

A NUMERICAL TOOLKIT FOR MULTIPROJECTIVE VARIETIES

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ABSTRACT. A numerical description of an algebraic subvariety of projective space is given by a general linear section, called a witness set. For a subvariety of a product of projective spaces (a multiprojective variety), the corresponding numerical description is given by a witness collection, whose structure is more involved. We build on recent work to develop a toolkit for the numerical manipulation of multiprojective varieties that operates on witness collections and use this toolkit in an algorithm for numerical irreducible decomposition of multiprojective varieties. The toolkit and decomposition algorithm are illustrated throughout in a series of examples.

INTRODUCTION

Numerical algebraic geometry [20] uses numerical analysis to manipulate and study algebraic varieties on a computer. In numerical algebraic geometry, a subvariety X of affine or projective space is represented by a witness set, which includes a finite set of points in a general linear section of X [16]. Algorithms to manipulate a variety operate on its witness sets. A fundamental algorithm is numerical irreducible decomposition [17], which uses monodromy [18] and a trace test [19] to partition a witness set of a reducible variety into witness sets for each irreducible component.

Oftentimes, a variety possesses additional structure, such as multihomogeneity, which is when its defining polynomials are separately homogeneous in disjoint subsets of variables. For example, the determinant $\det(x_{i,j})$ is separately linear in the variables of each column. Such a variety is naturally a subvariety of a product of projective spaces (a multiprojective variety). For the $n \times n$ determinant, this product is $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$ (n factors). We seek algorithms for multiprojective varieties that are adapted to their structure.

Algorithms for numerically solving systems of multihomogeneous polynomials are classical [13]. A useful notion of witness set—a witness collection—for multiprojective varieties, along with fundamental algorithms, was given in [6]. There, it was observed that the trace test could not be applied naively to a witness collection. Consequently, for numerical irreducible decomposition, a witness collection for a multiprojective variety must be transformed into a witness set for a projective or affine variety. Since a multiprojective variety is a projective variety under the Segre embedding, that could be used for numerical irreducible decomposition. In general, the Segre embedding dramatically increases

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the ambient dimension and degree as we show in Section 1.7. Passing instead to an affine patch in the product of projective spaces preserves the ambient dimension, but we do not know an algorithm of acceptable complexity to compute a witness set from a witness collection unless the variety is a curve. This is because for curves the Segre embedding can be used without increasing the degree so much as seen in Section 4.

A version of numerical irreducible decomposition was proposed for subvarieties in the product of two projective spaces [10]. This reduces numerical irreducible decomposition to that of a curve, decreasing the size of the witness sets. We extend that analysis to arbitrary multiprojective varieties. We present four geometric constructions and corresponding algorithms that operate on witness collections, and together provide a toolkit for the numerical manipulation of multiprojective varieties. A key ingredient is the support of a multiprojective variety [2], which is a multiprojective version of dimension.

The computation of this (multi)dimension locally at a point reduces to linear algebra. When the multidimension decomposes as a product, the corresponding variety is also a product as is a witness collection for it. We next explain how witness collections transform under birational maps that change the multiprojective structure, and finally how a witness collection behaves under slicing with a hyperplane. We also give an algorithm based on monodromy for computing a witness collection. The utility of this toolkit is illustrated in an algorithm for numerical irreducible decomposition of multiprojective varieties. We use these tools to reduce the numerical irreducible decomposition to that of a curve in affine space, to which we may apply an efficient trace test. This generalizes the method of [10], from two to arbitrarily many projective factors.

Algorithms in numerical algebraic geometry typically operate on affine varieties. A subvariety $X \subset \mathbb{P}^n$ of projective space is replaced by its intersection X_{aff} with a general affine patch $\mathbb{C}^n \subset \mathbb{P}^n$ where a general linear polynomial ℓ does not vanish. The same approach could be followed for a multiprojective variety $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by taking affine patches in each projective factor and combining them, giving $X_{\text{aff}} \subset \mathbb{C}^{n_1 + \cdots + n_k}$. This neglects the given structure and increases the size and complexity of the witness set, which is particularly significant when X is neither a curve nor a hypersurface. We will work with multiaffine varieties $X_{\text{aff}} \subset \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$, using algorithms that respect this decomposition and are compatible with the multiprojective structure of X .

This paper is structured as follows. In Section 1, we define witness collections of multiprojective and multiaffine varieties, and introduce some running examples. In Section 2, we give an algorithm to compute (multi)dimension locally and an algorithm based on monodromy to compute a witness collection. In Section 3, we show how to detect and exploit that a variety is a product. In Section 4, we show how to transform witness collections under the birational maps that correspond to changing the multiprojective structure of a variety. In Section 5, we show how to perform a dimension reduction based on intersections with linear spaces that preserves (ir)reducibility. In Section 6, we sketch an algorithm for numerical irreducible decomposition that uses this toolkit. In Section 7, we consider two examples based on fiber products which naturally yield multihomogeneous systems. Apart from showcasing our toolkit, these examples demonstrate that using multiprojective structure leads to significant reduction in the size of computations.

1. BACKGROUND

For a finite set F of polynomials, let $\mathcal{V}(F)$ be its variety, the subset of affine or projective space (or products thereof) where every polynomial in F vanishes. A variety will also be a union of irreducible components of such a set $\mathcal{V}(F)$. We first recall numerical homotopy continuation, then witness sets [20, Ch. 13], multiprojective varieties, witness collections [6], and multiaffine varieties. We end by introducing our running examples.

1.1. Numerical homotopy continuation. Many algorithms described herein are based on numerical homotopy continuation. A *homotopy* is a system of polynomials $H(\mathbf{x}; t)$ ($\mathbf{x} \in \mathbb{C}^n$, $t \in \mathbb{C}$), that interpolates between two systems—the *start system* when $t = 1$ and the *target system* when $t = 0$ —in a particular way. We require that $\mathcal{V}(H) \subset \mathbb{C}_{\mathbf{x}}^n \times \mathbb{C}_t$ contains a curve C that is a union of components of $\mathcal{V}(H)$ which projects dominantly to \mathbb{C}_t , and that $t = 1$ is a regular value of this projection $\pi: C \rightarrow \mathbb{C}_t$. We further require that C is bounded above a neighborhood of $1 \in \mathbb{C}_t$, and that $\mathcal{V}(H)$ is smooth at $W \subset \pi^{-1}(1)$. The start system $H(\mathbf{x}, 1) = 0$ has W among its isolated solutions. The target system is $H(\mathbf{x}, 0) = 0$ and its intended solutions are the points of C above $t = 0$.

Given a homotopy H , we restrict C to its points above the interval $[0, 1]$ or above an arc in \mathbb{C}_t with endpoints $\{0, 1\}$. This gives a set of $|W|$ arcs in $\mathbb{C}_{\mathbf{x}}^n \times \mathbb{C}_t$, one for each point of W . Each arc is either unbounded for t near 0 or it ends in a point of $\pi^{-1}(0)$. Starting with points of W and using numerical path-tracking to follow the corresponding arcs will recover the isolated points of $\pi^{-1}(0)$. In this way, we use the solutions W of the start system to compute the solutions of the target system. For more, see [12, 20].

1.2. Witness sets and numerical irreducible decomposition. Let Y be an irreducible subvariety of projective space \mathbb{P}^n . By Bertini’s Theorem [9], the dimension $\dim(Y)$ of Y is the maximum number of general linear polynomials that have a common zero on Y , and its degree $\deg(Y)$ is the number of such common zeroes. For a collection L of $\dim(Y)$ general linear polynomials, the set $Y \cap \mathcal{V}(L)$ of $\deg(Y)$ common zeroes is a *linear section* of Y , called a *witness point set* of Y . If F is a finite set of polynomials with Y an irreducible component of $\mathcal{V}(F)$, then the triple $(F, L, Y \cap \mathcal{V}(L))$ is a *witness set* for Y .

Suppose that $X \subset \mathbb{P}^n$ is a union of irreducible components of $\mathcal{V}(F)$. A witness set for X is composed of witness sets for each irreducible component of X . We assume for simplicity that the linear sections are chosen coherently: Let ℓ_1, \dots, ℓ_n be general linear polynomials on \mathbb{P}^n , and for each $e \in \{0, 1, \dots, n\}$, set $L^e := (\ell_1, \dots, \ell_e)$. For each dimension e , the e th witness set for X is the triple (F, L^e, P^e) where P^e is the set of isolated points in $X \cap \mathcal{V}(L^e)$. If X is *equidimensional* of dimension e (all components of X have dimension e) then $(F, L^e, X \cap \mathcal{V}(L^e))$ is a witness set for X . For this assertion/definition the generality of the ℓ_i is essential, by Bertini’s Theorem.

Remark 1.1. Given another collection L' of e linear polynomials, the convex combination $tL^e + (1 - t)L'$ may be used in a homotopy $H(t) = (F, tL^e + (1 - t)L')$ to transform the witness point set $P^e \subset X \cap \mathcal{V}(L^e)$ into one lying in $X \cap \mathcal{V}(L')$. This homotopy can be used, for example, to test membership. In particular, if X is equidimensional of dimension e , $\mathbf{x} \in \mathbb{P}^n$, and L' is e general linear polynomials vanishing at \mathbf{x} , then $\mathbf{x} \in X$ if and only if \mathbf{x} is an endpoint of the homotopy $H(t)$ with start points $X \cap \mathcal{V}(L^e)$. \diamond

A fundamental algorithm involving witness sets is numerical irreducible decomposition. It first decomposes a witness set for X into witness sets for X_0, \dots, X_n , where $\textcolor{brown}{X}_e \subset X$ is the union of the irreducible components of X of dimension e . When $X = X_e$, numerical irreducible decomposition computes the partition of $X \cap \mathcal{V}(L)$ ($L = L^e$) into subsets, each of which is a linear section $Y \cap \mathcal{V}(L)$ of an irreducible component Y of X .

Numerically following the points of $X \cap \mathcal{V}(L)$ as L varies in a loop gives a monodromy permutation ω of $X \cap \mathcal{V}(L)$. The points belonging to a cycle of ω lie in the same irreducible component of X , and thus the cycles of ω give a finer partition than the numerical irreducible decomposition. Computing additional monodromy permutations coarsens this partition. This monodromy break up algorithm [18] gives a partition $P_1 \sqcup \dots \sqcup P_s$ of $X \cap \mathcal{V}(L)$, where each $P_i \subset Y \cap \mathcal{V}(L)$ for some irreducible component Y of X .

The trace test [10, 19] is a heuristic stopping criterion for monodromy break up. In it, the points of some part P_i of the partition are numerically continued as L moves in a general linear pencil. The average of the points in P_i is collinear if and only if P_i is a witness point set of a component. Thus, when each part of the partition passes this *trace test*, we have computed the numerical irreducible decomposition.

This is unchanged if we replace projective varieties by affine varieties. In practice, the algorithm operates on affine varieties, working in a random affine patch of \mathbb{P}^n .

1.3. Multiprojective varieties. For more background, see [11, Ch. 8]. Let k, n_1, \dots, n_k be positive integers and let $\mathbb{P}^{n\bullet} := \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ be the indicated product of projective spaces. Writing \mathbf{x}_i for the indeterminates $x_{i,0}, \dots, x_{i,n_i}$, we have that $\mathbb{C}[\mathbf{x}_i]$ is the homogeneous coordinate ring of \mathbb{P}^{n_i} and $\mathbb{C}[\mathbf{x}] := \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_k]$ is the coordinate ring of $\mathbb{P}^{n\bullet}$. This ring is *multigraded*, its multihomogeneous elements $f(\mathbf{x})$ are separately homogeneous in each variable group $\mathbf{x}_1, \dots, \mathbf{x}_k$. Such an element has a *multidegree* which is a vector $(d_1, \dots, d_k) \in \mathbb{N}^k$ where d_i is the degree of $f(\mathbf{x})$ in the variable group \mathbf{x}_i .

A subvariety $X \subset \mathbb{P}^{n\bullet}$ (a *multiprojective variety*) is a union of irreducible components of a set $\mathcal{V}(F)$, where $F \subset \mathbb{C}[\mathbf{x}]$ is a finite set of multihomogeneous polynomials. Each irreducible component Y of X has an intrinsic dimension $\dim(Y)$ as an algebraic variety. As a subvariety of $\mathbb{P}^{n\bullet}$, its (extrinsic) dimension and degree are more involved than for projective varieties. This already occurs for hypersurfaces. A multihomogeneous linear polynomial in $\mathbb{C}[\mathbf{x}]$ has multidegree $(0, \dots, 1, \dots, 0)$: it is linear in one variable group \mathbf{x}_i and no other variables occur in it. In particular, there are k different types of ‘hyperplanes’.

There are similarly many different types of ‘linear’ sections of multiprojective varieties in $\mathbb{P}^{n\bullet}$. Set $[n_\bullet] := \{(e_1, \dots, e_k) \in \mathbb{N}^k \mid e_i \in \{0, 1, \dots, n_i\}\}$ and let $\mathbf{e} \in [n_\bullet]$. For each $i = 1, \dots, k$, let L_i be e_i general linear polynomials in $\mathbb{C}[\mathbf{x}_i]$ and write $\mathbf{L}^\mathbf{e} = (L_1, \dots, L_k)$. Then $\mathcal{V}(\mathbf{L}^\mathbf{e}) \subset \mathbb{P}^{n\bullet}$ is a product of linear subspaces in the factors of $\mathbb{P}^{n\bullet}$, where the linear subspace in \mathbb{P}^{n_i} has dimension $n_i - e_i$. When $Y \subset \mathbb{P}^{n\bullet}$ is an irreducible multiprojective variety with intrinsic dimension $\dim(Y)$, Bertini’s Theorem implies that $Y \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$ is nonempty and finite only if $\dim(Y) = e_1 + \dots + e_k =: |\mathbf{e}|$. Similarly, it is empty if $\dim(Y) < |\mathbf{e}|$ and, for $\dim(Y) > |\mathbf{e}|$, it is either empty or infinite.

