Communications in Information and Systems
Volume 21, Number 2, 297–315, 2021

Solution existence and uniqueness for degenerate SDEs with application to Schrödinger-equation representations

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Existence and uniqueness results for solutions of stochastic differential equations (SDEs) under exceptionally weak conditions are well known in the case where the diffusion coefficient is nondegenerate. Here, existence and uniqueness of strong solutions is obtained in the case of degenerate SDEs in a class that is motivated by diffusion representations for solutions of Schrödinger initial value problems. In such examples, the dimension of the range of the diffusion coefficient is exactly half that of the state. In addition to this degeneracy, two types of discontinuities and singularities in the drift are allowed, where these are motivated by the structure of the Coulomb potential. The first type consists of discontinuities that may occur on a possibly high-dimensional manifold. The second consists of singularities that may occur on a smoothly parameterized curve.

1. Introduction

Existence and uniqueness results for solutions of stochastic differential equations (SDEs) typically have weaker assumptions on the smoothness of the drift than those which are required in the case of the corresponding ordinary differential equations. The results with the weakest conditions on the drift have been those where the diffusion coefficient is assumed to be nondegenerate, cf. [1, 15, 28, 30].

In recent efforts on diffusion representations for solutions of Schrödinger initial value problems (IVPs) [3, 5, 18, 19, 20], the representation dynamics take the form of complex-valued SDEs. In particular, the SDEs are given as

$$d\xi_t = f(\xi_t)\,dt + \tfrac{1+i}{\sqrt{2}}\sigma\,dB_t,$$

*Research supported in part by JSPS KAKENHI Grant Number 20K03733.
†Research supported in part by AFOSR Grant FA9550-18-1-0268 and NSF Grant DMS 1908918.

where $B_t \in \mathbb{R}^m$, $\sigma \in \mathbb{R}$, $f : \mathbb{C}^m \to \mathbb{C}^m$, i denotes the imaginary unit, and $\mathbb{C}$ denotes the complex field. Breaking out the real and imaginary parts, one obtains an SDE with a $2m$–dimensional state and a $2m \times m$ degenerate diffusion coefficient. Hence, we have an iconic application class for which there were previously no solution existence and uniqueness results. This class motivates the effort here, and particular examples appear in Section 3. Further, this $2m$–dimensional real-valued class of interest allows for two types of nonsmoothness in the drift. The first consists of discontinuities that may occur on a possibly high-dimensional manifold. In the example class considered below, these occur on a manifold of dimension m. The second consists of singularities that may occur on a one-dimensional manifold.

We briefly indicate other recent results on existence and uniqueness for degenerate SDEs, so as to situate the result herein. Kumar [14] considers degenerate SDEs with non-Lipschitz coefficients and states taking values in the positive orthant, where in the particular case where the coefficients are Lipschitz, both existence and uniqueness of a strong solution is obtained. Figalli [7] employs known results for associated partial differential equations (PDEs), including the Fokker–Planck equation, as an aid in developing results on existence and uniqueness for degenerate SDEs. Chaudru de Raynal [26] also employs known results on associated PDEs to obtain pathwise uniqueness for degenerate SDEs with Hölder drift with exponents greater than $2/3$.

In Section 2, the assumptions defining the class of SDEs will be presented. Then, in Section 3, an iconic problem that motivates those assumptions will be described. The main result is obtained in two steps. The first step, discussed in Section 4, is to obtain the existence and uniqueness for a system where the discontinuities and singularities have been smoothed. Finally, in Section 5, a limit is taken, which yields the asserted existence and uniqueness for the original, desired class of SDEs.

2. The class of SDEs

We consider SDEs on $[0, T]$ of the form

$$d\eta_t = F(\eta_t, \zeta_t)\, dt + dB_t, \qquad \eta_0 = y^0 \in \mathbb{R}^m, \tag{1}$$

$$d\zeta_t = G(\eta_t, \zeta_t)\, dt, \qquad \zeta_0 = z^0 \in \mathbb{R}^m, \tag{2}$$

and we let $l \doteq 2m$. In order to describe the problem structure and assumptions, we make some additional definitions. These are largely described in the material leading up to, and including, Assumption $(A.3)$. Although these may appear odd at first, the motivation for this selection will become apparent in Section 3. For each $z \in \mathbb{R}^m$, let $\tilde{\mathcal{H}}_0(z) \subset \mathbb{R}^m$ be such that its Lebesgue measure is zero, i.e., $\mu(\tilde{\mathcal{H}}_0(z)) = 0$, and let

$$\mathcal{H}_0 \doteq \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{H}}_0(z)\}.$$

Note that $\mathcal{H}_0$ will be a set along which the drift may have discontinuities. There will also be a set $\mathcal{G}_0 \subset \mathbb{R}^l$ along which the drift may have singularities as well as discontinuities. For each $z \in \mathbb{R}^m$, we let $\tilde{\mathcal{G}}_0(z) \subset \mathbb{R}^m$, and define $\mathcal{G}_0 \doteq \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{G}}_0(z)\}$, where a more complete specification of $\tilde{\mathcal{G}}_0$ is given below in Assumption $(A.3)$. For $\delta > 0$, let $\tilde{\mathcal{G}}_\delta(z) \doteq \{y \in \mathbb{R}^m \,|\, d(y, \tilde{\mathcal{G}}_0(z)) \le \delta|z|\}$ and $\mathcal{G}_\delta \doteq \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathcal{G}}_\delta(z)\}$. We assume the following.

$F, G \in C^1([\mathcal{G}_0 \cup \mathcal{H}_0]^c)$. For each $\delta > 0$, F and G are bounded on $\mathcal{G}_\delta^c$. For each $\delta > 0$, $\nabla_{(y,z)}F$ and $\nabla_{(y,z)}G$ are bounded on $[\mathcal{G}_\delta \cup \mathcal{H}_0]^c$. Initial conditions $(y^0, z^0) \notin \mathcal{G}_0 \cup \mathcal{H}_0$, and $z^0 \neq 0$. $\quad (A.1)$

$\mathcal{H}_0 \subset \hat{\mathcal{H}}_0 \doteq \{(y,z) \in \mathbb{R}^l \mid h_0(y) = h_1(z)\}$, for some $h_0 \in C^2(\mathbb{R}^m; \mathbb{R})$ and $h_1 \in C^1(\mathbb{R}^m; \mathbb{R})$ satisfying $\nabla h_0(y) \neq 0$, $\forall y \neq 0$. $\quad (A.2)$

Let $\bar{\mathcal{L}}$ denote the space of nonsingular $m \times m$ matrices, and let $I_{m\times m} \in \bar{\mathcal{L}}$ denote the identity matrix.

