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Scaffolds: A graph-theoretic tool for tensor computations related to Bose-Mesner algebras

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A B S T R A C T

We introduce a pictorial notation for certain tensors arising in the study of association schemes, based on earlier ideas of Terwilliger, Neumaier and Jaeger. These tensors, which we call “scaffolds”, obey a simple set of rules which generalize common linear-algebraic operations such as trace, matrix product and entrywise product. We first study an elementary set of “moves” on scaffolds and illustrate their use in combinatorics. Next we re-visit results of Dickie, Suzuki and Terwilliger. The main new results deal with the relationships among vector spaces of scaffolds with edge weights chosen from a fixed coherent algebra and various underlying diagrams. As one consequence, we provide simple descriptions of the Terwilliger algebras of triply regular and dually triply regular association schemes. We finish with a conjecture connecting the duality of Bose-Mesner algebras to the graph-theoretic duality of circular planar graphs.

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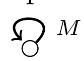
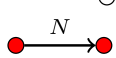
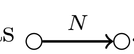
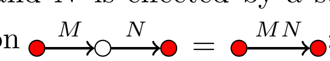
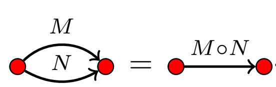
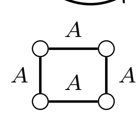
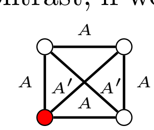
1. Introduction

Without definition, we begin with some simple examples of the diagrams considered in this paper. Once we arrive at a formal definition, the reader may check back to verify the calculations here as exercises. For example we will see how, in a precise way,  denotes the diagonal of matrix M while  is its trace. We will see matrix N, as a second-order tensor, represented as  and the sum of its entries as . The ordinary matrix product of M and N is effected by a series reduction on diagrams as evidenced by the scaffold equation ; likewise, entrywise multiplication is effected by a parallel reduction since we have . If Γ is a graph on vertex set X with adjacency matrix A, then  is an integer equal to the number of homomorphisms from the cycle of length four into graph Γ. By contrast, if we denote the adjacency matrix of the complement of Γ by $A' = J - I - A$, then  is a vector of length $|X|$ and, for each vertex v of Γ, the v entry of this vector is twice the number of induced cycles of length four passing through v.

Originating from unpublished notes of Neumaier (ca. 1989) and tensor calculations of Terwilliger [23], diagrams of this sort seem to have been shared informally in the community for several decades now, mostly to illustrate computations rather than as algebraic objects themselves. Precise algebraic formulations of the concept appear in the work of Dickie [8] and Suzuki [22,21]. An important special case arises in the state models for link invariants as seen, for example, in Jaeger [14]. The primary goal of this paper is to present the diagrammatic formalism as a rigorous alternative to the more cumbersome algebraic expressions that these diagrams represent. In [18] and [19], Penjić and Neumaier recently proposed a slightly different definition. To the author's knowledge, these papers are the first published record of Neumaier's diagrammatic notation.

Fix a nonempty set X and, for $x \in X$, let $\hat{x} \in \mathbb{C}^X$ denote the standard basis vector indexed by x. For a digraph $G = (V(G), E(G))$, an ordered multiset $R = \{r_1, \ldots, r_m\}$ of nodes in G, and a function w mapping the edges of G to matrices with rows and columns indexed by X, we study the tensor

$$\mathsf{S}(G, R; w) = \sum_{\varphi : V(G) \to X} \left(\prod_{\substack{e \in E(G) \\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} \right) \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)} \tag{1.1}$$

and a slight generalization thereof. Note here that $w(e)$ is a matrix and $w(e)_{\varphi(a),\varphi(b)}$ is simply the entry of that matrix which appears in row $\varphi(a)$, column $\varphi(b)$.

1.1. Basic notation

Let X be a nonempty finite set and let $\mathsf{Mat}_X(\mathbb{C})$ denote the vector space of matrices with rows and columns indexed by X and entries from the complex numbers. Take $V = \mathbb{C}^X$ with standard basis of column vectors $\{\hat{x} \mid x \in X\}$, equipped with the corresponding positive definite Hermitian inner product $\langle v, w \rangle = v^\dagger w$ (where $\cdot^\dagger$ denotes conjugate transpose) satisfying $\langle \hat{x}, \hat{y} \rangle = \delta_{x,y}$ for $x, y \in X$; this allows us to identify V with its dual space $V^\dagger$ of linear functionals. The objects of study belong to tensor products of this space of the form

$$V^{\otimes m} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{m}$$

with standard basis consisting of simple tensors of the form $\hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_m$ where $x_1, x_2, \ldots, x_m \in X$. In the case $m = 2$, we identify $V \otimes V$ with $\mathsf{Mat}_X(\mathbb{C})$ via

$$M = [M_{xy}] \;\longleftrightarrow\; \sum_{x,y \in X} M_{xy}\, \hat{x} \otimes \hat{y}.$$

Clearly $\mathsf{Mat}_X(\mathbb{C})$ forms an algebra both under matrix multiplication and under entrywise (Hadamard, or Schur) multiplication, which we denote by $\circ$, and contains the identities, I and J respectively, for these two multiplications. A vector subspace $\mathbb{A}$ of $\mathsf{Mat}_X(\mathbb{C})$ is said to be a *coherent algebra* if $\mathbb{A}$ is closed under the conjugate transpose operation, closed under both ordinary and entrywise multiplication, and contains both I and J.

We will use the term "subring" to mean a vector subspace of $\mathsf{Mat}_X(\mathbb{C})$ which is closed under ordinary matrix multiplication and we will use the term "$\circ$-subring" to mean a vector subspace of $\mathsf{Mat}_X(\mathbb{C})$ which is closed under entrywise multiplication. Our primary example for the vector space $\mathbb{A}$ will be the Bose-Mesner algebra $\mathbb{A} = \mathrm{span}_{\mathbb{C}}\{A_0, \ldots, A_d\}$ of a commutative d-class association scheme or, relaxing the commutativity condition, a coherent algebra. In this case, $\mathbb{A}$ is both a subring and a $\circ$-subring of $\mathsf{Mat}_X(\mathbb{C})$. But some of the tools presented here clearly extend to other settings; for example where $\mathbb{A} = \langle A \rangle$ is the adjacency algebra of a finite simple graph Γ with vertex set X and adjacency matrix A. (See [17].)

1.2. Scaffolds (or "star-triangle diagrams")

Suppose we are given

- A finite (di)graph $G = (V(G), E(G))$ possibly with loops and/or multiple edges, the *diagram* of the scaffold[1];
- An ordered multiset $R = \{r_1, \ldots, r_m\} \subseteq V(G)$ of "root" nodes (or "roots"). In the language of [15, p39], G, together with R, is a "k-multilabeled graph" (where $k = m$ here), but we will call (G, R) a *rooted diagram*;
- A finite set X and a map from edges of G to matrices in $\mathsf{Mat}_X(\mathbb{C})$: $w : E(G) \to \mathsf{Mat}_X(\mathbb{C})$ (*edge weights*);
- a subset $F \subseteq V(G)$ of *fixed nodes* and a fixed function $\varphi_0 : F \to X$.

The *(general) scaffold* $\mathsf{S}(G, R; w; F, \varphi_0)$ is defined as the quantity

$$\mathsf{S}(G, R; w; F, \varphi_0) = \sum_{\substack{\varphi: V(G) \to X \\ (\forall a \in F)(\varphi(a) = \varphi_0(a))}} \left(\prod_{\substack{e \in E(G) \\ e=(a,b)}} w(e)_{\varphi(a), \varphi(b)} \right) \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)}. \tag{1.2}$$

Each function $\varphi : V(G) \to X$ whose restriction to F is φ_0 is called a *state* of the scaffold. Observe that each state itself yields a scaffold with one summand by taking $F = X$ and $\varphi_0 = \varphi$, namely

$$w(\varphi)\ \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)}$$

where the *weight* of φ is defined as $w(\varphi) := \prod_{\substack{e \in E(G) \\ e=(a,b)}} w(e)_{\varphi(a), \varphi(b)}$.

[1] Note that we write $e = (a, b)$ to indicate that edge e has tail a and head b; this is a slight abuse of notation in the presence of parallel edges.

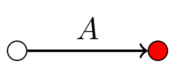
Reversing an arc $e = (a, b)$ in diagram G is equivalent to replacing $w(e)$ by its transpose. In the case where all edge weights are symmetric matrices, we may treat G as an undirected graph.

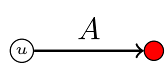
The scaffold $\mathsf{S}(G, R; w; F, \varphi_0)$ is an element of $V^{\otimes m}$, so we say $\mathsf{S}(G, R; w; F, \varphi_0)$ is a scaffold of *order* m. Scaffolds with $m = 0$ are simply complex numbers. In our discussion, X will typically denote the vertex set of some association scheme. To distinguish $V(G)$ from X, we will refer to elements of $V(G)$ as *nodes*. Viewed as elements of $\mathbb{C}^{X^m}$, we may take linear combinations of scaffolds with a common multiset R of roots as needed and, in parallel to terminology used for tensors, it would be natural to refer to these linear combinations as "scaffolds" as well.

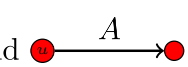
As the examples above and below show, a scaffold on a small number of nodes can often be concisely encoded pictorially as an edge-labeled diagram once a convention is established for the ordering $r_1, \ldots, r_m$ of root nodes. Let us make this precise. The data $((G, R), w)$ is given as a plane drawing (possibly with crossings) of graph G with root nodes highlighted in color and each edge e labeled with the matrix $w(e)$. Throughout, we identify this pictorial representation of the data $((G, R), w)$ with the tensor $\mathsf{S}(G, R; w)$. In the case of general scaffolds, each node $a \in F$ is labeled with the vertex $\varphi_0(a) \in X$ in this pictorial representation of the tensor.

Example 1.1. Suppose $X = \{u, v, w, x\}$ and

$$A = \begin{array}{c} \\ \\ \end{array}\begin{matrix} & u & v & w & x & \\ & \left[\begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right] & & & & \begin{matrix} u \\ v \\ w \\ x \end{matrix} \end{matrix}$$

Then we have 

$= \hat{u} + 2\hat{v} + 2\hat{w} + \hat{x}$, 

$= \hat{v} + \hat{w}$, and 

$= \hat{u} \otimes \hat{v} + \hat{u} \otimes \hat{w}$. Here, $G = (\{1, 2\}, \{(1, 2)\})$, $w(1, 2) = A$, $R = \{2\}$ in the first two examples and $R = V(G) = \{1, 2\}$ in the third. The set F is empty in the first tensor while $F = \{1\}$ with $\varphi_0(1) = u$ in the second and third. $\square$

Fixing the rooted diagram (G, R), we obtain (cf. [14]) a multilinear map from $\mathbb{A}^{E(G)}$ to $\mathbb{C}^{X^m}$, mapping each choice $w : E(G) \to \mathbb{A}$ of edge weights to the scaffold $\mathsf{S}(G, R; w)$ and extended linearly. In Section 3, we study the images of such maps.

In this paper, with few exceptions, the set F of fixed nodes is empty. In this case, the expression takes the simpler form given in (1.1) which we repeat here:

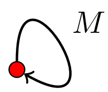
$$\mathsf{S}(G, R; w) = \sum_{\varphi: V(G) \to X} \left(\prod_{\substack{e \in E(G) \\ e = (a,b)}} w(e)_{\varphi(a), \varphi(b)} \right) \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)}.$$

Let us call tensors of this form *symmetric scaffolds* when the distinction is necessary.[2]

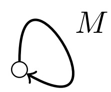
In the case where $R = \emptyset$, scaffolds evaluate to scalars and we recover Jaeger's definition of a *partition function* in [14, p. 107] and the original counting diagrams of Neumaier (cf. [18,19]).

1.3. Rules for scaffold manipulation

Throughout Section 1.3, we assume a nonempty finite set X is given and all edge weights are assumed to be matrices in $\mathsf{Mat}_X(\mathbb{C})$. Scaffolds with a single vertex and a single arc give us the diagonal of a matrix and its trace:



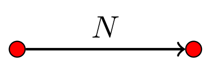
$$= \sum_{x\in X} M_{xx}\ \hat{x}\ ,$$



$$= \sum_{x\in X} M_{xx} = \mathsf{tr}(M)\ .$$

Note that, when edge weights are taken from a Bose-Mesner algebra, where all matrices have constant diagonal, loops can always be removed once this scalar is accounted for (see Rule $\mathsf{SR9}'$ in Appendix A).

Employing the canonical isomorphism between $V = \mathbb{C}^X$ and $V^\dagger$, we identify each matrix $N \in \mathsf{Mat}_X(\mathbb{C})$ with the corresponding second order tensor $\sum_{x,y\in X} N_{xy}\hat{x}\otimes\hat{y}$:



$$= \sum_{x,y\in X} N_{xy}\ \hat{x}\otimes\hat{y}\ =\ N$$

$$= \sum_{x,y\in X} N_{xy}\ \hat{x}\ =\ N\mathbf{1}$$

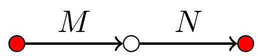
$$= \sum_{x,y\in X} N_{xy} = \mathsf{SUM}(N)\ =\ \mathbf{1}^\top N\mathbf{1}$$

where $\mathbf{1}$ denotes the vector of all ones in $\mathbb{C}^X$. A special case occurs when G is an edgeless graph; if $|V(G)| = n$ and R is a set of m (distinct) elements, then the corresponding

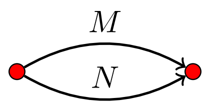
[2] One might consider a more general tensor of this sort as follows. Let $\hat{X}$ be some basis for $\mathbb{C}^X$ — one might choose the standard basis as we have done, choose an eigenbasis, or some other basis. Rather than sum over all functions from $V(G)$ to X, we may instead sum over all functions from $V(G)$ to $\hat{X}$ and define $\mathsf{S}(G,R;w) = \sum_{\varphi:V(G)\to\hat{X}} \left(\prod_{\substack{e\in E(G)\\ e=(a,b)}} \varphi(a)^\dagger w(e)\varphi(b)\right) \varphi(r_1)\otimes\cdots\otimes\varphi(r_m)$.

scaffold is $|X|^{n-m} \sum_{x_1,\ldots,x_m \in X} \hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_m$. For example, when $n = m = 2$, we obtain the all ones matrix J.

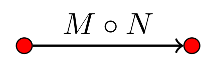
The fundamental scaffold identities below show that matrix product and Schur/entrywise product correspond, respectively, to series and parallel reductions on diagrams:



$$= \sum_{x,y,z \in X} M_{xy} N_{yz}\ \hat{x} \otimes \hat{z} = \sum_{x,z \in X} (MN)_{xz}\ \hat{x} \otimes \hat{z} =$$



$$= \sum_{x,y \in X} M_{xy} N_{xy}\ \hat{x} \otimes \hat{y} = \sum_{x,y \in X} (M \circ N)_{xy}\ \hat{x} \otimes \hat{y} =$$



.

Note that the first identity only makes sense when the middle (hollow) node in the top left digraph is incident only to the two edges shown but, as with all scaffold identities, the root nodes may be incident to edges of the diagram other than those shown. We present these identities, denoted in Appendix A as SR1 and SR1′, in the following lemma.

Lemma 1.2. *Let $G = (V(G), E(G))$ with $e_1, e_2 \in E(G)$ where $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$. Let*

$$E' = (E(G) \setminus \{e_1, e_2\}) \cup \{e' = (a_1, b_2)\} \ .$$

(i) If $b_1 \notin R$ and e_1 and e_2 are in series, i.e., $b_1 = a_2$ and no other edge is incident to b_1, then

$$\mathsf{S}(G, R; w) = \mathsf{S}(G', R; w')$$

where $G' = (V(G) \setminus \{b_1\}, E')$, $w'(e') = w(e_1)w(e_2)$ and $w'(e) = w(e)$ otherwise.

(ii) If e_1 and e_2 are in parallel, i.e., $a_1 = a_2$ and $b_1 = b_2$, then

$$\mathsf{S}(G, R; w) = \mathsf{S}(G', R; w')$$

where $G' = (V(G), E')$, $w'(e') = w(e_1) \circ w(e_2)$ and $w'(e) = w(e)$ otherwise. □

Another basic lemma:

Lemma 1.3. *Let $G = (V(G), E(G))$ with $e' = (a, b) \in E(G)$.*

(i) Let G' be obtained from G by deleting edge e'. If $w(e') = J$, then $\mathsf{S}(G, R; w) = \mathsf{S}(G', R; w')$ where $w'(e) = w(e)$ for all $e \in E(G) \setminus \{e'\}$.

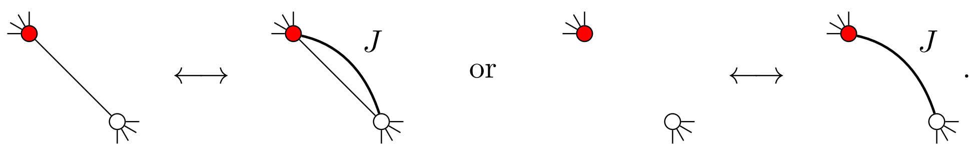
(ii) Let G' be obtained from G by contracting edge e': for $f(a) = f(b) = a'$ and $f(c) = c$ otherwise,

$$V(G') = (V(G) \setminus \{a,b\}) \cup \{a'\}, \quad E(G') = \{(f(u), f(v)) \mid (u,v) \in E(G) \setminus \{e'\}\}.$$

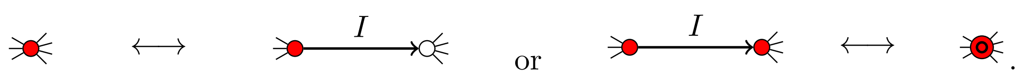
If $w(e') = I$, *then* $\mathsf{S}(G,R;w) = \mathsf{S}(G',R';w')$ *where* $w'(f(u),f(v)) = w(u,v)$ *for all* $e = (u,v) \in E(G) \setminus \{e'\}$, $R' = \{f(r_1), \ldots, f(r_m)\}$. □

This second lemma provides us with two more basic moves, $\mathsf{SR0}$ and $\mathsf{SR0}'$ in our notation, that preserve scaffolds:

- Insert an edge between two existing nodes, mapping the new edge to J



- Split a node in two, introducing a hollow node, mapping the new edge to I



  Note that the edges incident to the original node may be distributed among the two nodes on the right in any fashion. In the reverse direction, contracting an edge e with $w(e) = I$ may result in a multiset of root nodes as in the example on the right.

In the case where edge weights belong to a Bose-Mesner algebra, these steps are useful in conjunction with the linear expansions $I = \sum_j E_j$ and $J = \sum_i A_i$ given in Equation (2.2), respectively, which allow us to expand one scaffold as a linear combination of closely related scaffolds.

If our rooted diagram contains a hollow node of degree one and the incident edge weight has constant row and column sum, we may simplify the scaffold by deleting this node and scaling the resulting tensor by that constant (Rule $\mathsf{SR9}$).

Lemma 1.4. *Let* $G = (V(G), E(G))$ *be a digraph,* $w : E(G) \to \mathsf{Mat}_X(\mathbb{C})$, R *an ordered multiset of nodes from* $V(G)$, $a_0 \in V(G)$ *not in* R *and incident to just one edge in* $E(G)$ *(say* $e = (a_0, b_0)$ *or* $e = (b_0, a_0)$*). Let* $M = w(e)$. *Denote by* G' *the graph obtained from* G *by deletion of* a_0 *and* e*; denote by* w' *the restriction of* w *to edges other than* e.

(i) *if* $e = (b_0, a_0)$ *and* M *has constant row sum* α, *then* $\mathsf{S}(G,R;w) = \alpha\mathsf{S}(G',R;w')$;
(ii) *if* $e = (a_0, b_0)$ *and* M *has constant column sum* α, *then* $\mathsf{S}(G,R;w) = \alpha\mathsf{S}(G',R;w')$. □

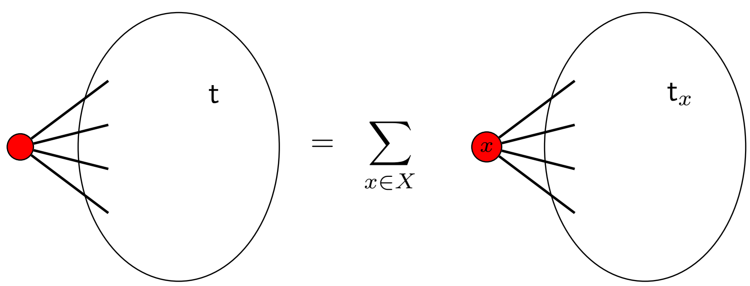
General scaffolds become useful when one's investigation differentiates individual vertices of an object or when one wishes to expand an expression as a sum over vertices. If

$$\mathsf{t} = \mathsf{S}(G, R; w)$$

is a symmetric scaffold with edge weights in $\mathsf{Mat}_X(\mathbb{C})$ and $a \in V(G)$, define, for $x \in X$,

$$\mathsf{t}_x = \mathsf{S}(G, R; w; \{a\}, \varphi_x)$$

where $\varphi_x(a) = x$. Then we have $\mathsf{t} = \sum_{x \in X} \mathsf{t}_x$: for $a \in R$, this is depicted as



Using the notion of a general scaffold, we obtain a straightforward proof that our manipulations of subdiagrams are valid operations on overall diagrams. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be digraphs with disjoint node sets and a bijection ξ pairing ℓ distinct nodes u_i of G to ℓ nodes r_i of H. Writing $\xi(u_i) = r_i$, define $G +_\xi H$ to be the digraph with vertex set

$$V(G +_\xi H) = (V(G) \setminus \{u_1, \ldots u_\ell\}) \cup V(H)$$

and edges

$$E(H) \cup \{(\xi(a), \xi(b)) \mid (a, b) \in E(G)\}$$

where we extend ξ to $V(G)$ defining $\xi(c) = c$ for $c \notin \{u_1, \ldots, u_\ell\}$. If (G, Q) and (H, R) are rooted diagrams with $R = \{r_1, \ldots, r_m\}$, the corresponding ordered multiset of roots for diagram $G +_\xi H$ is the union of the multiset of the roots of G not among $\{u_1, \ldots, u_\ell\}$ followed by $\{r_1, \ldots, r_m\}$. Extending this notation to symmetric scaffolds, if

$$\mathsf{s} = \mathsf{S}(G, Q; w) \qquad \text{and} \qquad \mathsf{t} = \mathsf{S}(H, R; w_1)$$

and ξ is given as above such that $\xi(c) = c$ whenever $c \notin Q$, define

$$\mathsf{s} +_\xi \mathsf{t} = \mathsf{S}\left(G +_\xi H, \xi(Q) \cup \bar{R}; w^\xi \cup w_1\right)$$

where $\bar{R} = \{r_{\ell+1}, \ldots, r_m\}$ and $w^\xi(\xi(a), \xi(b)) = w(a, b)$ for $(a, b) \in E(G)$. Here, as in the next proposition, we assume that $\xi(u_1), \ldots, \xi(u_\ell)$ are the first ℓ elements of R for notational convenience only.

