Sparse Sensing and Optimal Precision: Robust $\mathcal{H}_\infty$ Optimal Observer Design with Model Uncertainty

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Abstract— We present a framework which incorporates three aspects of the estimation problem, namely, sparse sensor configuration, optimal precision, and robustness in the presence of model uncertainty. The problem is formulated in the $\mathcal{H}_\infty$ optimal observer design framework. We consider two types of uncertainties in the system, i.e. structured affine and unstructured uncertainties. The objective is to design an observer with a given $\mathcal{H}_\infty$ performance index with minimal number of sensors and minimal precision values, while guaranteeing the performance for all admissible uncertainties. The problem is posed as a convex optimization problem subject to linear matrix inequalities. Numerical simulations demonstrate the application of the theoretical results presented in this work.

Index Terms— Sparse sensing, optimal sensor precision, robust estimation, $\mathcal{H}_\infty$ optimal observer, convex optimization.

I. INTRODUCTION

The problem of sparse sensor selection typically deals with an estimator's design while using a minimal number of sensors from the available set. This is a well-studied problem as there exist a number of works by various researchers [1]–[14]. This problem has been formulated for both continuous and discrete-time systems in different frameworks such as Kalman filter [2]–[4], $\mathcal{H}_2/\mathcal{H}_\infty$ optimal estimation [5]–[10], and in terms of CramrRao bound without any restrictions on the type of an estimator [12].

It is typical for most sparse sensing formulations to assume that the sensor precisions are known and fixed [1], [2], [8]–[12]. However, this assumption often limits the performance of the control and estimation algorithms designed for control systems. On the other hand, at the design time, it might be unclear how precise a sensor should be to achieve the pre-specified performance criterion with a plausible risk of using sensors with unnecessarily high precisions, which increases the economic cost. In [5], authors presented a framework in which sensor and actuator precisions are treated as variables with economic cost constraints on them. The problem is solved in a convex optimization framework with guaranteed steady-state covariance bounds. They also proposed an ad-hoc algorithm to reduce the number of sensors by iteratively removing those with the least precision. An extension of their work for uncertain plants in an $\mathcal{H}_2$ formulation is presented in [6].

Motivated by [5], in our recent work [7], we presented an integrated framework for sparse sensor selection, which also minimizes the required sensor precision in the context of $\mathcal{H}_2/\mathcal{H}_\infty$ optimal observer design. This paper generalizes the framework presented in [7] for systems with model uncertainties but limited to an $\mathcal{H}_\infty$ formulation.

In related work, the authors of [10] considered the problem of sparse sensing for uncertain systems for a specific application of battery temperature estimation. However, they assumed that the sensor precision is known and fixed, and they solved the problem via exhaustive search, which is a combinatorial problem and does not scale well for large-scale systems.

Contribution and novelty: In this paper, we present a theoretical framework that incorporates three aspects of the estimation problem: sparse sensor configuration, optimal precision, and robustness in model uncertainty. In particular, we discuss the $\mathcal{H}_\infty$-optimal observer design for uncertain systems. The objective here is threefold. First, we are interested in identifying a sparse sensor configuration. Second, we want to minimize the required sensor precision to realize the sparse configuration. And finally, the sparse observer should satisfy the specified $\mathcal{H}_\infty$ performance criterion for all admissible uncertainties. We consider the following two classes of uncertain systems: (i) systems with structured affine uncertainty in the system matrices, and (ii) systems with unstructured uncertainty which can be expressed in the linear fractional transformation (LFT) [15] framework. We present results to determine sparse and robust observers for both types of uncertainties.

The organization of this paper is as follows. The sparse robust $\mathcal{H}_\infty$ observer design problems are formulated in Section II, and solutions to these problems are presented in Section III. In Section IV, we show numerical simulations. Concluding remarks are discussed in Section V.

II. PROBLEM FORMULATION

A. Notation

The set of real numbers is denoted by $\mathbb{R}$. Bold uppercase (lowercase) letters denote matrices (column vectors). $\boldsymbol{I}$ and $\boldsymbol{0}$ respectively denote an identity matrix and a zero matrix of suitable dimensions. Define $\mathbf{sym}\left(\boldsymbol{X}\right) := \boldsymbol{X} + \boldsymbol{X}^T$, where $\boldsymbol{X}^T$ denotes transpose of $\boldsymbol{X}$. Symmetric positive (negative) definite matrices are denoted by the inequality $\boldsymbol{X} > 0$ ($\boldsymbol{X} < 0$). $\mathbf{diag}(\boldsymbol{x})$ denotes a diagonal matrix whose diagonal elements are the vector $\boldsymbol{x}$. Similarly, $\mathbf{diag}\left(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N\right)$ denotes a block diagonal matrix. All inequalities and exponents of a vector are to be interpreted elementwise.