Let $\mathcal{I} \doteq [0,1]$, and let $p \in C^1(\mathcal{I}^o; \mathbb{R}^m) \cap C(\mathcal{I}; \mathbb{R}^m)$ be such that there exists $M_p < \infty$ such that $|\frac{dp}{d\lambda}(\lambda)| \in [1/M_p, M_p]$ for all $\lambda \in \mathcal{I}^o$. Let $\bar{e} \in \mathbb{R}^m \setminus \{0\}$. Let $\mathcal{D}_J \doteq \mathbb{R}^m \setminus \{k_1 \bar{e} \,|\, k_1 \in \mathbb{R}\}$ and $J \in C^2(\mathcal{D}_J; \bar{\mathcal{L}})$ be given by $J(z) = (1/|z|)\Gamma(z)$ where $\Gamma : \mathbb{R}^m \setminus \{0\} \to \bar{\mathcal{L}}$ is such that $\Gamma(z)$ is orthonormal for all $z \in \mathbb{R}^m \setminus \{0\}$, and such that $J(z)z = \bar{e}$ for all $z \in \mathbb{R}^m \setminus \{0\}$, $[J(\bar{e})]^{-1} = I_{m\times m}$, and for each $\delta > 0$, $\frac{dJ}{dz}$ is bounded on $\mathbb{R}^m \setminus \{z \in \mathbb{R}^m \,|\, |z - (z\cdot\bar{e}/|\bar{e}|^2)\bar{e}| < \delta\}$. Finally, suppose $\tilde{\mathcal{G}}_0(z)$ is specifically given by $\tilde{\mathcal{G}}_0(z) \doteq \{\, y \in \mathbb{R}^m \,|\, \exists \lambda \in \mathcal{I} \text{ s.t. } y = [J(z)]^{-1} p(\lambda) \,\}$ for all $z \in \mathbb{R}^m$. $\quad (A.3)$

Remark 1. *The above structures for $\mathcal{G}_0$ and $\mathcal{H}_0$, which may at first seem unusual, were chosen for the case where the discontinuity and singular sets are defined in terms of η_t relative to ζ_t. A motivational example where these assumptions are satisfied is given in Section 3. In that example, the space dimension is $m = 3$, while $\mathcal{H}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0,\ y^T z > 0\}$ and $\mathcal{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0$ and $y^T z = 0\}$. Further, in that case, one may take $\bar{e}$ to be $(1,0,0)^T$ and $p(\cdot)$ to be a parameterization of the unit circle in the plane perpendicular to $\bar{e}$.*

Remark 2. *The assumptions may be weakened to allow for a finite number of both discontinuity and singularity manifolds, with no fundamental change in the proofs. For clarity of exposition, we do not include the details.*

An additional assumption will appear in Section 4, and it will be the final assumption. That assumption is more easily indicated there, after some additional definitions.

3. Motivation

One motivation for consideration of this large class of SDE problems is the staticization based diffusion representation for the solution of Schrödinger initial value problems (IVPs) [17, 18, 19, 20]. The case of the Coulomb potential was discussed in [17]. For $x \in \mathbb{C} \setminus \{0\}$, define the single-valued logarithm and square-root operations

$$\log_q(x) \doteq \log(r) + i\theta, \quad \sqrt{x} \doteq \exp\left[\tfrac{1}{2}\log_q(x)\right],$$

where $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$ are such that $x = re^{i\theta}$. We specifically look at the Maslov dequantization (cf. [16]) of the solution of a Schrödinger IVP associated to the lowest energy "electron shell" of the Bohr model, cf. [10], which may be extended to complex-valued states as $S^0 : [0, \infty) \times \mathbb{C}^3 \to \mathbb{C}$ given by

$$S^0(t, x) = \tfrac{-c_1^2}{2\overline{m}}t + ic_1\sqrt{x^T x},$$

where T denotes transpose (without conjugation), $c_1 = \frac{2\overline{m}C}{(m-1)\hbar} = \frac{\overline{m}C}{\hbar}$, $\overline{m}$ denotes mass, space dimension $m = 3$, $\hbar$ denotes Planck's constant, $C \doteq q_0 q_1/(4\pi\bar{\epsilon}_0)$, q_0 denotes the central (nucleus) charge, q_1 denotes the electron charge and $\bar{\epsilon}_0$ denotes the vacuum permittivity. One may check that $S_t^0(r, x) = \frac{-c_1^2}{2\overline{m}}$ and $S_x^0(r, x) = ic_1 x/\sqrt{x^T x}$, $\Delta S^0(r, x) = 2ic_1/\sqrt{x^T x}$, and further, that S^0 satisfies the dequantized, time-reversed form of the Schrödinger equation, given by

$$0 = S_t(r, x) + \tfrac{i\hbar}{2\overline{m}}\Delta S(r, x) - \tfrac{1}{2\overline{m}}(S_x(r, x))^T S_x(r, x) - V(x),$$

where $V(x) = -C/\sqrt{x^T x}$.

The dynamics of the diffusion process generating the solution as the associated stationary value function are given by [17, 20]

$$d\xi_r = (-1/\overline{m})S_x^0(r, \xi_r)\, dr + \sqrt{\hbar/\overline{m}}\tfrac{1+i}{\sqrt{2}}\, dB_r,$$

with $\xi_0 = x$. One may separate the three-dimensional complex state, ξ_r, into its real and imaginary parts as $\xi_r = \hat{\eta}_r + i\hat{\zeta}_r$. Similarly, letting $S^0(r,x) = R^0(r,\hat{y},\hat{z}) + iT^0(r,\hat{y},\hat{z})$ with $x = \hat{y} + i\hat{z}$, and employing the Cauchy-Riemann equations, the SDE system becomes

$$d\hat{\eta}_r = (-1/\overline{m}) R^0_y(r,\hat{\eta}_r,\hat{\zeta}_r)\,dr + \sqrt{\tfrac{\hbar}{2\overline{m}}}\,dB_r, \quad \hat{\eta}_0 = \hat{y},$$
$$d\hat{\zeta}_r = (1/\overline{m}) R^0_z(r,\hat{\eta}_r,\hat{\zeta}_r)\,dr + \sqrt{\tfrac{\hbar}{2\overline{m}}}\,dB_r, \quad \hat{\zeta}_0 = \hat{z}.$$