Proposition 1.5. *Let (H_1, R) and (H_2, R) be rooted diagrams with a common ordered multiset of roots. Let $w_j : E(H_j) \to \mathsf{Mat}_X(\mathbb{C})$ $(j \in \{1, 2\})$ such that the two symmetric scaffolds*

$$\mathsf{t}_1 = \mathsf{S}(H_1, R; w_1) \quad \textit{and} \quad \mathsf{t}_2 = \mathsf{S}(H_2, R; w_2)$$

are equal as tensors. If $R' = \{r_1, \dots, r_\ell\} \subseteq R$ is an ordered subset of (distinct) roots, then for any scaffold $\mathsf{s} = \mathsf{S}(G, Q; w)$ where $V(G)$ is disjoint from both $V(H_1)$ and $V(H_2)$, $w : E(G) \to \mathsf{Mat}_X(\mathbb{C})$ and any ordered set $u_1, \dots, u_\ell$ of ℓ distinct roots from Q, we have $\mathsf{s} +_\xi \mathsf{t}_1 = \mathsf{s} +_\xi \mathsf{t}_2$ where $\xi(u_i) = r_i$.

Proof. Write $R = \{r_1, \dots, r_m\}$, $m \geq \ell$, and order roots so that $\xi(Q) \cup \bar{R} = \{q_1, \dots, q_n, r_1, \dots, r_m\}$, $n \geq 0$. Since $\mathsf{t}_1 = \mathsf{t}_2$, we have, for each $\varphi_0 : R \to X$ and any choice of $y_1, \dots, y_n \in X$

$$\sum_{\substack{\varphi: V(H_1) \to X \\ (\forall a \in R)(\varphi(a) = \varphi_0(a))}} \left(\prod_{\substack{e \in E(H_1) \\ e = (a,b)}} w_1(e)_{\varphi(a), \varphi(b)} \right) \hat{y}_1 \otimes \cdots \otimes \hat{y}_n \otimes \widehat{\varphi(r_1)} \otimes \cdots \otimes \widehat{\varphi(r_m)}$$

$$= \sum_{\substack{\varphi: V(H_2) \to X \\ (\forall a \in R)(\varphi(a) = \varphi_0(a))}} \left(\prod_{\substack{e \in E(H_2) \\ e = (a,b)}} w_2(e)_{\varphi(a), \varphi(b)} \right) \hat{y}_1 \otimes \cdots \otimes \hat{y}_n \otimes \widehat{\varphi(r_1)} \otimes \cdots \otimes \widehat{\varphi(r_m)}$$

Taking linear combinations over all $\psi : V(G) \setminus \{u_1, \dots u_\ell\} \to X$, we have

$$\sum_{\substack{\varphi: V(G +_\xi H_1) \to X \\ (\forall a \in R)(\varphi(a) = \varphi_0(a))}} \left(\prod_{\substack{e \in E(G) \\ e = (a,b)}} w^\xi(e)_{\varphi(\xi(a)), \varphi(\xi(b))} \right) \left(\prod_{\substack{e \in E(H_1) \\ e = (a,b)}} w_1(e)_{\varphi(a), \varphi(b)} \right) \widehat{\varphi(q_1)} \otimes \cdots$$
$$\otimes \widehat{\varphi(q_n)} \otimes \widehat{\varphi(r_1)} \otimes \cdots \otimes \widehat{\varphi(r_m)}$$
$$= \sum_{\substack{\varphi: V(G +_\xi H_2) \to X \\ (\forall a \in R)(\varphi(a) = \varphi_0(a))}} \left(\prod_{\substack{e \in E(G) \\ e = (a,b)}} w^\xi(e)_{\varphi(\xi(a)), \varphi(\xi(b))} \right) \left(\prod_{\substack{e \in E(H_2) \\ e = (a,b)}} w_2(e)_{\varphi(a), \varphi(b)} \right) \widehat{\varphi(q_1)} \otimes \cdots$$
$$\otimes \widehat{\varphi(q_n)} \otimes \widehat{\varphi(r_1)} \otimes \cdots \otimes \widehat{\varphi(r_m)}.$$

Finally, summing over all $\varphi_0 : R \to X$, we obtain our result. □

This tool is denoted SR5 in Appendix A. Note that, while each node of attachment must be a root node in both s and t_i, the remaining nodes of (H_i, R) and (G, Q) may be any combination of root and non-root nodes. This proposition, combined with the following lemma, allows us to make the local moves on scaffolds that are used throughout this paper.

An *order reduction* operation $\mathsf{hollow}_{R'}$ replaces

$$\mathsf{S} = \mathsf{S}(G,R;w;F,\varphi_0) = \sum_{\substack{\varphi:V(G)\to X \\ (\forall a\in F)(\varphi(a)=\varphi_0(a))}} \prod_{\substack{e\in E(G) \\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} \widehat{\varphi(r_1)} \otimes \cdots \otimes \widehat{\varphi(r_m)}$$

by

$$\begin{aligned} \mathsf{hollow}_{R'}(\mathsf{S}) &= \mathsf{S}(G,R';w;F,\varphi_0) \\ &= \sum_{\substack{\varphi:V(G)\to X \\ (\forall a\in F)(\varphi(a)=\varphi_0(a))}} \prod_{\substack{e\in E(G) \\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} \widehat{\varphi(r_{i_1})} \otimes \cdots \otimes \widehat{\varphi(r_{i_\ell})} \end{aligned}$$

where $R' = \{r_{i_1}, \ldots, r_{i_\ell}\} \subseteq R = \{r_1, \ldots, r_m\}$. This maps m^{th} order tensors to tensors of order ℓ. In terms of diagrams, the solid nodes outside R' are converted to hollow nodes. The fact that this operation preserves scaffold identities is denoted in the appendix below as Rule $\mathsf{SR10}$.

Lemma 1.6. *If* $\mathsf{S}(G_1,R;w_1;F_1,\varphi_1) = \mathsf{S}(G_2,R;w_2;F_2,\varphi_2)$ *(defined on the same ordered multiset R of roots), then, for any ordered submultiset $R' \subseteq R$,* $\mathsf{S}(G_1,R';w_1;F_1,\varphi_1) = \mathsf{S}(G_2,R';w_2;F_2,\varphi_2)$*. In particular, if* $\mathsf{S}(G,R;w;F,\varphi_0) = \mathbf{0}$ *and* $R' \subseteq R$*, then* $\mathsf{S}(G,R';w;F,\varphi_0) = \mathbf{0}$ *also.* □

1.4. Partition functions and spin models

Before we delve into the main line of investigation, namely computations in Bose-Mesner algebras, we illustrate the broader utility of Neumaier's diagrams by mentioning just one connection to spin models and link invariants. This application motivated Jaeger's formulation in [14].

If X is a finite set of colors (or "spins"), we may employ a partition function to assign a complex number to each link diagram. The concept of a *spin model* plays a key role here in determining which partition functions are link invariants.

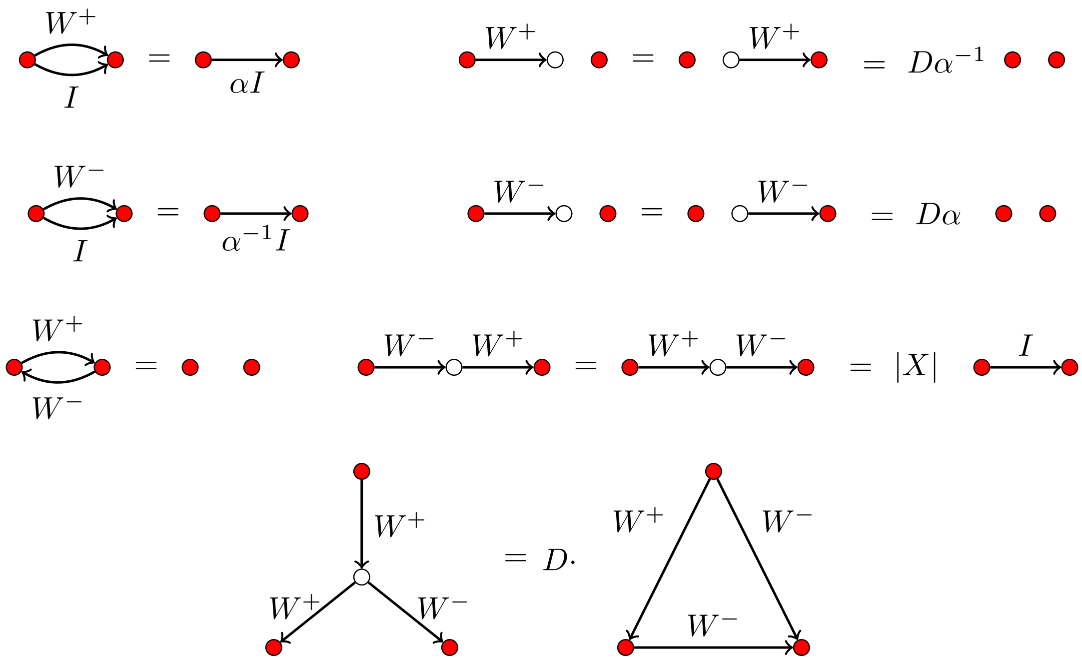
A spin model [14, Prop. 1] is a triple (X, W^+, W^-) where X is a nonempty finite set and matrices $W^+, W^- \in \mathsf{Mat}_X(\mathbb{C})$ satisfy, for $D^2 = |X|$ and some scalar α,

- ("Type I relation" for W^+) W^+ has constant diagonal α, constant row sum $D\alpha^{-1}$, and constant column sum $D\alpha^{-1}$;
- ("Type I relation" for W^-) W^- has constant diagonal α^{-1}, constant row sum $D\alpha$, and constant column sum $D\alpha$;
- ("Type II relation") $W^+ \circ (W^-)^\top = J$ while $W^-W^+ = W^+W^- = |X|I$

- ("Type III" or "Star-Triangle Relation") for every $a, b, c \in X$,

$$\sum_{x \in X} (W^+)_{a,x}(W^+)_{x,b}(W^-)_{x,c} = D\ (W^+)_{a,b}(W^-)_{a,c}(W^-)_{b,c}$$

In scaffold formalism, these conditions are written



2. Association schemes

A *(commutative) association scheme* [7,1,2,12,16] consists of a finite set X together with a collection $\mathcal{R} = \{R_0, \dots, R_d\}$ of nonempty binary relations (the *basis relations*) on X satisfying the following conditions:

(i) some relation in $\mathcal{R}$ is the identity relation on X; we denote this relation by R_0;
(ii) $R_i \cap R_j = \emptyset$ whenever $i \neq j$;
(iii) $R_0 \cup \cdots \cup R_d = X \times X$;
(iv) for each i, the relation $R_i^\top = \{(b, a) : (a, b) \in R_i\}$ also belongs to $\mathcal{R}$;
(v) there are *intersection numbers* p_{ij}^k $(0 \le i, j, k \le d)$ such that, whenever $(a, b) \in R_k$, we have exactly p_{ij}^k elements $c \in X$ for which both $(a, c) \in R_i$ and $(c, b) \in R_j$;
(vi) for each i, j and k, $p_{ij}^k = p_{ji}^k$.

An association scheme is *symmetric* if $R_i^\top = R_i$ for all $R_i \in \mathcal{R}$.

Let $(X, \mathcal{R})$ be an association scheme with basis relations $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ having adjacency matrices $A_0, A_1, \dots, A_d$ respectively where $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and $(A_i)_{x,y} = 0$ otherwise. It is well known [2, Thm. 2.6.1] that the vector space $\mathbb{A}$ spanned by these $d+1$ matrices is a Bose-Mesner algebra[3] and that $\{A_0, A_1, \ldots, A_d\}$ forms a basis of pairwise orthogonal idempotents with respect to the entrywise product: $A_i \circ A_j = \delta_{i,j} A_i$. By convention, we have $A_0 = I$ and the unique basis of orthogonal idempotents with respect to ordinary matrix multiplication is denoted by $\{E_0, E_1, \ldots, E_d\}$ with $E_0 = \frac{1}{|X|} J$. It follows that there exist structure constants, called *Krein parameters*, q_{ij}^k for which

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$$

for $0 \leq i, j \leq d$. Note that, for each $j \in \{0, \ldots, d\}$, there is some $j' \in \{0, \ldots, d\}$ with $E_{j'} = E_j^\top = \bar{E}_j$. The two bases $\{A_0, \ldots, A_d\}$ and $\{E_0, \ldots, E_d\}$ for algebra $\mathbb{A}$ are related by the *first and second eigenmatrices* P and Q defined by

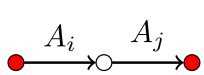
$$A_i = \sum_{j=0}^{d} P_{ji} E_j \ , \qquad E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{ij} A_i \ . \tag{2.1}$$

These satisfy the *orthogonality relations* [2, Sec. 2.2] $PQ = |X| I$ and $m_j P_{ji} = v_i \bar{Q}_{ij}$ where $m_j = q_{jj}^0$ is the rank of E_j and $v_i = p_{ii}^0$ is the valency of the graph (X, R_i) corresponding to the i^{th} basis relation. Though trivial, the specializations of (2.1) to A_0 and E_0 are used frequently:

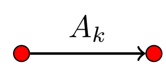
$$\sum_{j=0}^{d} E_j = I \ , \qquad \sum_{i=0}^{d} A_i = J \ . \tag{2.2}$$

Let us first consider scaffolds $\mathsf{S}(G, R; w)$ where $w(e) \in \{A_0, \ldots, A_d\}$ for every edge e in G. Note that we are abusing our conventions in that the set X is not specified: when an identity is given, we mean that it holds true (under the given hypotheses) for any association scheme $(X, \mathcal{R})$ whose Bose-Mesner algebra $\mathbb{A} \subseteq \mathsf{Mat}_X(\mathbb{C})$ contains all the edge weights under standard naming conventions for $A_0, \ldots, A_d, E_0, \ldots, E_d$.

Most basic identities for the parameters of the Bose-Mesner algebra can be stated as scaffold equations. For instance, one may define the intersection numbers p_{ij}^k and Krein parameters q_{ij}^k by the equations



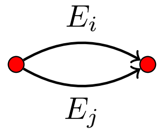
$$= \sum_{k=0}^{d} p_{ij}^k \sum_{(a,b) \in R_k} \hat{a} \otimes \hat{b} \quad = \sum_{k=0}^{d} p_{ij}^k$$

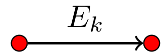


(2.3)

and

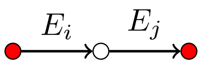
[3] I.e., the vector space is closed under transpose, closed under conjugation, closed and commutative under both ordinary and entrywise multiplication, and contains the identities, I and J, respectively, for these two operations.

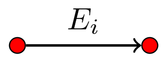


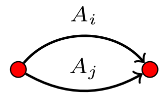
$$= \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^{k}$$


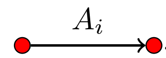
(2.4)

On the other hand, the orthogonality properties of the two fundamental bases $\{A_i\}_{i=0}^{d}$ and $\{E_j\}_{j=0}^{d}$ give us



$= \delta_{i,j}$


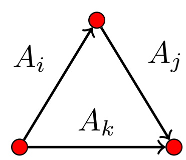
and


$= \delta_{i,j}$


. (2.5)

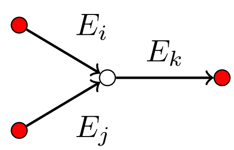
The first non-trivial relations we encounter are given by the following lemma, recorded in our appendix as rules SR2 and SR2′, respectively. The first of these well-known statements is obvious and one may derive short proofs for the second [3].

Lemma 2.1. *For any association scheme $(X, \mathcal{R})$ with intersection numbers p_{ij}^k and Krein parameters q_{ij}^k, we have $p_{ij}^k = 0$ if and only if*

$\mathbf{0} =$


(2.6)

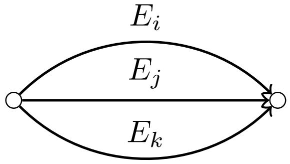
and $q_{ij}^k = 0$ if and only if

$\mathbf{0} =$


(2.7)

We also make frequent use of the following lemma.

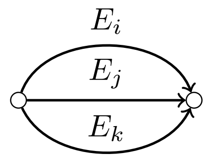
Lemma 2.2. *Let $(X, \mathcal{R})$ be an association scheme with Bose-Mesner algebra having primitive idempotents $\{E_0, \ldots, E_d\}$ where, for $0 \le h \le d$, h' is the index for which $E_{h'} = E_h^\top$. Then*

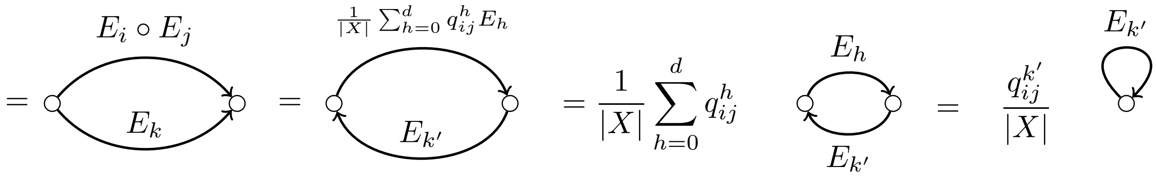


$$= \frac{q_{ij}^{k'}}{|X|} m_k = \frac{q_{ik}^{j'}}{|X|} m_j = \frac{q_{jk}^{i'}}{|X|} m_i$$

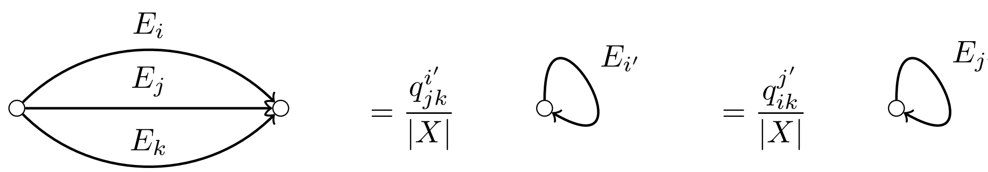
In particular, the zeroth order scaffold on the left is zero if and only if $q_{ij}^{k'} = 0$.

Proof. We compute, applying Lemma 1.2*(ii)*, (2.4) and (2.5),





and note that $\mathsf{tr}E_{k'} = \mathsf{tr}E_k = m_k$. By the same token,



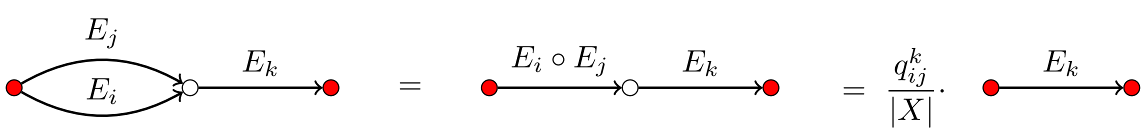
the three possible evaluations all seen to be equal using the basic identity $q_{ij}^{k'} m_k = q_{ik}^{j'} m_j$. $\square$

2.1. The pinched star and the hollowed delta

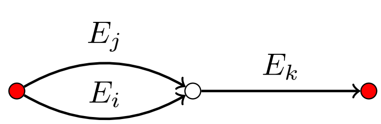
Since $E_i \circ E_j = \frac{1}{|X|}\sum_{h=0}^{d} q_{ij}^h E_h$ we have

$$(E_i \circ E_j)\, E_k = \frac{q_{ij}^k}{|X|} E_k. \tag{2.8}$$

This provides us with our Rule $\mathsf{SR3}$:



and

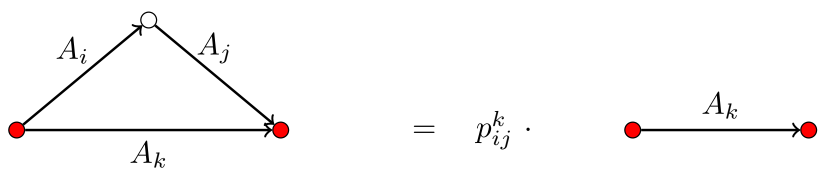


$= \mathbf{0}$ when $q_{ij}^k = 0$.

