathfrak{mnd}, \mathcal{X})_{(\mathrm{co})\mathrm{lax}};$$

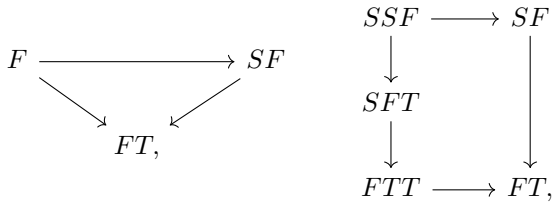
we denote the underlying ∞-categories by $\mathrm{Mnd}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}$.

Remark A.5.2. If T is a monad on $X \in \mathcal{X}$ and S is a monad on $Y \in \mathcal{X}$, then

(1) a *lax morphism* $T \to S$ consists of a morphism $F\colon X \to Y$ and a natural transformation $SF \to FT$ — in other words, a lax square



compatible with multiplication and units through commutative diagrams



and so on for iterated composites of S and T.

[7] This is defined in [RV16] as a full sub-2-category of the universal 2-category containing an adjunction; it can also be described as the one-object 2-category corresponding to the monoidal envelope of the non-symmetric associative operad.

(2) a *colax morphism* $T \to S$ consists of a morphism $F\colon X \to Y$ and a natural transformation $FT \to SF$ — in other words, a colax square

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle T}\downarrow & \swarrow & \downarrow{\scriptstyle S} \\ X & \xrightarrow[F]{} & Y, \end{array}$$

compatible with multiplication and units through commutative diagrams

$$\begin{array}{ccc} F & \longrightarrow & FT \\ & \searrow \quad \swarrow & \\ & SF, & \end{array} \qquad \begin{array}{ccc} FTT & \longrightarrow & FT \\ \downarrow & & \downarrow \\ SFT & & \\ \downarrow & & \\ SSF & \longrightarrow & SF, \end{array}$$

and so on for iterated composites of S and T.

For ordinary 2-categories, Street [Str72] showed that the (2-)category of monads and lax morphisms is equivalent to that of monadic right adjoints and commutative squares between them. One of the main results of [Hau20] uses work of Riehl–Verity [RV16] and Zaganidis [Zag17] to upgrade this to an equivalence of $(\infty, 2)$-categories, in the case of monads in the $(\infty, 2)$-category CAT_∞ of ∞-categories:

Theorem A.5.3 ([Hau20, Corollary 5.7])**.** *Let* $\mathrm{FUN}(\Delta^1, \mathrm{CAT}_\infty)_{\mathrm{mndradj}}$ *denote the full sub-*$(\infty, 2)$*-category of* $\mathrm{FUN}(\Delta^1, \mathrm{CAT}_\infty)$ *spanned by the monadic right adjoints. There is an equivalence of* $(\infty, 2)$*-categories*

$$\mathrm{MND}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{FUN}(\Delta^1, \mathrm{CAT}_\infty)_{\mathrm{mndradj}}$$

taking a monad to the right adjoint of its monadic adjunction.

Corollary A.5.4 ([Hau20, Corollary 5.10])**.** *The inclusion*

$$\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)_{\mathrm{mndradj}} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)_{\mathrm{radj}}$$

of the full subcategory of monadic right adjoints into that of all right adjoints in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$*, has a left adjoint, which takes a right adjoint functor to the right adjoint of the associated monadic adjunction.*

Definition A.5.5. Let $\mathfrak{end}$ denote the universal category containing an endomorphism, i.e. the pushout

$$\begin{array}{ccc} \partial\Delta^1 & \hookrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \mathfrak{end}, \end{array}$$

or the delooping $B\mathbb{N}$ of the natural numbers under addition. If $\mathcal{X}$ is an $(\infty, 2)$-category, we write

$$\mathrm{END}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}} := \mathrm{FUN}(\mathfrak{end}, \mathcal{X})_{(\mathrm{co})\mathrm{lax}}$$

for the $(\infty, 2)$-category of endomorphisms and (co)lax transformations between them; we denote the underlying ∞-categories by $\mathrm{End}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}$.