B. Systems with structured affine uncertainty

1) Plant: Consider the following LTI system

$$\dot{x} = (A + \Delta A)x + (B_d + \Delta B_d)d, \tag{1a}$$

$$y = C_y x + D_d d + D_n S_n n, \tag{1b}$$

$$z = C_z x, \tag{1c}$$

where, $x \in \mathbb{R}^{N_x}$ is the state vector, $y \in \mathbb{R}^{N_y}$ is the vector of measured outputs, and $z \in \mathbb{R}^{N_z}$ is the output vector we are interested in estimating. The process noise $d \in \mathbb{R}^{N_d}$ and the sensor noise $n \in \mathbb{R}^{N_y}$ are $\mathcal{L}_2$-norm bounded signals. The process equation (1a) is assumed to be independent of the sensor noise. The matrices $\Delta A \in \mathcal{A}$ and $\Delta B_d \in \mathcal{B}$ denote uncertainty in the system defined as

$$\mathcal{A} := \{\Delta A \mid \Delta A = M_1 F_1 N_1, F_1^T F_1 \le I\}, \tag{2a}$$

$$\mathcal{B} := \{\Delta B \mid \Delta B = M_2 F_2 N_2, F_2^T F_2 \le I\}, \tag{2b}$$

where M_1, N_1, M_2, N_2 are known deterministic matrices.

The nominal system matrices A, B_d, C_y, C_z, D_d are known constant real matrices of appropriate dimensions. The diagonal matrix $0 < S_n \in \mathbb{R}^{N_y \times N_y}$ is an unknown scaling matrix to be determined. As discussed below in Section II-D, S_n is related to the precision of sensors. We also assume that the individual sensor channels are independent of each other, i.e., $D_n = I$. All other weightings or scaling matrices are assumed to be known and absorbed in the system matrices.

2) Observer and error system: Now, let us consider the state observer for the system (1) given by

$$\dot{\hat{x}} = (A + LC_y)\hat{x} - Ly, \quad \hat{z} = C_z \hat{x}, \tag{3}$$

where, $\hat{x} \in \mathbb{R}^{N_x}$ denotes the estimate of the state vector, $\hat{z} \in \mathbb{R}^{N_z}$ denotes the estimate of z, and the $L \in \mathbb{R}^{N_x \times N_y}$ is the unknown observer gain. Let us define the state estimation error e, and the observer error ε as

$$e := x - \hat{x}, \quad \varepsilon := z - \hat{z}. \tag{4}$$

Therefore, from (1) and (3),

$$\dot{e} = (A + LC_y)e + \Delta A x + (B_d + \Delta B)d$$
$$+ LD_d d + LS_n n \tag{5}$$

The observation error dynamics follows from (1) and (5) as

$$\dot{\tilde{x}} = (\tilde{A} + \Delta\tilde{A})\tilde{x} + (\tilde{B}\tilde{S} + \Delta\tilde{B})\tilde{w}, \quad \varepsilon = \tilde{C}\tilde{x}, \tag{6}$$

where $\tilde{x} := [x^T, e^T]^T$ is the augmented state vector, and $\tilde{w} := [d^T, n^T]^T$ is the augmented vector of exogenous noises. And the augmented system matrices are as follows

$$\left.\begin{aligned}
\tilde{A} &:= \begin{bmatrix} A & 0 \\ 0 & A + LC_y \end{bmatrix}, \\
\Delta\tilde{A} &:= \begin{bmatrix} \Delta A & 0 \\ \Delta A & 0 \end{bmatrix} = \tilde{M}_1 F_1 \tilde{N}_1, \\
\tilde{B} &:= \begin{bmatrix} B_d & 0 \\ B_d + LD_d & L \end{bmatrix}, \tilde{S} := \begin{bmatrix} I & 0 \\ 0 & S_n \end{bmatrix}, \\
\Delta\tilde{B} &:= \begin{bmatrix} \Delta B_d & 0 \\ \Delta B_d & 0 \end{bmatrix} = \tilde{M}_2 F_2 \tilde{N}_2, \\
\tilde{M}_i &:= [M_i^T \quad M_i^T]^T, \tilde{N}_i := [N_i \quad 0], i = 1, 2.
\end{aligned}\right\} \tag{7}$$

The objective is to determine the observer gain L such that the error system (6) is stable, and the effect of $\tilde{w}$ on ε is bounded by the specified performance index. The sparse robust observer design problem will be stated in Section II-D. Now lets consider the systems with unstructured uncertainty.