We refer to the diagram on the left as the *pinched star*. Dual to this is the following identity for intersection numbers: since

$$(A_i A_j) \circ A_k = \left(\sum_{h=0}^{d} p_{ij}^h A_h \right) \circ A_k = p_{ij}^k A_k, \tag{2.9}$$

we have the following identity, denoted SR3′ in the appendix:

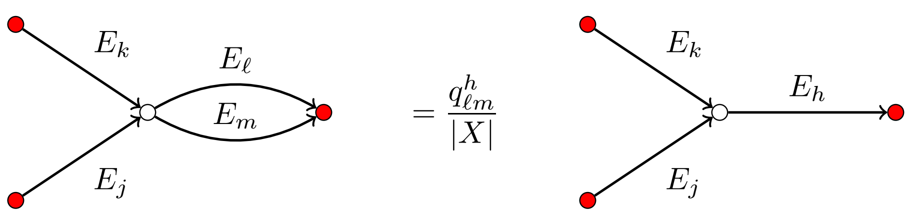


2.2. Isthmuses

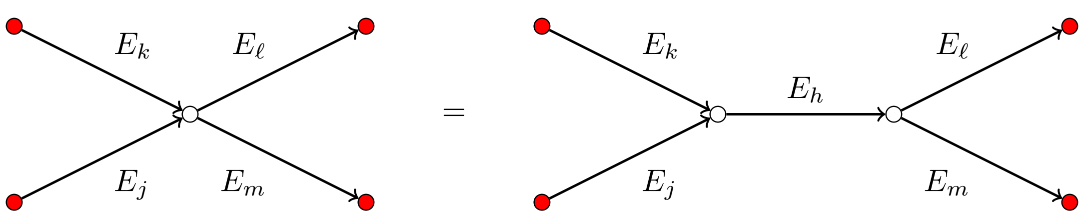
The fundamental identity of Cameron, Goethals and Seidel [3] given in Equation (2.7) extends, using Rule SR0, to give us information about scaffolds of higher order. Suzuki [21] proved the symmetric version of the following "Isthmus Lemma", based on ideas of Dickie (Cf. [8, Lemma 4.2.2]). We extend Suzuki's result to the case of (commutative) association schemes using scaffold notation and we denote this as Rule SR4.

Lemma 2.3 *(Lemma 4, [21]). Let $(X, \mathcal{R})$ be an association scheme.*

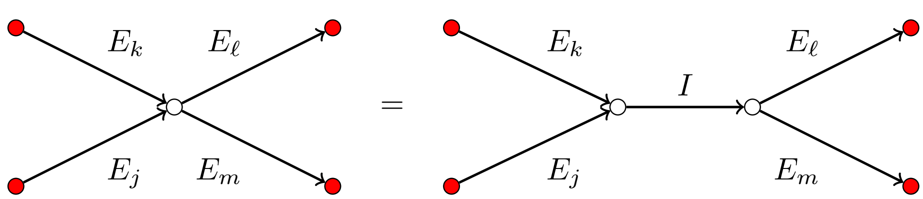
(I) *If $q_{jk}^e \cdot q_{\ell m}^e = 0$ for all $e \neq h$, then*



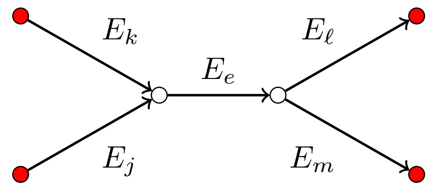
(II) *If $q_{jk}^e \cdot q_{\ell m}^e = 0$ for all $e \neq h$, then*



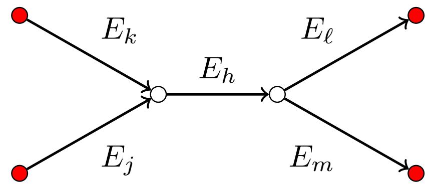
Proof. We provide a proof of (II) only:



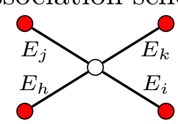
(SR0)

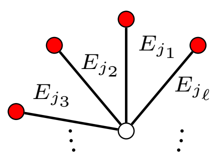
$$= \sum_{e=0}^{d}$$ 

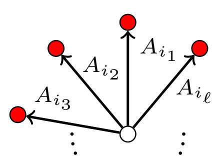
by (2.2)

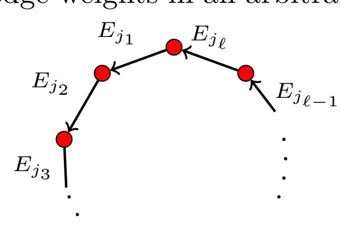
$$=$$ 

where the last equation follows from (2.7) and Proposition 1.5 using the hypothesis. □

For example, in any cometric association scheme (see Section 2.3) with Q-polynomial ordering $E_0, E_1, E_2, \ldots$, we have 

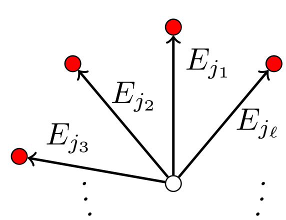
$= \mathbf{0}$ whenever $|k - j| > h + i$ and 

$= \mathbf{0}$ for any Q-bipartite association scheme in which $j_1 + \cdots + j_\ell$ is odd. On the other hand, it is easy to see that, with edge weights in an arbitrary Bose-Mesner algebra, scaffolds of the form 



can never be zero.

Proposition 2.4. *Let* $(X, \mathcal{R})$ *be an association scheme having basis of primitive idempotents* $E_0, E_1, \ldots, E_d$ *and second eigenmatrix* Q. *If* $j_1, \ldots, j_\ell \in \{0, 1, \ldots, d\}$ *satisfy* $Q_{ij_1} \cdots Q_{ij_\ell} \geq 0$ *for all* $0 \leq i \leq d$*, then*



$$\neq \mathbf{0} \quad .$$

Proof. Let $y \in X$. We simply show that the coefficient of $\hat{y} \otimes \hat{y} \otimes \cdots \otimes \hat{y}$ is strictly positive. This coefficient is

$$\sum_{x \in X} (E_{j_1})_{x,y} (E_{j_2})_{x,y} \cdots (E_{j_\ell})_{x,y}$$

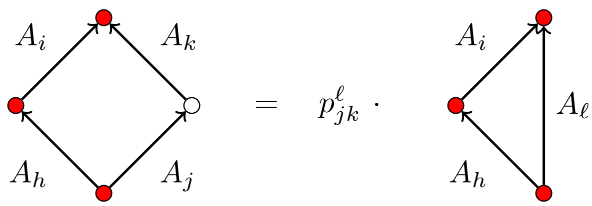
$$= \frac{1}{|X|^\ell} \left(Q_{0,j_1} \cdots Q_{0,j_\ell} + \sum_{i=1}^{d} v_i Q_{ij_1} \cdots Q_{ij_\ell} \right) > 0. \quad \square$$

The author does not know of a nonzero scaffold of this form in which the coefficients of $\hat{y} \otimes \hat{y} \otimes \cdots \otimes \hat{y}$ are all zero.

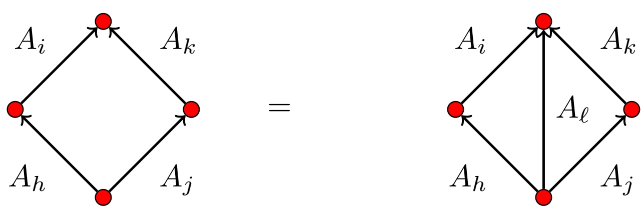
The dual to Lemma 2.3, denoted SR4′, is given without proof.

Lemma 2.5. *Assume $(X, \mathcal{R})$ is an association scheme.*

(I) *If $p^e_{hi} \cdot p^e_{jk} = 0$ for all $e \neq \ell$, then*



(II) *If $p^e_{hi} \cdot p^e_{jk} = 0$ for all $e \neq \ell$, then*



$\square$

2.3. Theorems of Dickie and Suzuki

Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$. An ordering $E_0, E_1, \ldots, E_d$ of its basis of primitive idempotents is a *Q-polynomial ordering* if the following conditions are satisfied:

- $q^k_{ij} = 0$ whenever one of the indices i, j, k exceeds the sum of the remaining two, and
- $q^k_{ij} > 0$ when $i, j, k \in \{0, 1, \ldots, d\}$ and one of the indices equals the sum of the remaining two.

We say that $(X, \mathcal{R})$ is a *cometric* (or *Q-polynomial*) association scheme when such an ordering of its primitive idempotents exists. Now suppose that $E_0, E_1, \ldots, E_d$ is a Q-polynomial ordering and recall the standard abbreviations for cometric scheme parameters

$$a^*_j = q^j_{1,j} \qquad b^*_j = q^j_{1,j+1} \qquad c^*_j = q^j_{1,j-1} \ .$$

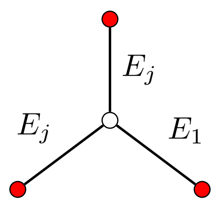
Note that $b_j^* > 0$ for $0 \le j < d$ and $c_j^* > 0$ for $1 \le j \le d$. A cometric association scheme is *Q-bipartite*, with respect to a given Q-polynomial ordering of its primitive idempotents, if its Krein parameters satisfy $q_{ij}^k = 0$ whenever $i + j + k$ is odd.

To further illustrate the algebraic manipulation of pictorial representations of scaffolds, we now give a proof of a theorem from the 1995 dissertation of Dickie [8].

Theorem 2.6 *(Dickie, Thm. 4.1.1). Suppose* $(X, \mathcal{R})$ *is a cometric association scheme with* Q*-polynomial ordering* $E_0, E_1, \ldots, E_d$*. If* $0 < j < d$ *and* $a_j^* = 0$*, then* $a_1^* = 0$*.*

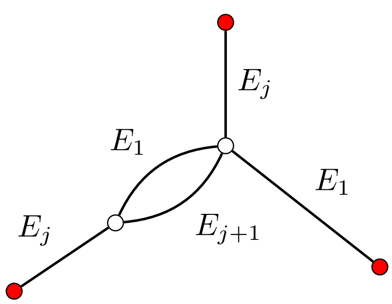
Proof. This proof consists almost entirely of a sequence of scaffolds all equal to the zero tensor $\mathbf{0} \in V^{\otimes 3}$ or the zero scalar. We begin with our assumption that $a_j^* = 0$:

$\mathbf{0} =$



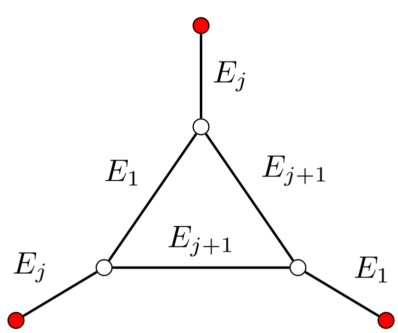
by (2.7)

$\mathbf{0} = \frac{|X|}{b_j^*} \cdot$



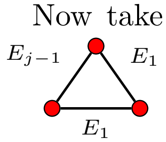
We may apply SR3 since $(E_1 \circ E_{j+1})E_j = \frac{q_{1,j+1}^j}{|X|} E_j$ with $q_{1,j+1}^j = b_j^* > 0$

$\mathbf{0} = \frac{|X|}{b_j^*} \cdot$

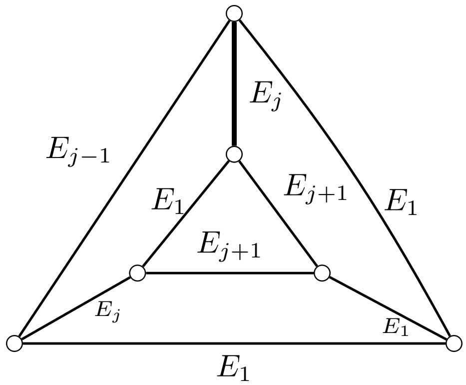


using Lemma 2.3(II) since $q_{j,1}^e \cdot q_{1,j+1}^e = 0$ for any $e \ne j + 1$

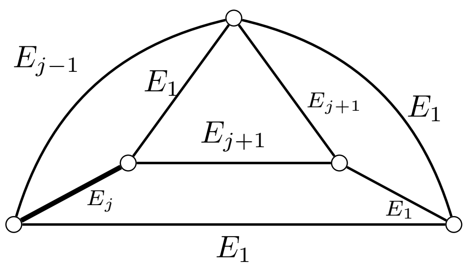
Now take the scalar product of this vanishing scaffold with the third-order tensor



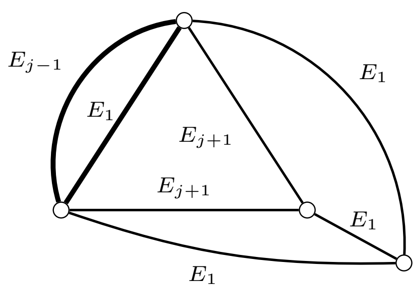
to obtain

0 =  (◇)

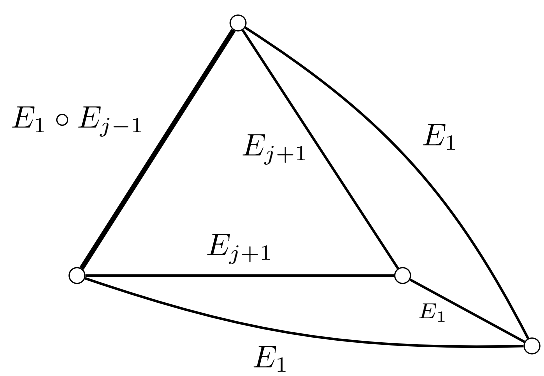
(Note that here and below we use a bold edge to indicate the location in the diagram where the next simplification will be applied.) Applying the Isthmus Lemma,

0 =  since $q^e_{1,j-1} \cdot q^e_{1,j+1} = 0$ for any $e \neq j$.

Likewise,

0 =  since $q^e_{1,j-1} \cdot q^e_{1,j+1} = 0$ for any $e \neq j$.

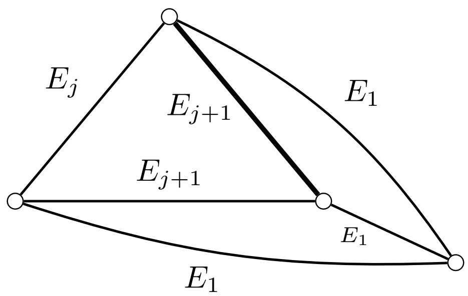
Using the entrywise product,

0 = 

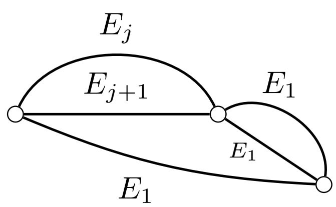
Now we expand

$$E_1 \circ E_{j-1} = \frac{b^*_{j-2}}{|X|} E_{j-2} + \frac{a^*_{j-1}}{|X|} E_{j-1} + \frac{c^*_j}{|X|} E_j$$

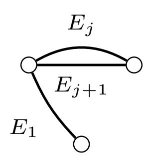
and observe $q^e_{1,j+1} = 0$ for $e < j$. Since $c^*_j \neq 0$, we have

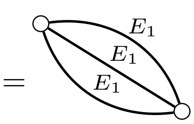
$0 =$ 

and so, again applying Lemma 2.3(II),

$0 =$ 

since $q^e_{1,j} \cdot q^e_{1,j+1} = 0$ for any $e \neq j+1$.

Now we have a pinched star 

and we know $q^1_{j,j+1} \neq 0$ by the cometric property since $j < d$. So, using SR3, we have $0 =$ 

and Lemma 2.2 tells us that $q^1_{11} = 0$, or $a^*_1 = 0$. □

We find the same sort of proof structure in Proposition 3 of Suzuki's paper [21], which specializes to Dickie's Theorem in the case $h = 0$ and $j = 1$.

Theorem 2.7 *(Suzuki [21]). In a cometric association scheme $(X, \mathcal{R})$ with Q-polynomial ordering $E_0, \ldots, E_d$ of its primitive idempotents, if indices $j \leq i \leq i + j \leq h + i + j \leq d$ satisfy $q^e_{j,h+i} \cdot q^e_{i-j,h+j} = 0$ for all $e \neq h + i - j$ and $q^{h+i}_{i,h+j} = 0$, then $q^{h+j}_{j,h+j} = 0$.* □

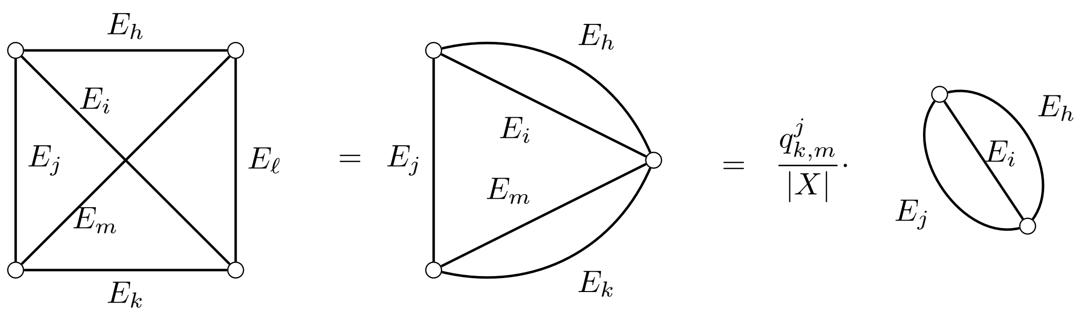
In [22], Suzuki uses these and other ideas to prove that a cometric association scheme which is not a polygon admits at most two Q-polynomial orderings. In order to do this, he narrows down the possibilities for a second Q-polynomial ordering given that $\{E_0, E_1, \ldots, E_d\}$ is such an ordering. Here is a lemma from that paper.

Lemma 2.8 *(Suzuki [22]). Let $(X, \mathcal{R})$ be a symmetric association scheme. Let h, i, j, k, ℓ, m be indices satisfying $q^h_{i,j} \neq 0$ and*

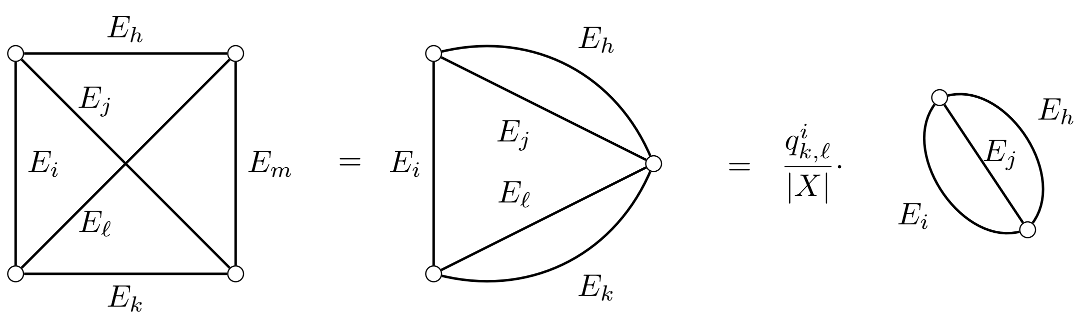
$$(\forall e \neq \ell)\left(q^e_{h,m} \cdot q^e_{i,k} = 0\right) \qquad \textit{and} \qquad (\forall e \neq m)\left(q^e_{h,\ell} \cdot q^e_{j,k} = 0\right).$$

Then $q^i_{k,\ell} = q^j_{k,m}$.

Proof. We manipulate, in two ways, a scaffold built on K_4: applying Lemma 2.3(II),



and, at the same time,



Since the scalars on the left of the two equations are equal, we have

$$\frac{1}{|X|} q^j_{k,m} q^h_{i,j} m_h \;=\; \frac{1}{|X|} q^i_{k,\ell} q^h_{i,j} m_h \;.$$

As $q^h_{i,j} \neq 0$, we obtain $q^i_{k,\ell} = q^j_{k,m}$. $\square$

A similar proof, using Lemma 2.5(II), gives us the dual result[4]: if $p^t_{h,\ell}\, p^t_{j,k} = 0$ for all $t \neq m$ and $p^t_{h,m} p^t_{i,k} = 0$ for all $t \neq \ell$, then $p^i_{k,\ell} = p^j_{k,m}$ unless $p^j_{h,i} = 0$.

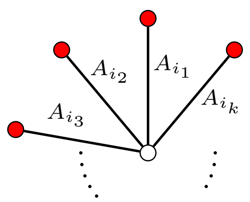
2.4. Generalized intersection numbers

Following Coolsaet and Jurišić [4], we now define generalized intersection numbers for an arbitrary association scheme. For $a_1, \ldots, a_k \in X$ and $i_1, \ldots, i_k \in \{0, \ldots, d\}$, define

[4] This holds for any commutative association scheme in which $A_h = A_h^\top$.