Performing the change of coordinates given by $\eta_r = (1/\sqrt{2})[\hat{\eta}_r + \hat{\zeta}_r]$, $\zeta_r = (1/\sqrt{2})[-\hat{\eta}_r + \hat{\zeta}_r]$ yields

$$d\eta_r = (1/\sqrt{2}\overline{m})[-R^0_y + R^0_z](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\,dr + \sqrt{\tfrac{\hbar}{\overline{m}}}\,dB_r,$$
$$d\zeta_r = (1/\sqrt{2}\overline{m})[R^0_y + R^0_z](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\,dr,$$

with $\eta_0 = y^0 \doteq (1/\sqrt{2})[\hat{y} + \hat{z}]$ and $\zeta_0 = z^0 \doteq (1/\sqrt{2})[-\hat{y} + \hat{z}]$. Using the specific form of S^0 in this example, this reduces to

$$\begin{aligned} &(3) \qquad d\eta_r = F(\eta_r,\zeta_r)\,dr + \sigma\,dB_r \\ &\qquad\qquad \doteq \frac{c_1}{\overline{m}\sqrt{\tilde{R}_r}}[\sin(\tilde{\theta}_r)\eta_r - \cos(\tilde{\theta}_r)\zeta_r]\,dr + \sqrt{\tfrac{\hbar}{\overline{m}}}\,dB_r, \\ &(4) \qquad d\zeta_r = G(\eta_r,\zeta_r)\,dr \doteq \frac{-c_1}{\overline{m}\sqrt{\tilde{R}_r}}[\cos(\tilde{\theta}_r)\eta_r + \sin(\tilde{\theta}_r)\zeta_r]\,dr, \end{aligned}$$

where $\tilde{R}_r \doteq \bar{R}(\eta_r,\zeta_r) \doteq \left[(-2\eta_r^T\zeta_r)^2 + (|\eta_r|^2 - |\zeta_r|^2)^2\right]^{1/2}$, $\cos(2\tilde{\theta}_r) = \frac{-2\eta_r^T\zeta_r}{\tilde{R}_r}$ and $\sin(2\tilde{\theta}_r) = \frac{|\eta_r|^2 - |\zeta_r|^2}{\tilde{R}_r}$ with $\tilde{\theta}_r \in (-\pi/2, \pi/2]$.

In this case, $\mathcal{H}_0$ corresponds to the branch cut induced by $\sqrt{x^Tx}$, which is taken at $|\hat{y}|^2 - |\hat{z}|^2 < 0$, $\hat{y}^T\hat{z} = 0$, or equivalently, at $y^Tz > 0$, $|y|^2 - |z|^2 = 0$. That is, $\mathcal{H}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y| = |z|,\ \ y^Tz > 0\,\}$ (i.e., $y \in \hat{\mathcal{H}}_0(z)$ iff $|y| = |z|$ and $y^Tx > 0$). For $\hat{\mathcal{H}}_0$ in $(A.2)$, one can take $h_0(y) = |y|^2$ and $h_1(z) = |z|^2$. From this, one may easily verify Assumption $(A.2)$. Also, we see that the singularities occur on

$$\mathcal{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, \bar{R}(y,z) = 0\,\} = \{(y,z) \in \mathbb{R}^l \,|\, y^Tz = 0 \ \text{ and } \ |y| = |z|\,\}.$$

One easily finds that Assumption $(A.1)$ is satisfied. To see that Assumption $(A.3)$ is satisfied, one may take $\tilde{\mathcal{G}}_0(z) \doteq \{\,y \in \mathbb{R}^m \,|\, y^Tz = 0 \ \text{ and } \ |y| = |z|\,\}$. Note that if $z = (0,0,1)^T$, then $\tilde{\mathcal{G}}_0(z)$ is the unit circle in the (z_2, z_3)–plane.

Hence, one may take $p(\lambda) \doteq (\cos(2\pi\lambda), \sin(2\pi\lambda), 0)$ and $\bar{e} = (0,0,1)^T$. Then, for $z \in \mathbb{R}^m \setminus \{0\}$, one may then let

$$\Gamma(z) \doteq \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}, \quad \text{where } u \doteq \frac{z}{|z|},$$

$$\hat{v} \doteq \begin{cases} u \times \bar{e} & \text{if } u \neq \lambda\bar{e} \text{ for some } \lambda \in \mathbb{R}, \\ (1,0,0)^T & \text{if } u = \lambda\bar{e} \text{ for some } \lambda \in (0,\infty), \\ (-1,0,0)^T & \text{if } u = \lambda\bar{e} \text{ for some } \lambda \in (-\infty,0), \end{cases}$$

$$v = \frac{\hat{v}}{|\hat{v}|}, \quad w \doteq \frac{u \times v}{|u \times v|}.$$

One may easily verify Assumption $(A.3)$ for this Γ.

4. The $\delta > 0$ prelimit

We smooth the dynamics as follows. For $\delta > 0$, we let $g^\delta \in C^\infty(\mathbb{R})$ and $\tilde{g}^{\delta/4,\delta}(\cdot;\bar{e}) \in C^\infty(\mathbb{R}^m)$ be given by

(5)
$$g^\delta(\rho) \doteq \begin{cases} 1 - \exp\left\{\frac{1}{\delta^2} + \frac{1}{\rho^2-\delta^2}\right\} & \text{if } |\rho| \in [0,\delta), \\ 1 & \text{if } |\rho| \geq \delta, \end{cases}$$

(6)
$$\tilde{g}^{\delta/4,\delta}(z;\bar{e}) \doteq \begin{cases} 0 & \text{if } |z - (z\cdot\bar{e}/|\bar{e}|^2)\bar{e}| \in [0,\delta/4], \\ g^{3\delta/4}(|z - (z\cdot\bar{e}/|\bar{e}|^2)\bar{e}| - \delta/4) & \text{if } |z - (z\cdot\bar{e}/|\bar{e}|^2)\bar{e}| > \delta/4. \end{cases}$$

We also let

$$\hat{\mathcal{G}}_\delta \doteq \mathcal{G}_\delta \cup [\mathbb{R}^m \times \tilde{\mathcal{C}}(\delta;\bar{e})],$$

where $\tilde{\mathcal{C}}(\delta;\bar{e}) \doteq \{z \in \mathbb{R}^m \,|\, |z - (z\cdot\bar{e}/|\bar{e}|^2)\bar{e}| \leq \delta/4\}$. Next, defining $\hat{R}(y,z) \doteq d\big(y, \tilde{\mathcal{G}}_0(z)\big)/|z|$ for $|z| > 0$, we let

$$F^\delta(y,z) \doteq g^\delta(\hat{R}(y,z))F(y,z), \tag{7}$$

$$G^\delta(y,z) \doteq \tilde{g}^{\delta/4,\delta}(z;\bar{e})g^\delta(\hat{R}(y,z))G(y,z) \tag{8}$$

for all $(y,z) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$. Note that

$$F^\delta = F \quad \text{and} \quad G^\delta = G \quad \text{on} \quad (\hat{\mathcal{G}}_\delta)^c. \tag{9}$$

Our final assumption is that for each $\delta > 0$,

$$F^\delta, G^\delta \in C^1\big(\mathcal{H}_0^c \cap [\mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}]\big),\ F^\delta \text{ and } G^\delta \text{ are bounded on } [\mathbb{R}^m \times (B_{\delta/4}(0))^c], \text{ and } \nabla_{(y,z)}F^\delta, \nabla_{(y,z)}G^\delta \text{ are bounded on } \mathcal{H}_0^c \cap [\mathbb{R}^m \times (B_{\delta/4}(0))^c]. \tag{A.4}$$

Note that $(A.4)$ holds for the example given in Section 3, and that it will hold more generally when the dynamics are bounded by the multiplicative inverse of appropriate polynomial forms.