C. Systems with unstructured uncertainty

1) Plant: Consider the uncertain plant

$$\dot{x} = A(\Delta)x + B_d(\Delta)d, \tag{8a}$$

$$y = C_y(\Delta)x + D_d(\Delta)d + D_n S_n n, \tag{8b}$$

$$z = C_z x, \tag{8c}$$

where the coefficient matrices are dependent on the uncertain parameters Δ. As in (1), here also we assume that the process is independent of the sensor noise, and $D_n = I$. Systems such as (8) can be expressed in linear fractional transformation (LFT) framework [15] as shown in Fig. (1).

The signals w_Δ and z_Δ are so-called fictitious input and output of the plant due to uncertainty block Δ. Let Δ be uncertainty block such that $\|\Delta\|_\infty \le 1$. The system in Fig.
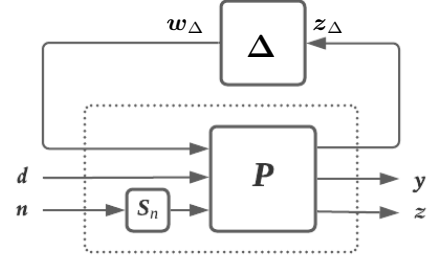

Fig. 1. Plant with unstructured uncertainty.

(1) can be equivalently written the following state-space form

$$\dot{x} = Ax + \begin{bmatrix} B_\Delta & B_d & 0 \end{bmatrix}\tilde{w}, \tag{9a}$$

$$z_\Delta = C_\Delta x + \begin{bmatrix} E_\Delta & E_d & 0 \end{bmatrix}\tilde{w}, \tag{9b}$$

$$y = C_y x + \begin{bmatrix} D_\Delta & D_d & S_n \end{bmatrix}\tilde{w}, \tag{9c}$$

$$z = C_z x, \tag{9d}$$

$$w_\Delta = \Delta z_\Delta, \tag{9e}$$

where $\tilde{w} := \begin{bmatrix} w_\Delta^T & d^T & n^T \end{bmatrix}^T$, and A, B_d etc. denote nominal values of the uncertain matrices $A(\Delta), B_d(\Delta)$. Top four equations in (9) denote the state-space equations for the open loop system shown in the dotted box in Fig. (1).

2) Observer and error system: We consider a state observer of the same form as (3) for the system (9), with state estimation error and observer error as defined in (4). Therefore, error dynamics equations follow as

$$\dot{e} = (A + LC_y)e + \tilde{B}\tilde{S}\tilde{w}, \quad \varepsilon = C_z e, \tag{10}$$

where, $\tilde{B} := \begin{bmatrix} B_\Delta & B_d & 0 \end{bmatrix} + L\begin{bmatrix} D_\Delta & D_d & I \end{bmatrix}, \tag{11a}$

$$\tilde{S} := \mathbf{diag}\,(I, I, S_n). \tag{11b}$$

The objective here is to determine the observer gain L such that the transfer function from $\tilde{w}$ to $[z_\Delta^T, \varepsilon^T]^T$ is stable and bounded, and the effect of $\tilde{w}$ on ε should be minimal. A formal problem statement will be presented in the following section.

D. Sensor precision and observer design

As mentioned earlier, d and n are $\mathcal{L}_2$-norm bounded but arbitrary signals. Let us denote the sensor noise entering the system by $\tilde{n} := S_n n$, and let $\tilde{n}_i$ denote the ith component of the signal $\tilde{n}$. We define sensor precision to be the reciprocal of square of $\mathcal{L}_2$-norm (or *energy*) of a signal, i.e. precision of the ith sensor channel is $1/\|\tilde{n}_i\|_2^2$.

Since $S_n > 0$ is a diagonal matrix, let us define $0 < \beta = \left[\beta_1, \cdots, \beta_{N_y}\right]^T \in \mathbb{R}^{N_y}$ such that

$$\mathbf{diag}(\beta) := (S_n S_n)^{-1}. \tag{12}$$

Therefore, precision of the ith sensor then becomes $1/\|\tilde{n}_i\|_2^2 = \beta_i/\|n_i\|_2^2$. Further, without loss of generality, we assume that $\|n_i\|_2^2 = 1$. Therefore, precision of the ith sensor is simply β_i, and β is the precision vector.