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ i_1 & i_2 & \cdots & i_k \end{bmatrix} = \left|\left\{ b \in X \mid (\forall j)\left((a_j, b) \in R_{i_j}\right)\right\}\right|;$$

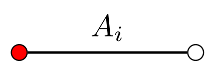
this is the number of vertices i_j-related to a_j for $j = 1, \ldots, k$. Restricting to the symmetric case for convenience, we immediately recognize these values as the coefficients of elementary tensors in the following star scaffold:



$$= \sum_{a_1,\cdots,a_k \in X} \begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ i_1 & i_2 & \cdots & i_k \end{bmatrix} \hat{a}_1 \otimes \hat{a}_2 \otimes \cdots \otimes \hat{a}_k$$

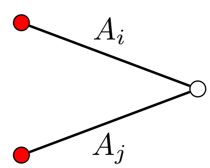
When k is very small, this number does not depend on the choice of the vertices a_j but only on the $\binom{k}{2}$ relations joining them. Here are the encoding of the most basic forms in the language of scaffolds:

$\begin{bmatrix} a \\ i \end{bmatrix} = v_i$:

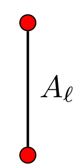


$= \quad v_i$

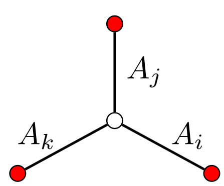
$\begin{bmatrix} a & b \\ i & j \end{bmatrix} = p^\ell_{ij}$ for $(a, b) \in R_\ell$:



$= \quad \sum_{\ell=0}^{d} \quad p^\ell_{ij}$

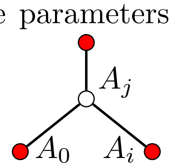


Let $\mathbb{A}$ be the Bose-Mesner algebra of a distance-regular graph Γ with P-polynomial ordering $A_0, A_1, \ldots, A_d$ of its Schur idempotents. Then

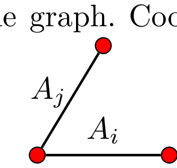


$$= \sum_{a,b,c \in X} \begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix} \hat{a} \otimes \hat{b} \otimes \hat{c}$$

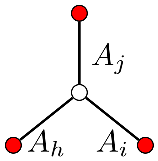
where $\begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix}$ is the number of vertices at distance i from a, distance j from b and distance k from c in Γ for each $a, b, c \in X$. While such numbers typically depend on the choice of vertices a, b, and c, they must satisfy a natural system of equations entirely determined by the parameters of the graph. Coolsaet and Jurišić [4] observe, in their Equation (5), that

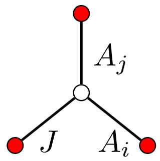


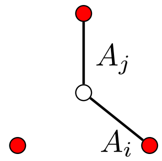
$=$

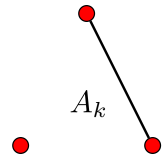


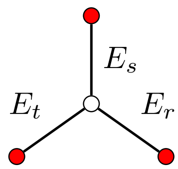
. Their Equation (6) can be expressed

$$\sum_{h=0}^{d}$$ 

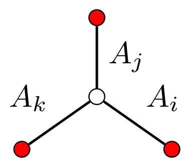
$$=$$ 

$$=$$ 

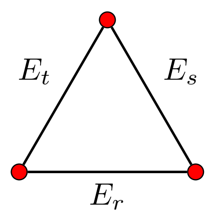
$$= \sum_{k=0}^{d} p_{ij}^{k}$$ 

Several researchers (e.g., [4,10]) have ruled out feasible parameter sets for distance-regular graphs by analyzing linear relations that these numbers must satisfy. A key insight in [4] is the following. If we know that a Krein parameter $q_{r,s}^{t}$ vanishes, then we have, by (2.7), $\mathbf{0} =$ 

and using (2.1) to expand $E_r = \frac{1}{|X|}\sum_{i=0}^{d} Q_{ir} A_i$, as well as E_s and E_t, we find

$$q_{rs}^{t} = 0 \qquad \Rightarrow \qquad \sum_{i,j,k=0}^{d} Q_{ir} Q_{js} Q_{kt} \;\cdot$$  $$= \mathbf{0}\ .$$

The same logic works on the dualized diagrams to give us the following apparently new identity:

$$p_{ij}^{k} = 0 \qquad \Rightarrow \qquad \sum_{r,s,t=0}^{d} P_{ri} P_{sj} P_{tk} \;\cdot$$  $$= \mathbf{0}\ .$$

3. Vector spaces of scaffolds

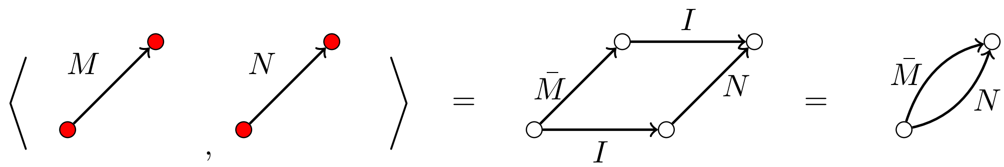
Given a subspace or subalgebra $\mathbb{A}$ of $\mathsf{Mat}_X(\mathbb{C})$, we now investigate various spaces contained in the vector space spanned by all m^{th} order scaffolds with edge weights in $\mathbb{A}$. We obtain interesting results when we fix the rooted diagram or simply fix the number m of root nodes.

3.1. Inner products

The standard scalar product on tensors can be described easily as a gluing operation on diagrams. Our most familiar inner product on tensors is the Frobenius product of two matrices:

$$M = \sum_{a,b\in X} M_{ab}\ \hat{a}\otimes\hat{b}, \qquad N = \sum_{a,b\in X} N_{ab}\ \hat{a}\otimes\hat{b}, \qquad \langle M, N\rangle = \sum_{a,b\in X} \bar{M}_{ab} N_{ab}\ .$$

In terms of scaffolds, this is expressed



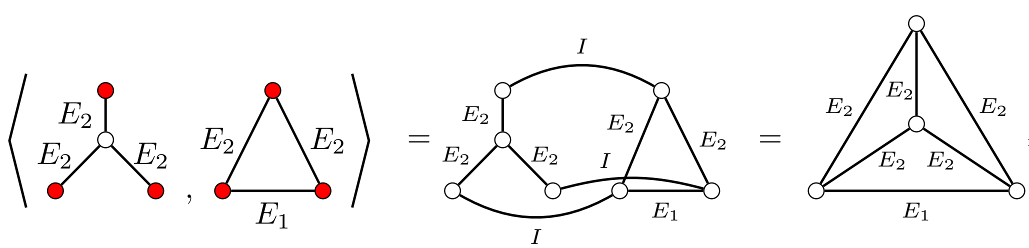
where $\bar{M}$ is the matrix obtained by conjugating each entry of M.

We extend this to scaffolds s and t of order m: we assume a consistent ordering of the root nodes in the two diagrams, indicated by their spatial arrangement, and simply join each pair of corresponding root nodes v_s, v_t by an edge $e = (v_s, v_t)$ with weight $w(e) = I$, conjugate the edge weights coming from the left argument and, in the new scaffold, make all nodes hollow. This is simply the linear extension of the product defined by

$$\langle \mathbf{y}_1 \otimes \mathbf{y}_2 \otimes \cdots \otimes \mathbf{y}_m \ , \ \mathbf{z}_1 \otimes \mathbf{z}_2 \otimes \cdots \otimes \mathbf{z}_m \rangle = \langle \mathbf{y}_1, \mathbf{z}_1 \rangle \langle \mathbf{y}_2, \mathbf{z}_2 \rangle \cdots \langle \mathbf{y}_m, \mathbf{z}_m \rangle \tag{3.1}$$

for $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m \in \mathbb{C}^X$.

Example 3.1. Let X denote the vertex set of the Petersen graph with adjacency matrix A_1 and primitive idempotents E_0, E_1, E_2 satisfying $A_1E_1 = E_1$ and $A_1E_2 = -2E_2$. Although every edge weight below is positive semidefinite, straightforward computation shows



$$= \frac{-2}{243} \ .$$

3.2. Important subspaces

Let $(X, \mathcal{R})$ be a (commutative) association scheme with standard module $V = \mathbb{C}^X$ and automorphism group[5] Σ. The group Σ acts componentwise on elementary basis tensors $\hat{x}_1 \otimes \cdots \otimes \hat{x}_m$: if $\sigma \in \Sigma$ sends $x \in X$ to x^σ, then

$$\sigma : \hat{x}_1 \otimes \cdots \otimes \hat{x}_m \mapsto \widehat{x_1^\sigma} \otimes \cdots \otimes \widehat{x_m^\sigma}.$$

Each scaffold of order m is an element of the tensor product $V^{\otimes m}$ and it is easy to see that, in full generality, the m^{th} order scaffolds span this space. But the space spanned

[5] The automorphism group of an association scheme $(X, \{R_0, \ldots, R_d\})$ is defined to be the subgroup of $\mathrm{Sym}(X)$ which preserves all relations R_i, $1 \le i \le d$.

by the symmetric scaffolds is almost always much smaller. The vector space of m^{th} order symmetric scaffolds contains an ascending chain of subspaces $S_m \subseteq S_{m+1} \subseteq S_{m+2} \subseteq \cdots$ where S_t is the vector space spanned by m^{th} order symmetric scaffolds on t nodes.

Theorem 3.2. *Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$ and automorphism group Σ. The vector space $\mathsf{W}(m; \mathbb{A})$ of all linear combinations of m^{th} order scaffolds with edge weights in $\mathbb{A}$ has dimension equal to the number of orbits of Σ on ordered m-tuples of vertices. If $\mathcal{O}_1, \ldots, \mathcal{O}_N$ is a full list of orbits of Σ on ordered m-tuples of vertices, then the tensors*

$$\left\{ \sum_{(x_1,\ldots,x_m)\in\mathcal{O}_h} \hat{x}_1 \otimes \cdots \otimes \hat{x}_m \,\middle|\, 1 \le h \le N \right\}$$

form a basis for this vector space.

Proof. It is not hard to see that every element $\sigma \in \Sigma$ preserves every symmetric scaffold:

$$\mathsf{S}(G, R; w) = \sum_{\varphi : V(G)\to X} \left(\prod_{\substack{e\in E(G)\\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} \right) \widehat{\varphi(r_1)} \otimes \widehat{\varphi(r_2)} \otimes \cdots \otimes \widehat{\varphi(r_m)} =$$

$$\sum_{\varphi : V(G)\to X} \left(\prod_{\substack{e\in E(G)\\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} \right) \widehat{\varphi(r_1)^\sigma} \otimes \widehat{\varphi(r_2)^\sigma} \otimes \cdots \otimes \widehat{\varphi(r_m)^\sigma}.$$

To see that the two spaces are equal, let $\mathcal{O}_h$ be any orbit on m-tuples with orbit representative $(y_1, \ldots, y_m)$. Let G be the complete graph with vertex set $V(G) = X$, edge weights $w(e) = A_i$ whenever $e = (a, b) \in R_i$, $0 \le i \le d$, and root nodes $R = \{y_1, \ldots, y_m\}$. Then, for $\varphi : X \to X$,

$$\prod_{\substack{e\in E(G)\\ e=(a,b)}} w(e)_{\varphi(a),\varphi(b)} = \begin{cases} 1, & \text{if } \varphi \in \Sigma; \\ 0 & \text{otherwise.} \end{cases}$$

So, applying the Orbit-Stabilizer Theorem, $\mathsf{S}(G, R; w) = \dfrac{|\Sigma|}{|\mathcal{O}_h|} \sum_{(x_1,\ldots,x_m)\in\mathcal{O}_h} \hat{x}_1 \otimes \cdots \otimes \hat{x}_m$. □

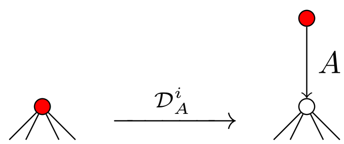
For example, if $(X, \mathcal{R})$ is *Schurian* (i.e., there is a group Σ acting on X whose orbitals are precisely $R_0, R_1, \ldots, R_d$), then the space of second order scaffolds is no larger than the Bose-Mesner algebra, which is the vector space of single-edge scaffolds of order two.

For each $m \ge 1$, we have various actions of matrices on $V^{\otimes m}$ given by Jaeger [14].

I (NODE ACTION) Here we view $\mathsf{Mat}_X(\mathbb{C})$ as an algebra of matrices under ordinary matrix multiplication. For $A \in \mathsf{Mat}_X(\mathbb{C})$, $1 \leq i \leq m$ and $\hat{x}_1 \otimes \cdots \otimes \hat{x}_m \in V^{\otimes m}$, define

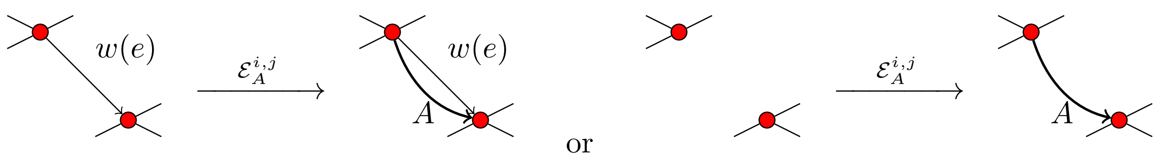
$$\mathcal{D}^i_A : \hat{x}_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_m \;\mapsto\; \hat{x}_1 \otimes \cdots \otimes A\hat{x}_i \otimes \cdots \otimes \hat{x}_m$$

Diagrammatically, this adds a node of degree one to a scaffold as follows:



II (EDGE ACTION) Here we view $\mathsf{Mat}_X(\mathbb{C})$ as a $\circ$-subring of itself: i.e., the product is entrywise. Assume $m \geq 2$. For $A \in \mathsf{Mat}_X(\mathbb{C})$, $1 \leq i, j \leq m$, and standard basis element $\hat{x}_1 \otimes \cdots \otimes \hat{x}_m \in V^{\otimes m}$, define

$$\mathcal{E}^{i,j}_A : \hat{x}_1 \otimes \cdots \otimes \hat{x}_m \;\mapsto\; A_{x_i,x_j}\, \hat{x}_1 \otimes \cdots \otimes \hat{x}_m$$



The first investigation of these actions on tensors appears in the work of Terwilliger [23] who considered the case $m = 3$. Let $(X, \mathcal{R})$ be an association scheme with Bose-Mesner algebra $\mathbb{A}$ acting on the standard module. The inner product space

$$V^{\otimes 3} = V \otimes V \otimes V = \text{span}\left\{\hat{a} \otimes \hat{b} \otimes \hat{c} \mid a, b, c \in X\right\}$$

of all third order tensors can be viewed as an $\mathbb{A}^{\otimes 3}$-module in two important ways:

$$(M_1 \otimes M_2 \otimes M_3)\mathsf{s} = \left(\mathcal{D}^1_{M_1} \bullet \mathcal{D}^2_{M_2} \bullet \mathcal{D}^3_{M_3}\right)(\mathsf{s}), \tag{3.2}$$

where $\bullet$ here denotes composition of functions, or

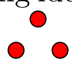
$$\{M_1 \otimes M_2 \otimes M_3\}\mathsf{s} = \left(\mathcal{E}^{1,2}_{M_3} \bullet \mathcal{E}^{1,3}_{M_2} \bullet \mathcal{E}^{2,3}_{M_1}\right)(\mathsf{s}), \tag{3.3}$$

each extended linearly. For the moment, let us denote the action given in (3.2) by $(\cdot)$ and the action given in (3.3) by $\{\cdot\}$. The space $V^{\otimes 3}$ is then a module for the algebra of linear operators generated by $(\cdot)$ and $\{\cdot\}$ where

$$(M_1 \otimes M_2 \otimes M_3) \bullet (N_1 \otimes N_2 \otimes N_3) = ((M_1N_1) \otimes (M_2N_2) \otimes (M_3N_3)) \tag{3.4}$$

and

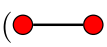
$$\{M_1 \otimes M_2 \otimes M_3\} \bullet \{N_1 \otimes N_2 \otimes N_3\} = \{(M_1 \circ N_1) \otimes (M_2 \circ N_2) \otimes (M_3 \circ N_3)\}\,. \tag{3.5}$$

An interesting idea of Terwilliger is to consider the smallest subspace $\mathbb{T}$ of $V^{\otimes 3}$ containing the scaffold  which is both a $(\cdot)$-module and a $\{\cdot\}$-module. More generally, we are interested in subspaces of $V^{\otimes 3}$ invariant under the node action $(\cdot)$, the edge action $\{\cdot\}$, or both.

Definition 3.3. Given a finite (di)graph $G = (V(G), E(G))$, an ordered multiset of m root nodes $R \subseteq V(G)$, and a vector subspace $\mathbb{A}$ of $\mathsf{Mat}_X(\mathbb{C})$, we denote by $\mathbf{W}((G,R);\mathbb{A})$ the vector space of all m^{th} order tensors spanned by scaffolds defined on rooted diagram (G,R) having edge weights in $\mathbb{A}$.

Observe that we may treat G as an undirected graph when $\mathbb{A}$ is closed under the transpose map. For the remainder of this section, we assume that edge weights are chosen from a coherent algebra.

Examples: Assume $\mathbb{A}$ is a coherent algebra.

- For $G = K_2$ with $R = V(G)$, $\mathbf{W}((G,R);\mathbb{A}) = \mathbf{W}($$;\mathbb{A}) = \mathbb{A}$;
- When G connected with $|E(G)| \le 4$ and R consists of two distinct nodes, $\mathbf{W}((G,R);\mathbb{A}) = \mathbb{A}$. In the case where $\mathbb{A}$ is a Bose-Mesner algebra, this may be verified using scaffold manipulation rules $\mathsf{SR0}$, $\mathsf{SR0'}$, $\mathsf{SR1}$, $\mathsf{SR1'}$, and $\mathsf{SR9}$.

In Lemma 1.6, we saw that equality of scaffolds defined on a given diagram G is preserved when, on both sides of such an equation, we replace the ordered multiset of root nodes R by any ordered submultiset R' of R. We note here that the operation

$$\mathsf{hollow}_{R'} : \mathsf{S}(G,R;w) \mapsto \mathsf{S}(G,R';w)$$

naturally extends to a linear map from $\mathbf{W}((G,R);\mathbb{A})$ to $\mathbf{W}((G,R');\mathbb{A})$. As a consequence, each statement of the form $\mathbf{W}((G,R);\mathbb{A}) = \mathbf{W}((H,R);\mathbb{A})$ for diagrams G and H with a common (or identified) ordered multiset of root nodes R also tells us that $\mathbf{W}((G,R');\mathbb{A}) = \mathbf{W}((H,R');\mathbb{A})$ for each $R' \subseteq R$.

Problem 3.4. Given an association scheme $(X,\mathcal{R})$ with Bose-Mesner algebra $\mathbb{A}$ and an integer $m \ge 2$, observe that

$$\mathbf{W}(m;\mathbb{A}) = \sum_{\substack{(G,R)\\ |R|=m}} \mathbf{W}((G,R);\mathbb{A}).$$

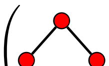
For this given $(X, \mathcal{R})$ what is a smallest diagram G with root nodes $R \subseteq V(G)$ satisfying $|R| = m$ such that $\mathbf{W}((G, R); \mathbb{A}) = \mathbf{W}(m; \mathbb{A})$? Is there a unique minimal rooted diagram with this property?

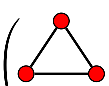
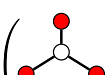
Definition 3.5. Let G and H be finite undirected graphs. An *H-minor* in G is a set $\{G_v \mid v \in V(H)\}$ of pairwise disjoint connected subgraphs of G indexed by the nodes of H such that there is an injection $\iota : E(H) \to E(G)$ that maps each edge (x, y) of H to some edge (x', y') in G with $x' \in G_x$ and $y' \in G_y$. Given H with a specified ordered multiset $R = \{r_1, \ldots, r_m\}$ of nodes in $V(H)$ and G with a specified ordered multiset $R' = \{r'_1, \ldots, r'_m\}$ of nodes in $V(G)$, a *rooted H-minor* with respect to (H, R) and (G, R') is an H-minor in G satisfying $r'_i \in V(G_{r_i})$ for each $1 \le i \le m$.

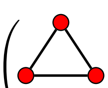
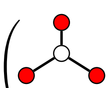
Theorem 3.6. *Assume $\mathbb{A}$ is a coherent algebra. If there is a rooted H-minor in G with respect to (H, R) and (G, R'), then $\mathbf{W}((H, R); \mathbb{A}) \subseteq \mathbf{W}((G, R'); \mathbb{A})$.*

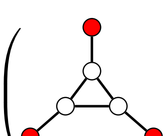
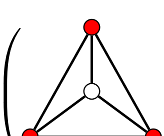
Proof. Consider a scaffold $\mathsf{S}(H, R; w)$ defined on rooted diagram (H, R) and assume $\{G_v \mid v \in V(H)\}$ and $\iota : E(H) \to E(G)$ are given as in Definition 3.5 above. Define $w' : E(G) \to \mathbb{A}$ as follows. For $e \in E(H)$, set $w'(\iota(e)) = w(e)$; set $w'(e) = I$ for any edge e with both ends in the same subgraph G_v; finally set $w'(e) = J$ for all remaining edges of G. Then, by Lemma 1.3, $\mathsf{S}(H, R; w) = |X|^{-\ell} \cdot \mathsf{S}(G, R'; w')$ where $\ell = |V(G)| - |\cup_{v \in V(H)} G_v|$. $\square$

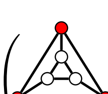
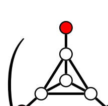
An immediate consequence of Theorem 3.6 is the following result.