Consider the system with modified dynamics given in integral form as

$$\eta_t^\delta = y^0 + \int_0^t F^\delta(\eta_r^\delta, \zeta_r^\delta)\,dr + B_t, \tag{10}$$

$$\zeta_t^\delta = z^0 + \int_0^t G^\delta(\eta_r^\delta, \zeta_r^\delta)\,dr, \tag{11}$$

for $t \in [0,T]$. We demonstrate existence and uniqueness of a strong solution via application of the Girsanov transform approach to first obtain existence of a weak solution, followed by a demonstration of pathwise uniqueness to then obtain the strong-solution assertion.

Lemma 3. *Suppose η^δ is an m-dimensional $\{\mathcal{F}_t\}$-Brownian motion on probability space $(\Omega, \mathcal{F}, P)$ where Ω, $\mathcal{F}$ and P denote a sample space, σ-algebra and probability measure, respectively, and with filtration denoted by $\mathcal{F} = \{\mathcal{F}_t\}$. Let ζ^δ be an $\{\mathcal{F}_t\}$-adapted $\mathbb{R}^m$-valued continuous process on $[0,T]$ with bounded variation. Then, for a.e. $\omega \in \Omega$, $\mu(\{t \in [0,T] \,|\, (\eta_t^\delta, \zeta_t^\delta) \in \mathcal{H}_0\}) = 0$, where μ denotes Lebesgue measure on $\mathbb{R}$.*

Proof. Since $\mathcal{H}_0 \subset \hat{\mathcal{H}}_0$ by $(A.2)$, it suffices to show that for a.e. $\omega \in \Omega$, $\mu(\{t \in [0,T] \,|\, (\eta_t^\delta, \zeta_t^\delta) \in \hat{\mathcal{H}}_0\}) = 0$. Let $X_t \doteq h_0(\eta_t^\delta) - h_1(\zeta_t^\delta)$. We note that

$$\{t \in [0,T] \,|\, (\eta_t^\delta, \zeta_t^\delta) \in \hat{\mathcal{H}}_0\} = \{t \in [0,T] \,|\, X_t = 0\}.$$

Using the Itô formula, we have the following semi-martingale decomposition:

$$X_t = X_0 + \int_0^t \nabla h_0(\eta_s^\delta) \cdot d\eta_s^\delta + \int_0^t \frac{1}{2}\Delta h_0(\eta_s^\delta) ds - \int_0^t \nabla h_1(\zeta_s^\delta) \cdot d\zeta_s^\delta.$$

Let $L_t^a(X)$ be a local time of the continuous semi-martingale $X = \{X_t\}$. By the occupation time formula for continuous semi-martingales (cf. [27, (1.6) Corollary of Chap.VI]), we have for a.e. $\omega \in \Omega$,

$$\int_0^T 1_{\{0\}}(X_t)|\nabla h_0(\eta_t^\delta)|^2 dt = \int_{-\infty}^{\infty} 1_{\{0\}}(a) L_T^a(X) da = 0,$$

which implies that there exists $N_1 \subset [0,T]$ with $\mu(N_1) = 0$ such that

$$1_{\{0\}}(X_t)|\nabla h_0(\eta_t^\delta)|^2 = 0, \ \forall t \in [0,T] \backslash N_1. \tag{12}$$

Let $\eta_t^{\delta,i}$ be the ith component of $\eta_t^\delta = (\eta_t^{\delta,1}, \ldots, \eta_t^{\delta,m})$ and $L_t^a(\eta^{\delta,i})$ be a local time of one-dimensional Brownian motion $\eta^{\delta,i} = \{\eta_t^{\delta,i}\}$. By the occupation time formula for one-dimensional Brownian motion, we have for a.e. $\omega \in \Omega$,

$$\mu(\{t \in [0,T] \,|\, \eta_t^{\delta,i} = 0\}) = \int_0^T 1_{\{0\}}(\eta_t^{\delta,i})dt = \int_{-\infty}^{\infty} 1_{\{0\}}(a)L_T^a(\eta^{\delta,i})da = 0.$$

Thus, for a.e. $\omega \in \Omega$, there exists $N_2 \subset [0,T]$ with $\mu(N_2) = 0$ such that

$$\eta_t \neq 0, \ \forall t \in [0,T] \backslash N_2,$$

which implies by $(A.2)$

$$\nabla h_0(\eta_t^\delta) \neq 0, \ \forall t \in [0,T] \backslash N_2. \tag{13}$$

Let $N = N_1 \cup N_2$. By (12) and (13), we see that for a.e. $\omega \in \Omega$, $\mu(N) = 0$ and

$$1_{\{0\}}(X_t) = 0, \ \forall t \in [0,T] \backslash N.$$

Hence we can obtain $\Omega^0 \subset \Omega$ with $P(\Omega^0) = 1$ such that

$$\mu(\{t \in [0,T] \,|\, X_t = 0\}) = \int_0^T 1_{\{0\}}(X_t)dt = \int_{[0,T]\backslash N} 1_{\{0\}}(X_t)dt = 0$$

for all $\omega \in \Omega^0$. □

Lemma 4. *For a.e. $\omega \in \Omega$, There exists absolutely continuous, unique $\zeta_\cdot^\delta(\omega)$ satisfying (11).*

Proof. The proof follows the standard successive approximations approach. We indicate the main steps. By Assumption $(A.4)$, there exists $L < \infty$ such that $|\nabla_{(y,z)} G^\delta(y,z)| \leq L$ for all $(y,z) \in \mathcal{H}_0^c$. Let $0 = t_0 < t_1 < \ldots t_J = T$ where $t_{j+1} - t_j \in (0, 1/(2L))$ for all j. Fix $\omega \in \Omega^0$, where Ω^0 is defined in the proof of Lemma 3. Suppose we have a unique, absolutely continuous solution, $\zeta_\cdot^\delta(\omega)$, up to t_j (where j may be zero), and let $z^j = \zeta_{t_j}^\delta(\omega)$. We extend the solution to $[t_j, t_{j+1}]$. Let $\tilde{\zeta}^{\delta,0}$ be absolutely continuous (and hence of bounded variation) on $[t_j, t_{j+1}]$, with $\tilde{\zeta}^{\delta,0}(t_j) = z^j$. For $k \geq 0$, let

$$\tilde{\zeta}_t^{\delta,k+1} \doteq z^j + \int_{t_j}^t G^\delta(\eta_r^\delta(\omega), \tilde{\zeta}_r^{\delta,k})\, dr \quad \forall\, t \in [t_j, t_{j+1}].$$

Noting Assumption $(A.4)$, we find that $\tilde{\zeta}^{\delta,k}$ is absolutely continuous for all $k \geq 0$. Letting $\mathcal{A}^k(\omega) \doteq \{t \in [t_j, t_{j+1}] \,|\, (\eta_t^\delta(\omega), \tilde{\zeta}_t^{\delta,k}) \in \mathcal{H}_0\}$, we see by Lemma 3 that $\mu(\mathcal{A}^k(\omega)) = 0$ for all k. Then, $\|\zeta^{\delta,k+1} - \zeta^{\delta,k}\|_{L_\infty(t_j,t_{j+1})} \leq \frac{1}{2}$. Application of the Banach Fixed Point Theorem then yields a unique, absolutely continuous extension of the solution, $\zeta_\cdot^\delta(\omega)$, to $[0, t_{j+1}]$. □