Since β is interpreted as the precision vector, a sparse sensor configuration can be characterized by a sparse vector β. If $\beta_i = 0$ for some sensor, it implies that the ith sensor noise channel contains infinite energy, or equivalently, that sensor is not used.

Minimizing the number of non-zero elements in β, i.e. $\|\beta\|_0$, would yield a sparse configuration. However, minimization of l_0-norm is a non-convex problem, and in general very difficult to solve, especially for large-scale systems. A natural relaxation for the sparse configuration problem is minimization of l_1-norm instead. The minimization of $\|\beta\|_1$ promotes sparsity, and as discussed in Section III-C, iterative reweighting techniques can be used to arrive at a sparse configuration. Such iterative techniques minimize weighted l_1-norm defined as

$$\|\beta\|_{1,\rho} := \rho^T |\beta|,$$

where $0 < \rho \in \mathbb{R}^{N_y}$ is a specified weighting vector. Next, we formally define the sparse sensing problem for uncertain systems as follows.

1) Observer design problem for system (1): For the observer error system (6), let $\mathcal{G}_{\tilde{w}\varepsilon}(s)$ be the transfer function matrix from $\tilde{w}$ to ε. We wish to minimize the effect of $\tilde{w}$ on the observation error ε, which can be achieved by ensuring $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty < \gamma$, i.e. $\|\varepsilon\|_2 \leq \gamma \|\tilde{w}\|_2$, for some $\gamma > 0$. Therefore, the robust sparse sensing problem is formally stated as:

Given uncertain system (1) and error system (6), given $\gamma > 0$ and $\rho > 0$, determine optimal L, β that minimize $\|\beta\|_{1,\rho}$ such that $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty \leq \gamma$ for all admissible ΔA and ΔB_d. (13)

2) Observer design problem for the system in (9): Consider the system in (9) and the error system in (10). As mentioned earlier, the transfer function from $\tilde{w}$ to $[z_\Delta^T, \varepsilon^T]^T$ should be stable and bounded. This is guaranteed if we ensure that the $\mathcal{H}_\infty$ norm of the transfer function from $\tilde{w}$ to ε is less than a specified performance $\gamma > 0$, i.e. $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty \leq \gamma$, and for the transfer function from $\tilde{w}$ to z_Δ we require $\|\mathcal{G}_{\tilde{w}z_\Delta}\|_\infty \leq 1$ for the overall stability. Therefore, the robust sparse sensing problem is then:

Given the uncertain system in (9), and the error system in (10), given $\gamma > 0$ and $\rho > 0$, determine optimal L, β that minimize $\|\beta\|_{1,\rho}$ such that $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty \leq \gamma$ and $\|\mathcal{G}_{\tilde{w}z_\Delta}\|_\infty \leq 1$ for all admissible Δ. (14)

In the following section, we present the solutions to the problems defined in (13) and (14).

III. ROBUST SPARSE $\mathcal{H}_\infty$ OBSERVERS

Before proceeding to the main results, we present the following lemmas, which will be useful in completing the proofs.

A. Preliminaries

Lemma 1 (Schur complement [16]): Let X be a well-partitioned matrix defined as

$$X := \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix},$$

then, $X < 0$, if and only if, $R < 0$ and $P - QR^{-1}Q^T < 0$.

Lemma 2 (Variable elimination [16]): Let Q, X, Y, F be real matrices of appropriate dimensions. Then the following statements are equivalent.

(a) $Q + XFY + Y^T F^T X^T < 0$ for all F that satisfy $F^T F \leq I$.

(b) There exists a scalar $\delta > 0$ such that $Q + \delta XX^T + \delta^{-1} Y^T Y < 0$.

Lemma 3 (Wang et al. [17]): Let P, Q, X, Y, F be real matrices of appropriate dimensions such that $P > 0$ and $F^T F \leq I$. If there exists a scalar $\delta > 0$ such that $P^{-1} - \delta^{-1} XX^T > 0$, then the following is true.

$$(Q + XFY)^T P(Q + XFY)$$
$$\leq Q^T (P^{-1} - \delta^{-1} XX^T)^{-1} Q + \delta Y^T Y.$$

Lemma 4 (Bounded real [16]): Consider the LTI system,

$$\dot{x} = Ax + Bw, \ y = Cx + Dw,$$

and its transfer function matrix $\mathcal{G}(s) := C(sI - A)^{-1}B + D$. Then for a given scalar $\gamma > 0$, $\|\mathcal{G}(s)\|_\infty \leq \gamma$ if and only if there exists a matrix $X > 0$ such that

$$\begin{bmatrix} XA + A^T X + C^T C & XB + C^T D \\ * & D^T D - \gamma^2 I \end{bmatrix} < 0.$$

Proof of the following lemma can be easily established by starting with Lemma 4, followed by manipulations using the results of Lemmas 1, 2 and 3.