Corollary 3.7. *The vector space* $\mathbf{W}\Big($  $; \mathbb{A}\Big)$ *is contained in*

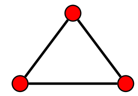
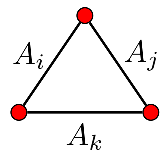
$\mathbf{W}\Big($  $; \mathbb{A}\Big) \bigcap \mathbf{W}\Big($  $; \mathbb{A}\Big)$.

The spaces $\mathbf{W}\Big($  $; \mathbb{A}\Big)$ *and* $\mathbf{W}\Big($  $; \mathbb{A}\Big)$ *are both contained in the spaces*

$\mathbf{W}\Big($  $; \mathbb{A}\Big)$, $\quad \mathbf{W}\Big($  $; \mathbb{A}\Big)$.

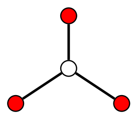
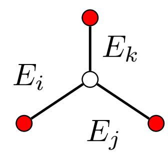
These spaces, in turn, are both contained in $\mathbf{W}\Big($  $; \mathbb{A}\Big) \bigcap \mathbf{W}\Big($  $; \mathbb{A}\Big)$.

If $\mathbb{A}$ is the Bose-Mesner algebra of an association scheme with standard bases $\{A_0, \ldots, A_d\}$ and $\{E_0, \ldots, E_d\}$, intersection numbers p_{ij}^k, and Krein parameters q_{ij}^k, then one may apply Lemma 2.1 to see that

W ( ; $\mathbb{A}$) = span { 

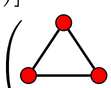
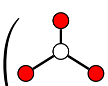
| $p_{ij}^k > 0$ }

and

W ( ; $\mathbb{A}$) = span { 

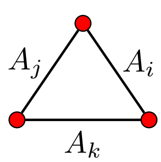
| $q_{ij}^k > 0$ }

These are, respectively, the images of Jaeger's triangle projection [14, (40)] and star projection [14, (39)].

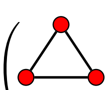
The space **W** ( ; $\mathbb{A}$) is invariant under the action $\{\cdot\}$ while the space **W** ( ; $\mathbb{A}$) is invariant under the action $(\cdot)$.

Theorem 3.8 *(Terwilliger [25, Lemma 87]). For any symmetric association scheme,*

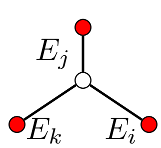
(i) *the set*

{ 

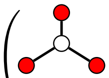
| $p_{ij}^k > 0$ }

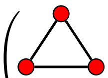
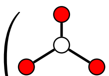
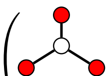
is an orthogonal basis for the subspace **W** ( ; $\mathbb{A}$) *and*

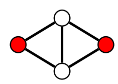
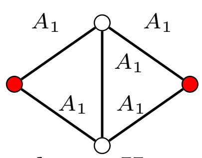
(ii) *the set*

{ 

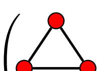
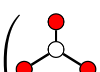
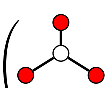
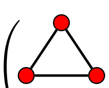
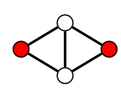
| $q_{ij}^k > 0$ }

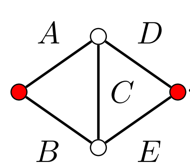
is an orthogonal basis for the subspace $\mathsf{W}\big($  $;\mathbb{A}\big)$. □

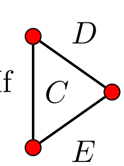
Example 3.9. Consider the association scheme of the Petersen graph, with Bose-Mesner algebra $\mathbb{A}$. Theorem 3.8 tells us that dim $\mathsf{W}\big($  $;\mathbb{A}\big) = 14$, the number of non-zero intersection numbers, and dim $\mathsf{W}\big($  $;\mathbb{A}\big) = 15$, the number of non-zero Krein parameters. Straightforward calculation verifies that the automorphism group of the Petersen graph has only 15 orbits on triples, so $\mathsf{W}(3;\mathbb{A}) =$ $\mathsf{W}\big($  $;\mathbb{A}\big)$ by Theorem 3.2.

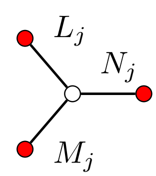
Example 3.10. The *Doob graph* $\mathsf{Doob}(s,t)$ is the Cartesian product of s copies of K_4 and t copies of the Shrikhande graph [2, Section 9.2B]. If $\mathbb{A}$ is the Bose-Mesner algebra of a Doob graph $\mathsf{Doob}(s,t)$, then $\mathsf{W}\big($  $;\mathbb{A}\big) = \mathbb{A}$ if and only if $t = 0$; the scaffold 

belongs to the Bose-Mesner algebra only in the case where the Doob graph is a Hamming graph.

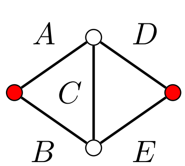
Theorem 3.11. *Let* $\mathbb{A}$ *be a coherent algebra. If either* $\mathsf{W}\big($  $;\mathbb{A}\big) \subseteq \mathsf{W}\big($  $;\mathbb{A}\big)$ *or* $\mathsf{W}\big($  $;\mathbb{A}\big) \subseteq \mathsf{W}\big($  $;\mathbb{A}\big)$, *then* $\mathsf{W}\big($  $;\mathbb{A}\big) = \mathbb{A}$.

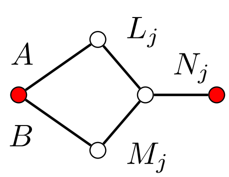
Proof. Suppose we are given a scaffold 

. If 

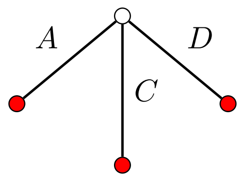
$= \sum_{j=1}^{\ell}$ 

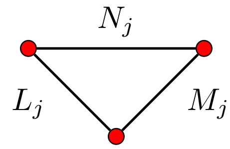
then



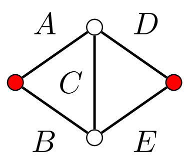
$= \sum_{j=1}^{\ell}$ 

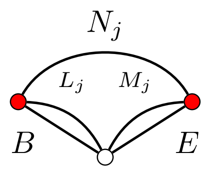
$= \sum_{j=1}^{\ell}[(AL_j)\circ(BM_j)]N_j \in \mathbb{A}$

Likewise, if 

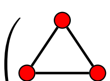
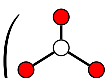
$= \sum_{j=1}^{\ell}$ 

then



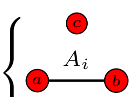
$= \sum_{j=1}^{\ell}$ 

$= \sum_{j=1}^{\ell} [(B \circ L_j)(E \circ M_j)] \circ N_j \in \mathbb{A}.$ □

Problem 3.12. Let $(X, \mathcal{R})$ be a symmetric association scheme. Determine necessary and sufficient conditions on $(X, \mathcal{R})$ for $\mathbf{W}\Big($  $; \mathbb{A}\Big) = \mathbf{W}\Big($  $; \mathbb{A}\Big)$ to hold. (Theorem 3.8 tells us that the number of non-vanishing intersection numbers must equal the number of non-vanishing Krein parameters.)

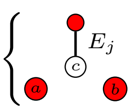
3.3. Bases for spaces of third order scaffolds

The space $V^{\otimes 3}$, endowed with the inner product given in (3.1), admits

$$\left(V^{\otimes 3}\right)_i = \text{span}\left\{ \right.$$ 

$$\left. \middle| \; a, b, c \in X \right\}$$

as a $\{\cdot\}$-submodule and admits

$$\left(V^{\otimes 3}\right)^*_j = \text{span}\left\{ \right.$$ 

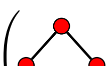
$$\left. \middle| \; a, b, c \in X \right\}$$

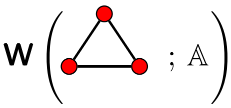
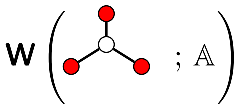
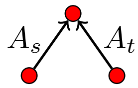
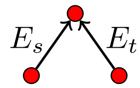
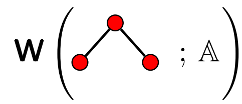
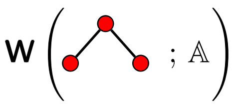
as a $(\cdot)$-submodule. (These spaces were introduced in [23, Definition 2.7].) The orthogonal projection $p_i : V^{\otimes 3} \to \left(V^{\otimes 3}\right)_i$ is given by

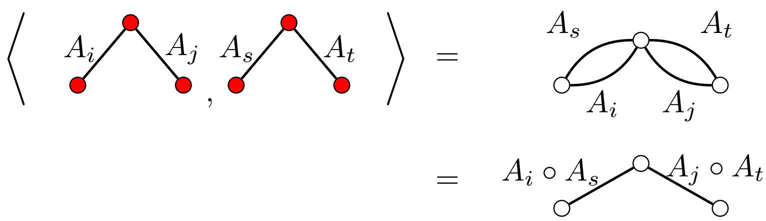
$$p_i(\hat{a} \otimes \hat{b} \otimes \hat{c}) = \begin{cases} \hat{a} \otimes \hat{b} \otimes \hat{c} & \text{if } (a, b) \in R_i; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Dually, the orthogonal projection $p^*_j : V^{\otimes 3} \to \left(V^{\otimes 3}\right)^*_j$ is given by

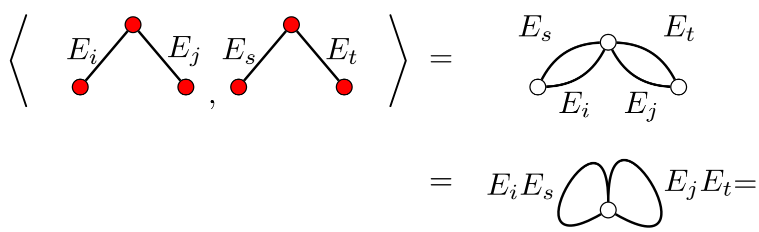
$$p^*_j(\hat{a} \otimes \hat{b} \otimes \hat{c}) = \hat{a} \otimes \hat{b} \otimes E_j \hat{c} \, .$$

We recognize these as the edge action $\mathcal{E}^{1,2}_{A_i}$ and node action $\mathcal{D}^3_{E_j}$ introduced in Section 3.2. The two linear transformations $\mathcal{E}^{1,2}_{A_i}$ and $\mathcal{D}^3_{E_j}$ map the space $\mathbf{W}\Big($  $; \mathbb{A}\Big)$

into  and , respectively. Terwilliger [23] introduces scaffolds e_{st} =  and[6] e^*_{st} =  belonging to  and proves that the sets $\{\mathsf{e}_{st} \mid 0 \le s,t \le d\}$ and $\{\mathsf{e}^*_{st} \mid 0 \le s,t \le d\}$ form two orthogonal bases for  : the relevant inner products are ([23, Lemma 2.11])

$\langle \mathsf{e}_{ij}, \mathsf{e}_{st} \rangle =$  $= \delta_{i,s}\delta_{j,t}|X|v_i v_j$

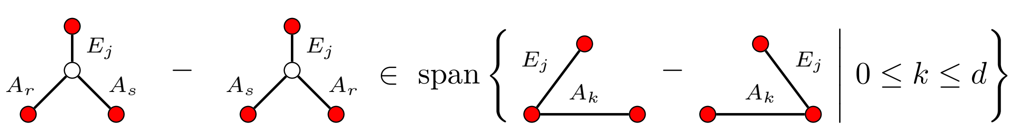
and

$\langle \mathsf{e}^*_{ij}, \mathsf{e}^*_{st} \rangle =$  $= \delta_{i,s}\delta_{j,t}|X|^{-1}m_i m_j$

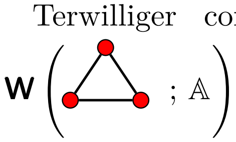
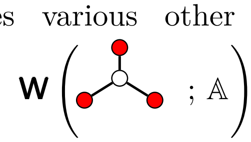
In [23], the subspace

$$\mathrm{span}\{\mathsf{e}_{st} - \mathsf{e}_{ts} \mid 0 \le s,t \le d\} \;=\; \mathrm{span}\{\mathsf{e}^*_{st} - \mathsf{e}^*_{ts} \mid 0 \le s,t \le d\}$$

is then introduced in the study of the *balanced set condition*: for a given j, the idempotent E_j satisfies the balanced set condition if, for all $0 \le r,s \le d$, we have



Da Zhao [27] expressed the balanced set condition more precisely in scaffold notation and introduced a dual concept which he calls the “dual balanced condition”.

Terwilliger computes various other inner products of pairs of scaffolds in  and . Identities (B.1)–(B.4) in Appendix B are

[6] Our definition of e^*_{st} differs from the definition in [23] by a constant factor of $1/|X|$.

simply restatements of Lemmas 2.13, 2.14 and 2.16 in [23] omitting the assumption that the scheme is symmetric.

3.4. Terwilliger algebras

Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$ having bases $\{A_0, \ldots, A_d\}$ satisfying $A_i \circ A_j = \delta_{i,j} A_i$ and $\{E_0, \ldots, E_d\}$ satisfying $E_i E_j = \delta_{i,j} E_i$ as usual. Fix $x \in X$ and define $E_i^*(x)$ to be the diagonal matrix with $(E_i^*(x))_{a,a} = (A_i)_{x,a}$; i.e., the (a,b)-entry of $E_i^*(x)$ is equal to one if $a = b$ with $(x, a) \in R_i$ and equal to zero otherwise. The *Terwilliger algebra* of $(X, \mathcal{R})$ with respect to base point x is the matrix algebra generated by the matrices A_i and the matrices $E_i^*(x)$:

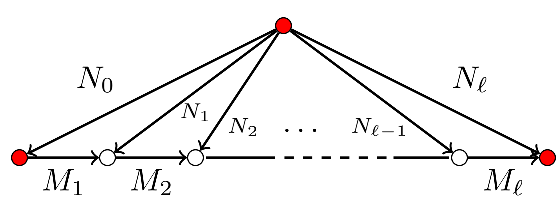
$$\mathbb{T}_x = \langle A_0, \ldots, A_d,\ E_0^*(x), \ldots, E_d^*(x) \rangle = \langle E_0, \ldots, E_d,\ A_0^*(x), \ldots, A_d^*(x) \rangle$$

where

$$(A_j^*(x))_{a,b} = \begin{cases} |X|(E_j)_{x,a} & \text{if } a = b; \\ 0 & \text{otherwise.} \end{cases}$$

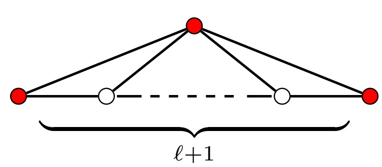
Beginning with [24], an extensive theory of Terwilliger algebras, particularly for symmetric association schemes that are both metric and cometric, i.e., for Q-polynomial distance-regular graphs, has developed over the past three decades. See [6] for a relatively recent survey, and see [25] for a more substantive update. Our goal here is to identify scaffolds encoding the matrices in such algebras and to explore subspaces of $\mathbf{W}(3; \mathbb{A})$ that contain all such scaffolds under various conditions.

Fix an association scheme $(X, \mathcal{R})$ and corresponding Bose-Mesner algebra $\mathbb{A}$. Let $\mathcal{T}$ denote the vector space of all linear combinations of scaffolds of the form

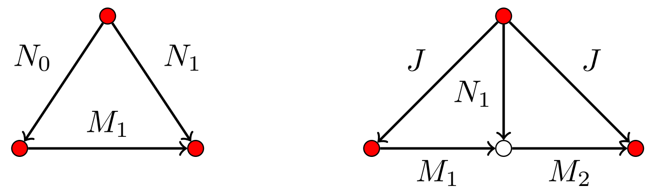


where $\ell \geq 1$ and $M_1, \ldots, M_\ell, N_0, \ldots, N_\ell \in \mathbb{A}$. That is,

$$\mathcal{T} = \bigcup_{\ell=1}^{\infty} \mathbf{W}\left(\quad ; \ \mathbb{A} \right).$$

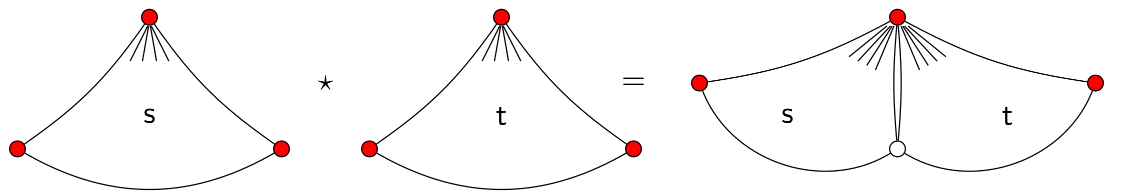


The most basic third-order tensors of this form are the triangle and star



where $M_1, M_2, N_0, N_1 \in \mathbb{A}$ and J is the all ones matrix.

We define a product on elements of $\mathcal{T}$ by gluing diagrams as follows:



Here, spacial arrangement is important and the root nodes at the top are identified while the rightmost root node of s is identified with the leftmost root node of t and made hollow.

Let us make this precise.

Given scaffolds

$$\mathsf{s} = \mathsf{S}(G, \{b_0, a, b_\ell\}; w) \qquad \text{and} \qquad \mathsf{t} = \mathsf{S}(H, \{b'_0, a', b'_m\}; w')$$

where

$$V(G) = \{a, b_0, b_1, \ldots, b_\ell\}, \quad E(G) = \{(b_0, b_1), \ldots, (b_{\ell-1}, b_\ell),\ (a, b_0), \ldots, (a, b_\ell)\}$$
$$V(H) = \{a', b'_0, b'_1, \ldots, b'_m\}, \quad E(H) = \{(b'_0, b'_1), \ldots, (b'_{m-1}, b'_m),\ (a', b'_0), \ldots, (a', b'_m)\}$$
$$w(b_{h-1}, b_h) = M_h, \quad w(a, b_h) = N_h, \qquad w'(b'_{h-1}, b'_h) = M'_h, \quad w'(a', b'_h) = N'_h,$$

we define

$$\mathsf{s} \star \mathsf{t} = \mathsf{S}(K, \{b_0, a, b'_m\}; \hat{w})$$

where

$$V(K) = \{a, b_0, b_1, \ldots, b_\ell, b'_1, \ldots, b'_m\}$$
$$E(K) = \{(b_0, b_1), \ldots, (b_{\ell-1}, b_\ell),\ (b_\ell, b'_1), \ldots, (b'_{m-1}, b'_m),\ (a, b_0), \ldots, (a, b_\ell),$$
$$(a, b'_1), \ldots, (a, b'_m)\}$$

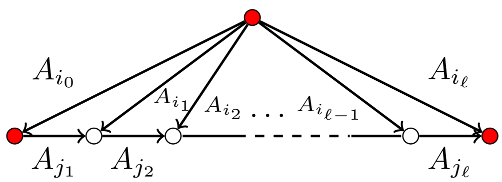
with edge weights $\hat{w}(a, b_\ell) = N_\ell \circ N'_0$ and

$$\hat{w}(b_{h-1}, b_h) = M_h, \quad \hat{w}(a, b_h) = N_h,$$
$$\hat{w}(b_\ell, b'_1) = M'_1,\ \hat{w}(b'_{h-1}, b'_h) = M'_h, \quad \hat{w}(a, b'_h) = N'_h\ .$$

The product is extended linearly to $\mathcal{T}$.

We state the following theorem without proof:

Theorem 3.13. *The map* $\zeta : \mathcal{T} \to \bigoplus_{x\in X} \mathbb{T}_x$ *defined by*



$$\mapsto \bigoplus_{x\in X} E^*_{i_0}(x) A_{j_1} E^*_{i_1}(x) A_{j_2} \cdots A_{j_\ell} E^*_{i_\ell}(x)$$

and extended linearly is an injective linear map satisfying

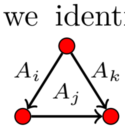
$$\zeta(\mathsf{s} \star \mathsf{t}) = \zeta(\mathsf{s})\zeta(\mathsf{t})$$

where the product on the right is ordinary matrix product of block diagonal matrices. □

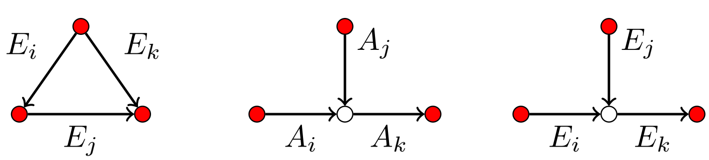
This gives a natural interpretation of certain third-order scaffolds as elements of the direct sum of all Terwilliger algebras $\mathbb{T}_x$ as x ranges over the elements of X. For example, since

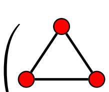
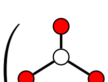
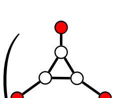
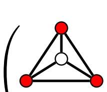
$$E^*_i(x) A_j E^*_k(x) = \sum_{\substack{y,z\in X \\ (x,y)\in R_i, (y,z)\in R_j, (z,x)\in R_k}} \hat{y} \otimes \hat{z}$$

we identify this matrix with $(E^*_i(x) A_j E^*_k(x)) \otimes \hat{x}$ and sum over $x \in X$ to obtain

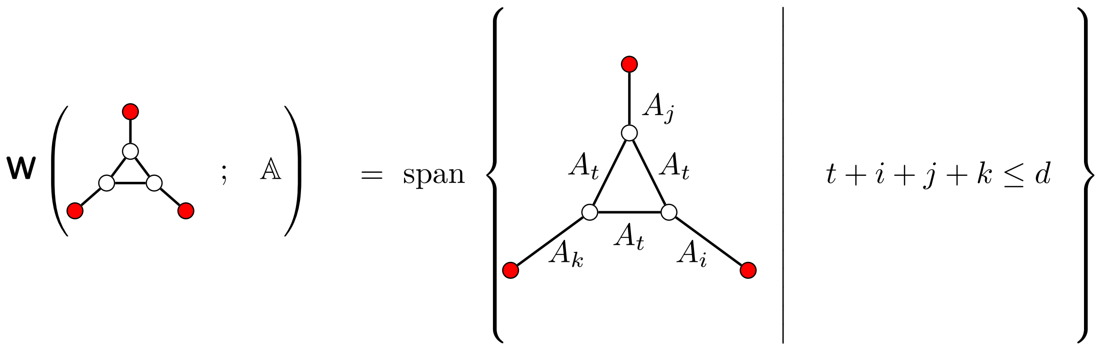


. Likewise, this isomorphism associates $\oplus_x A^*_i(x) E_j A^*_k(x)$, $\oplus_x A_i E^*_j(x) A_k$, and $\oplus_x E_i A^*_j(x) E_k$, respectively, to the following scaffolds:



While $\mathcal{T}$ contains $\mathbf{W}$( ; $\mathbb{A}$) and $\mathbf{W}$( ; $\mathbb{A}$), it does not necessarily contain $\mathbf{W}$( ; $\mathbb{A}$) or $\mathbf{W}$( ; $\mathbb{A}$).