The (*multi*)dimension $\text{Dim}(Y)$ of an irreducible multiprojective variety $Y \subset \mathbb{P}^{n\bullet}$ is the set of vectors $\mathbf{e} \in [n_\bullet]$ such that $Y \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$ is finite and nonempty. In [2] this is called the support of Y . Unlike for projective varieties, $\text{Dim}(Y)$ is a set. Note that $\mathbf{e} \in \text{Dim}(Y)$

implies that $|\mathbf{e}| = \dim(Y)$. The *(multi)degree* of Y is the map $\text{Deg}_Y : \text{Dim}(Y) \rightarrow \mathbb{N}$, where $\text{Deg}_Y(\mathbf{e})$ is the number of points in the linear section $Y \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$. For convenience, we extend the domain Deg_Y to $[n_\bullet]$, where if $\mathbf{e} \notin \text{Dim}(Y)$, then $\text{Deg}_Y(\mathbf{e}) = 0$.

If $X \subset \mathbb{P}^{n_\bullet}$ has irreducible decomposition $X = Y_1 \cup \dots \cup Y_s$, then we define Deg_X by

$$\text{Deg}_X(\mathbf{e}) := \sum_{j=1}^s \text{Deg}_{Y_j}(\mathbf{e}) \quad \text{for } \mathbf{e} \in [n_\bullet].$$

Likewise, the dimension of X is the support of Deg_X ,

$$\text{Dim}(X) = \{\mathbf{e} \in [n_\bullet] \mid \text{Deg}_X(\mathbf{e}) > 0\} = \bigcup_{j=1}^s \text{Dim}(Y_j).$$

When $k = 1$, this reduces to the dimension and degree of a projective variety $X \subset \mathbb{P}^n$, where the dimension of X is the set of dimensions of its irreducible components and the degree sends e to the degree of the equidimensional part of X of dimension e .

Remark 1.2. The structure of the extrinsic dimension and degree of a multiprojective variety is a consequence of the structure of the homology groups of \mathbb{P}^{n_\bullet} [4]. The homology of \mathbb{P}^n has a \mathbb{Z} -basis \mathbf{T}^e for $e = 0, \dots, n$, where \mathbf{T}^e is the class $[\Lambda^e]$ of a linear subspace Λ^e of dimension e . Then the class of a subvariety $X \subset \mathbb{P}^n$ is

$$[X] = \sum_{e=0}^n \text{Deg}_X(e) \mathbf{T}^e.$$

The homology of \mathbb{P}^{n_\bullet} has a \mathbb{Z} -basis $\mathbf{T}^\mathbf{e} := [\Lambda_1^{e_1} \times \dots \times \Lambda_k^{e_k}]$ for $\mathbf{e} \in [n_\bullet]$, where $\Lambda_i^{e_i} \subset \mathbb{P}^{n_i}$ is a linear space of dimension e_i . Then the class of a multiprojective variety $X \subset \mathbb{P}^{n_\bullet}$ is

$$[X] = \sum_{\mathbf{e} \in [n_\bullet]} \text{Deg}_X(\mathbf{e}) \mathbf{T}^\mathbf{e}. \quad \diamond$$

A *witness collection* for an irreducible multiprojective variety $Y \subset \mathbb{P}^{n_\bullet}$ that is a component of $\mathcal{V}(F)$ is a map that assigns each $\mathbf{e} \in \text{Dim}(Y)$ to $(F, \mathbf{L}^\mathbf{e}, Y \cap \mathcal{V}(\mathbf{L}^\mathbf{e}))$. This triple is an \mathbf{e} -*witness set* of Y with $Y \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$ an \mathbf{e} -*witness point set* of Y . As with ordinary witness sets, we assume that the linear polynomials are chosen coherently. That is, for each $i \in \{1, \dots, k\}$, let $\ell_{i,1}, \dots, \ell_{i,n_i} \in \mathbb{C}[\mathbf{x}_i]$ be general linear polynomials. For $\mathbf{e} \in [n_\bullet]$, set $L_i^{e_i} := (\ell_{i,1}, \dots, \ell_{i,e_i})$ and $\mathbf{L}^\mathbf{e} := (L_1^{e_1}, \dots, L_k^{e_k})$. If X is a union of components of $\mathcal{V}(F)$, then a witness collection for X is the map that sends $\mathbf{e} \in \text{Dim}(X)$ to $(F, \mathbf{L}^\mathbf{e}, P^\mathbf{e})$, where $P^\mathbf{e}$ is the set of isolated points of $X \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$.

Remark 1.3. Given another collection \mathbf{L}' of \mathbf{e} linear polynomials with e_i in $\mathbb{C}[\mathbf{x}_i]$, the convex combination $t\mathbf{L}^\mathbf{e} + (1-t)\mathbf{L}'$ may be used in a homotopy $H(t) = (F, t\mathbf{L}^\mathbf{e} + (1-t)\mathbf{L}')$ to transform the witness point set $P^\mathbf{e} \subset X \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$ into one lying in $X \cap \mathcal{V}(\mathbf{L}')$. Similar to Remark 1.1, this homotopy can be used, for example, to test membership. \diamond

This membership test for multiprojective varieties relies on the result that if X is irreducible and $\mathbf{x} \in \mathbb{P}^{n_\bullet}$, then $\mathbf{x} \in X$ if and only if there exists $\mathbf{e} \in \text{Dim}(X)$ such that \mathbf{x} is an endpoint of the homotopy $H(t) = (F, t\mathbf{L}^\mathbf{e} + (1-t)\mathbf{L}')$ with start points $X \cap \mathcal{V}(\mathbf{L}^\mathbf{e})$, where \mathbf{L}' is \mathbf{e} general linear polynomials vanishing at \mathbf{x} .

Algorithm 1.4 (Membership test for multiprojective varieties [6, Alg. 3]).

Input: Witness collection for an irreducible multiprojective variety $X \subset \mathbb{P}^{n_\bullet}$ and $\mathbf{x} \in \mathbb{P}^{n_\bullet}$.

Output: A boolean $B_{\mathbf{x}}$ which answers if $\mathbf{x} \in X$.

Do: For each $\mathbf{e} \in \text{Dim}(X)$, choose \mathbf{L}' to be \mathbf{e} general linear polynomials vanishing at \mathbf{x} and return “true” if \mathbf{x} is an endpoint of the homotopy $H(t) = (F, t\mathbf{L}^{\mathbf{e}} + (1-t)\mathbf{L}')$ with start points $X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$. Return “false” after testing all possible $\mathbf{e} \in \text{Dim}(X)$.

1.4. Multiaffine varieties. Let $X \subset \mathbb{P}^{n_\bullet}$ be a multiprojective variety. Choosing an affine patch $\mathbb{C}^{n_i} \subset \mathbb{P}^{n_i}$ in each factor, $X_{\text{aff}} := X \cap (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k})$ is an affine variety that retains much information about X . To keep track of its multiprojective origins, we retain the decomposition $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ from the factors of \mathbb{P}^{n_\bullet} . Write \mathbb{C}^{n_\bullet} for $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ and call a subvariety of \mathbb{C}^{n_\bullet} a *multiaffine variety*. Algorithms for a multiprojective variety X operate locally on a corresponding multiaffine variety X_{aff} .

Let $i \in \{1, \dots, k\}$. The affine patch \mathbb{C}^{n_i} has coordinate ring the polynomial ring $\mathbb{C}[\mathbf{y}_i]$ with variables $\mathbf{y}_i := (y_{i,1}, \dots, y_{i,n_i})$. This ring is not graded. The coordinate ring of \mathbb{C}^{n_\bullet} is $\mathbb{C}[\mathbf{y}] := \mathbb{C}[\mathbf{y}_1, \dots, \mathbf{y}_k]$. This is an ordinary polynomial ring whose only structure is the indicated grouping of its variables. A multihomogeneous polynomial $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ (multi)dehomogenizes to a polynomial $f(\mathbf{y}) \in \mathbb{C}[\mathbf{y}]$.

The dimension of a multiaffine variety $X \subset \mathbb{C}^{n_\bullet}$ is a set $\text{Dim}(X) \subset [n_\bullet]$. Its degree is a map $\text{Deg}_X : [n_\bullet] \rightarrow \mathbb{N}$. These are defined in the same way as for multiprojective varieties, except that a homogeneous linear polynomial $\ell(\mathbf{x}_i) \in \mathbb{C}[\mathbf{x}_i]$ is replaced by its dehomogenization $\ell(\mathbf{y}_i) \in \mathbb{C}[\mathbf{y}_i]$, which is a degree one polynomial, or *affine form*. When the multiaffine patch $\mathbb{C}^{n_\bullet} \subset \mathbb{P}^{n_\bullet}$ is general, $\text{Dim}(X) = \text{Dim}(X_{\text{aff}})$ and $\text{Deg}_X = \text{Deg}_{X_{\text{aff}}}$.

There is a second and more important reason (besides that our algorithms operate on them) to introduce multiaffine varieties. A key step in our numerical irreducible decomposition for multiprojective varieties in \mathbb{P}^{n_\bullet} , called coarsening and described in Section 4, requires passing to a multiaffine variety (multi-dehomogenizing) and then rehomogenizing it into a different multiprojective variety in a different multiprojective space.

1.5. Monodromy and partial witness collections. In [6], algorithms based on regeneration [8] were given to compute a witness collection of a multiprojective variety. We describe an alternative method based on monodromy. Let $Y \subset \mathbb{P}^{n_\bullet}$ be an irreducible component of $\mathcal{V}(F)$, where $F \subset \mathbb{C}[\mathbf{x}]$ is a finite set of multihomogeneous polynomials. Suppose that $\ell_{ij} \in \mathbb{C}[\mathbf{x}_i]$ are general linear polynomials as in Subsection 1.3. A *partial witness collection* for Y is a map $\text{Dim}(Y) \ni \mathbf{e} \mapsto (F, \mathbf{L}^{\mathbf{e}}, W_{\mathbf{e}})$, where $W_{\mathbf{e}} \subset Y \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$ and at least one set $W_{\mathbf{e}}$ is nonempty.

The monodromy solving algorithm [3] gives a method to complete a partial witness set to a witness set. If in Subsection 1.2, we have a variety $X \subset \mathbb{P}^n$ of pure dimension e and a partial witness set $W \subset X \cap \mathcal{V}(L)$ (L consists of e linear polynomials, with the intersection transverse), following points of W as L varies along loops both finds more points of $X \cap \mathcal{V}(L)$ and computes a putative numerical irreducible decomposition, with the caveat that the points found and subsequent decomposition will only lie on the irreducible components of X that contained points in the original set W . The transversality of $X \cap \mathcal{V}(L)$ at points of W is necessary for there to be a homotopy starting at points of W .

This also may begin with a nonempty partial \mathbf{e} -witness set $W_{\mathbf{e}} \subset X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$ of a multiprojective or multiaffine variety X with $\mathbf{e} \in \text{Dim}(X)$. That is, monodromy may be used to complete $W_{\mathbf{e}}$ to a full \mathbf{e} -witness set $X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$, at least for the components of X that contain points of $W_{\mathbf{e}}$. In Section 2, we explain a more general procedure.

1.6. Examples. We give the dimension and multidegree of some multiaffine varieties. Subsequent sections use these examples to demonstrate the numerical toolkit.

Example 1.5. Let $Y \subset (\mathbb{P}^1)^k$ be irreducible of intrinsic dimension e . Then $\text{Dim}(Y)$ consists of 01-vectors with e 1s and $k-e$ 0s; the positions of the 1s give a subset of $\{1, \dots, k\}$ of cardinality e . For each such subset \mathbf{e} , let $\pi_{\mathbf{e}}: (\mathbb{P}^1)^k \rightarrow (\mathbb{P}^1)^e$ be the surjection onto the factors corresponding to \mathbf{e} . Our definitions imply that for $|\mathbf{e}| = e$, $\mathbf{e} \in \text{Dim}(Y)$ if and only if $\pi_{\mathbf{e}}: Y \rightarrow (\mathbb{P}^1)^e$ is surjective. Thus $\text{Dim}(Y)$ is the algebraic matroid [14, p. 211] of $Y_{\text{aff}} \subset \mathbb{C}^k$. Its bases are subsets y_{i_1}, \dots, y_{i_e} of cardinality e of the variables y_1, \dots, y_k that are algebraically independent in the coordinate ring of Y_{aff} . \diamond

Example 1.6. We consider two multiaffine varieties in $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. Suppose that its coordinates are x, y, z, w and consider the three polynomials

$$\begin{aligned} f &= 1 + 2x + 3y^2 + 4z^3 + 5w^4, \\ g &= 1 + 2x + 3y + 5z + 7w, \text{ and} \\ h &= 1 + 2x + 3y + 5z + 7w + 11xy + 13xz + 17xw + 19yz + 23yw + 29zw \\ &\quad + 31xyz + 37xyw + 41xzw + 43yzw + 47xyzw. \end{aligned}$$

Let $X := \mathcal{V}(f, g)$ and $Y := \mathcal{V}(f, h)$, which are surfaces. Both have the same dimension, $\{1100, 1010, 1001, 0110, 0101, 0011\}$ (we omit commas). These form the second hypersimplex, which is an octahedron in their affine span. We display this in Figure 1.