Lemma 5. *Let $\delta > 0$. There exists a weak solution to (10)–(11).*

Proof. Let η^δ be a Brownian motion as in Lemma 3, and let ζ^δ be the corresponding solution of (11) given by Lemma 4. Let $\nu_t^\delta(\omega) \doteq F^\delta(\eta_t^\delta, \zeta_t^\delta)$ for all $\omega \in \Omega^0$ (indicated in the proof of Lemma 3) and all $t \in [0, T]$. By Assumption $(A.4)$, there exists $D_1 < \infty$ such that $|\nu_t^\delta(\omega)| \leq D_1$ for all $\omega \in \Omega^0$ and $t \in [0, T]$. Let $B_t^\delta \doteq \eta_t^\delta - \int_0^t \nu_r^\delta \, dr$ for all $\omega \in \Omega^0$ and $t \in [0, T]$. We note that the Novikov condition is satisfied, and letting $\hat{P}(\mathcal{C}) \doteq \mathbb{E}^P[1_{\mathcal{C}} Z_T(\nu^\delta)]$ for $\mathcal{C} \in \mathcal{F}_T$, with $Z_t(\nu^\delta) \doteq 1 + \sum_{j=1}^3 \int_0^t Z_r(\nu^\delta)[\nu_r^\delta]_j \, d[\nu_r^\delta]_j$, B^δ is a Brownian motion on $(\Omega, \mathcal{F}_T, \hat{P})$, with filtration $\mathcal{F}_\cdot$. Then $(\eta^\delta, \zeta^\delta)$ forms a solution to (10)–(11) with Brownian motion B^δ and probability space $(\Omega, \mathcal{F}_T, \hat{P})$. □

Theorem 6. *Let $\delta > 0$. There exists a unique strong solution to (10)–(11).*

Proof. The strong solution will follow from a demonstration of pathwise uniqueness (cf. [11, Cor. 5.3.23]). Let $\gamma_t^\delta \doteq (\eta_t^{\delta^T}, \zeta_t^{\delta^T})^T$ for all $t \in [0, T]$, $H^\delta \doteq ([F^\delta]^T, [G^\delta]^T)^T$ and $\bar{\gamma}^0 \doteq ([y^0]^T, [z^0]^T)^T$, in which case,

$$\gamma_t^\delta = \bar{\gamma}^0 + \int_0^t H_r^\delta \, dr + \begin{bmatrix} \mathcal{I}_{m \times m} \\ 0 \end{bmatrix} B_t^\delta \quad \forall\, t \in [0, T]. \tag{14}$$

Letting γ^δ and $\tilde{\gamma}^\delta$ be two solutions of (14), one sees from Assumption $(A.4)$ that there exists $\bar{L} < \infty$ such that

$$|\gamma_t^\delta - \tilde{\gamma}_t^\delta| \leq \bar{L} \int_0^t |\gamma_r^\delta - \tilde{\gamma}_r^\delta| \, dr \quad \forall\, t \in [0, T].$$

Hence, by the Gronwall inequality, $\gamma^\delta = \tilde{\gamma}^\delta$, and we have pathwise uniqueness. □

5. Taking $\delta \downarrow 0$

We obtain the limit result in the case where the dimension satisfies $m \geq 3$. This restriction is related to the form of $\tilde{\mathcal{G}}_0$, which takes the form of a curve in $\mathbb{R}^m$. It is expected that in the case where $\tilde{\mathcal{G}}_0$ is a point, the result would follow for $m \geq 2$.

Fix a probability space, $(\Omega, \bar{\mathcal{F}}, \bar{P})$, and Brownian motion, $B.$, with filtration $\mathcal{F}.$ generated by $B.$. As by $(A.1)$, $(y^0, z^0) \notin \mathcal{G}_0$, there exists $\bar{\delta} > 0$ such that $(y^0, z^0) \notin \mathcal{G}_\delta$ for all $\delta \in [0, \bar{\delta}]$. Let $\delta_n \downarrow 0$ with $\delta_1 \in (0, \bar{\delta})$. Let the corresponding strong solutions of (10)–(11) be denoted by (η^n, ζ^n). Then, note that $G^{\delta_n}(y, z) = 0$ for all $z \in B_{\delta_n/4}(0)$, and hence

$$|\zeta_t^n| \geq \delta_n/4 \quad \forall\, t \in [0, T],\, \omega \in \Omega,\, n \in \mathbb{N}. \tag{15}$$

For $n \in \mathbb{N}$, let

$$\mathcal{A}_n \doteq \{\, \omega \in \Omega \,|\, \nexists t \in [0, T] \text{ s.t. } (\eta_t^n, \zeta_t^n) \in \mathcal{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n}(0)]\,\}. \tag{16}$$

Recalling that $F^\delta = F$ on $\mathcal{G}_\delta^c$ and $G^\delta = G$ on $\mathcal{G}_\delta^c \cap B_{\delta/4}(0)^c$, we see that

$$(\eta^m, \zeta^m) = (\eta^n, \zeta^n) \quad \forall\, \omega \in \mathcal{A}_n \text{ and } m \geq n \geq 1. \tag{17}$$

Lastly, let

$$\tilde{\eta}_t^n = J(\zeta_t^n)\eta_t^n, \quad \tilde{\zeta}_t^n = J(\zeta_t^n)\zeta_t^n = \bar{e}, \tag{18}$$

for all $t \in [0, T]$.

Lemma 7. *For each $\omega \in \Omega$, $(\eta_t^n, \zeta_t^n)(\omega) \in \mathcal{G}_\delta$ if and only if $\eta_t^n(\omega) \in \tilde{\mathcal{G}}_\delta(\zeta_t^n(\omega))$ if and only if $\tilde{\eta}_t^n(\omega) \in \tilde{\mathcal{G}}_{|\zeta_t^n(\omega)|\delta}(\tilde{\zeta}_t^n(\omega))$ if and only if there exists $\lambda_t^n(\omega) \in \mathcal{I}$ such that $|\tilde{\eta}_t^n(\omega) - p(\lambda_t^n(\omega))| \leq \delta$.*

Proof. The first assertion is by definition. Noting Assumption $(A.3)$, that assertion is true if and only if $d(\eta_t^n, \tilde{\mathcal{G}}_0(\zeta_t^n)) \leq \delta|\zeta_t^n|$, or equivalently, if and only if $\min_{\lambda \in \mathcal{I}} |\eta_t^n - J^{-1}(\zeta_t^n)p(\lambda)| \leq \delta|\zeta_t^n|$. Using the orthonormality of $\Gamma(\zeta_t^n)$, one finds that this is true if and only if $\min_{\lambda \in \mathcal{I}} |J(\zeta_t^n)\eta_t^n - p(\lambda)| \leq \delta$, which by (18), is equivalently, $\min_{\lambda \in \mathcal{I}} |\tilde{\eta}_t^n - p(\lambda)| \leq \delta$, which yields the remaining two assertions. □