Paul Terwilliger [private communication] conjectures the following: For a Q-polynomial bipartite distance-regular graph, the space of third order tensors of the form depicted on the left below is spanned by the subset of scaffolds with inner edges all having weight A_t, outer edges having weights A_i, A_j and A_k. Further we obtain a basis when we include only the scaffolds of this sort where $t + i + j + k \le d$.

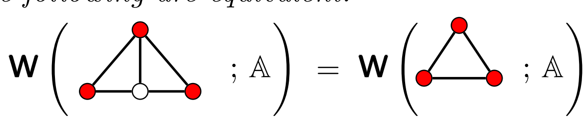


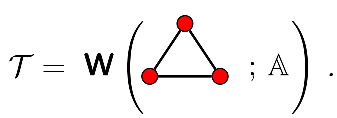
More importantly, Terwilliger conjectures that this space is both $(\cdot)$-invariant and $\{\cdot\}$-invariant; i.e., each of the maps $\mathcal{E}_{A_i}^{k,\ell}$ and $\mathcal{D}_{E_j}^{i}$ map this space into itself. Terwilliger claims that, interpreted as in Theorem 3.13, this space is the full algebra $\mathcal{T}$ when $\mathbb{A}$ is the Bose-Mesner algebra of a Q-polynomial bipartite distance-regular graph.

Association schemes for which the Terwilliger algebra takes a simpler form are of interest for two reasons: one can often prove more about the combinatorial structure of the scheme when $\mathcal{T}$ is generated as a vector space by a relatively small set of tensors; and some of the most important families of association schemes, such as the Hamming schemes, enjoy this property. The following lemma follows from an easy induction argument.

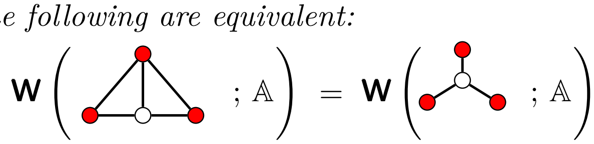
Lemma 3.14. *Let $\mathbb{A}$ be the Bose-Mesner algebra of a symmetric association scheme.*

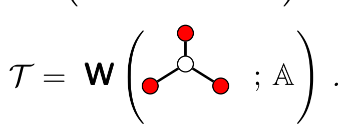
(a) The following are equivalent:

- 

- 

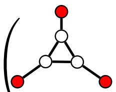
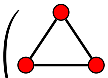
(b) The following are equivalent:

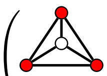
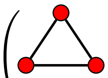
- 

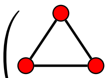
- 
 □

Theorem 3.15. *Let $\mathbb{A}$ be the Bose-Mesner algebra of a symmetric association scheme.*

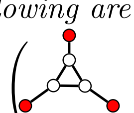
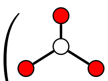
(a) The following are equivalent:

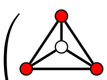
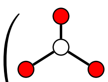
(i) $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$

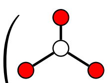
(ii) $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$

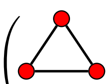
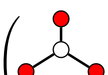
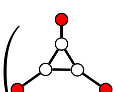
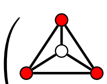
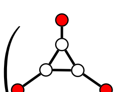
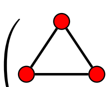
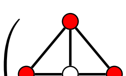
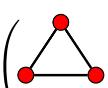
(iii) $\mathcal{T} = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$.

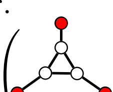
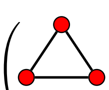
(b) The following are equivalent:

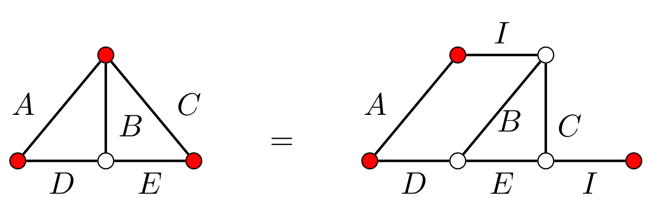
(i) $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$

(ii) $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$

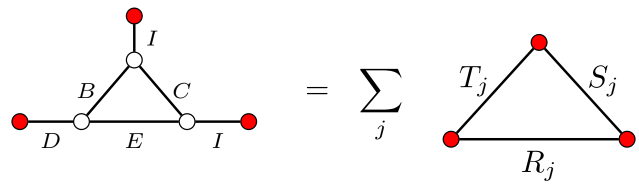
(iii) $\mathcal{T} = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$.

Proof. By Theorem 3.6, both $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ and $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ are contained in both $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ and $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ as well as in $\mathcal{T}$. Here, we prove that $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ if and only if $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ and apply Lemma 3.14 to obtain *(a)(i)* $\Leftrightarrow$ *(a)(iii)*. The remaining three equivalences *(a)(ii)* $\Leftrightarrow$ *(a)(iii)*, *(b)(i)* $\Leftrightarrow$ *(b)(iii)*. *(b)(ii)* $\Leftrightarrow$ *(b)(iii)* are proved in a similar manner.

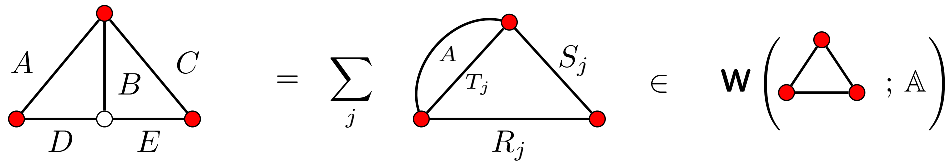
Assume $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) \subseteq \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ and consider a scaffold of the form



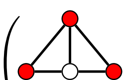
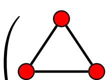
(using $\mathsf{SR0}$). By our assumption, there exist matrices $R_1, S_1, T_1, \ldots$ in $\mathbb{A}$ satisfying



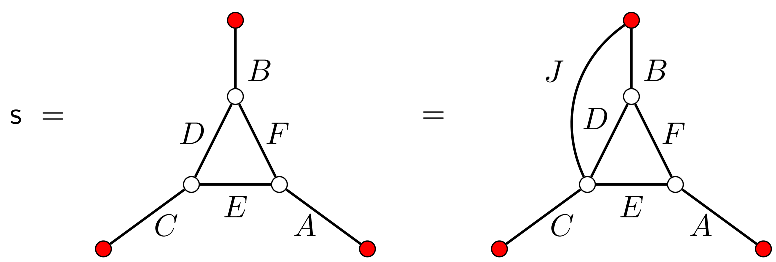
Applying Proposition 1.5, we substitute to find

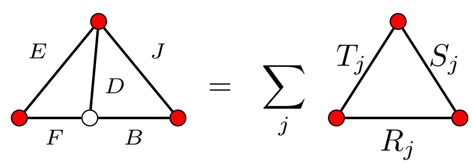


since 𝔸 is closed under the entrywise product.

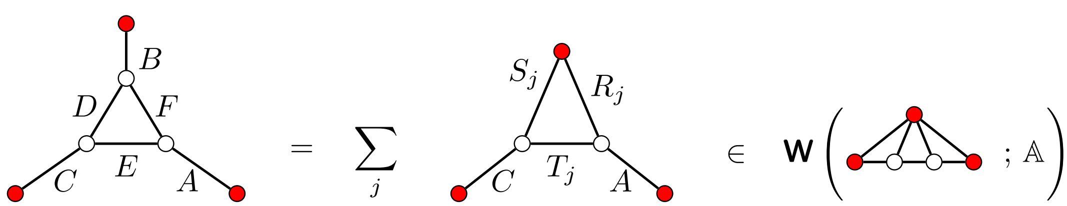
In the other direction, assume now that **W** ( ; 𝔸) ⊆ **W** ( ; 𝔸)

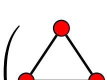
and consider the scaffold

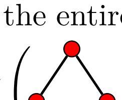
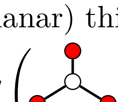


There exist matrices $R_1, S_1, T_1, \ldots$ in 𝔸 satisfying  .

Substitute this into the above to find

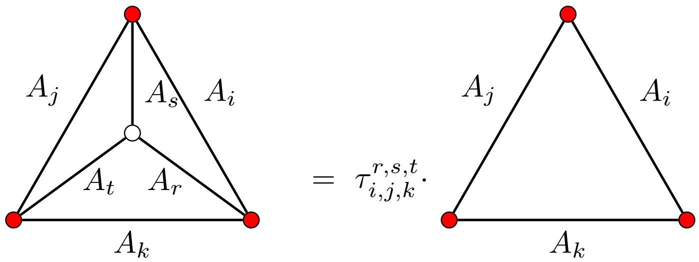


and one more application of our hypothesis gives $\mathsf{s} \in$ **W** ( ; 𝔸) . □

Problem 3.16. Under what conditions on 𝔸 does the entire space of (planar) third order scaffolds with edge weights in 𝔸 coincide with **W** ( ; 𝔸) or **W** ( ; 𝔸) ?

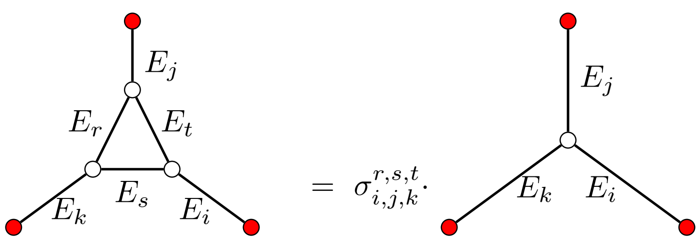
3.5. Triply regular association schemes

Following Jaeger [14, Section 5.3], we call a d-class symmetric association scheme *triply regular* if, for all $0 \le i, j, k, r, s, t \le d$, the following identity of tensors holds for some scalar $\tau_{i,j,k}^{r,s,t}$:



In other words, for all $x, y, z \in X$ and all indices r, s, t the number $\begin{bmatrix} x & y & z \\ r & s & t \end{bmatrix}$ of vertices w which are r-related to x, s-related to y and t-related to z depends only on r, s, t and those indices i, j, k for which $(x, y) \in R_i$, $(y, z) \in R_j$ and $(z, x) \in R_k$ and not on the choice of vertices x, y, z themselves.

Dually, let's call a symmetric association scheme *dually triply regular* if, for all i, j, k, r, s, t, there exists a scalar $\sigma_{i,j,k}^{r,s,t}$ such that



In other words, for all indices $0 \le i, j, k, r, s, t \le d$ and all $x, y, z \in X$

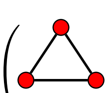
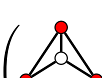
$$\sum_{u,v,w \in X} (E_i)_{x,u} (E_j)_{y,v} (E_k)_{z,w} \ (E_r)_{v,w} (E_s)_{w,u} (E_t)_{u,v}$$

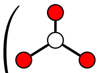
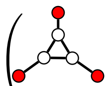
is a scalar multiple of

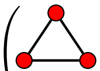
$$\sum_{w' \in X} (E_i)_{x,w'} (E_j)_{y,w'} (E_k)_{z,w'}$$

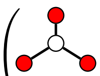
independent of the choice of x, y, z.

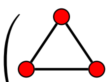
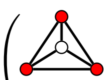
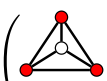
Theorem 3.17. *Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$. Then*

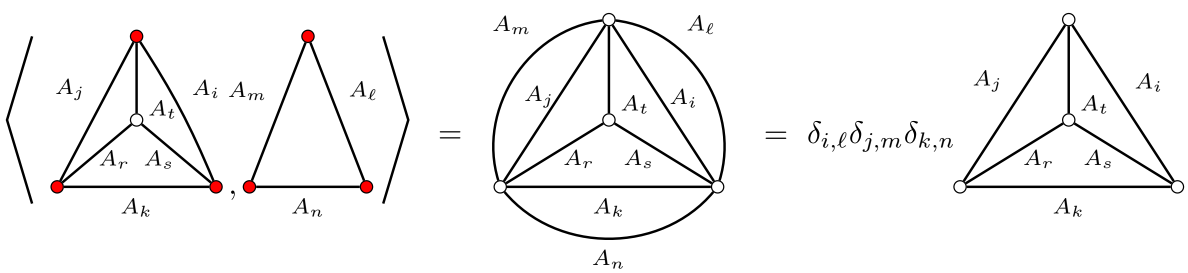
(a) $(X, \mathcal{R})$ is triply regular if and only if $\mathbf{W}\Big($  $; \mathbb{A}\Big) = \mathbf{W}\Big($  $; \mathbb{A}\Big)$ *;*

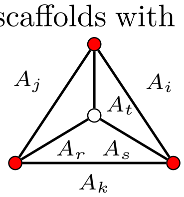
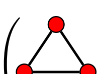
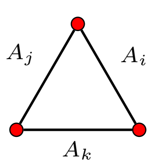
(b) $(X, \mathcal{R})$ *is dually triply regular if and only if* $\mathbf{W}\Big($  $; \mathbb{A}\Big) = \mathbf{W}\Big($  $; \mathbb{A}\Big)$.

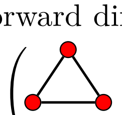
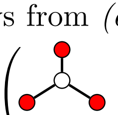
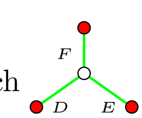
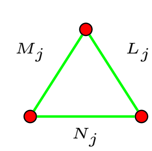
(c) $(X, \mathcal{R})$ *is both triply regular and dually triply regular if and only if* $\mathbf{W}\Big($  $; \mathbb{A}\Big) =$

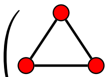
$\mathbf{W}\Big($  $; \mathbb{A}\Big)$.

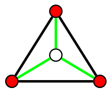
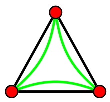
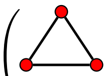
Proof. Clearly $\mathbf{W}\Big($  $; \mathbb{A}\Big)$ is a subspace of $\mathbf{W}\Big($  $; \mathbb{A}\Big)$. If the triply regular condition holds, then we obviously have containment in the other direction as well since the scaffolds in $\mathbf{W}\Big($  $; \mathbb{A}\Big)$ with all edge weights in $\{A_0, \ldots, A_d\}$ span the space. Conversely, observe that

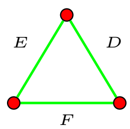


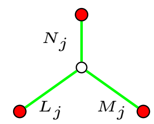
and that the "Delta" scaffolds with edge weights in $\{A_0, \ldots, A_d\}$ are pairwise orthogonal by Theorem 3.8. So, if  belongs to $\mathbf{W}\Big($  $; \mathbb{A}\Big)$, then it must be a scalar multiple of . The second claim is proved in a similar manner.

For part *(c)*, the forward direction follows from *(a)* and *(b)* by mutual containment. Suppose now that $\mathbf{W}\Big($  $; \mathbb{A}\Big) = \mathbf{W}\Big($  $; \mathbb{A}\Big)$. Then, given any $D, E, F \in \mathbb{A}$, there exist $L_1, M_1, N_1, \ldots, L_k, M_k, N_k \in \mathbb{A}$ for which  $= \sum_j$ .

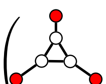
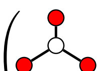
Applying Proposition 1.5 and using the fact that $\mathbb{A}$ is closed under entrywise multiplication, we see that any scaffold built on K_4 lies in $\mathsf{W}\Big($  $;\ \mathbb{A}\Big)$ as follows:

 $=\ \sum_j$  $\in\ \mathsf{W}\Big($  $;\ \mathbb{A}\Big).$

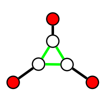
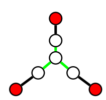
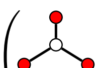
Likewise, given $D, E, F \in \mathbb{A}$, we may write 

$= \sum_j$ 

for some $L_j, M_j, N_j \in \mathbb{A}$. Making such a substitution and using closure under matrix multiplication, we see that

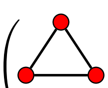
$\mathsf{W}\Big($  $;\ \mathbb{A}\Big) \subseteq \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$

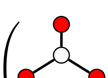
via

 $=\ \sum_j$  $\in\ \mathsf{W}\Big($  $;\ \mathbb{A}\Big).$ □

We now have another way to interpret Theorem 3.15.

Theorem 3.18. *Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$. Then*

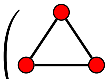
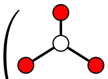
(a) $(X, \mathcal{R})$ is triply regular if and only if $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathcal{T}$;

(b) $(X, \mathcal{R})$ is dually triply regular if and only if $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = \mathcal{T}$. □

Corollary 3.19. *Let $(X, \mathcal{R})$ be a symmetric association scheme with Bose-Mesner algebra $\mathbb{A}$, intersection numbers $p^k_{i,j}$ and Krein parameters $q^k_{i,j}$ $(0 \le i, j, k \le d)$. Let*

$$N_p = \left|\left\{(i,j,k) \in \{0,\ldots,d\}^3 \mid p^k_{i,j} > 0\right\}\right|, \quad N_q = \left|\left\{(i,j,k) \in \{0,\ldots,d\}^3 \mid q^k_{i,j} > 0\right\}\right|.$$

(a) If $(X, \mathcal{R})$ is triply regular, then $N_p \ge N_q$;
(b) If $(X, \mathcal{R})$ is dually triply regular, then $N_q \ge N_p$.

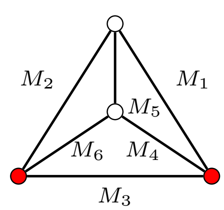
Proof. We prove *(a)* and leave the proof of part *(b)* to the reader. By Theorem 3.8, we have dim $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = N_p$ and dim $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) = N_q$. Since each of these spaces is contained in $\mathcal{T}$, the inequalities both follow from Theorem 3.18. □

Following Hestenes and Higman [13], a strongly regular graph Γ is said to enjoy the *t-vertex condition* if, for any graph G on at most t nodes and any two distinguished nodes $a, b \in V(G)$ the number of graph homomorphisms from G to Γ mapping a to x and b to y depends only on whether x and y are equal, adjacent, or non-adjacent. A recent investigation on this topic is Reichard's paper [20].

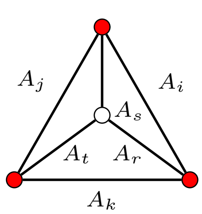
Inspired by this, we say an association scheme $(X, \mathcal{R})$ with Bose-Mesner algebra $\mathbb{A}$ enjoys the *t-vertex condition* if every second order scaffold $\mathsf{S}(G, R; w)$ with t or fewer nodes and edge weights in $\mathbb{A}$ belongs to $\mathbb{A}$.

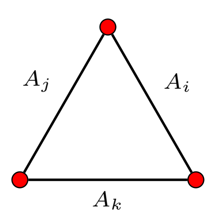
Proposition 3.20. *Every triply regular symmetric association scheme satisfies the 4-vertex condition. Every dually triply regular symmetric association scheme satisfies the 4-vertex condition.*

Proof. Let $(X, \mathcal{R})$ be a triply regular association scheme with Bose-Mesner algebra $\mathbb{A}$. We must prove that, for every choice of edge weights $M_1, \ldots, M_6 \in \mathbb{A}$, the scaffold

$\mathsf{s} =$ 

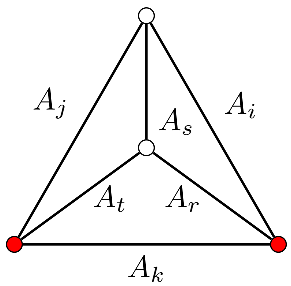
belongs to $\mathbb{A}$. By linearity, we may assume each $M_i \in \{A_0, \ldots, A_d\}$ and employ the triply regular property to write



$= \tau^{r,s,t}_{i,j,k} \cdot$


.