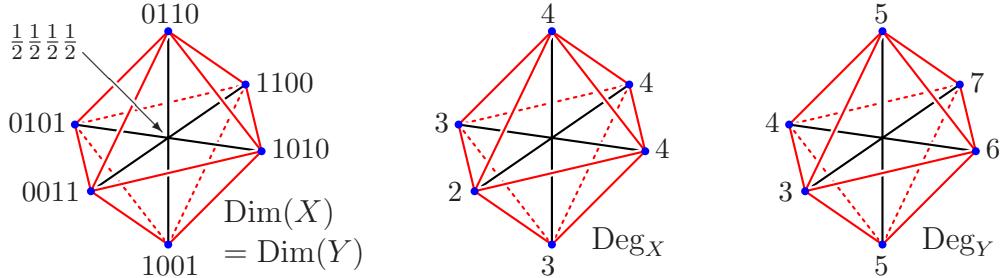


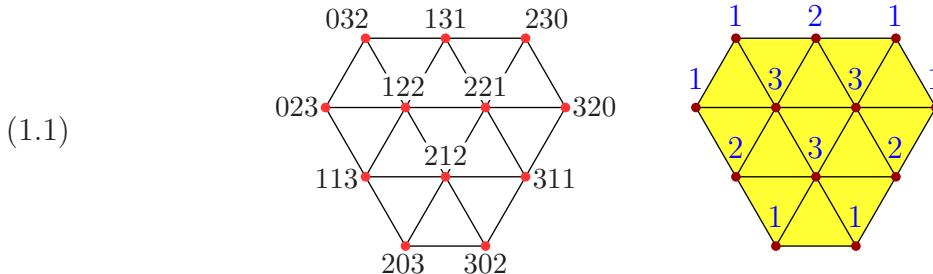
FIGURE 1. Dimension and degree of multiaffine varieties.

Both g and h are the dehomogenization of multilinear polynomials on $(\mathbb{P}^1)^4$. The difference between Deg_X and Deg_Y is that $\mathcal{V}(f, g)$ is a reducible variety in $(\mathbb{P}^1)^4$ which has components not meeting the given multiaffine patch so that X is a component of $\mathcal{V}(f, g)$ in $(\mathbb{P}^1)^4$. In contrast, h is sufficiently general so that Y is dense in $\mathcal{V}(f, h)$ in $(\mathbb{P}^1)^4$. \diamond

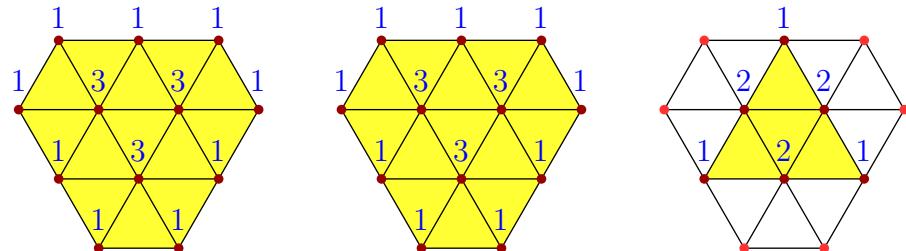
Example 1.7. Suppose that $n_{\bullet} = (3, 3, 3)$. Let $M = (y_{i,j})_{i,j=1}^3$ be a 3×3 matrix with rows the variable groups $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ of $\mathbb{C}^{n_{\bullet}}$. Set $C := (I_3 \mid M)^T$, a 6×3 matrix, and let N_1, N_2 be general complex 6×2 matrices. The conditions $\text{rank}(C \mid N_i) \leq 4$ for

$i = 1, 2$ define an irreducible subvariety Y of \mathbb{C}^9 of dimension five. (Taking the column span of C parameterizes a dense open subset of the Grassmannian $G(3, 6)$, each condition $\text{rank}(C \mid N_i) \leq 4$ gives a codimension two Schubert variety, and these are in general position by the choice of the N_i . Thus Y is an open subset of a Richardson variety.)

Each condition $\text{rank}(C \mid N_i) \leq 4$ is given by cubic determinants (minors) of the six 5×5 matrices obtained by removing a row of $(C \mid N_i)$. Let $f_{i,j}$ be the minor when row j is removed. It has degree one in each variable group $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, and so Y is a multiaffine subvariety of \mathbb{C}^{n_\bullet} . Its dimension is the set $\{\mathbf{e} \in [n_\bullet] \mid |\mathbf{e}| = 5\}$, which consists of the twelve integer points in the hexagon on the left below. On the right is its multidegree, where $\text{Deg}_Y(\mathbf{e})$ is displayed adjacent to \mathbf{e} .



Replacing the twelve minors $f_{i,j}$ defining the rank conditions by the subset $f_{1,3}, f_{1,5}, f_{2,4}, f_{2,6}$ gives a complete intersection with four components, one of which is Y . Two have the same dimension as Y and one has a different dimension. We display their multidegrees below.



1.7. Numerical irreducible decomposition for multiprojective varieties. An algorithm for computing witness set collections was given in [6]. There, Example 20 showed that the trace test cannot be applied to a witness set collection for $X \subset \mathbb{P}^{n_\bullet}$. We must embed X into an affine or projective space and transform the witness set collection into a witness set for the embedded X , and then apply the trace test.

This poses several problems. Under the Segre embedding, $X \subset \mathbb{P}^{n_\bullet}$ becomes a subvariety $\sigma(X)$ of \mathbb{P}^N , where $N + 1 = (n_1 + 1) \cdots (n_k + 1)$. Following [5, Exer. 19.2], if X has dimension d , then $\sigma(X)$ has degree

$$(1.2) \quad \sum_{|\mathbf{e}|=d} \binom{d}{\mathbf{e}} \text{Deg}_X(\mathbf{e}),$$

where $\binom{d}{\mathbf{e}}$ is the multinomial coefficient $\frac{d!}{e_1! \cdots e_k!}$. Thus, both the ambient dimension and size of a witness set increases dramatically.

Replacing X by its intersection with an affine patch $X_{\text{aff}} \subset \mathbb{C}^{n_1+\dots+n_k}$, does not increase its ambient dimension. Unlike the Segre embedding, it is not clear how to efficiently transform a witness collection for X into a witness set for X_{aff} . The Richardson variety Y of Example 1.7 has degree 450 under the Segre map and degree eight as an affine variety.

2. COMPUTING DIMENSION AND COMPLETING A PARTIAL WITNESS SET

Suppose that $X \subset \mathbb{C}^n$ is an irreducible affine variety that is a component of $\mathcal{V}(F)$, for a collection $F = (f_1, \dots, f_m)$ of polynomials. We assume that $\mathcal{V}(F)$ is reduced along X in that there is a point $x \in X$ such that the differential $d_x F := (d_x f_1, \dots, d_x f_m)$ (a linear map $\mathbb{C}^n \rightarrow \mathbb{C}^m$) has rank $n - \dim(X)$. Then X is smooth at x with tangent space $T_x X$ the kernel of $d_x F$. The smooth points of X form a nonempty Zariski open subset.

The differential $d_x F$ at a general smooth point $x \in X$ is given by the Jacobian matrix of F ,

$$DF := (\partial f_i / \partial x_j)_{i=1, \dots, m}^{j=1, \dots, n},$$

evaluated at x . Thus $\dim(X) = n - \text{rank}(DF(x))$.

2.1. Dimension of an irreducible multiprojective variety. Let X be an irreducible subvariety of \mathbb{P}^{n_\bullet} of intrinsic dimension e . Its dimension $\text{Dim}(X)$ is a subset of

$$(2.1) \quad \{\mathbf{e} \in [n_\bullet] \mid |\mathbf{e}| = e\}.$$

Castillo et al. [2] characterized $\text{Dim}(X)$ as follows. For $I = \{i_1, \dots, i_s\} \subset \{1, \dots, k\}$, define $\mathbb{P}^{n_I} := \mathbb{P}^{n_{i_1}} \times \dots \times \mathbb{P}^{n_{i_s}}$, and let $\pi_I: \mathbb{P}^{n_\bullet} \rightarrow \mathbb{P}^{n_I}$ be the projection onto the factors indexed by I . Let $\dim_I(X)$ be the intrinsic dimension of $\pi_I(X) \subset \mathbb{P}^{n_I}$. Dimension counting implies that if $\mathbf{e} \in \text{Dim}(X)$, then

$$(2.2) \quad e_{i_1} + \dots + e_{i_s} \leq \dim_I(X).$$

This follows because if $\ell(\mathbf{x}_i)$ is a linear polynomial in the variable group \mathbf{x}_i , then $\mathcal{V}(\ell(\mathbf{x}_i))$ is $\pi_{\{i\}}^{-1}(\mathcal{V}(\ell(\mathbf{x}_i)))$, with the second variety $\mathcal{V}(\ell(\mathbf{x}_i))$ a hyperplane in \mathbb{P}^{n_i} .

Proposition 2.1 (Thm. A in [2]). *Suppose that $X \subset \mathbb{P}^{n_\bullet}$ is an irreducible multiprojective variety. Then $\mathbf{e} \in [n_\bullet]$ lies in $\text{Dim}(X)$ if and only if $|\mathbf{e}| = \dim(X)$ and for all proper subsets I of $\{1, \dots, k\}$, the inequality (2.2) holds.*

These inequalities in \mathbb{R}^k define a lattice polytope of dimension at most $k-1$, which is a polymatroid polytope (called a generalized permutohedron in [15]).

Example 2.2. We continue Example 1.7. Suppose that in addition to the four minors defining the reducible complete intersection $X \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$, defining polynomials F include the quadrics

$$y_{1,1}y_{2,2} - y_{1,2}y_{2,1} \quad \text{and} \quad y_{1,1}y_{2,3} - y_{1,3}y_{2,1}.$$

Then $\mathcal{V}(F)$ has intrinsic dimension three with twelve irreducible components—the four components of X giving rise to 2, 3, 3, and 4 irreducible components, respectively. The i th row of Figure 2 displays the dimension and multidegree of the irreducible decomposition of $Y \cap \mathcal{V}(y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, y_{1,1}y_{2,3} - y_{1,3}y_{2,1})$, where Y is the i th component of X from Example 1.7. The first row also shows the set $\{\mathbf{e} \in [(3, 3, 3)] \mid |\mathbf{e}| = 3\}$ from (2.1).

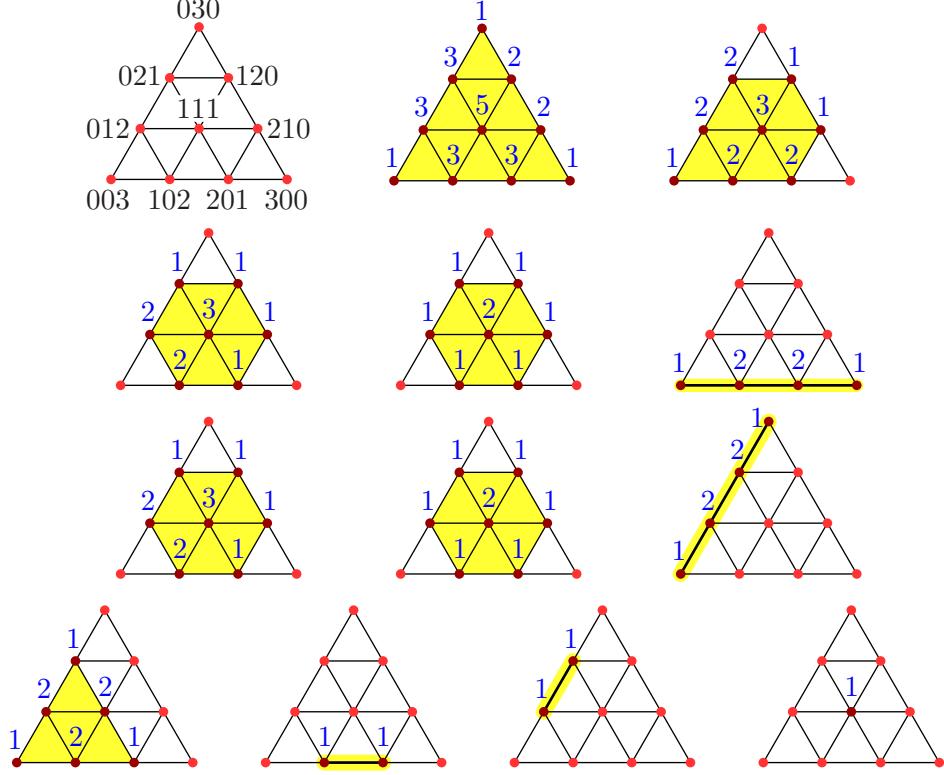


FIGURE 2. Decomposition of $X \cap \mathcal{V}(y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, y_{1,1}y_{2,3} - y_{1,3}y_{2,1})$.

As $k = 3$, the dimension of a polymatroid polytope $\text{Dim}(Z)$ is at most 2. For seven components this is a polygon, for four, it is a line segment, and for one, it is a point. \diamond

Let x be a point on an irreducible multiprojective variety $X \subset \mathbb{P}^{n\bullet}$ and suppose that $I \subset \{1, \dots, k\}$. We assume that x is general in that the map π_I is regular at x . (That is, x is a smooth point of X and the projection map $d_x\pi_I: T_x X \rightarrow T_{\pi_I(x)}\mathbb{P}^{n_I}$ has maximal rank among all smooth points of X .) Then, $\dim_I(X)$ is equal to the dimension of $d_x\pi_I(T_x X)$.

This leads to a method to compute these dimensions in local coordinates. Suppose that $F = (f_1, \dots, f_m)$ are polynomials in $\mathbb{C}[\mathbf{y}_1, \dots, \mathbf{y}_k]$ which are the dehomogenization of multihomogeneous polynomials defining $X \subset \mathbb{P}^{n\bullet}$ in some multiaffine patch $\mathbb{C}^{n\bullet} \subset \mathbb{P}^{n\bullet}$ containing x . Suppose that Y is the component of $\mathcal{V}(F)$ containing x and Y is smooth at x . Then the intrinsic dimension $\dim(Y)$ of Y (the local dimension of $\mathcal{V}(F)$ at x) is the dimension of the tangent space $T_x Y$, which is the kernel of the Jacobian $DF(x)$ of F at x . Thus, $\dim(Y) = \dim_x(Y) = \dim \ker DF(x)$.