Lemma 5: Consider the LTI system,

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)w, \ y = Cx + Dw,$$

such that $\Delta A \in \mathcal{A}$ and $\Delta B \in \mathcal{B}$ as defined in (2), and let $\mathcal{G}(s)$ be the associated transfer function matrix. Then for a given scalar $\gamma > 0$, $\|\mathcal{G}(s)\|_\infty \leq \gamma$ for all admissible ΔA and ΔB, if there exist a matrix $X > 0$, scalars $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{bmatrix} Z & XB + C^T D & XM_1 & XM_2 \\ * & D^T D - \gamma^2 I + \delta_2 N_2^T N_2 & 0 & 0 \\ * & * & -\delta_1 I & 0 \\ * & * & * & -\delta_2 I \end{bmatrix} < 0.$$

where $Z := XA + A^T X + C^T C + \delta_1 N_1^T N_1$.

B. Main result

Next, we present the result for solving the $\mathcal{H}_\infty$-optimal robust observer design problem (13).

Theorem 6: The optimal observer gain L and sensor precision β for the sparse $\mathcal{H}_\infty$-optimal robust observer design problem (13) is determined by solving the following optimization problem, and if the problem is feasible then the gain is recovered as $L = X_2^{-1}Y$.

$$\left. \begin{array}{c} \min_{X_1>0,X_2>0,Y,\beta>0,\delta_1>0,\delta_2>0} \|\beta\|_{1,\rho} \\[2mm] \text{such that} \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ * & Z_{22} & 0 & 0 \\ * & * & -\delta_1 I & 0 \\ * & * & * & -\delta_2 I \end{bmatrix} < 0, \end{array} \right\} \quad (15)$$

$$Z_{11} = \begin{bmatrix} \mathbf{sym}(X_1 A) + \delta_1 N_1^T N_1 & 0 \\ 0 & \mathbf{sym}(X_2 A + Y C_y) + C_z^T C_z \end{bmatrix}$$

$$Z_{12} = \begin{bmatrix} X_1 B_d & 0 \\ X_2 B_d + Y D_d & Y \end{bmatrix},$$

$$Z_{13} = \begin{bmatrix} X_1 M_1 \\ X_2 M_1 \end{bmatrix}, \ Z_{14} = \begin{bmatrix} X_1 M_2 \\ X_2 M_2 \end{bmatrix},$$

$$Z_{22} = \begin{bmatrix} -\gamma^2 I + \delta_2 N_2^T N_2 & 0 \\ 0 & -\gamma^2 \, \mathbf{diag}(\beta) \end{bmatrix}.$$

Proof: As a direct application of Lemma 5 for the system (6), the condition $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty \leq \gamma$ in (13) is satisfied if there exist a symmetric matrix $\tilde{X} > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{bmatrix} Z_{11} & \tilde{X}\tilde{B}\tilde{S} & \tilde{X}\tilde{M}_1 & \tilde{X}\tilde{M}_2 \\ * & W & 0 & 0 \\ * & * & -\delta_1 I & 0 \\ * & * & * & -\delta_2 I \end{bmatrix} < 0, \quad (16)$$

where, $Z_{11} := \mathbf{sym}\left(\tilde{X}\tilde{A}\right) + \tilde{C}^T \tilde{C} + \delta_1 \tilde{N}_1^T \tilde{N}_1, \quad (17)$

$$W := -\gamma^2 I + \delta_2 \tilde{N}_2^T \tilde{N}_2.$$

Using the result of Lemma 1 successively, the inequality (16) becomes

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ * & Z_{22} & 0 & 0 \\ * & * & -\delta_1 I & 0 \\ * & * & * & -\delta_2 I \end{bmatrix} < 0, \quad (18)$$

where, $Z_{12} := \tilde{X}\tilde{B}$, $Z_{13} := \tilde{X}\tilde{M}_1$, $Z_{14} := \tilde{X}\tilde{M}_2$, and

$$Z_{22} := \tilde{S}^{-1} W \tilde{S}^{-1}$$
$$= \mathbf{diag}\left((-\gamma^2 I + \delta_2 N_2^T N_2), \ -\gamma^2 (S_n S_n)^{-1} \right)$$
$$= \mathbf{diag}\left((-\gamma^2 I + \delta_2 N_2^T N_2), \ -\gamma^2 \, \mathbf{diag}(\beta) \right),$$