Lemma 8. *For each $n \in \mathbb{N}$, there exists a probability measure, P_n, mutually absolutely continuous with respect to $\bar{P}$, such that η^n is a Brownian motion with respect to P_n.*

Proof. By the boundedness of F^{δ_n} and (10), one finds that the Novikov condition is satisfied, and hence the assertion follows from the Girsanov theorem, cf. [11]. □

Let

$$\hat{\mathcal{A}}_n \doteq \{\, \omega \in \Omega \,|\, \nexists t \in [0, T] \text{ s.t. either } \tilde{\eta}_t^n \in \tilde{\mathcal{G}}_{\delta_n}(\bar{e}) \text{ or } \zeta_t^n \in B_{\delta_n}(0)\,\}. \tag{19}$$

Using Lemma 7 and (15), we see that

$$\mathcal{A}_n = \hat{\mathcal{A}}_n. \tag{20}$$

Lemma 9. *There exists a probability measure, $\breve{P}_n$, mutually absolutely continuous with respect to P_n, such that*

$$d\tilde{\eta}^n_t = J(\zeta^n_t)\, d\breve{\eta}^n_t,$$

where $\breve{\eta}^n_t$ is a Brownian motion under $\breve{P}_n$.

Proof. Applying Itô's rule to $\tilde{\eta}^n$, and noting that one has $d\langle[\zeta^n]_k, [\zeta^n]_j\rangle_t \equiv 0$ for all $k, j \in]1, m[$, one sees that

$$d\tilde{\eta}^n_t = \bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)\, dt + J(\zeta^n_t)\, d\eta^n_t = J(\zeta^n_t)\Big[(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)\, dt + d\eta^n_t\Big], \tag{21}$$

where, component-wise,

$$\bar{F}^n_k(\tilde{\eta}^n_t, \zeta^n_t) \doteq \sum_{j=1}^m \Big(\sum_{l=1}^m \big[\frac{\partial J}{\partial z_j}(\zeta^n_t)\big]_{k,l}[\tilde{\eta}^n_t]_l\Big)\big[G^{\delta_n}\big([J(\zeta^n_t)]^{-1}\tilde{\eta}^n_t, \zeta^n_t\big)\big]_j \tag{22}$$

for all $k \in]1, m[$. We examine $\bar{F}^n$. By Assumption $(A.4)$, there exists $M^1_n < \infty$ such that

$$|G^{\delta_n}\big([J(\zeta^n_t)]^{-1}\tilde{\eta}^n_t, \zeta^n_t\big)| \le M^1_n \quad \forall\, t \in [0, T],\ \omega \in \Omega. \tag{23}$$

Also, by (15) and Assumption $(A.3)$, there exists $M^2_n < \infty$ such that

$$\max\big\{|J(\zeta^n_t)|,\ \big|\tfrac{\partial J}{\partial z_j}(\zeta^n_t)\big|\big\} \le M^2_n \quad \forall\, j \in]1, m[,\ t \in [0, T],\ \omega \in \Omega. \tag{24}$$

Lastly, by (11), (23) and Assumption $(A.3)$ one sees that

$$|(J(\zeta^n_t))^{-1}| = |\zeta^n_t| \le |z^0| + M^1_n T \doteq M^3_n < \infty \tag{25}$$

for all $t \in [0, T]$ and $\omega \in \Omega$. By (22)–(25), we see that there exists $\bar{M}_n < \infty$ such that

$$|(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t, \zeta^n_t)| \le \bar{M}_n|\tilde{\eta}^n_t| \quad \forall\, t \in [0, T],\ \omega \in \Omega. \tag{26}$$

For integers $0 \le k \le K-1 < \infty$, let $\Delta_K \doteq T/K$ and $t_k \doteq k\Delta_K$. By (26),

$$\mathbb{E}\Big\{\exp\big[\tfrac{1}{2}\int\limits_{t_k}^{t_{k+1}} |(J(\zeta_t^n))^{-1}\bar{F}^n(\tilde{\eta}_t^n,\zeta_t^n)|^2\,dt\big]\Big\}$$
$$\le \exp\big[(\bar{M}_n)^2/2\big]\mathbb{E}\Big\{\exp\big[\tfrac{1}{2}\int\limits_{t_k}^{t_k+\Delta_K} |\tilde{\eta}_t^n|^2\,dt\big]\Big\}$$

for all $0 \le k \le K < \infty$ and $n \in \mathbb{N}$. However, recalling that $\tilde{\eta}^n$ is a Brownian motion on measure P_n, this is finite for sufficiently large K. Hence, a weak Novikov condition is satisfied, cf. [11, Cor. 3.5.14], and we may apply a Girsanov transformation, yielding measure $\breve{P}_n$, mutually absolutely continuous with respect to P_n, given by $d\breve{P}_n \doteq \tilde{\mu}_T^n dP_n$, where

$$\tilde{\mu}_T^n \doteq \exp\Big[-\int\limits_0^T (v_t^n)^T\,d\eta_t^n - \tfrac{1}{2}\int\limits_0^T |v_t^n|^2\,dt\Big],$$

with $v_t^n \doteq (J(\zeta_t^n))^{-1}\bar{F}^n(\tilde{\eta}_t^n,\zeta_t^n)$, and such that under $\breve{P}_n$, the process $\breve{\eta}_t^n \doteq \int_0^t v_r^n\,dr + \eta_t^n$ is a Brownian motion. Recalling (21), we have $d\tilde{\eta}_t^n = J(\zeta_t^n)\,d\breve{\eta}_t^n$. □

We define $\{\beta_t^n\}_{t\ge 0}$ and $\{\alpha_s^n\}_{s\ge 0}$ by

$$\beta_t^n \doteq \int_0^{t\wedge T} \frac{dr}{|\zeta_r^n|^2}, \qquad \alpha_s^n \doteq \inf\{t \in [0,\infty) \,|\, \beta_t^n > s\},$$

where the infimum of the empty set is taken to be ∞.