The variable groups $\mathbf{y}_1, \dots, \mathbf{y}_k$ partition the columns of the Jacobian matrix

$$DF = \left(\begin{array}{c|c|c|c} \frac{\partial F}{\partial \mathbf{y}_1} & \frac{\partial F}{\partial \mathbf{y}_2} & \cdots & \frac{\partial F}{\partial \mathbf{y}_k} \end{array} \right),$$

where for each $l \in \{1, \dots, k\}$, $\partial F / \partial \mathbf{y}_l = (\partial f_i / \partial y_{l,j})_{i=1, \dots, m}^{j=1, \dots, n_l}$ is the Jacobian matrix with respect to the variables \mathbf{y}_l . Denote by DF_{I^c} the submatrix of DF obtained by omitting

the blocks $\partial F / \partial \mathbf{y}_i$ for $i \in I$. In other words,

$$(2.3) \quad DF_{I^c} := \left(\begin{array}{c|c|c} \frac{\partial F}{\partial \mathbf{y}_{j_1}} & \cdots & \frac{\partial F}{\partial \mathbf{y}_{j_r}} \end{array} \right), \quad \text{where} \quad I^c = \{j_1, \dots, j_r\}.$$

Since the intrinsic dimension of the image of Y under π_I equals the intrinsic dimension of Y minus the intrinsic dimension of the fiber over a general point, it follows that $\dim_I(Y) = \dim \ker DF(x) - \dim \ker DF_{I^c}(x)$.

By Proposition 2.1, if Y is the irreducible component of a multiprojective variety $\mathcal{V}(F) \subset \mathbb{P}^{n^*}$ containing the point x , then $\text{Dim}(Y)$ is determined by the numbers $\dim(Y)$ and $\dim_I(Y)$ for all proper subsets I of $\{1, \dots, k\}$. Let $\text{Dim}_x(F)$ be these numbers, which may be computed in local coordinates by determining the ranks of the Jacobian matrices $DF(x)$ and $DF_{I^c}(x)$. This leads to two algorithms to classify the dimension of components of $\mathcal{V}(F)$ given points of $\mathcal{V}(F)$.

Algorithm 2.3 (Dimension at a smooth point).

Input: A general smooth point $x \in \mathcal{V}(F) \subset \mathbb{P}^{n^*}$.

Output: $\text{Dim}_x(F)$.

Do: Dehomogenize F and compute $\dim \ker DF(x)$. For each proper subset I of $\{1, \dots, k\}$ compute $\dim \ker DF_{I^c}(x)$ to determine the difference $\dim \ker DF(x) - \dim \ker DF_{I^c}(x)$.

If y is smooth but not general, then it can be perturbed via a homotopy to a general point. (Recall that witness points are smooth.) Given $F \subset \mathbb{C}[\mathbf{y}_1, \dots, \mathbf{y}_k]$ defining a multiaffine variety $\mathcal{V}(F) \subset \mathbb{C}^{n^*}$, this algorithm simply skips the dehomogenization.

A multiprojective variety X is *equidimensional* if all irreducible components have the same multidimension. A multiprojective variety has a unique decomposition into equidimensional pieces. Given a collection W of general smooth points of $\mathcal{V}(F)$, by computing the local dimension via Algorithm 2.3 one can sort the points by the equidimensional component of $\mathcal{V}(F)$ on which they lie. Let $\text{Dim}(W)$ be the set of dimensions of components of $\mathcal{V}(F)$ containing points of W . For $\Delta \in \text{Dim}(W)$, define $W_\Delta := \{w \in W \mid \text{Dim}_w(F) = \Delta\}$. These sets partition W and form the *equidimensional decomposition* of W ,

$$W = \bigsqcup \{W_\Delta \mid \Delta \in \text{Dim}(W)\}.$$

Algorithm 2.4 (Equidimensional decomposition).

Input: A finite set $W \subset \mathcal{V}(F)$ of general smooth points.

Output: $\text{Dim}(W)$ and the equidimensional partition of W .

Do: For each $w \in W$, compute the local dimension $\text{Dim}_w(F)$ of $\mathcal{V}(F)$ at w to get $\text{Dim}(W)$ and for each $\Delta \in \text{Dim}(W)$ let $W_\Delta = \{w \in W \mid \text{Dim}_w(F) = \Delta\}$.

It is important that the points of W be general so that the maps π_I are regular on W .

2.2. Completing a partial witness collection. A partial witness collection (F, \mathbf{L}^e, W_e) for a multiprojective variety Y may be completed to a witness collection using monodromy. While this was sketched in Subsection 1.5, it needs the definitions given in this section.

Algorithm 2.5 (Completing a witness collection from a single point).

Input: A general smooth point y on an irreducible multiprojective variety Y that is a

component of $\mathcal{V}(F)$.

Output: A witness collection for Y .

Do: Use Algorithm 2.3 to compute $\text{Dim}(Y)$. Choose linear polynomials $\ell_{ij} \in \mathbb{C}[\mathbf{x}_i]$ for $i \in \{1, \dots, k\}$ and $j = 1, \dots, n_i$ that are general given that they vanish at y . Using the ℓ_{ij} gives a partial witness collection $\{(F, \mathbf{L}^e, \{y\}) \mid e \in \text{Dim}(Y)\}$ for Y . Use monodromy as in Subsection 1.5 to complete each partial e -witness point set $\{y\}$ to the complete e -witness point set $Y \cap \mathcal{V}(\mathbf{L}^e)$.

Proof of correctness. We note that this does not have a stopping criterion, and is therefore technically not an algorithm. Nevertheless, by the choice of ℓ_{ij} , each intersection $Y \cap \mathcal{V}(\mathbf{L}^e)$ is transverse and contains $\{y\}$. Thus, letting the \mathbf{L}^e vary in a loop gives a homotopy. The rest follows from the discussion in Subsection 1.5. \square

3. CARTESIAN PRODUCTS

Of the twelve irreducible components Y of the variety $\mathcal{V}(F)$ of Example 2.2, $\text{Dim}(Y)$ was a line segment for four and a point for one. In these five cases, $\text{Dim}(Y)$ was decomposable as a product of polymatroid polytopes. We will show that if $Y \subset \mathbb{P}^{n_\bullet}$ is an irreducible multiprojective variety for which $\text{Dim}(Y)$ is such a product, then $Y = Y' \times Y''$ is a Cartesian product of irreducible varieties in disjoint factors of \mathbb{P}^{n_\bullet} , and the witness sets for Y are also products of witness sets for Y' and Y'' .

Fix $1 \leq l < k$. Let $n'_\bullet := (n_1, \dots, n_l)$ and $n''_\bullet := (n_{l+1}, \dots, n_k)$ so that $\mathbb{P}^{n_\bullet} = \mathbb{P}^{n'_\bullet} \times \mathbb{P}^{n''_\bullet}$. If $Y' \subset \mathbb{P}^{n'_\bullet}$ and $Y'' \subset \mathbb{P}^{n''_\bullet}$ are irreducible varieties, then so is $Y' \times Y'' \subset \mathbb{P}^{n_\bullet}$. Its intrinsic dimension is the sum of the intrinsic dimensions of its factors, $\dim(Y' \times Y'') = \dim(Y') + \dim(Y'')$. Its multidimension has a similar decomposition,

$$\begin{aligned} \text{Dim}(Y' \times Y'') &= \text{Dim}(Y') \times \text{Dim}(Y'') \\ &= \{\mathbf{e} \in [n_\bullet] \mid \mathbf{e} = (\mathbf{e}', \mathbf{e}'') \text{ for } \mathbf{e}' \in \text{Dim}(Y') \text{ and } \mathbf{e}'' \in \text{Dim}(Y'')\}, \end{aligned}$$

as $[n_\bullet] = [n'_\bullet] \times [n''_\bullet]$. This is a consequence of the definition given in Subsection 1.3 for the multidimension of a multiprojective variety, applied to such a product.

For $(\mathbf{e}', \mathbf{e}'') \in \text{Dim}(Y' \times Y'')$, suppose that $\mathbf{L}^{\mathbf{e}'} \subset \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_l]$ and $\mathbf{L}^{\mathbf{e}''} \subset \mathbb{C}[\mathbf{x}_{l+1}, \dots, \mathbf{x}_k]$ are general linear polynomials with corresponding witness point sets $W_{\mathbf{e}'} = Y' \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$ for Y' and $W_{\mathbf{e}''} = Y'' \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}''})$ for Y'' . Then

$$W_{\mathbf{e}'} \times W_{\mathbf{e}''} = (Y' \times Y'') \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'}, \mathbf{L}^{\mathbf{e}''})$$

is an $(\mathbf{e}', \mathbf{e}'')$ -witness point set for the product $Y' \times Y''$.

More generally, let $I \subset \{1, \dots, k\}$ be a proper subset with complement J so that $\mathbb{P}^{n_\bullet} = \mathbb{P}^{n_I} \times \mathbb{P}^{n_J}$. Given irreducible multiprojective varieties $Y \subset \mathbb{P}^{n_I}$ and $Z \subset \mathbb{P}^{n_J}$, their product is a multiprojective variety $Y \times Z \subset \mathbb{P}^{n_\bullet}$. We similarly have $\text{Dim}(Y \times Z) = \text{Dim}(Y) \times \text{Dim}(Z)$, and witness point sets for $Y \times Z$ are products of witness point sets for Y and for Z . This reduces to the previous discussion after reordering the factors of \mathbb{P}^{n_\bullet} .

Theorem 3.1. *An irreducible multiprojective variety $X \subset \mathbb{P}^{n_\bullet}$ is a Cartesian product $X = Y \times Z$ of multiprojective varieties $Y \subset \mathbb{P}^{n_I}$ and $Z \subset \mathbb{P}^{n_J}$ in disjoint factors of \mathbb{P}^{n_\bullet} if and only if $\text{Dim}(X)$ is the product of polymatroid polytopes $P \subset [n_I]$ and $Q \subset [n_J]$ with $\text{Dim}(Y) = P$ and $\text{Dim}(Z) = Q$.*

When this occurs, the multidegree $\text{Deg}_X(\mathbf{e}', \mathbf{e}'')$ for $\mathbf{e}' \in \text{Dim}(Y)$ and $\mathbf{e}'' \in \text{Dim}(Z)$ is the product $\text{Deg}_Y(\mathbf{e}') \cdot \text{Deg}_Z(\mathbf{e}'')$ of multidegrees and any $(\mathbf{e}', \mathbf{e}'')$ -witness point set for X is the product of corresponding witness point sets for Y and for Z .

Proof. The forward direction of the first part is a consequence of the preceding discussion, as is the second part of the theorem (which follows from the cartesian product $X = Y \times Z$). For the reverse direction of the first part, suppose that $\text{Dim}(X) = P \times Q$, where $P \subset [n_I]$ and $Q \subset [n_J]$ are polymatroid polytopes in disjoint factors of $[n_\bullet]$, so that $I \sqcup J = \{1, \dots, k\}$. Since P and Q are polymatroid polytopes, there are integers p and q such that if $\mathbf{e}' \in P$ and $\mathbf{e}'' \in Q$, then $|\mathbf{e}'| = p$, $|\mathbf{e}''| = q$, and $p + q = \dim(X)$.

Let us study the map $\pi_I: X \rightarrow \mathbb{P}^{n_I}$ whose image $\pi_I(X)$ has dimension $\dim_I(X)$. Let $\mathbf{e}' \in \text{Dim}(\pi_I(X)) \subset [n_I]$ and let $\mathbf{L}^{\mathbf{e}'} \subset \mathbb{C}[\mathbf{x}_i \mid i \in I]$ be $|\mathbf{e}'| = \dim_I(X)$ general linear polynomials so that $\pi_I(X) \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$ consists of $d = \text{Deg}_{\pi_I(X)}(\mathbf{e}')$ points. Since in \mathbb{P}^{n_\bullet} , we have $\mathcal{V}(\mathbf{L}^{\mathbf{e}'}) = \pi_I^{-1}(\mathcal{V}(\mathbf{L}^{\mathbf{e}'}))$, the intersection $X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$ is nonempty and it consists of d fibers of the map $\pi_I: X \rightarrow \pi_I(X)$. By the generality of $\mathbf{L}^{\mathbf{e}'}$, each fiber has dimension $\dim(X) - \dim_I(X)$. Then there is some $\mathbf{e}'' \in [n_J]$ such that if $\mathbf{L}^{\mathbf{e}''} \subset \mathbb{C}[\mathbf{x}_j \mid j \in J]$ are $|\mathbf{e}''| = \dim(X) - \dim_I(X)$ general linear polynomials, then $X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'}) \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}''})$ is nonempty.

This implies that $(\mathbf{e}', \mathbf{e}'') \in \text{Dim}(X)$ and in particular that $\mathbf{e}' \in P$ and $\mathbf{e}'' \in Q$ and that $\dim_I(X) = \dim(\pi_I(X)) = p$. Similarly, $\dim_J(X) = \dim(\pi_J(X)) = q$. Since $X \subset \pi_I(X) \times \pi_J(X)$ and both are irreducible of dimension $p + q$, they are equal. \square

Example 3.2. Let us look at the last two components in the bottom row of Figure 2. The third component Y has $\text{Dim}(Y) = \{0\} \times \{12, 21\}$. Its ideal is generated by

$$\begin{aligned} & y_{1,1}, \quad y_{1,2}, \quad y_{1,3}, \quad 19y_{2,2} + 46y_{2,3}, \quad 19y_{3,2} + 46y_{3,3} + 34, \\ & 243y_{2,3}y_{3,1} - 243y_{2,1}y_{3,3} - 306y_{2,1} + 1020y_{2,3} - 342y_{3,1} + 1194y_{3,3} + 68 \end{aligned}$$

The first three define the point $\{(0, 0, 0)\}$ in the first \mathbb{C}^3 factor and the next two define a plane in each of the last two factors. Thus $Y = \{(0, 0, 0)\} \times Z$, where $Z \subset \mathbb{C}^2 \times \mathbb{C}^2$ is the hypersurface defined by the last bilinear polynomial. This explains $\text{Dim}(Y)$ and Deg_Y .