wherein we have used (12) and the definitions of $\tilde{N}_2$ and $\tilde{S}$ from (7). We partition $\tilde{X}$ using $X_1 > 0$, $X_2 > 0$, such that $\tilde{X} := \mathbf{diag}(X_1, X_2)$. Let us define $Y := X_2 L$. Then Z_{11} follows from (17) and (7) as follows

$$Z_{11} = $$
$$\begin{bmatrix} \mathbf{sym}(X_1 A) + \delta_1 N_1^T N_1 & 0 \\ 0 & \mathbf{sym}(X_2 A + Y C_y) + C_z^T C_z \end{bmatrix}$$

Similarly, $\quad Z_{12} = \begin{bmatrix} X_1 B_d & 0 \\ X_2 B_d + Y D_d & Y \end{bmatrix},$

$$Z_{13} = \begin{bmatrix} X_1 M_1 \\ X_2 M_1 \end{bmatrix}, \ Z_{14} = \begin{bmatrix} X_1 M_2 \\ X_2 M_2 \end{bmatrix}.$$

Note that the inequality (18) is linear in unknown variables $X_1, X_2, Y, \beta, \delta_1$ and δ_2, and defines a feasibility condition for (13). The optimal solution is determined by minimizing the weighted l_1-norm $\|\beta\|_{1,\rho}$, and the observer gain can be recovered as $L = X_2^{-1}Y$. ∎

The following theorem presents a solution to the problem (14).

Theorem 7: The optimal observer gain L and sensor precision β for the sparse $\mathcal{H}_\infty$-optimal robust observer design problem (14) is determined by solving the following optimization problem, and if the problem is feasible then the gain is recovered as $L = X_2^{-1}Y$.

$$\left. \begin{array}{c} \min_{X_1>0,X_2>0,Y,\beta>0} \|\beta\|_{1,\rho} \ \text{such that} \\[2mm] \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ * & E_\Delta^T E_\Delta - I & E_\Delta^T E_d \\ * & * & E_d^T E_d - I \end{bmatrix} < 0, \\[2mm] \begin{bmatrix} W_{11} & W_{12} & W_{13} & Y \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -\gamma^2 \, \mathbf{diag}(\beta) \end{bmatrix} < 0, \end{array} \right\} \quad (19)$$

$$Z_{11} = \mathbf{sym}(X_1 A) + C_\Delta^T C_\Delta,$$
$$Z_{12} = X_1(B_\Delta + C_\Delta^T E_\Delta), \ Z_{13} = X_1(B_d + C_\Delta^T E_d),$$
$$W_{11} = \mathbf{sym}(X_2 A + Y C_y) + C_z^T C_z,$$
$$W_{12} = X_2 B_\Delta + Y B_\Delta, \ W_{13} = X_2 B_d + Y B_d.$$

Proof: Consider the transfer function from $\tilde{w}$ to z_Δ from (9). The first LMI in the theorem statement follows directly from the Lemma 4 for the condition $\|\mathcal{G}_{\tilde{w}z_\Delta}\|_\infty \leq 1$.

Now consider the condition $\|\mathcal{G}_{\tilde{w}\varepsilon}(s)\|_\infty \leq \gamma$ for the error system (10). Using Lemma 4, it becomes, for $X_2 > 0$

$$\begin{bmatrix} \mathbf{sym}\left(X_2(A + LC_y)\right) + C_z^T C_z & X_2 \tilde{B}\tilde{S} \\ * & -\gamma^2 I \end{bmatrix} < 0.$$

Using the result of Lemma 1 successively, we get

$$\begin{bmatrix} \mathbf{sym}\left(X_2(A + LC_y)\right) + C_z^T C_z & X_2 \tilde{B} \\ * & -\gamma^2(\tilde{S}\tilde{S})^{-1} \end{bmatrix} < 0.$$

Finally, by defining $Y := X_2 L$, using the definitions of $\tilde{B}$, $\tilde{S}$ and β from (11) and (12), we arrive at the second LMI in the theorem statement, which concludes the proof. ∎

C. Iterative reweighted l_1-minimization

We use an iterative reweighting scheme presented in [18] to achieve a sparse sensor configuration. We perform multiple iterations of solving the optimization problems (15) or (19), and the weights for $(k + 1)^{\text{th}}$ iteration are defined in terms of the previous iterate as

$$\rho_i^{(k+1)} = \left(\epsilon + |\beta_i^{(k)}|\right)^{-1}, \tag{20}$$

where $\epsilon > 0$ is a small number which ensures that the weights are well defined. Initial weights are chosen to be equal, i.e. without loss of generality, $\rho_i^{(0)} = 1$. These iterations are stopped if the convergence criterion is met or the maximum number of iterations is reached [18].