Lemma 10. *There exists a Brownian motion $\{w_s\}_{s\ge 0}$ on an enlarged probability space of $(\Omega, \bar{\mathcal{F}}, \breve{P}_n)$, which we denote by $(\tilde{\Omega}, \tilde{\bar{\mathcal{F}}}, \tilde{\breve{P}}_n)$, such that $w_s = \tilde{\eta}^n_{\alpha_s^n}$ for $0 \le s \le \beta_T^n$. Moreover there exist $0 \le \underline{\alpha} \le \bar{\alpha} < \infty$ such that $\alpha_{s+r}^n - \alpha_s^n \in [\,\underline{\alpha} r, \bar{\alpha} r\,]$ for all $0 \le s, r < \infty$.*

Proof. The asserted bounds on $\alpha_\cdot^n$ follow from Assumption $(A.3)$, (15) and (25). We extend $\{\tilde{\eta}_t^n\}_{0\le t\le T}$ and $\{\mathcal{F}_t\}_{0\le t\le T}$ to $[0,\infty)$ by

$$\hat{\eta}_t^n \doteq \tilde{\eta}_{t\wedge T}^n, \;\; \hat{\mathcal{F}}_t \doteq \mathcal{F}_{t\wedge T}, \;\; t \ge 0.$$

Noting that $\{\hat{\eta}_t^n\}_{t\ge 0}$ is a continuous $\{\hat{\mathcal{F}}_t\}$-martingale and $J(\zeta_t^n)J^T(\zeta_t^n) = |\zeta_t^n|^{-2} I_{m\times m}$ for all $t \in [0,T]$, $\omega \in \Omega$, we have

$$\langle \hat{\eta}^{n,i}, \hat{\eta}^{n,j}\rangle_t = \langle \tilde{\eta}^{n,i}, \tilde{\eta}^{n,j}\rangle_{t\wedge T} = \delta_{ij}\beta_t^n.$$

Thus, by [27, (1.10) Theorem of Chap.V], there exists a Brownian motion $\{w_s\}_{s\geq 0}$ on an enlarged probability space of $(\Omega, \bar{\mathcal{F}}, \breve{P}_n)$ satisfying $w_s = \tilde{\eta}^n_{\alpha^n_s}$ for $s \in [0, \beta^n_T]$. To clarify the enlargement procedure and the construction of $\{w_s\}$ in the above theorem, let $(\Omega', \mathcal{F}', P')$ be a probability space with a filtration $\{\mathcal{F}'_s\}$ and $\{b_s\}_{s\geq 0}$ be an m-dimensional $\{\mathcal{F}'_s\}$-Brownian motion with $b_0 = 0$. Define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\breve{P}}_n)$ and $\{\tilde{\mathcal{F}}_s\}$ by

$$\tilde{\Omega} \doteq \Omega \times \Omega', \;\; \tilde{\mathcal{F}} \doteq \bar{\mathcal{F}} \otimes \mathcal{F}', \;\; \tilde{P}_n \doteq \breve{P}_n \otimes P', \;\; \tilde{\mathcal{F}}_s \doteq \hat{\mathcal{F}}_{\alpha^n_s} \otimes \mathcal{F}'_s.$$

Then $\{w_s\}_{s\geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_n)$ is given by

$$\begin{aligned} w_s(\tilde{\omega}) &\doteq \begin{cases} \hat{\eta}^n_{\alpha^n_s(\omega)}(\omega), & 0 \leq s \leq \beta^n_\infty(\omega), \\ \hat{\eta}^n_\infty(\omega) + b_{s-\beta^n_\infty(\omega)}(\omega'), & s > \beta^n_\infty(\omega), \end{cases} \\ &= \begin{cases} \tilde{\eta}^n_{\alpha^n_s(\omega)}(\omega), & 0 \leq s \leq \beta^n_T(\omega), \\ \tilde{\eta}^n_T(\omega) + b_{s-\beta^n_T(\omega)}(\omega'), & s > \beta^n_T(\omega), \end{cases} \end{aligned} \tag{27}$$

where we denote $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega} = \Omega \times \Omega'$. □

For $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$, we let $\breve{\zeta}^n_s(\tilde{\omega}) \doteq \zeta^n_s(\omega)$ for all $s \in [0, T]$ and $n \in \mathbb{N}$. By (27), we note that

$$\begin{aligned} \hat{\mathcal{A}}_n \times \Omega' = \{\tilde{\omega} \in \tilde{\Omega} \,|\, &\nexists s \in [0, \beta^n_T(\omega)] \; s.t. \; either \;\; \tilde{\eta}^n_{\alpha^n_s(\omega)} \in \tilde{\mathcal{G}}_{|\breve{\zeta}^n_s(\tilde{\omega})|\delta_n}(\bar{e}) \\ &or \;\; \breve{\zeta}^n_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\ = \{\tilde{\omega} \in \tilde{\Omega} \,|\, &\nexists s \in [0, \beta^n_T(\omega)] \; s.t. \; either \;\; w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}^n_s(\tilde{\omega})|\delta_n}(\bar{e}) \\ &or \;\; \breve{\zeta}^n_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\ \supseteq \{\tilde{\omega} \in \tilde{\Omega} \,|\, &\nexists s \in [0, \infty) \; s.t. \; either \;\; w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}^n_s(\tilde{\omega})|\delta_n}(\bar{e}) \\ &or \;\; \breve{\zeta}^n_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\ \doteq \mathcal{C}_n \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{28}$$

For $n \in \mathbb{N}$, let

$$\tau_n \doteq \inf\{s \geq 0 \,|\, (\eta^n_s, \zeta^n_s)(\omega) \in \mathcal{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n}(0)]\},$$

and let

$$(\bar{\eta}_s, \bar{\zeta}_s) = (\eta^n_s, \zeta^n_s) \quad \forall\, s \leq \tau_n,$$

which is

$$= (\eta_s^k, \zeta_s^k) \quad \forall\, k \geq n, \forall s \leq \tau_n. \tag{29}$$

Next, let $\breve{\bar{\zeta}}_s(\tilde{\omega}) \doteq \bar{\zeta}_s(\omega)$ for all $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$. Then,

$$\begin{aligned}
\mathcal{C}_n =& \{\tilde{\omega} \in \tilde{\Omega} \,|\, \nexists s \in [0,\infty)\, s.t.\ either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_{|\breve{\zeta}_s^n(\tilde{\omega})|\delta_n}(\bar{e}) \\
&\qquad or\ \ \breve{\bar{\zeta}}_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \\
=& \{\tilde{\omega} \in \tilde{\Omega} \,|\, \nexists s \in [0,\infty)\, s.t.\ either\ \ \min_{\lambda \in \mathcal{I}} |w_s(\tilde{\omega}) - p(\lambda)| \leq \delta \\
&\qquad or\ \ \breve{\bar{\zeta}}_s(\tilde{\omega}) \in B_{\delta_n}(0)\,\} \quad \forall\, n \in \mathbb{N}.
\end{aligned}$$

Lastly, define

$$\bar{\mathcal{C}} \doteq \{\tilde{\omega} \in \tilde{\Omega} \,|\, \ \nexists s \in [0,\infty)\, s.t.\ either\ \ w_s(\tilde{\omega}) \in \tilde{\mathcal{G}}_0(\bar{e})\ \ or\ \ \breve{\bar{\zeta}}_s(\tilde{\omega}) = 0\,\}.$$

Note that $\mathcal{C}_k \subset \mathcal{C}_n$ for $k \leq n$. and that $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n = \bar{\mathcal{C}}$, which implies $\bar{\mathcal{C}} = \lim_{n\to\infty} \mathcal{C}_n$.