The last component Y has $\text{Dim}(Y) = \{1\} \times \{1\} \times \{1\}$. Its ideal is generated by

$$\begin{aligned} & 57y_{1,1} - 199y_{1,3}, \quad 19y_{1,2} + 46y_{1,3}, \quad 57y_{2,1} - 199y_{2,3}, \\ & 19y_{2,2} + 46y_{2,3}, \quad 171y_{3,1} - 597y_{3,3} - 34, \quad 19y_{3,2} + 46y_{3,3} + 34. \end{aligned}$$

As there are two affine forms in each variable group, Y is isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$, which again explains its multidegree. \diamond

Membership testing in Cartesian products can be simplified since one can consider membership in each factor independently.

Algorithm 3.3 (Membership test in Cartesian product).

Input: A witness collection for an irreducible multiprojective variety $X \subset \mathbb{P}^{n_\bullet}$ which is a Cartesian product $X = Y \times Z$ of multiprojective varieties $Y \subset \mathbb{P}^{n_I}$ and $Z \subset \mathbb{P}^{n_J}$ in disjoint factors of \mathbb{P}^{n_\bullet} and a point $\mathbf{x} = (\mathbf{y}, \mathbf{z}) \in \mathbb{P}^{n_\bullet}$.

Output: A triple $(B_{\mathbf{x}}, B_{\mathbf{y}}, B_{\mathbf{z}})$ of booleans such that B_ω answers if $\omega \in \Omega$.

Do: Select $\mathbf{e} \in \text{Dim}(X)$ and fix a point $(\mathbf{y}^*, \mathbf{z}^*)$ from the \mathbf{e} -witness point set for X .

Construct witness collections for Y and Z from the given witness collection for X following Theorem 3.1 with polynomial systems $F(\mathbf{y}, \mathbf{z}^*)$ and $F(\mathbf{y}^*, \mathbf{z})$, respectively. Apply Algorithm 1.4 to Y and Z yielding B_Y and B_Z , respectively. Set $B_{\mathbf{x}} = B_Y \times B_Z$

Proof of correctness. Since $X = Y \times Z$, we know $\mathbf{x} \in X$ if and only if $\mathbf{y} \in Y$ and $\mathbf{z} \in Z$. Let $\mathbf{e} = (\mathbf{e}', \mathbf{e}'') \in \text{Dim}(X)$ be the selection that yielded $(\mathbf{y}^*, \mathbf{z}^*)$ in the \mathbf{e} -witness point set for X with corresponding $\mathbf{L}^{\mathbf{e}} = (\mathbf{L}^{\mathbf{e}'}, \mathbf{L}^{\mathbf{e}''})$. Then, $Y \times \{z^*\}$ and $\{y^*\} \times Z$ are irreducible components of $\mathcal{V}(F, \mathbf{L}^{\mathbf{e}''})$ and $\mathcal{V}(F, \mathbf{L}^{\mathbf{e}'})$, respectively. Hence, by selecting a representative of \mathbf{y}^* and \mathbf{z}^* , it follows that Y and Z are irreducible components of $F(\mathbf{y}, \mathbf{z}^*)$ and $F(\mathbf{y}^*, \mathbf{z})$, respectively. Hence, Algorithm 3.3 decides membership of y in Y and z in Z which immediately decides membership of x in X . \square

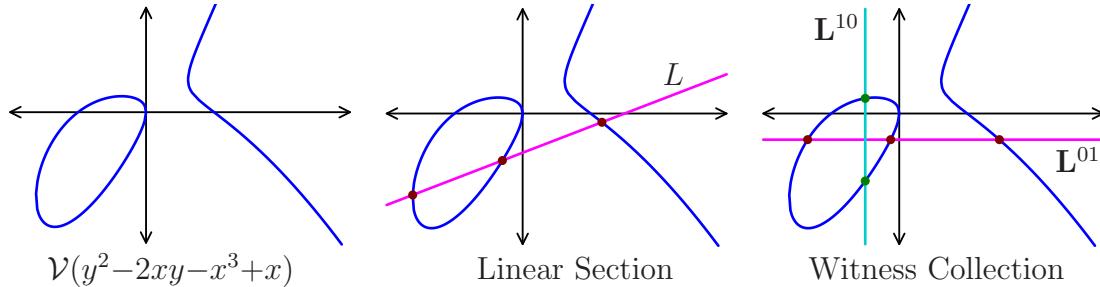
A natural recursion applies when X is a Cartesian product of more than two varieties.

4. REFINING AND COARSENING WITNESS COLLECTIONS

Algorithms for computing witness sets and witness collections operate on affine patches of projective and multiprojective varieties. Changing the multiaffine structure is straightforward in such patches and corresponds to a birational map on the underlying (multi)projective variety. We describe this and investigate how it affects witness collections.

A multiaffine variety $X_{\text{aff}} \subset \mathbb{C}^{n_{\bullet}}$ is simply a variety in the affine space $\mathbb{C}^{n_1 + \dots + n_k}$ whose coordinates have been partitioned into subsets of sizes n_1, \dots, n_k . Changing the partition does not change the variety X_{aff} , but it does change its multiaffine structure, that is, its multidimension and multidegrees. In particular, repartitioning changes how X_{aff} is represented using a witness collection. Any repartitioning is a composition of two operations, *refining*, in which one variable group is split into two, and *coarsening*, in which two variable groups are merged into one. We describe the geometry of refining and coarsening, and give algorithms for transforming witness collections for both.

Example 4.1. The polynomial $y^2 - 2xy - x^3 + x$ defines a plane cubic curve. As a multiaffine variety in $\mathbb{C}_x^1 \times \mathbb{C}_y^1$ its multidimension is $\{10, 01\}$ with corresponding multidegrees 2 for 10 and 3 for 01. In \mathbb{C}^2 , it is represented by a witness set which uses a linear section such as shown at center below. In $\mathbb{C}_x^1 \times \mathbb{C}_y^1$, it is represented by a witness collection, which are its intersections with a vertical and with a horizontal line as at right below. \diamond



4.1. Refining. Suppose that $k = 2$, so that $n_{\bullet} = (n_1, n_2)$ and set $n := n_1 + n_2$. Let $Y \subset \mathbb{P}^n$ be an irreducible variety of dimension e and degree d . Let Λ be a linear polynomial that does not vanish identically on Y and set $Y_{\text{aff}} := Y \setminus \mathcal{V}(\Lambda)$, which is an affine variety in

the affine patch $\mathbb{C}^n = \mathbb{P}^n \setminus \mathcal{V}(\Lambda)$. For the splitting $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, let Y_{n_\bullet} be the closure of Y_{aff} in the compactification $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ of $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$.

Proposition 4.2. *The multiprojective variety Y_{n_\bullet} and multiaffine variety Y_{aff} are irreducible and have dimension e . For any $(e_1, e_2) \in [n_\bullet]$ with $e = e_1 + e_2$, the (e_1, e_2) -multidegree of Y_{n_\bullet} (and also of Y_{aff}) is at most the degree of Y .*

This agrees with Example 4.1, where the size of each set in the witness collection was bounded above by the degree of the plane curve.

Proof. As Y_{aff} is a nonempty open subset of the irreducible variety Y , it is irreducible and of the same dimension. The same arguments imply that Y_{n_\bullet} is irreducible of dimension e . A general multilinear section $\mathcal{V}(\mathbf{L}^{(e_1, e_2)}) \cap Y_{n_\bullet}$ will be a subset of Y_{aff} . In the affine space $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, the variety $\mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ is a (non-general) linear subspace of codimension e , and thus $\mathcal{V}(\mathbf{L}^{(e_1, e_2)}) \cap Y_{\text{aff}}$ consists of at most $\deg(Y)$ points. \square

This gives a homotopy algorithm for computing witness collections under a refinement of a coordinate partition. Let $Y \subset \mathbb{C}^n$ be an equidimensional affine variety of dimension e and degree d , given as a union of components of a variety $\mathcal{V}(F)$ and let $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ be a splitting of \mathbb{C}^n with $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ the corresponding partition of variables for \mathbb{C}^n .

Suppose that $Y \subset \mathbb{C}^n$ is represented by a witness set $(F, L^e, Y \cap \mathcal{V}(L^e))$ where $L^e \subset \mathbb{C}[\mathbf{y}]$ consists of e general affine forms. Let $\mathbf{L}^{(e_1, e_2)}$ be e affine forms with e_1 from $\mathbb{C}[\mathbf{y}_1]$ and e_2 from $\mathbb{C}[\mathbf{y}_2]$, but otherwise general. Then, $Y \cap \mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ is a (e_1, e_2) -witness point set for the multiaffine variety $Y \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$. The system

$$(4.1) \quad H(t) := (F, tL^e + (1-t)\mathbf{L}^{(e_1, e_2)})$$

is a homotopy that connects the solutions $Y \cap \mathcal{V}(L^e)$ of the start system $H(1)$ to solutions $Y \cap \mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ of the target system $H(0)$.

Algorithm 4.3 (Transforming witness sets under refinement).

Input: A witness set $(F, L^e, Y \cap \mathcal{V}(L^e))$ for an equidimensional affine variety $Y \subset \mathbb{C}^n$ of dimension e , a splitting $\mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, and integers $0 \leq e_1, e_2$ with $e_1 + e_2 = e$.

Output: An (e_1, e_2) -witness point set for the multiaffine variety $Y \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$.

Do: Form the homotopy (4.1) and follow the points of $Y \cap \mathcal{V}(L^e)$ along H from $t = 1$ to $t = 0$, keeping those whose paths are bounded near $t = 0$.

Executing Algorithm 4.3 for each $(e_1, e_2) \in \text{Dim}(Y)$ computes the witness point sets for the full witness collection of the multiaffine variety $Y \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$. In Example 4.1 Algorithm 4.3 amounts to rotating the line L in the middle picture to either a horizontal or a vertical line.

Proof of correctness. Since L^e is general, the intersection $Y \cap \mathcal{V}(L^e)$ is transverse and consists of $d = \deg(Y)$ points. Thus, for general t , the intersection $Y \cap \mathcal{V}(tL^e + (1-t)\mathbf{L}^{(e_1, e_2)})$ is also transverse and consists of d points, and so (4.1) is a homotopy. As the affine forms in $\mathbf{L}^{(e_1, e_2)}$ are general given their variables, the intersection $Y \cap \mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ is transverse and consists of $\text{Deg}_Y(e_1, e_2)$ points. Thus $\text{Deg}_Y(e_1, e_2)$ paths in the homotopy end at the points of $Y \cap \mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ and $d - \text{Deg}_Y(e_1, e_2)$ paths diverge as t approaches 0. \square

Remark 4.4. Suppose that $Y \subset \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ is a multiaffine variety with $k > 1$ and that $\mathbb{C}^{n_i} = \mathbb{C}^{n'_i} \times \mathbb{C}^{n''_i}$ is a refinement splitting the i th factor \mathbb{C}^{n_i} . We may use the ideas in Algorithm 4.3 to transform an \mathbf{e} -witness point set for Y into one for this refinement. Given an \mathbf{e} -witness point set $Y \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$, we wish to compute a \mathbf{e}' -witness point set for the refinement, where the component e_i of \mathbf{e} is split into $e'_i + e''_i$ in \mathbf{e}' . For this, let $\mathbf{L}_i^{(e'_i, e''_i)}$ be e_i general affine forms with e'_i in $\mathbb{C}[\mathbf{y}_{i'}]$ and e''_i in $\mathbb{C}[\mathbf{y}_{i''}]$, where $\mathbf{y}_i = (\mathbf{y}_{i'}, \mathbf{y}_{i''})$ is the corresponding split of the variable group \mathbf{y}_i . Replacing the e_i affine forms of $L_i \subset \mathbb{C}[\mathbf{y}_i]$ in $\mathbf{L}^{\mathbf{e}}$ by the convex combination $tL_i + (1-t)\mathbf{L}_i^{(e'_i, e''_i)}$ gives a homotopy as in Algorithm 4.3 that transforms $Y \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$ into $Y \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$.

4.2. Coarsening. Suppose that $n_{\bullet} = (n_1, n_2)$ and set $\mathbf{n} := n_1 + n_2$. Let $Y_{n_{\bullet}} \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ be an irreducible multiprojective variety of intrinsic dimension e . For each $i = 1, 2$, let $\Lambda_i \in \mathbb{C}[\mathbf{x}^{(i)}]$ be a general linear polynomial. Then

$$Y_{\text{aff}} := Y_{n_{\bullet}} \setminus \mathcal{V}(\Lambda_1 \cdot \Lambda_2) \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \setminus \mathcal{V}(\Lambda_1 \cdot \Lambda_2) \simeq \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$$

is a multiaffine variety with the same multidimension and multidegree as $Y_{n_{\bullet}}$. Regarding $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} = \mathbb{C}^n$ as an affine patch in \mathbb{P}^n , let Y be the closure of Y_{aff} in \mathbb{P}^n . We investigate how to transform a witness collection for $Y_{n_{\bullet}}$ into a witness set for Y . In particular, we describe Algorithm 4.5, which transforms witness sets under coarsening. In this algorithm we first construct a witness set for the Segre embedding $\sigma(Y)$ from witness points of $Y_{n_{\bullet}}$ and then degenerate this witness set into another witness set of $\sigma(Y)$ whose pullback is a witness set for Y . In fact, all steps of Algorithm 4.5 operate in local coordinates for $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, not in the ambient space for the Segre map.