The final refined solution is determined by removing sensors with very small precisions and re-solving (15) or (19) with equal weights $\rho_i = 1$.

IV. EXAMPLE

Let us consider a serially connected spring-mass-damper system on a frictionless surface as shown in Fig. (2).
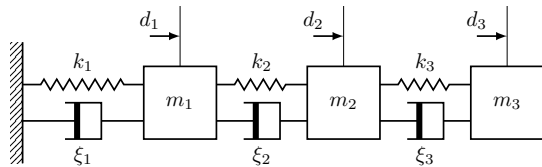

Fig. 2.   Serially connected spring-mass-damper system.

Let x_i denote the distance of the ith mass from the wall. Define state vector to be $\boldsymbol{x} := [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3]^T$. The nominal values of masses m_i, spring constants k_i, and damper coefficients ξ_i are all assumed to be unity. Disturbances d_i enters the system in the form of external forces acting independently on all masses. Also, we assume that sensors measure position and velocity of each mass. Therefore, there are six sensors. The first three sensors measure positions, and the last three sensors measure velocities

of the masses. The nominal system matrices are given by

$$A = \begin{bmatrix} \mathbf{0} & I \\ H & H \end{bmatrix}, H = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, B_d = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} S_d,$$

$$C_y = I, D_d = \mathbf{0}, D_n = I, C_z = I,$$

where S_d is a known matrix which represents a scaling for the disturbance signal. Next we consider the uncertainty of the form (1) and (9), and determine the robust sparse observers using results of Theorems 6 and 7.

Uncertainty of the form (1): The uncertainty in the system matrices is assumed to of the following form

$$\Delta A = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} F_1 \begin{bmatrix} c_0 H & \mathbf{0} \\ \mathbf{0} & c_1 H \end{bmatrix}, \Delta B_d = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix} F_2(c_2 I).$$

where $F_1^T F_1 \leq I$, $F_2^T F_2 \leq I$ and c_0, c_1, c_2 are known non-negative constants which quantify the magnitude of uncertainty in the system, i.e., larger values of these parameters would imply uncertainty of larger magnitude, and on the other extreme end, zero-valued parameters correspond to the nominal plant with no uncertainty. For the system defined as above, we can directly apply the result of Theorem 6 with iterative reweighting (20) to determine sparse sensor configuration and corresponding optimal precision β, which is discussed next.

The optimization problems are solved using the solver SDPT3 [19] with CVX [20] as a parser. In the figures of this section, solid circle indicates that a sensor is required and the number above it shows the required precision β, while cross indicates that a sensor is not used.

First, we consider the effect of specified performance parameter γ, for uncertainty of a fixed magnitude quantified by the parameters c_0, c_1, c_2. In particular, in Fig. (3), we set c_0, c_1, c_2 to non-zero values, and vary the specified performance parameter γ. We observer that as we decrease γ, i.e. demand better performance, the number of required sensors and their associated precisions increase. For $\gamma = 1$, only two sensors are needed, whereas for $\gamma = 0.25$, all six sensors are needed with relatively higher precisions.
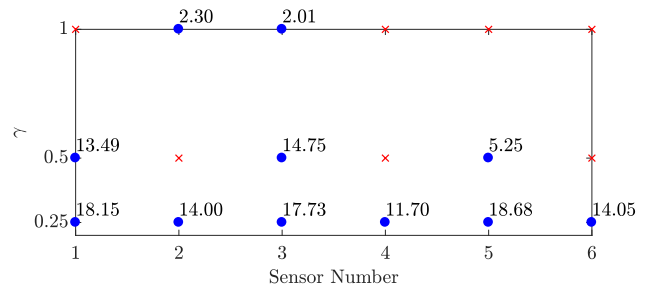

Fig. 3.   Sensor configuration and precision for different values of specified γ, and fixed $c_0 = 0.01$, $c_1 = 0.02$, $c_2 = 0.03$, and $S_d = I$.