Lemma 11. *Let $m \geq 3$. $P_n(\bar{\mathcal{C}}) = 1$.*

Proof. By (15),

$$P_n(\{\omega \in \Omega \,|\, \exists s \in [0,\infty)\, s.t.\ \ \breve{\bar{\zeta}}_s = 0\,\}) = 0. \tag{30}$$

We next employ classical potential theory and its relationship to Brownian motion hitting-time problems (cf. [4, 22, 25]). As we were unable to find an already-existing proof of assertion (33) below, a reasonably detailed proof follows. Fix $\bar{y} \in \mathbb{R}^m$, and for $y \neq \bar{y}$, let $v(y;\bar{y}) \doteq \frac{1}{|y-\bar{y}|}$. One may easily verify that for $m \geq 3$,

$$\Delta v(y;\bar{y}) \leq 0 \quad \forall\, y \in \mathbb{R}^m \setminus \{\bar{y}\}, \tag{31}$$

(i.e., $v(\cdot;\bar{y})$ is superharmonic), and that $v(y;\bar{y}) \to +\infty$ as $y \to \bar{y}$.

Fix $z \in \mathbb{R}^m \setminus \{0\}$. Let $V(y) \doteq \int_0^1 v(y;[J(z)]^{-1}p(\lambda))\,d\lambda$ for all $y \in \mathbb{R}^m \setminus \tilde{\mathcal{G}}_0(z)$. Consider $y \in \mathbb{R}^m \setminus \tilde{\mathcal{G}}_0(z)$. By repeated application of the Dominated Convergence Theorem, and (31), one obtains

$$\Delta_y V(y) = \int_0^1 \Delta_y v(y;[J(z)]^{-1}p(\lambda))\,d\lambda \leq 0 \quad \forall\, y \in \mathbb{R}^m \setminus \tilde{\mathcal{G}}_0(z). \tag{32}$$

Let $y_n \to \tilde{\mathcal{G}}_0(z)$ as $n \to \infty$ be such that $V(y_n) \not\to +\infty$. By compactness, there exists $\bar{y} = p(\bar{\lambda}) \in \tilde{\mathcal{G}}_0(z)$ and subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to \bar{y}$.

Without loss of generality, suppose there exists $\delta > 0$ such that $[\bar\lambda, \bar\lambda+\delta] \subset \mathcal{I}$. Letting $\hat y_{n_k} \doteq J(z) y_{n_k}$ for all k,

$$\begin{aligned} V(y_{n_k}) &\geq \int_{\bar\lambda}^{\bar\lambda+\delta} |y_{n_k} - [J(z)]^{-1} p(\lambda)|^{-1}\, d\lambda = |z| \int_{\bar\lambda}^{\bar\lambda+\delta} |\hat y_{n_k} - p(\lambda)|^{-1}\, d\lambda \\ &\geq |z| \int_{\bar\lambda}^{\bar\lambda+\delta} \left(|\hat y_{n_k} - p(\bar\lambda)| + M_p(\lambda - \bar\lambda) \right)^{-1} d\lambda \\ &= \frac{|z|}{M_p} \log(1 + M_p \delta / |\hat y_{n_k} - p(\bar\lambda)|) \to +\infty \ \text{ as } k \to \infty. \end{aligned}$$

By contradiction, we find that $V(y) \to +\infty$ as $y \to \tilde{\mathcal{G}}_0(z)$ Combining this with (32), we see that $\tilde{\mathcal{G}}_0(z)$ is polar [4, Th. 1.V.4]. Hence, by [4, Th. 2.IX.5],

$$\tilde P_n(\{\omega \in \Omega \,|\, \exists s \in [0,\infty) \ s.t. \ w_s \in \tilde{\mathcal{G}}_0(z)\,\}) = 0. \tag{33}$$

By (30), (33) and the mutual absolute continuity of P_n with respect to $\tilde P_n$, one obtains the result. ☐

Note that $\hat{\mathcal{A}}_k = \mathcal{A}_k \subset \mathcal{A}_n = \hat{\mathcal{A}}_n$ for all $k \leq n$, and let $\bar{\mathcal{A}} \doteq \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$.

Lemma 12. *Let $m \geq 3$. $\bar P(\bar{\mathcal{A}}) = \lim_{n\to\infty} \bar P(\mathcal{A}_n) = 1$.*

Proof. Recall from (28), that $\hat{\mathcal{A}}_n \times \Omega' \supseteq \mathcal{C}_n$. One then finds that $\bigcup_{n\in\mathbb{N}}[\hat{\mathcal{A}}_n \times \Omega'] \supseteq \bigcup_{n\in\mathbb{N}} \mathcal{C}_n = \bar{\mathcal{C}}$. By Lemma 11, this yields $P_n(\bigcup_{n\in\mathbb{N}}[\hat{\mathcal{A}}_n \times \Omega']) = 1$, or equivalently, $\breve P_n(\bigcup_{n\in\mathbb{N}} \hat{\mathcal{A}}_n) = 1$. Then, as $\breve P_n$ is mutually absolutely continuous with respect to P_n, $P_n(\bar{\mathcal{A}}) = P_n(\bigcup_{n\in\mathbb{N}} \hat{\mathcal{A}}_n) = 1$. Further, as P_n is mutually absolutely continuous with respect to $\bar P$, we have $\bar P(\bar{\mathcal{A}}) = 1$. ☐

Theorem 13. *Let $m \geq 3$. $(\bar\eta, \bar\zeta)$ is a unique strong solution of (1)–(2).*

Proof. We first address existence. Note that $\tau_n = T$ for all $\omega \in \mathcal{A}_n \subset \mathcal{A}_k$ for all $k \geq n \geq 1$. By (9) and (29), $(\bar\eta, \bar\zeta)$ satisfies (1)–(2) on $[0,T]$ for a.e. $\omega \in \mathcal{A}_n$. Then, by the definition of $\bar{\mathcal{A}}$ and Lemma 12, $(\bar\eta, \bar\zeta)$ satisfies (1)–(2) on $[0,T]$ a.s., where we note that $(\bar\eta, \bar\zeta)$ is $\mathcal{F}_t$-adapted by construction, hence yielding existence.

Fix $(\Omega, \bar{\mathcal{F}}, \bar P)$. Suppose $(\bar\eta, \bar\zeta)$ and $(\bar\eta', \bar\zeta')$ are two strong solutions of (1)–(2) on $[0,T]$. Let $n \in \mathbb{N}$. Then, $(\bar\eta, \bar\zeta)$ and $(\bar\eta', \bar\zeta')$ satisfy (10), (11), (2) on $[0,\tau_n]$ a.s. Then, by Theorem 6, $(\bar\eta_t, \bar\zeta_t) = (\bar\eta'_t, \bar\zeta'_t)$ on $[0,\tau_n]$. As this is true for all $n \in \mathbb{N}$, we have uniqueness on $[0,T]$. ☐

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