The multiprojective space $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ is a projective variety under the Segre map

$$\sigma : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \hookrightarrow \mathbb{P}^{n_1 n_2 + n_1 + n_2} =: \mathbb{P}^N.$$

A linear polynomial on \mathbb{P}^N pulls back to a bilinear form B on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. Writing \mathbb{P}^N as $\mathbb{P}(\mathbb{C}^{n_1+1} \otimes \mathbb{C}^{n_2+1}) = \mathbb{P}(\text{Mat}_{(n_1+1) \times (n_2+1)}(\mathbb{C}))$, a linear form ℓ on \mathbb{P}^N corresponds to a matrix M . When M has rank one, the pullback $\sigma^*(\ell) = \ell^{10}\ell^{01}$ is a product of linear polynomials, one in each set of variables. Thus $\mathcal{V}(t\ell^{10}\ell^{01} + (1-t)B)$ is a family (of hyperplane sections of $\sigma(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$) that transforms the union $Y_{n_{\bullet}} \cap (\mathcal{V}(\ell^{10}) \cup \mathcal{V}(\ell^{01}))$ of multilinear sections into the bilinear section $Y_{n_{\bullet}} \cap \mathcal{V}(B)$.

Passing from $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ to \mathbb{P}^n through affine patches, both B and $\ell^{10}\ell^{01}$ remain bilinear forms. Given a linear polynomial ℓ on \mathbb{P}^n and a choice z_0 of coordinate for the hyperplane at infinity, $z_0\ell$ is another bilinear form whose variety in the affine patch $\mathbb{P}^n \setminus \mathcal{V}(z_0)$ is the hyperplane $\mathcal{V}(\ell)$. Thus $tB + (1-t)z_0\ell$ or better $t\ell^{10}\ell^{01} + (1-t)z_0\ell$ is a family that may be used to transform the union $Y_{n_{\bullet}} \cap (\mathcal{V}(\ell^{10}) \cup \mathcal{V}(\ell^{01}))$ of sections of $Y_{n_{\bullet}}$ into the union of the section $Y \cap \mathcal{V}(\ell)$ with its part $Y \cap \mathcal{V}(z_0)$ at infinity.

This may be used to transform a multilinear section $Y_{n_{\bullet}} \cap \mathcal{V}(\mathbf{L}^{(e_1, e_2)})$ into a subset of a linear section $Y \cap \mathcal{V}(L^e)$, but only if we work in an affine patch $\mathbb{P}^n \setminus \mathcal{V}(z_0)$, as the bilinear forms coming from linear polynomials in L^e all have z_0 as a factor. By the inequality among degree and multidegree in Proposition 4.2, we typically obtain a subset of $Y \cap \mathcal{V}(L^e)$ (a partial witness set).

Let us describe a homotopy for this. Let $\ell_1^{10}, \dots, \ell_e^{10} \in \mathbb{C}[\mathbf{y}_1]$, $\ell_1^{01}, \dots, \ell_e^{01} \in \mathbb{C}[\mathbf{y}_2]$, and $\ell_1, \dots, \ell_e \in \mathbb{C}[\mathbf{y}]$ be general affine forms. Set $\mathbf{M} = (\ell_1^{10}\ell_1^{01}, \dots, \ell_e^{10}\ell_e^{01})$ and $\mathbf{L}^e = (\ell_1, \dots, \ell_e)$. Form the homotopy

$$(4.2) \quad H(t) := (F, t\mathbf{M} + (1-t)\mathbf{L}^e).$$

We describe the start points for $H(t)$ at $t = 1$. For a partition $S \sqcup T$ of $\{1, \dots, e\}$ with $(|S|, |T|) \in \text{Dim}(Y_{\text{aff}})$, let $\mathbf{L}^{S,T} := \{\ell_i^{10} \mid i \in S\} \cup \{\ell_j^{01} \mid j \in T\}$, a subset of linear forms involved above. Then, $(F, t\mathbf{L}^{(|S|,|T|)} + (1-t)\mathbf{L}^{S,T})$ is a homotopy transforming the witness point set $Y_{\text{aff}} \cap \mathcal{V}(\mathbf{L}^{(|S|,|T|)})$ into the multilinear section $W_{S,T} := Y_{\text{aff}} \cap \mathcal{V}(\mathbf{L}^{S,T})$, which is a transverse intersection as the affine forms are general. Let W be the disjoint union of all the $W_{S,T}$. These are disjoint as the affine forms are general.

Algorithm 4.5 (Transforming witness sets under coarsening).

Input: A witness collection $\{(F, \mathbf{L}^e, Y \cap \mathcal{V}(\mathbf{L}^e)) \mid e \in \text{Dim}(Y)\}$ for an equidimensional multiaffine variety $Y_{\text{aff}} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$.

Output: A witness point set $Y^\circ \cap \mathcal{V}(L^e)$ for the affine variety $Y^\circ \subset \mathbb{C}^n = \mathbb{P}^n \setminus \mathcal{V}(z_0)$.

Do: Recall from above that W is the union of all the $W_{S,T}$. Compute the points of W and use the homotopy (4.2) to follow the points of W along H from $t = 1$ to $t = 0$, keeping those whose paths are bounded near $t = 0$.

Proof of correctness. Observe that in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ we have

$$\mathcal{V}(\mathbf{M}) = \mathcal{V}(\ell_1^{10}\ell_1^{01}, \dots, \ell_e^{10}\ell_e^{01}) = \bigcup_{S \sqcup T = \{1, \dots, e\}} \mathcal{V}(\mathbf{L}^{S,T}).$$

As $Y_{n_\bullet} \cap \mathcal{V}(L^{S,T}) = W_{S,T}$, we have

$$Y_{n_\bullet} \cap \mathcal{V}(\mathbf{M}) = \bigcup_{S \sqcup T = \{1, \dots, e\}} Y_{n_\bullet} \cap \mathcal{V}(\mathbf{L}^{S,T}) = \bigsqcup_{S \sqcup T = \{1, \dots, e\}} W_{S,T} = W.$$

By (1.2), we have

$$(4.3) \quad \deg_{\mathbb{P}^N}(\sigma(Y_{n_\bullet})) = \sum_{e_1+e_2=e} \binom{e}{e_1} \text{Deg}_{Y_{n_\bullet}}(e_1, e_2) = |W|,$$

as $|W_{S,T}| = \text{Deg}_{Y_{n_\bullet}}(|S|, |T|)$. Thus $Y_{n_\bullet} \cap \mathcal{V}(\mathbf{M})$ is a transverse intersection consisting of $\delta := \deg_{\mathbb{P}^N}(\sigma(Y_{n_\bullet}))$ points. Since $W \subset Y_{\text{aff}}$, for general t the intersection

$$(4.4) \quad Y_{\text{aff}} \cap \mathcal{V}(t\mathbf{M} + (1-t)\mathbf{L}^e)$$

is also transverse and consists of δ points. Thus, (4.2) is a homotopy.

Consider the variety in $\mathbb{P}^n \times \mathbb{C}_t$ defined by

$$(4.5) \quad (Y \times \mathbb{C}_t) \bigcap \mathcal{V}(t\mathbf{M} + (1-t)(z_0\ell_1, \dots, z_0\ell_e)).$$

(Note the homogenizing variable z_0 .) Since (4.4) is transverse and consists of δ points for general t , the components of (4.5) that map onto \mathbb{C}_t form a curve C whose general fiber

over t is δ points. Restricting C to an arc in \mathbb{C}_t with endpoints $\{0, 1\}$ gives δ arcs that start (when $t = 1$) at the points of W and end (at $t = 0$) in

$$Y \cap \mathcal{V}(z_0\ell_1, \dots, z_0\ell_e) = Y \cap \mathcal{V}(L^e) \bigcup Y \cap \mathcal{V}(z_0).$$

As the affine forms ℓ_1, \dots, ℓ_e are general, $Y \cap \mathcal{V}(L^e) \subset Y^\circ$ is transverse and consists of $d = \deg_{\mathbb{P}^n}(Y)$ points. Thus d paths in the homotopy end at the points of $Y^\circ \cap \mathcal{V}(L^e)$ and $\delta - d$ paths diverge as t approaches 0. \square

Algorithm 4.5 may use up to 2^e witness sets and tracks $\delta = \deg_{\mathbb{P}^N}(\sigma(Y_{n,\bullet}))$ as in (4.3) paths. While δ is typically enormous, when $Y_{n,\bullet}$ is a curve, $\delta = \text{Deg}_{Y_{n,\bullet}}(1, 0) + \text{Deg}_{Y_{n,\bullet}}(0, 1)$, which is the cardinality of the witness collection for $Y_{n,\bullet}$. In Example 4.1, Algorithm 4.5 starts with five points on the intersection of the cubic curve with the horizontal and vertical lines at right, passes through a family of hyperbolas, and ends at the intersection of the cubic curve with the line L in the middle, with two paths diverging to infinity.

Remark 4.6. Suppose that $Y \subset \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k}$ is a multiaffine variety with $k \geq 2$. Let $Y' \subset \mathbb{C}^{n_1+n_2} \times \mathbb{C}^{n_3} \times \dots \times \mathbb{C}^{n_k}$ be the variety Y with the multiaffine structure induced by merging the first two factors. As in Remark 4.4, Algorithm 4.5 may be used to transform a witness collection for Y into one for Y' .

For each $i = 1, \dots, k$, let \mathbf{y}_i be n_i indeterminates—these are the indeterminates for $\mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k}$. Write $\mathbf{x} := (\mathbf{y}_1, \mathbf{y}_2)$. Then the partition of the indeterminates for $\mathbb{C}^{n_1+n_2} \times \mathbb{C}^{n_3} \times \dots \times \mathbb{C}^{n_k}$ is $(\mathbf{x}, \mathbf{y}_3, \dots, \mathbf{y}_k)$. We explain how to compute an $\mathbf{e}' = (e, e_3, \dots, e_k)$ -witness point set for Y' given a witness collection for Y . (The indexing in \mathbf{e}' is intended.)

An \mathbf{e}' -witness point set for Y' is an intersection $Y' \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$, where $\mathbf{L}^{\mathbf{e}'} = (L^e, L_3, \dots, L_k)$ with L^e consisting of e general affine forms $\ell_1, \dots, \ell_e \in \mathbb{C}[\mathbf{x}]$ and for $i \geq 3$, L_i consists of e_i general affine forms in $\mathbb{C}[\mathbf{y}_i]$. As in the discussion preceding Algorithm 4.5, let $\ell_1^{10}, \dots, \ell_e^{10} \in \mathbb{C}[\mathbf{y}_1]$ and $\ell_1^{01}, \dots, \ell_e^{01} \in \mathbb{C}[\mathbf{y}_2]$ be general affine forms and set \mathbf{M} to be $(\ell_1^{10}\ell_1^{01}, \dots, \ell_e^{10}\ell_e^{01})$. Following the same discussion, for each $S \sqcup T = \{1, \dots, e\}$ construct $\mathbf{L}^{S,T}$, substitute this for L^e in $\mathbf{L}^{\mathbf{e}'}$, use the $(|S|, |T|, e_3, \dots, e_k)$ th witness set for Y to compute $W_{S,T}$, and set W to be the union of the $W_{S,T}$.

Let $\mathbf{L}(t)$ be the convex combination $(t\mathbf{M} + (1-t)L^e, L_3, \dots, L_k)$. Then, as in Algorithm 4.3, we will have a homotopy that transforms the union of witness point sets $W = Y \cap \mathcal{V}(\mathbf{M}, L_3, \dots, L_k)$ into the \mathbf{e}' -witness point set $Y' \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}'})$. \diamond

Example 4.7. Let us revisit Example 1.6, which involved varieties $\mathcal{V}(f, g)$ and $\mathcal{V}(f, h)$ in $\mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z \times \mathbb{C}_w$. Both have their multidimension the vertices of an octahedron. Of the many coarsenings, we consider four, merging either the last two factors, the first two, both the first and the last two, and finally the last three. Table 1 displays the multidegrees of the original varieties in $\mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z \times \mathbb{C}_w$ and after merging. \diamond

5. SLICING

While the dimension of an equidimensional affine or projective variety X is reduced by 1 under a general linear section (a *slice*), its degree is preserved—if $\dim(X) \geq 1$. Similarly, the (ir)reducibility of X is preserved when $\dim(X) \geq 2$, by the classical Bertini Theorem.

TABLE 1. Coarsenings of $\mathcal{V}(f, g)$ and $\mathcal{V}(f, h)$ from Example 1.6.

	$\mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z \times \mathbb{C}_w$	$\mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_{zw}^2$	$\mathbb{C}_{xy}^2 \times \mathbb{C}_z \times \mathbb{C}_w$	$\mathbb{C}_{xy}^2 \times \mathbb{C}_{zw}^2$	$\mathbb{C}_x \times \mathbb{C}_{yzw}^3$
Dim					
$\mathcal{V}(f, g)$					
$\mathcal{V}(f, h)$					

Consequently decomposing a variety into irreducible components is reduced to the case of curves.

When X is a multiprojective or multiaffine variety, information about its multidimension and multidegrees may be lost under a general linear section, and its (ir)reducibility may not be preserved, even when X has dimension at least 2. However, this may be quantified and it leads to useful reductions. The subsequent reductions will be exploited in our algorithm for numerical irreducible decomposition in Section 6.

For $i = 1, \dots, k$ let $\pi_{\{i\}}$ be the projection onto the i th factor in multiprojective or multiaffine space. Let $\epsilon_i \in \mathbb{N}^k$ be the vector whose i th component is 1 and others are 0.