In Fig. (4), we show a complementary case to Fig. (3), i.e. we vary the magnitude of uncertainty for a fixed performance parameter γ. In particular, we vary c_0, for fixed parameters $c_1 = 0, c_2 = 0$, and $\gamma = 1$. As we increase the magnitude of uncertainty, i.e c_0, we see that the number of sensors
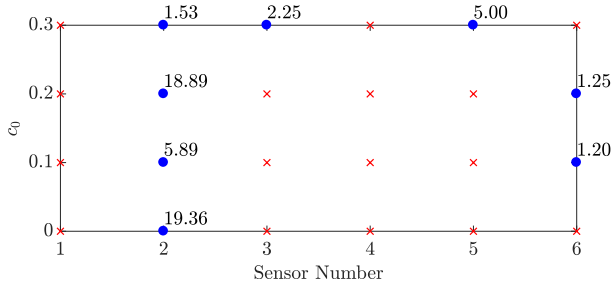
Fig. 4. Sensor configuration and precision for different magnitudes of uncertainty quantified by c_0, and fixed $c_1 = 0$, $c_2 = 0$, $\gamma = 1$, and $S_d = I$.

required and the precision values increase. For the nominal plant corresponding to $c_0 = 0$, only one sensor is required, whereas for $c_0 = 0.3$, three sensors are required.

Uncertainty of the form (9): We assume that the spring constants and damper coefficients can take values in the intervals $[c_0 - 1, c_0 + 1]$ and $[c_1 - 1, c_1 + 1]$ respectively, where c_0, c_1 are known non-negative constants which quantify the magnitude of uncertainty. We assume no uncertainty in the masses. With such uncertainty, the system can be written in an LFT form as in Fig. (1) and (9) such that $\|\Delta\|_\infty \leq 1$.

Analogous to Fig. (3) and Fig. (4), Fig. (5) and Fig. (6) below respectively show the effect of decreasing performance parameter γ for a fixed magnitude of uncertainty, and the effect of increasing magnitude of uncertainty for a fixed performance parameter γ. Similar observations as in the previous case can also be made for Fig. (5) and Fig. (6).
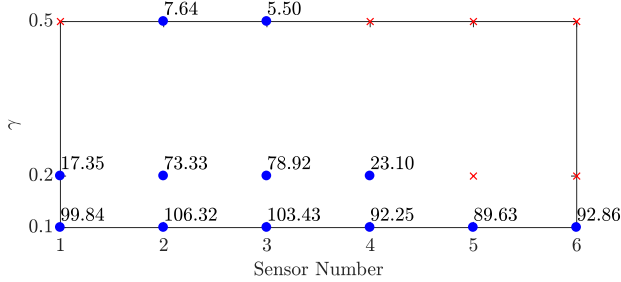

Fig. 5. Sensor configuration and precision for different values of specified γ, and fixed $c_0 = 0.1$, $c_1 = 0.1$, $S_d = 0.2I$.
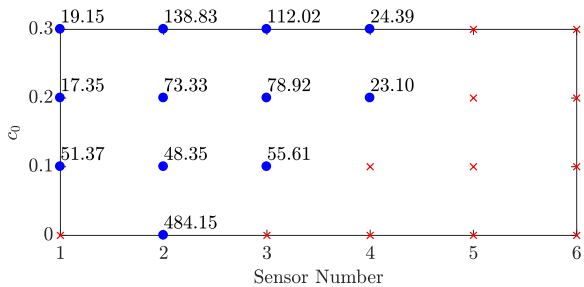

Fig. 6. Sensor configuration and precision for different magnitudes of uncertainty quantified by c_0, and fixed $c_1 = 0$, $\gamma = 0.2$, and $S_d = 0.2I$.

V. CONCLUSION

Herein, we present a unified theoretical framework to address the problem of sparse sensor configuration in the presence of model uncertainty while simultaneously minimizing the required sensor precisions. We consider two types of model uncertainties: structured affine uncertainty in system matrices and unstructured uncertainty. For both the cases, the robust sparse sensing problem is formulated in the context of $\mathcal{H}_\infty$-optimal observer design and posed as a convex optimization problem subject to linear matrix inequalities. We minimize l_1-norm of the precision vector to promote sparsity, and an iterative reweighting scheme is used to refine the solution.

The convex optimization problems formulated in this work are semi-definite programs (SDPs). General-purpose solvers that we used in numerical simulations to solve the SDPs, in general, do not scale well for large-scale systems. These SDPs require customized solvers that can exploit the optimization problem's local structure, e.g., [11]. However, for discussion brevity, this paper is limited to only the theoretical development of the framework. The development of customized algorithms for large-scale systems is a topic of our ongoing research.

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