Now the fundamental scaffold at right is simply the sum of elementary tensors $\hat{x} \otimes \hat{y} \otimes \hat{z}$ over all ordered triples $(x, y, z) \in X^3$ with $(x, z) \in R_i$, $(z, y) \in R_j$, $(x, y) \in R_k$. Summing over $z \in X$, we find

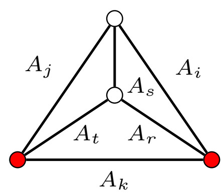


$$= \sigma \sum_{\substack{x,y\in X \\ (x,y)\in R_k}} \hat{x}\otimes\hat{y} \qquad\qquad \text{where} \quad \sigma = \tau^{r,s,t}_{i,j,k}\, p^k_{ij}$$

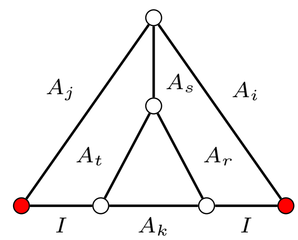
so that the tensor at left is a scalar multiple of A_k. Replacing various M_i in s by matrices I and J, as needed, one obtains the result for any second order scaffold on at most four nodes.

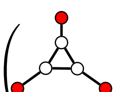
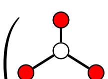
To prove the second statement, we first apply Lemma 1.3 to write

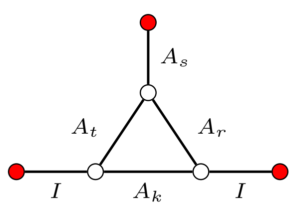
s =



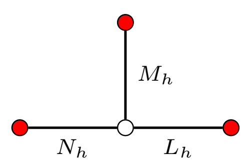
=

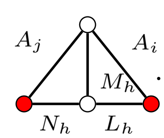


Assuming $\mathsf{W}\Big($  $;\ \mathbb{A}\Big) \subseteq \mathsf{W}\Big($  $;\ \mathbb{A}\Big)$, there exist $L_h, M_h, N_h \in \mathbb{A}$ such that



$= \sum_h$



Substituting this into the above using Proposition 1.5, we find $\mathsf{s} = \sum_h$  .

And now we may apply Theorem 3.11 to show that $\mathsf{s} \in \mathbb{A}$. □

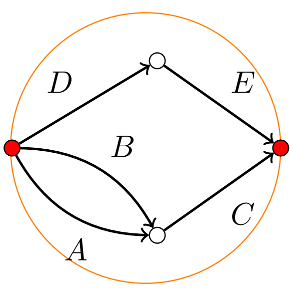
3.6. *The vector space of scaffolds of order two and planarity*

We consider the vector space of scaffolds of order two and various subspaces of this space. In particular, we wish to know, in the setting where edge weights belong to a Bose-Mesner algebra $\mathbb{A}$, when such a subspace is no larger than $\mathbb{A}$ itself. Assume in this subsection that all edge weights belong to the Bose-Mesner algebra $\mathbb{A}$ of some association scheme $(X, \mathcal{R})$.

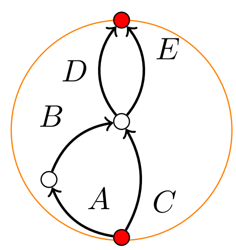
First, since $\mathbb{A}$ is a nonzero subspace of $\mathsf{Mat}_X(\mathbb{C})$, the vector space of scaffolds of order zero is simply $\mathbb{C}$. As we learned in Theorem 3.2, the space of first order scaffolds has dimension equal to the number of orbits (on vertices) of the automorphism group of the scheme. The author does not know of an example where the space of first order scaffolds defined on planar diagrams is strictly smaller than this space.

A *circular planar graph* [5,11] is an ordered pair (G, R) where G is a graph embedded in the plane (i.e., a *plane graph* [9, Section 4.2]) with a distinguished set $R \subseteq V(G)$ of nodes all appearing on the outer face. Let us say that a scaffold $\mathsf{S}(G, R; w)$ is *planar* if (G, R) is a circular planar graph. For fixed m, the vector subspace $\mathsf{P}(m; \mathbb{A})$ spanned by all m^{th} order planar scaffolds with edge weights in $\mathbb{A}$ is worthy of study. An *m-terminal series-parallel graph* is a graph with a distinguished set of m nodes which can be reduced via some sequence of series and parallel edge reductions to a graph on those m nodes only.

Example 3.21. A circular planar graph and its circular planar dual contain equally many terminal nodes. The following pair of examples illustrates the relationship between planar duality and duality in association schemes:



$= [(A \circ B)C] \circ (DE),$

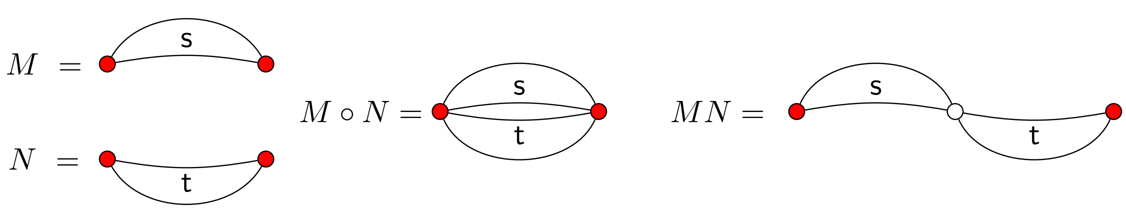


$= [(AB) \circ C]\,(D \circ E).$

Theorem 3.22. *Let $\mathcal{G}$ denote the set of all ordered pairs (G, R) of two-terminal series parallel graphs with root nodes $R = \{r_1, r_2\}$. For any coherent algebra $\mathbb{A}$, we have*

$$\mathbb{A} = \sum_{(G,R) \in \mathcal{G}} \mathbf{W}((G, R); \mathbb{A}).$$

Proof. To prove forward containment is trivial: each $M \in \mathbb{A}$ is expressible as a scaffold whose underlying diagram G is the complete graph on two nodes. If matrices $M, N \in \mathsf{Mat}_X(\mathbb{C})$ correspond to second order planar scaffolds s and t, respectively, then both their matrix product and their entrywise product are expressible as second order planar scaffolds as well.



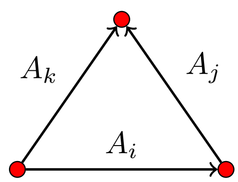
Applying these operations repeatedly, we see that any second order scaffold s having all edge weights in $\mathbb{A}$ whose underlying diagram can be constructed from K_2 by successive subdivision and doubling of edges belongs to $\mathbb{A}$. □

It is well known (cf. [9, Exer. 32,p191]) that a multigraph is series-parallel if and only if it contains no K_4 minor. It is easy to check that $K_{3,3}$ contains a K_4 minor. So it follows by Kuratowski's Theorem that every series-parallel graph is planar. In general, the space of second order planar scaffolds with edge weights in $\mathbb{A}$ can properly contain $\mathbb{A}$. The Doob graphs are convenient examples.

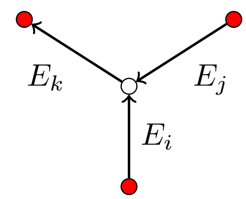
4. Duality of planar scaffolds

The obvious duality between the following two conditions on scaffold identities

$p_{ij}^k = 0 \quad \Leftrightarrow \quad \mathbf{0} =$



and $q_{ij}^k = 0 \quad \Leftrightarrow \quad \mathbf{0} =$



extends to dual pairs of theorems in some cases, as we've seen. We claim that these are instances of a much more general phenomenon.

A *scaffold equation* of order m is an equation of the form

$$\sum_{k=1}^{n} \alpha_k \mathsf{S}_k = \mathbf{0}$$

where each α_k is a scalar, each S_k is a scaffold of order m, $\mathbf{0}$ is the zero tensor of order m, and a bijection $\zeta_{j,k}$ is specified (or understood) between the root nodes of S_k and S_j for each j and k in a consistent manner; i.e., we assume $\zeta_{i,j} \circ \zeta_{j,k} = \zeta_{i,k}$ for each i, j, k and $\zeta_{k,k}$ is the identity map. We note that, throughout this paper, this correspondence of tensor components has been conveniently indicated pictorially by consistent spatial placement of the root nodes. Note that, for fixed d, the P-polynomial condition and the Q-polynomial condition can both be encoded as finite systems of scaffold equations and inequalities.

Some circular planar graphs admit multiple, inequivalent, embeddings in a disk. We may define an augmented graph G^+ by adding an additional node ∞ whose neighbors are exactly those nodes $v \in R$; it is immediate that (G, R) is a circular planar graph if and only if G^+ is a planar graph. Moreover, by a theorem of Whitney, if G^+ is 3-connected, then this planar embedding is essentially unique [9, Theorem 4.3.2].

Let $\mathsf{s} = \mathsf{S}(G, R; w)$ be a planar symmetric scaffold with $m = |R|$ distinct root nodes. Assume the underlying rooted diagram (G, R) is given with a fixed embedding in a closed disk where all root nodes appear on the boundary. Assume, for simplicity, that

$w(e) \in \{A_0, \ldots, A_d\}$ for each edge e. In order to define the *dual scaffold* $\mathsf{s}^\dagger$, we first construct a dual graph $G^\dagger$ which has one node for each face of this embedding. (This circular planar dual can be obtained from the planar dual of graph G^+ by deleting the edges dual to those edges of G^+ incident to the node ∞. So, in contrast to the planar dual of G, this graph has $m = |R|$ nodes on the infinite face, which has been subdivided by the m segments of the boundary of the disk between consecutive root nodes of G.) Each directed edge e of G is rotated 90° counterclockwise to give a directed edge $e^\dagger$ of $G^\dagger$ joining the two faces having e on their boundary (where $e^\dagger$ is a loop if only one face of the original embedding is incident to e). The distinguished ("root") nodes of the dual scaffold are those m faces incident to the bounding disk. The edge weights are then given by $w(e^\dagger) = E_j$ where $w(e) = A_j$. This map from $\mathsf{P}(m; \mathbb{A})$ to $V^{\otimes m}$ is extended linearly as in the notion of a duality map.

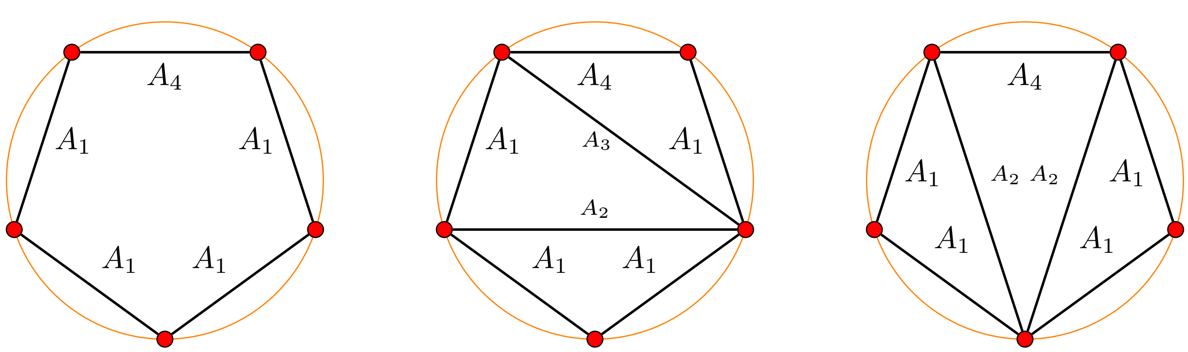
Conjecture 4.1. Suppose we have a collection $\{\mathsf{S}_k\}_{k=1}^n$ of m^{th} order planar scaffolds where all edge weights belong to the set of symbols $A_0, A_1, \ldots, A_\delta$. Assume that roots are paired up via a set of maps $\{\zeta_{j,k} \mid 1 \leq j, k \leq n\}$ where $\zeta_{j,k}$ maps the root nodes of S_k bijectively to the roots S_j according to the consistency rules $\zeta_{i,j} \circ \zeta_{j,k} = \zeta_{i,k}$ for all i, j, k where $\zeta_{k,k}$ is the identity map for each k. Assume that, for all association schemes with $d \geq \delta$ classes, the scaffold equations $\sum_{k=1}^n \alpha_{jk} \mathsf{S}_k = \mathbf{0}$ $(1 \leq j \leq N)$ together imply the scaffold equation $\sum_{k=1}^n \beta_k \mathsf{S}_k = \mathbf{0}$. Then, for any association scheme with $d \geq \delta$ classes, the dual scaffold equations

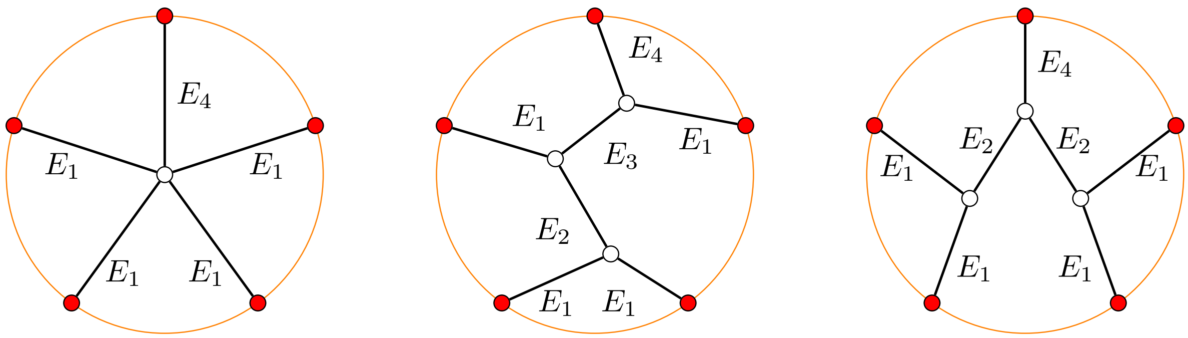
$$\sum_{k=1}^n \alpha_{jk} \mathsf{S}_k^\dagger = \mathbf{0} \qquad (1 \leq j \leq N)$$

together imply the dual scaffold equation

$$\sum_{k=1}^n \beta_k \mathsf{S}_k^\dagger = \mathbf{0}. \quad \square$$

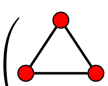
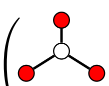
This conjecture allows us to map identities to identities. As an example, we now give three obviously equal scaffolds for P-polynomial schemes and the dual scaffolds which are equal for all Q-polynomial schemes. (Equality is easily shown using the Isthmus Lemma.)





As Jaeger [14, Prop. 5] points out, Epifanov's Theorem establishes that every connected undirected plane graph can be reduced to the trivial graph with one node and no edge via some finite sequence of Δ–Y and Y–Δtransformations, together with extended series-parallel reductions. In the proof of the following theorem, we use a variant of this result which applies to two-terminal planar graphs (see [26]).

In conjunction with Theorem 3.17, our next theorem shows that, for association schemes that are both triply regular and dually triply regular, every second order planar scaffold lies within the Bose-Mesner algebra.

Theorem 4.2. *Let $(X, \mathcal{R})$ be an association scheme with Bose-Mesner algebra $\mathbb{A}$. If* $\mathsf{W}\Big($  $; \mathbb{A}\Big) = \mathsf{W}\Big($  $; \mathbb{A}\Big)$*, then the vector space spanned by all second order planar scaffolds with edge weights in $\mathbb{A}$ is equal to $\mathbb{A}$.*

Proof. We deal only with the case where G is 2-connected, leaving the degenerate cases to the reader. Epifanov's Theorem tells us that any 2-connected two-terminal planar graph is reducible to a single edge joining those two terminals via a finite sequence of operations, each being of one of the following types:

- deletion of a loop
- deletion of a non-terminal node of degree one
- series reduction
- parallel reduction
- Δ–Y transformation
- Y–Δ transformation

We will show that given a second order planar scaffold s with underlying rooted diagram (G, R) and a rooted diagram (H, R) obtained from (G, R) by any one of the above operations, we may express s as a linear combination of scaffolds each of which has (H, R) as its rooted diagram. First consider the case where (H, R) differs from (G, R) by deletion of a loop e'. Since $w(e')$ belongs to Bose-Mesner algebra, it has constant diagonal, σ say, so $\mathsf{S}(G, R; w) = \sigma\mathsf{S}(H, R; w')$ where $w'(e) = w(e)$ for $e \neq e'$. The case where H is obtained from G by deletion of a non-root vertex of degree one is handled similarly via Lemma 1.4. If H is obtained from G via a series reduction or a parallel reduction, we may apply Lemma 1.2. Finally, our hypothesis, combined with Proposition 1.5, gives the desired result in the case where H differs from G by either a Δ–Y transformation or a Y–Δ transformation. □

Note: Note that this does not imply that all triply regular association schemes are Schurian. Theorem 3.2 requires the entire space of second order scaffolds to have dimension $d+1$ and this theorem only considers the space spanned by planar scaffolds.

Declaration of competing interest

There is no competing interest.

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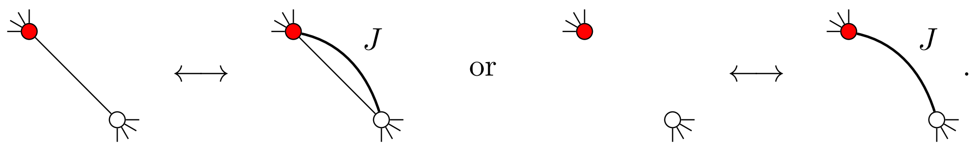
Appendix A. Rules for scaffold manipulation

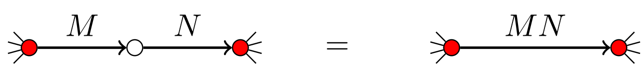
In this appendix, we summarize the rules for the manipulation of scaffolds and give each a label for future reference. In the case of symmetric association schemes, all references to directed edges may be replaced by equivalent language referring to edges. The rules here are given informally with reference to their precise statement in the body of the paper.

SR0 (split node rule) Lemma 1.3(ii): We may split a node, solid or hollow, introducing a new hollow vertex and choosing I as the new edge weight. (Alternatively, we may contract an edge e with $w(e) = I$.)

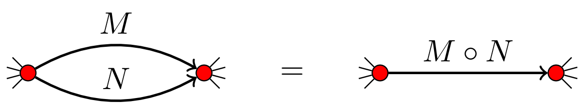
SR0′ (superfluous edge rule) Lemma 1.3(i): Between any two nodes of our diagram, we may insert a new edge e with $w(e) = J$, the all ones matrix. Conversely, edges with weight J may be deleted.



SR1 (series reduction) Lemma 1.2(i): We may suppress a hollow node of degree two by taking the matrix product of the two edge weights.

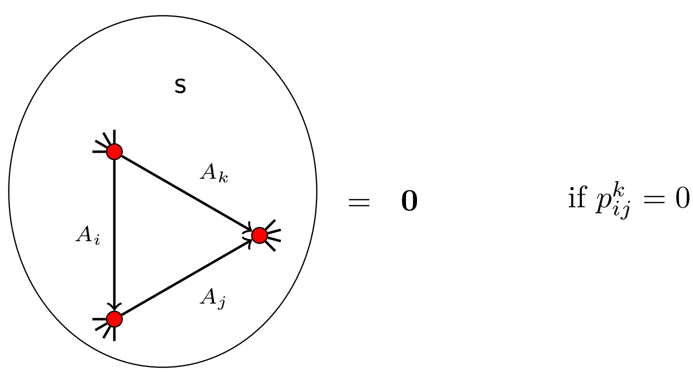


SR1′ (parallel reduction) Lemma 1.2(ii): We may replace two parallel edges by a single edge by taking the entrywise product of the two edge weights.

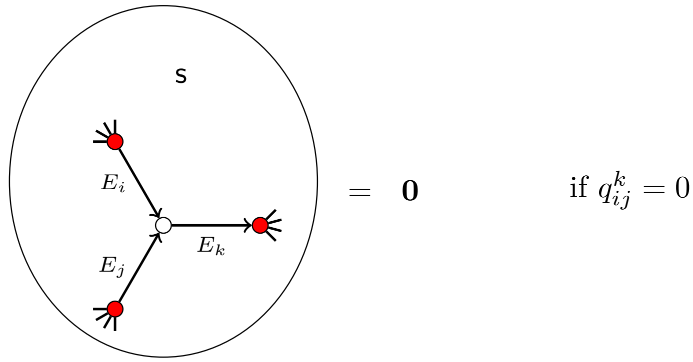


Note: Scaffold manipulation rules SR2 through SR4′ apply within the scope of Bose-Mesner algebras; edge weights follow standard notational conventions for association schemes.

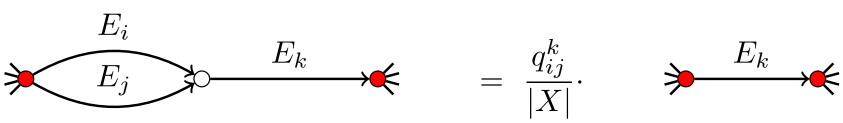
SR2 (vanishing intersection number) Lemma 2.1: Any scaffold containing a directed triangle a, b, c with $w(a, b) = A_i$, $w(b, c) = A_j$, $w(a, c) = A_k$ where $p^k_{ij} = 0$ is the zero tensor.



SR2′ (vanishing Krein parameter) Lemma 2.1: Any scaffold containing a hollow node x of degree three with neighbors a, b, c such that $w(a, x) = E_i$, $w(b, x) = E_j$, $w(x, c) = E_k$ where $q^k_{ij} = 0$ is the zero tensor.