Lemma 5.1. *Let $X \subset \mathbb{P}^{n_0}$ (or $X \subset \mathbb{C}^{n_0}$) be an equidimensional multiprojective or multiaffine variety and suppose that ℓ is a general linear polynomial/affine form in the i th variable group.*

- (1) *If $\dim \pi_{\{i\}}(X) = 0$, then $X \cap \mathcal{V}(\ell) = \emptyset$. This is equivalent to $\mathbf{e} \in \text{Dim}(X) \Rightarrow e_i = 0$.*
- (2) *If $\dim \pi_{\{i\}}(X) \geq 1$, then*

$$\begin{aligned} \text{Dim}(X \cap \mathcal{V}(\ell)) &= \{\mathbf{e} - \epsilon_i \mid \mathbf{e} \in \text{Dim}(X) \text{ and } e_i > 0\}, \\ \text{Deg}_{X \cap \mathcal{V}(\ell)}(\mathbf{e}) &= \text{Deg}_X(\mathbf{e} + \epsilon_i). \end{aligned}$$

- (3) *If $\dim \pi_{\{i\}}(X) \geq 2$, then X is (ir)reducible if and only if $X \cap \mathcal{V}(\ell)$ is (ir)reducible.*

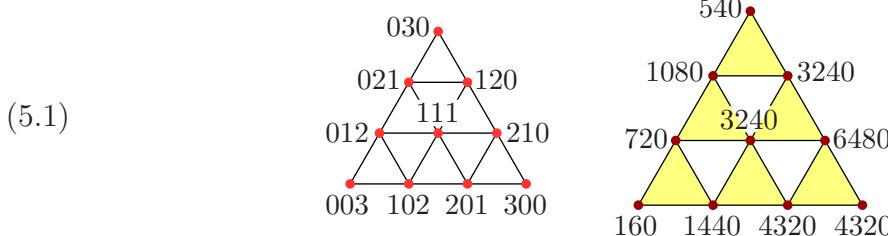
Proof. Statements (1) and (2) follow from the definitions given in Subsection 1.3, and (3) follows from the Bertini Theorem for maps to projective space [9, Thm. 6.3 (4)]. \square

Example 5.2. Let $n_\bullet = (3, 3, 3)$ and consider a subvariety X of \mathbb{P}^{n_0} given by six general multihomogeneous polynomials, each of multidegree $(1, 2, 3)$. Then X is irreducible and has intrinsic dimension 3. Following Remark 1.2 and [4], its multidimension and

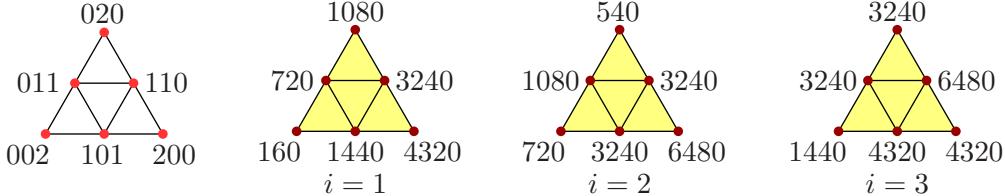
multidegree are computed as follows. Its cohomology class in $\mathbb{P}^{n\bullet}$ is the normal form of $(s_1 + 2s_2 + 3s_3)^6$ in $\mathbb{Z}[s_1, s_2, s_3]/\langle s_1^4, s_2^4, s_3^4 \rangle$, which is

$$\begin{aligned} 160s_1^3s_2^3 + 720s_1^3s_2^2s_3 + 1440s_1^2s_2^3s_3 + 1080s_1^3s_2s_3^2 + 3240s_1^2s_2^2s_3^2 \\ + 4320s_1s_2^3s_3^2 + 540s_1^3s_3^3 + 3240s_1^2s_2s_3^3 + 6480s_1s_2^2s_3^3 + 4320s_2^3s_3^3. \end{aligned}$$

Its homology class is obtained by replacing $s_1^{3-a}s_2^{3-b}s_3^{3-c}$ by \mathbf{T}^{abc} . We display its multidimension and multidegree below. (The central point in $\text{Dim}(X)$ is 111.)



Slicing with a general linear polynomial $\mathcal{V}(\ell)$ in the i th variable group gives a variety of dimension two in a product $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$ (permuted so that \mathbb{P}^2 is the i th factor) with multidimension and multidegrees as shown below.



Slicing with another general linear polynomial gives an irreducible curve in either $\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3$ or $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$ (with possibly permuted factors) of multidimension $\{001, 010, 100\}$, and multidegrees corresponding to one of the six upright shaded triangles in (5.1). \diamond

A consequence of Lemma 5.1 is that obtaining a witness collection for a linear slice $X \cap \mathcal{V}(\ell)$ from one for X is a matter of bookkeeping. We recall the definitions from Subsection 1.3. Let X be an equidimensional union of components of $\mathcal{V}(F)$. Choose general linear polynomials $\ell_{i,1}, \dots, \ell_{i,n_i} \in \mathbb{C}[\mathbf{x}_i]$ for each $i \in \{1, \dots, k\}$, and for $\mathbf{e} \in [n_\bullet]$, set $L_i^{\mathbf{e}} := (\ell_{i,1}, \dots, \ell_{i,e_i})$ and $\mathbf{L}^{\mathbf{e}} := (L_1^{\mathbf{e}_1}, \dots, L_k^{\mathbf{e}_k})$. Then, for $\mathbf{e} \in \text{Dim}(X)$, the \mathbf{e} -witness set is $(F, \mathbf{L}^{\mathbf{e}}, W_{\mathbf{e}})$, where $W_{\mathbf{e}} := X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$. We use the same notation for a multiaffine variety $X \subset \mathbb{C}^{n\bullet}$.

Algorithm 5.3 (Witness collection of a slice).

Input: A witness collection $\{(F, \mathbf{L}^{\mathbf{e}}, W_{\mathbf{e}})\}$ for an equidimensional multiprojective or multiaffine variety $X \subset \mathbb{P}^{n\bullet}$ (or $X \subset \mathbb{C}^{n\bullet}$) and an index $i \in \{1, \dots, k\}$.

Output: A witness collection for $X \cap \mathcal{V}(\ell_{i,1})$.

Do: Return $\{(F \cup \{\ell_{i,1}\}, \mathbf{L}^{\mathbf{e}} \setminus \{\ell_{i,1}\}, W_{\mathbf{e}}) \mid \mathbf{e} \in \text{Dim}(X) \text{ and } e_i > 0\}$.

Proof of correctness. By Lemma 5.1, $\text{Dim}(X \cap \mathcal{V}(\ell_{i,1})) = \{\mathbf{e} - \epsilon_i \mid \mathbf{e} \in \text{Dim}(X) \text{ and } e_i > 0\}$, as $\ell_{i,1}$ is general. Moreover, $W_{\mathbf{e}} = X \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}}) = X \cap \mathcal{V}(\ell_{i,1}) \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}} \setminus \{\ell_{i,1}\})$, so that $W_{\mathbf{e}}$ is both an \mathbf{e} -witness point set for X and an $(\mathbf{e} - \epsilon_i)$ -witness point set for $X \cap \mathcal{V}(\ell_{i,1})$. \square

Remark 5.4. As indicated in Example 5.2, both Lemma 5.1 and Algorithm 5.3 may be applied in succession to a variety and its witness collection. If the projection of the variety to the i th factor has dimension at least two, this preserves the irreducible components. The choice of slice affects the size of the output. In Example 5.2, slicing twice with a linear polynomial in \mathbf{x}_1 gives a witness collection with $160 + 720 + 1440 = 2320$ points, while slicing twice with a linear polynomial in \mathbf{x}_3 yields $4320 + 4320 + 6480 = 15120$ points.

6. NUMERICAL DECOMPOSITIONS OF ALGEBRAIC VARIETIES

6.1. Affine and projective varieties. Any variety has a unique (irredundant) decomposition into irreducible components. For subvarieties of \mathbb{C}^n or \mathbb{P}^n , a numerical irreducible decomposition mirrors the irreducible decomposition by producing a formal union of witness sets, one for each irreducible component. As described in Subsection 1.2, we summarize a well-known approach for computing a numerical irreducible decomposition for equidimensional varieties in \mathbb{C}^n .

Algorithm 6.1 (Equidimensional numerical irreducible decomposition in \mathbb{C}^n).

Input: A witness set (F, L, W) for equidimensional $X \subset \mathcal{V}(F)$.

Output: A numerical irreducible decomposition of X .

Do: Perform monodromy loops to partition the witness point set W into subsets of points $P_1 \sqcup \dots \sqcup P_s$ where all points in each P_i lie on the same irreducible component. Repeat until the trace test confirms that each P_i is a witness point set for some irreducible component yielding the numerical irreducible decomposition $\sqcup_i (F, L, P_i)$.

For varieties that are not equidimensional, one simply performs a numerical irreducible decomposition on each of its equidimensional components.

6.2. Multiprojective varieties. For a numerical irreducible decomposition of a multiprojective variety $\mathcal{V}(F)$, it makes sense to set a goal to partition an *arbitrary* set of general points according to membership in the irreducible components. Algorithm 6.2 (below) still applies to points in witness collections and can be modified to look similar to Algorithm 2.4 in a special case, which is explained in Remark 6.3.

The underlying idea is to systematically loop through the given points and determine if a point lies on a previously computed irreducible component. If not, information about this new irreducible component must be computed and it is then added to the list of known components. At a minimum, a membership test for this new irreducible component must be developed. To that end, one uses the given point p to compute the (multi)dimension of the corresponding irreducible component. This determines a sequence of slices that preserve irreducibility (Lemma 5.1) and coarsenings (Subsection 4.2) which can be used to produce a system of linear polynomials L vanishing at p such that the irreducible component becomes an irreducible affine curve $C_L \subset \mathcal{V}(F) \cap \mathcal{V}(L)$ containing p . Then, a complete witness point set for C_L can be constructed from p via monodromy (Algorithm 2.5), with the trace test used as a stopping criterion.

Given another point q from the given set of general points, one first checks if p and q have the same (multi)dimension. If so, then one produces a system L' of similar structure to L but vanishing at q . After computing a witness point set of the corresponding $C_{L'}$

from C_L , testing membership (Remark 1.1) of q in $C_{L'}$ is equivalent to determining if p and q lie on the same irreducible component. Note that genericity assumptions on p and q are needed here to avoid losing transversality as in [6, Ex. 3.2].

Algorithm 6.2 (Numerical irreducible decomposition in $\mathbb{C}^{n\bullet}$).

Input: A finite set $W \subset \mathcal{V}(F) \subset \mathbb{C}^{n\bullet}$ of general smooth points.

Output: A partition of W into sets corresponding to irreducible components.

Do: Make use of the developed toolkit to represent irreducible components of $\mathcal{V}(F)$ containing points W with curves in an affine space and use this representation to sort the points into the respective components as follows:

- Initialize the numerical irreducible decomposition N to be the empty set.
- While W is nonempty:
 - (1) Select a point $p \in W$, and let X_p denote the irreducible component of $\mathcal{V}(F)$ containing p .
 - (2) Determine $\text{Dim}(X_p)$ using Algorithm 2.3.
 - (3) Let \mathbf{m} denote a coordinatewise maximal integer vector in $[n_\bullet]$ such that $\mathbf{L}^{\mathbf{m}}$ is slice preserving the irreducibility of X_p .
 - (4) Choose an element \mathbf{e} in $\text{Dim}(X_p \cap \mathcal{V}(\mathbf{L}^{\mathbf{m}})) \subset \{0, 1\}^k$.
 - (5) Let $I = \{i_1, \dots, i_{|\mathbf{e}|}\}$ denote the positions of \mathbf{e} with a nonzero entry indexed such that $j = \dim \pi_{\{i_1, \dots, i_j\}}(X_p)$.
 - (6) For $j \in \{1, \dots, |\mathbf{e}| - 1\}$, let ℓ_j be a general linear polynomial in $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_{j+1}}$.
 - (7) Let $L := \mathbf{L}^{\mathbf{e}} \cup \{\ell_1, \dots, \ell_{|\mathbf{e}|-1}\}$. Use Algorithm 2.5 to compute a witness set for $C_L := X_p \cap \mathcal{V}(L)$, thereby deriving a membership test for X_p .
 - (8) Let M_p denote the points of W which are members of X_p .
 - (9) Replace W with $W \setminus M_p$ and append to N the set M_p .
- Return the numerical irreducible decomposition N of W .

Proof of correctness. Steps (3), (4), and (7) are the only steps needing further justification. The existence of \mathbf{m} in Step (3) is a consequence of iterating Lemma 5.1. Furthermore, for $i \in \{1, \dots, k\}$ the projection $\pi_{\{i\}}(X \cap \mathcal{V}(\mathbf{L}^{\mathbf{m}}))$ is a point or a curve because \mathbf{m} is maximal. Then it follows $\text{Dim}(X_p \cap \mathcal{V}(\mathbf{L}^{\mathbf{m}})) \subset \{0, 1\}^k$ as stated in Step (4). Finally, by Remark 1.3, a witness set $(F \cup L, \ell, S)$ for C_L yields a membership test for q in the variety X_p using the homotopy $H(t) = (F, L - (1-t)L(q), \ell - (1-t)\ell(q))$ with start points S . \square

Theorem 3.1 can be used to simplify computation for the points that belong to components that are Cartesian products by an obvious divide-and-conquer procedure.

Remark 6.3. Given a complete witness collection for an equidimensional variety $X \subset \mathcal{V}(F) \subset \mathbb{C}^{n\bullet}$, one can apply monodromy as in Algorithm 6.1 as a heuristic for numerical irreducible decomposition. However, without slicing and coarsening, the trace test cannot be used to ensure the completion of such an algorithm.

Coarsening changes the geometry, rendering the prior witness collection irrelevant. One can create a new witness collection using the old one in the fashion of Algorithm 4.5 to avoid using monodromy to reconstruct witness points in the hope of reducing the computational cost. In some cases, the completeness of the new witness collection is guaranteed. For a curve in a product of two projective spaces considered in detail in [10],

the original witness collection is linked to the new witness set with an optimal (one-to-one) homotopy. Hence, a decomposition via Algorithm 6.1 on the new witness set induces a decomposition on the original witness collection.

In general, however, a witness set produced by Algorithm 4.5 is incomplete and thus, this one-to-one correspondence is lost. One can choose to complete the witness set and decompose the result yielding a decomposition of the original witness collection. Whether this approach has an advantage over the method outlined in the beginning of this subsection depends on the number and nature of coarsenings taken.