Remark A.5.6. If $T\colon X \to X$ and $S\colon Y \to Y$ are endomorphisms in $\mathcal{X}$, then a *lax* morphism from T to S is given by a morphism $F\colon X \to Y$ and a lax square

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle T}\downarrow & \swarrow & \downarrow{\scriptstyle S} \\ X & \xrightarrow[F]{} & Y, \end{array}$$

while a *colax* morphism is again given by a morphism $F\colon X \to Y$ but now with a colax square

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle T}\downarrow & \nearrow & \downarrow{\scriptstyle S} \\ X & \xrightarrow[F]{} & Y. \end{array}$$

Remark A.5.7. There is an inclusion $\mathfrak{end} \to \mathfrak{mnd}$ picking out the underlying endomorphism of the universal monad, which induces natural functors of $(\infty,2)$-categories

$$\mathrm{MND}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}} \to \mathrm{END}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}.$$

Remark A.5.8. To reduce confusion regarding our conventions for lax vs. colax, let us point out explicitly that there is a functor

$$\mathrm{End}(\mathcal{X})_{\mathrm{lax}} \to \mathrm{Fun}(\Delta^1, \mathcal{X})_{\mathrm{lax}} = \mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})_1,$$

and hence a functor

$$\mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}} \to \mathrm{Sq}^{\mathrm{colax}}(\mathcal{X})_1.$$

Another key result from [Hau20] identifies the fibres of the underlying functors of ∞-categories:

Theorem A.5.9 ([Hau20, Corollary 8.9])**.** *For $\mathcal{X}$ an $(\infty,2)$-category and X an object of $\mathcal{X}$, there are natural identifications*

$$\begin{array}{ccc} \mathrm{Alg}(\mathrm{End}_{\mathcal{X}}(X)) & \xrightarrow{\sim} & \mathrm{Mnd}(\mathcal{X})_{\mathrm{colax},X} \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathcal{X}}(X) & \xrightarrow{\sim} & \mathrm{End}(\mathcal{X})_{\mathrm{colax},X}, \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}(\mathrm{End}_{\mathcal{X}}(X))^{\mathrm{op}} & \xrightarrow{\sim} & \mathrm{Mnd}(\mathcal{X})_{\mathrm{lax},X} \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathcal{X}}(X)^{\mathrm{op}} & \xrightarrow{\sim} & \mathrm{End}(\mathcal{X})_{\mathrm{lax},X}, \end{array}$$

where $\mathrm{End}_{\mathcal{X}}(X)$ is the monoidal ∞-category of endomorphisms of X in $\mathcal{X}$ under composition.

Remark A.5.10. Combining this with Theorem A.5.3, we get equivalences

$$\mathrm{Alg}(\mathrm{End}(\mathcal{C}))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{mndradj}},$$

where $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{mndradj}}$ denotes the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the monadic right adjoints. This equivalence has also been obtained by Heine [Hei17] by a different method.

Together with Corollary A.5.4, we get:

Corollary A.5.11. *The functor*

$$\mathrm{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{radj}}$$

that takes a monad to the associated right adjoint, is fully faithful, with image the monadic right adjoints. (Here $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{radj}}$ denotes the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the right adjoints.)

We end by recalling two further results from [Hau20] that we will make use of:

Proposition A.5.12 ([Hau20, Proposition 6.4])**.**

(i) The projection $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{Cat}_\infty$ has locally cocartesian morphisms and locally cartesian morphisms over functors that are right adjoints.

(ii) The projection $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{Cat}_\infty$ has locally cocartesian morphisms over functors that are right adjoints.

(iii) The forgetful functor $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ preserves these locally cocartesian morphisms.

Definition A.5.13. Let $\mathrm{Cat}_\infty^{\mathrm{radj}}$ denote the subcategory of Cat_∞ containing only the morphisms that are right adjoints. Then we define $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}^{\mathrm{radj}}$ and $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}^{\mathrm{radj}}$ by pulling back $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ and $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ along the inclusion $\mathrm{Cat}_\infty^{\mathrm{radj}} \to \mathrm{Cat}_\infty$.

Corollary A.5.14 ([Hau20, Corollary 6.6])**.** *There is a commuting diagram*

$$\begin{array}{ccc} \mathrm{Mnd}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}} & \longrightarrow & \mathrm{End}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}} \\ & \searrow \quad \swarrow & \\ & \mathrm{Cat}^{\mathrm{radj}}_\infty, & \end{array}$$

where the two downward functors are cocartesian fibrations, and the horizontal functor preserves cocartesian morphisms. Moreover, the right-hand functor is also a cartesian fibration.

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