Example 6.4. Consider $Y := \mathcal{V}(f, h) \subset \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z \times \mathbb{C}_w$ from Example 1.6. We discuss three of the ways to reduce to an affine curve preserving irreducibility.

One could simply coarsen to \mathbb{C}^4 and then intersect with a general hyperplane in \mathbb{C}^4 . This yields an irreducible curve of degree 15 showing that Y is irreducible.

Another option is to coarsen $\mathbb{C}_y \times \mathbb{C}_z \times \mathbb{C}_w$ to \mathbb{C}^3 , intersect with a general hyperplane in this \mathbb{C}^3 , and then coarsen $\mathbb{C}_x \times \mathbb{C}^3$ to \mathbb{C}^4 . This also yields an irreducible curve of degree 15.

A final option that we will consider is to coarsen $\mathbb{C}_y \times \mathbb{C}_z$ to \mathbb{C}^2 , intersect with a general hyperplane in this \mathbb{C}^2 , and then coarsen $\mathbb{C}_x \times \mathbb{C}_w \times \mathbb{C}^2$ to \mathbb{C}^4 . This yields an irreducible curve of degree 12. \diamond

As demonstrated in Example 6.4, different reductions to affine curves can yield different degrees. We leave it as a possible topic for future research to consider finding combinations of coarsening, slicing, and potential factoring that result in the smallest degree.

7. FIBER PRODUCT EXAMPLES

Computing exceptional sets using fiber products [21] yields multihomogeneous systems. We illustrate some of the tools from the toolkit on two examples: rulings of a hyperboloid and exceptional planar pentads.

7.1. Rulings of a hyperboloid. Motivated by [21, § 4], we use fiber products to compute the two rulings of the hyperboloid $H \subset \mathbb{C}^3$ defined by

$$h(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 - 1.$$

Since a generic line meets H in two points, a line which meets H in three points must be

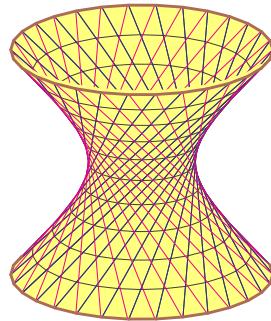


FIGURE 3. A hyperboloid with its two rulings.

contained in H . For each $\lambda \in \mathbb{C}_{\lambda}^4$, we associate a line $L_{\lambda} = \mathcal{V}(\ell_{\lambda}) \subset \mathbb{C}^3$ where

$$\ell_{\lambda}(\mathbf{x}) = (\lambda_1 x_1 + \lambda_2 x_2 - x_3, \lambda_3 x_1 + \lambda_4 x_2 - 1) .$$

Consider the following system on $\mathbb{C}_{\lambda}^4 \times \mathbb{C}_{\mathbf{x}_1}^3 \times \mathbb{C}_{\mathbf{x}_2}^3 \times \mathbb{C}_{\mathbf{x}_3}^3$,

$$F(\lambda, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\ell_{\lambda}(\mathbf{x}_1), \ell_{\lambda}(\mathbf{x}_2), \ell_{\lambda}(\mathbf{x}_3), h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)) .$$

A ruling of H corresponds to a four-dimensional irreducible component $X \subset \mathcal{V}(F)$ such that there exists an irreducible curve $C \subset \mathbb{C}_{\lambda}^4$ where

$$(7.1) \quad X = \overline{\bigcup_{\lambda \in C} \{(\lambda, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) : \mathbf{x}_i \in \mathcal{V}(\ell_{\lambda})\}}.$$

In particular, $\overline{\pi_1(X)} = C$ and $\overline{\pi_i(X)} = H$ for $i = 2, 3, 4$.

For $\mathbf{e} = (1, 1, 1, 1)$, the witness point set $W_{\mathbf{e}} := \mathcal{V}(F) \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$ consists of 16 isolated points. The following uses our toolkit to determine the irreducible components corresponding to the rulings.

Using Algorithms 2.3 and 2.4, we compute the dimensions of components containing points of $W_{\mathbf{e}}$ under different projections. The following table records the relevant information up to symmetry.

# points in $W_{\mathbf{e}}$	$\dim \overline{\pi_{\{1\}}(X)}$	$\dim \overline{\pi_{\{2\}}(X)}$	$\dim \overline{\pi_{\{1,2\}}(X)}$	$\dim \overline{\pi_{\{1,2,3\}}(X)}$
4	1	2	2	3
12	4	2	4	4

The first column shows that the points of $W_{\mathbf{e}}$ lie on components of $\mathcal{V}(F)$ having two distinct multidimensions. Let $W'_{\mathbf{e}}$ consist of the four points from the first row of this table. For each $(\lambda, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in W'_{\mathbf{e}}$, the last column implies that the fiber over $(\lambda, \mathbf{x}_{i_1}, \mathbf{x}_{i_2})$ for distinct $i_1, i_2 \in \{2, 3, 4\}$ is one-dimensional. The trace test shows that each irreducible component of the fiber is linear as expected from (7.1). We note that the number of witness points when treating F as system in \mathbb{C}^{13} is 120 and much larger than the number of points in the witness sets in the previous table.

Finally, we compute the irreducible components of $\overline{\pi_{\{1\}}(\mathcal{V}(F))}$. Using monodromy and the trace test in \mathbb{C}^4 , we partition $W'_{\mathbf{e}}$ into two sets of size two, each corresponding to a distinct ruling of the hyperboloid. The rulings correspond to the two irreducible curves

$$\{(\lambda_1, \lambda_2, -\lambda_2, \lambda_1) \mid \lambda_1^2 + \lambda_2^2 = 1\} \quad \text{and} \quad \{(\lambda_1, \lambda_2, \lambda_2, -\lambda_1) \mid \lambda_1^2 + \lambda_2^2 = 1\} .$$

The symbolic expressions of these curves are classically known, but one can also recover them directly from the computed witness points via [1].

7.2. Exceptional planar pentads. A planar pentad is a 3-RR mechanism constructed by connecting the vertices of two triangles by three legs with revolute joints as shown in Figure 4. We fix one of the triangles in the plane to remove the trivial motion of the entire mechanism. A generic mechanism can be assembled in six different configurations, which is the degree of $\text{SE}(2)$ as described in [7, Table 1]. Since a generic planar pentad is rigid (i.e., does not move), a planar pentad is **exceptional** when it exhibits motion.

Using isotropic coordinates, the parameters in Figure 4 are

$$([u_0, u_1, u_2, u_3, u_4, v_0, v_4], [\bar{u}_0, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{v}_0, \bar{v}_4]) \in \mathbb{P}^6 \times \mathbb{P}^6$$

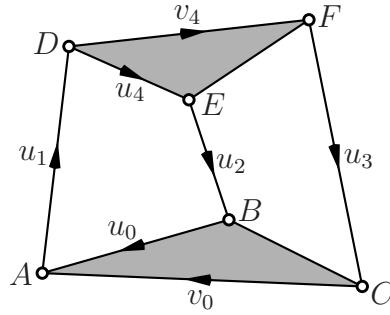


FIGURE 4. A planar pentad mechanism

with the following assemblability restrictions:

$$u_0 + u_1 + u_2 + u_4 = \bar{u}_0 + \bar{u}_1 + \bar{u}_2 + \bar{u}_4 = v_0 + u_1 + u_3 + v_4 = \bar{v}_0 + \bar{u}_1 + \bar{u}_3 + \bar{v}_4 = 0.$$

A mechanism is **degenerate** if a parameter, $u_0 - v_0$, $\bar{u}_0 - \bar{v}_0$, $u_4 - v_4$, or $\bar{u}_4 - \bar{v}_4$ is zero. Since nondegenerate mechanisms are desired, we dehomogenize the parameter space by setting $v_0 = \bar{v}_0 = 1$. Hence, the parameter space becomes

$$(\mathbf{u}, \bar{\mathbf{u}}) = (u_1, u_2, u_3, u_4, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in \mathbb{C}^4 \times \mathbb{C}^4,$$

where

$$u_0 = -(u_1 + u_2 + u_4), \quad \bar{u}_0 = -(\bar{u}_1 + \bar{u}_2 + \bar{u}_4), \quad v_4 = -(u_1 + u_3 + 1), \quad \bar{v}_4 = -(\bar{u}_1 + \bar{u}_3 + 1).$$

A mechanism is **physically meaningful** if $\bar{\mathbf{u}} = \text{conjugate}(\mathbf{u})$.

For $(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \in \mathbb{C}^4 \times \mathbb{C}^4$, the poses corresponding to link lengths $(\mathbf{u}, \bar{\mathbf{u}})$ satisfy

$$G(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) := \begin{pmatrix} \theta_1 \bar{\theta}_1 - 1 \\ \theta_2 \bar{\theta}_2 - 1 \\ \theta_3 \bar{\theta}_3 - 1 \\ \theta_4 \bar{\theta}_4 - 1 \\ u_0 + u_1 \theta_1 + u_2 \theta_2 + u_4 \theta_4 \\ \bar{u}_0 + \bar{u}_1 \bar{\theta}_1 + \bar{u}_2 \bar{\theta}_2 + \bar{u}_4 \bar{\theta}_4 \\ 1 + u_1 \theta_1 + u_3 \theta_3 + v_4 \theta_4 \\ 1 + \bar{u}_1 \bar{\theta}_1 + \bar{u}_3 \bar{\theta}_3 + \bar{v}_4 \bar{\theta}_4 \end{pmatrix} = 0.$$

The six configurations of a general planar pentad correspond with the six points in the witness point set $W_{\mathbf{e}} = \mathcal{V}(F) \cap \mathcal{V}(\mathbf{L}^{\mathbf{e}})$ for $\mathbf{e} = (4, 4, 0, 0)$.

The only family of nondegenerate exceptional planar pentads are the double-parallel-gram linkages [22], namely

$$\mathcal{U} := \{(\mathbf{u}, \bar{\mathbf{u}}) \mid u_1 + u_2 = u_1 + u_3 = \bar{u}_1 + \bar{u}_2 = \bar{u}_1 + \bar{u}_3 = 0\} \subset \mathbb{C}^4 \times \mathbb{C}^4.$$

We aim to compute \mathcal{U} directly from G by using fiber products. Since \mathcal{U} has codimension four, Corollary 2.14 of [21] shows that \mathcal{U} will correspond with an irreducible component

of the fourth fiber product system, namely

$$F(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}_1, \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2, \bar{\boldsymbol{\theta}}_2, \boldsymbol{\theta}_3, \bar{\boldsymbol{\theta}}_3, \boldsymbol{\theta}_4, \bar{\boldsymbol{\theta}}_4) = \begin{pmatrix} G(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}_1, \bar{\boldsymbol{\theta}}_1) \\ G(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}_2, \bar{\boldsymbol{\theta}}_2) \\ G(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}_3, \bar{\boldsymbol{\theta}}_3) \\ G(\mathbf{u}, \bar{\mathbf{u}}, \boldsymbol{\theta}_4, \bar{\boldsymbol{\theta}}_4) \end{pmatrix} = 0.$$

The irreducible component $X \subset \mathcal{V}(F) \subset (\mathbb{C}^4)^{10}$ corresponding to \mathcal{U} is called a *main component* in [21]. In fact, X is a Cartesian product of \mathcal{U} with four copies of

$$\{(\alpha, \alpha, \alpha, 1, \alpha^{-1}, \alpha^{-1}, \alpha^{-1}, 1) \in \mathbb{C}^4 \times \mathbb{C}^4 \mid \alpha \in \mathbb{C}^*\}.$$

This corresponds with rotating $\triangle ABC$ about a fixed $\triangle DEF$.

A necessary condition for locating such an exceptional component of codimension four consisting of nondegenerate and physically meaningful linkages with one dimension of motion is that there exists $\mathbf{e} \in \text{Dim}(\mathcal{V}(F))$ such that $e_1 = e_2 = 2$ and $e_3 + e_4 = e_5 + e_6 = e_7 + e_8 = e_9 + e_{10} = 1$. A sufficient condition for such a component to exist is that one of the isolated points in $\mathcal{V}(F) \cap \mathcal{V}(L^\mathbf{e})$ is a general point of X . To that end, we first compute the \mathbf{e} -witness point set $W_\mathbf{e}$ for $\mathcal{V}(F)$ where $\mathbf{e} = (2, 2, 1, 0, 1, 0, 1, 0, 1, 0)$ resulting in 14,828 isolated nonsingular points in $\mathcal{V}(F) \cap \mathcal{V}(L^\mathbf{e})$. The results of using Algorithm 2.3 to compute the the local dimension at each of these 14,828 witness points along with the local fiber dimensions obtained by fixing the $(\mathbf{u}, \bar{\mathbf{u}})$ coordinates and coarsening the fiber to the natural $(\mathbb{C}^8)^4$ are summarized in the following.

# points in $W_\mathbf{e}$	local dimension $\text{Dim}_x(F)$	local fiber dimension over $(\mathbf{u}, \bar{\mathbf{u}})$
14,144	(4, 4, 4, 4, 4, 4, 4, 4, 4, 4)	(0, 0, 0, 0)
678	(2, 2, 3, 3, 3, 3, 3, 3, 3, 3)	(1, 1, 1, 1)
6	(2, 2, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1, 1)

The first collection of 14,144 witness points correspond with rigid planar pentads. The second collection of 678 witness points correspond with degenerate planar pentads. The final collection of 6 witness points correspond with five degenerate planar pentads arising from one of the five nonconstant edges of $\triangle ABC$ or $\triangle DEF$ being zero (recall $v_0 = \bar{v}_0 = 1$). The other witness point in this collection is the unique point in $W_\mathbf{e}$ that is a general point on X thereby confirming $(2, 2, 1, 0, 1, 0, 1, 0, 1, 0) \in \text{Dim}(\mathcal{V}(F))$ and the existence of nondegenerate exceptional planar pentads. On the other hand, working without the multihomogeneous structure, leads to approximately 10^8 witness points which we can't robustly compute.

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