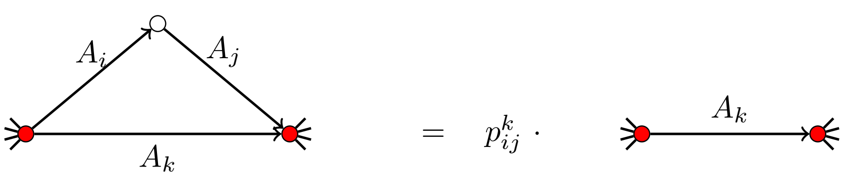


SR3 (pinched star) Equation (2.8):



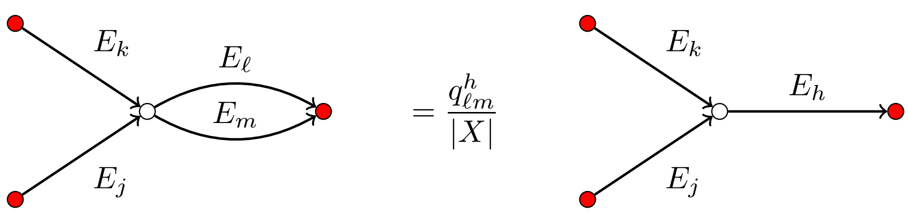
and this is the zero tensor if $q_{ij}^k = 0$.

SR3′ (hollow triangle) Equation (2.9):

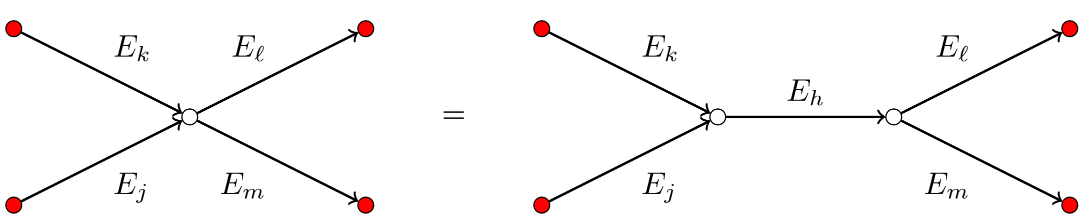


and this is the zero tensor if $p_{ij}^k = 0$.

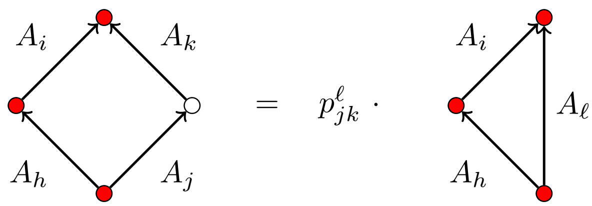
SR4 (Isthmus) Lemma 2.3: If $q_{jk}^e \cdot q_{\ell m}^e = 0$ for all $e \neq h$, then



and



SR4′ (Dual isthmus) Lemma 2.5: If $p^e_{hi} \cdot p^e_{jk} = 0$ for all $e \neq \ell$, then



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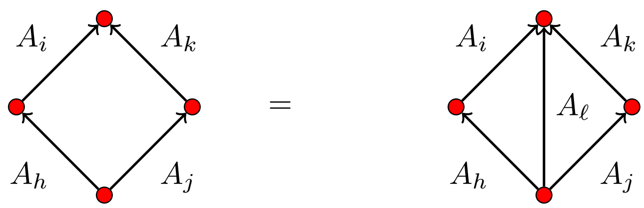
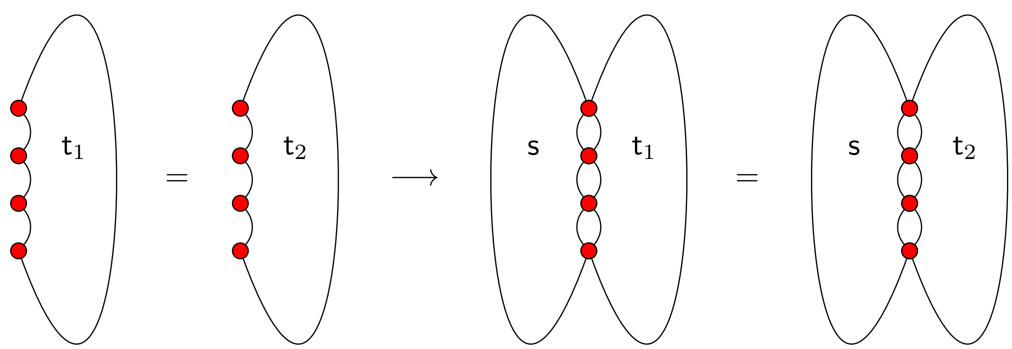
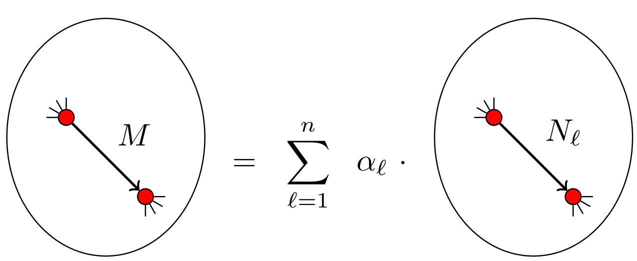


SR5 (substitution) Proposition 1.5: If t_1 and t_2 are scaffolds on the same ordered multiset R of roots such that $\mathsf{t}_1 = \mathsf{t}_2$ and $R' = \{r_1, \ldots, r_\ell\} \subseteq R$, then for any scaffold s and any root nodes $u_1, \ldots, u_\ell$ in the rooted diagram of s, we have $\mathsf{s} +_\xi \mathsf{t}_1 = \mathsf{s} +_\xi \mathsf{t}_2$ where $\xi(u_i) = r_i$.

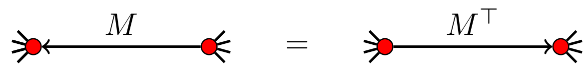


SR6 (multilinearity): If scaffolds s and $\mathsf{s}_1, \ldots, \mathsf{s}_n$ are identical except in their weight on one edge e where $w(e) = M$ in s and $w(e) = N_\ell$ in s_ℓ $(1 \leq \ell \leq n)$ where $M = \sum_{\ell=1}^n \alpha_\ell N_\ell$, then $\mathsf{s} = \sum_{\ell=1}^n \alpha_\ell \mathsf{s}_\ell$.

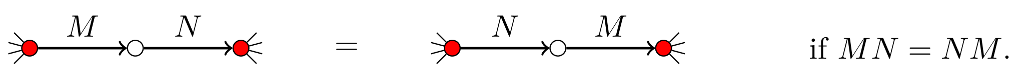


if $M = \sum_{\ell=1}^n \alpha_\ell N_\ell$

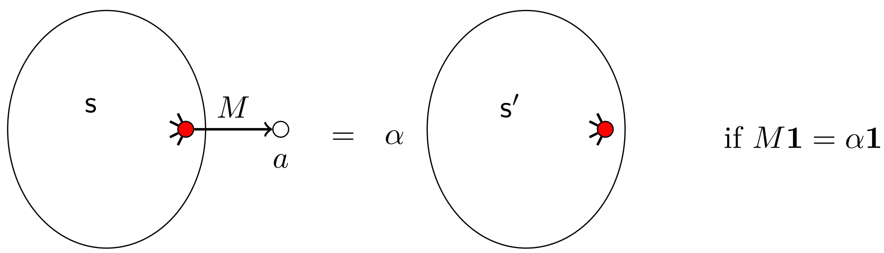
SR7 (Transpose property): Reversing the direction of an edge in diagram G is equivalent to replacing the weight of that edge by its transpose.



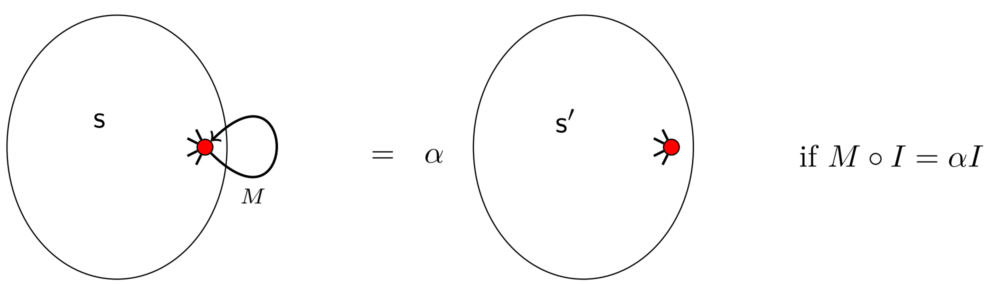
SR8 (Commutative property): If $a \in V(G)$ is a hollow node incident to just two edges e and e' where $w(e)w(e') = w(e')w(e)$, then swapping the weights on these edges leaves the scaffold unchanged.



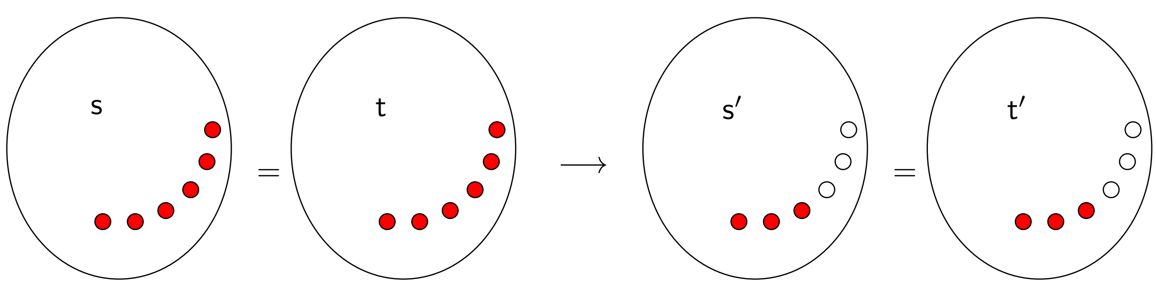
SR9 (Degree one vertices): Lemma 1.4: Assuming constant row sum or column sum (as appropriate) on the edge weight, a hollow node of degree one may be deleted.



SR9′ (Loops): If e' is a loop in G and $w(e') \circ I = \alpha I$, then $\mathsf{s} = \alpha \mathsf{s}'$ where s' is obtained from s by deletion of edge e':

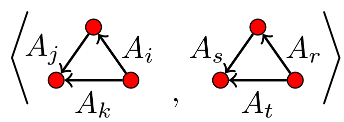


SR10 (Order Reduction) Lemma 1.6: Equality is preserved in passing from root nodes R to a proper submultiset R'.

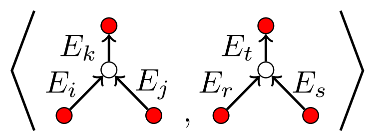


Appendix B. Inner products of common third order scaffolds

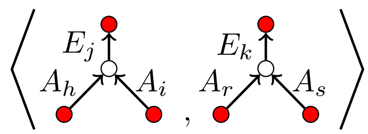
Here we use (3.1) to compute some inner products of third order scaffolds attached to (commutative) association schemes without proof. Note that $\bar{E}_j = E_j^\top = E_{j'}$ and that, in the symmetric case, edge orientations may be ignored. By an abuse of notation, we write $A_i^\top = A_{i'}$ with this definition of i' operative only in equations not involving any element from $\{E_0, \ldots, E_d\}$. Only in special cases is an inner product of two scaffolds expressible in terms of association scheme parameters.

(B.1)


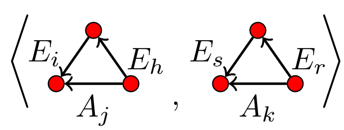
$$= |X| \; \delta_{i,r} \; \delta_{j,s} \; \delta_{k,t} \; v_k \; p_{ij}^k$$

(B.2)


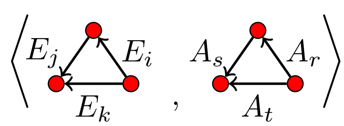
$$= \frac{1}{|X|} \delta_{i,r} \; \delta_{j,s} \; \delta_{k,t} \; m_k \; q_{ij}^k$$

(B.3)


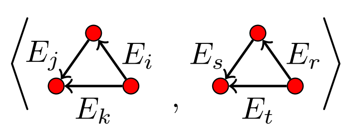
$$= \delta_{j,k} \; m_j \sum_{\ell=0}^{d} P_{j\ell} \; p_{h'r}^{\ell} \; p_{i's}^{\ell}$$

(B.4)


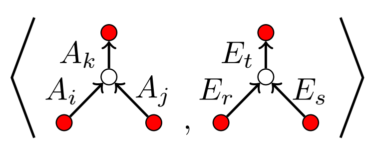
$$= \delta_{j,k} \; v_j \sum_{\ell=0}^{d} Q_{j\ell} \; q_{h'r}^{\ell} \; q_{i's}^{\ell}$$

(B.5)


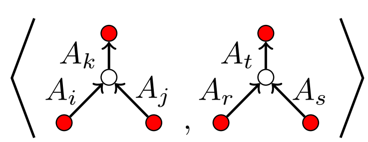
$$= |X|^{-2} \; Q_{ri'} \; Q_{sj'} \; Q_{tk'} \; v_t \; p_{rs}^t$$

(B.6)


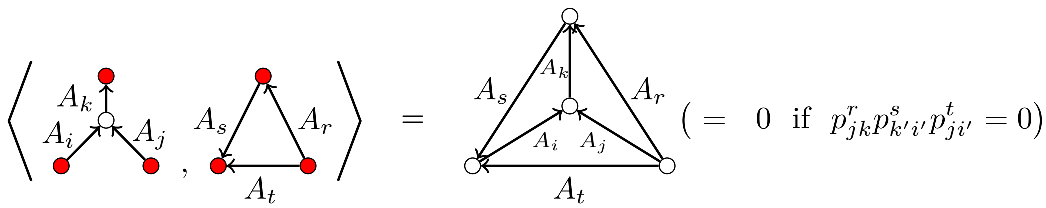
$$= |X|^{-3} \sum_{\ell=0}^{d} m_\ell \; q_{i',r}^{\ell} \; q_{j',s}^{\ell} \; q_{k',t}^{\ell'}$$

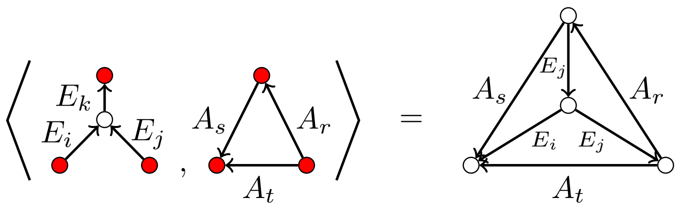
(B.7)


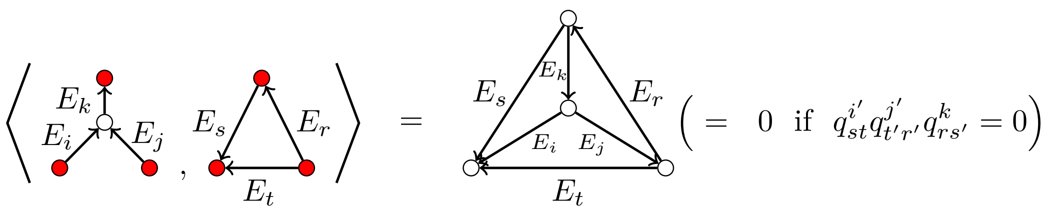
$$= |X|^{-1} \; P_{ri'} \; P_{sj'} \; P_{tk'} \; m_t \; q_{rs}^t$$

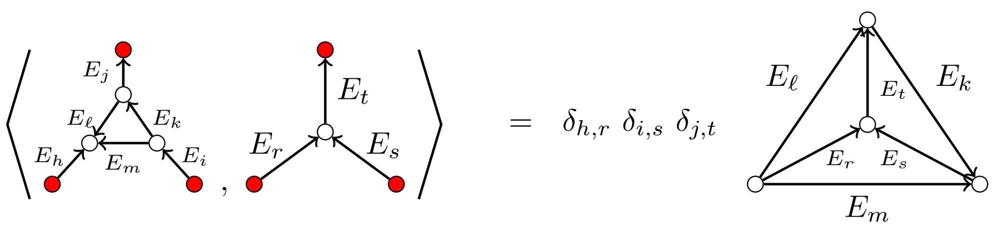
(B.8)


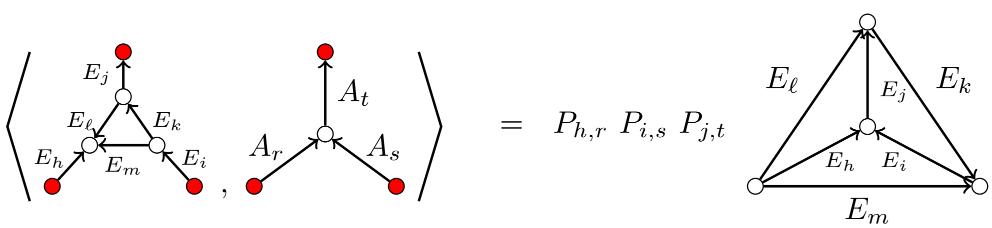
$$= |X| \sum_{\ell=0}^{d} v_\ell \; p_{i',r}^{\ell} \; p_{j',s}^{\ell} \; p_{k',t}^{\ell'}$$

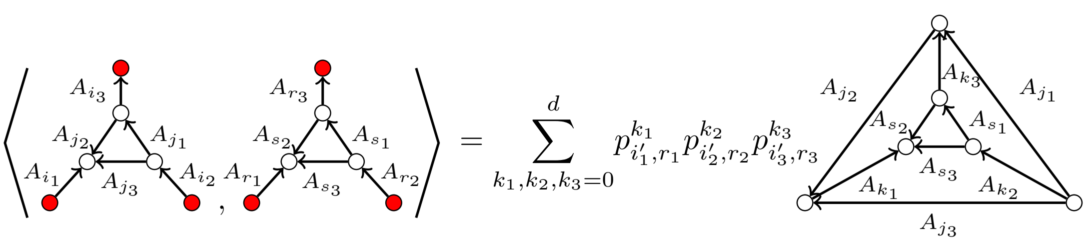
(B.9)


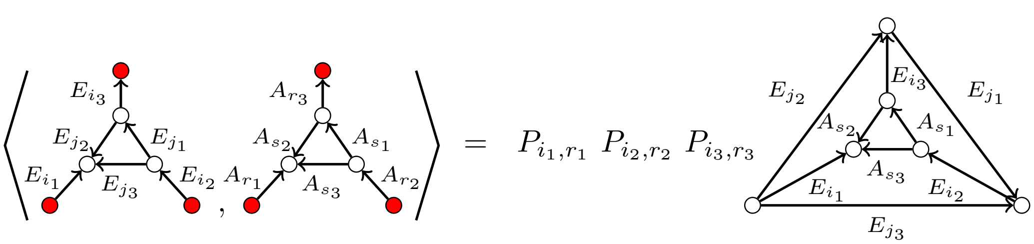
(B.10)


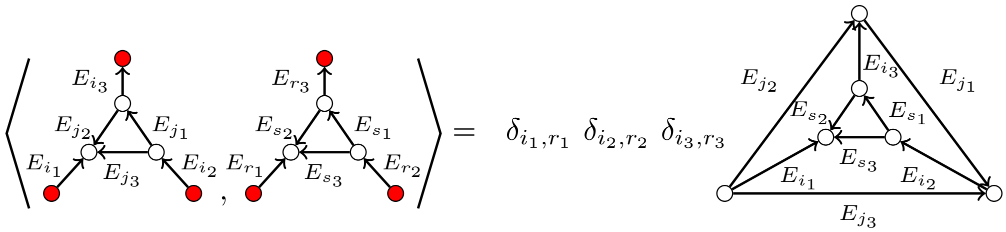
(B.11)


(B.12)


(B.13)


(B.14)


(B.15)


(B.16) 

NOTE: Many of these identities are not new. For example, (B.1) can be found in [14, Equation (43)] and follows from [1, Theorem II.3.6(ii)], [24, Lemma 3.2], while (B.2) stems from [3, Proposition 5.1], can be found in [14, Equation (42)], [1, Theorem II.3.6(i)], [24, Lemma 3.2], and [24, Lemma 3.2] as well.

References

[1] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin-Cummings, Menlo Park, 1984.
[2] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.
[3] P.J. Cameron, J.-M. Goethals, J.J. Seidel, The Krein condition, spherical designs, Norton algebras and permutation groups, Proc. K. Ned. Akad. Wet., Indag. Math. 40 (2) (1978) 196–206.
[4] K. Coolsaet, A. Jurišić, Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs, J. Comb. Theory, Ser. A 115 (6) (2008) 1086–1095.
[5] E.B. Curtis, D. Ingerman, J.A. Morrow, Circular planar graphs and resistor networks, Linear Algebra Appl. 283 (1998) 115–150.
[6] E.R. van Dam, J.H. Koolen, H. Tanaka, Distance-regular graphs, Electron. J. Comb. DS22 (2016).
[7] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep., Suppl. 10 (1973).
[8] G.A. Dickie, Q-polynomial structures for association schemes and distance-regular graphs, Ph.D. thesis, University of Wisconsin, 1995.
[9] R. Diestel, Graph Theory, Springer-Verlag, Heidelberg, 1997, electronic edition, 2005.
[10] A.L. Gavrilyuk, S. Suda, J. Vidali, On tight 4-designs in Hamming association schemes, Combinatorica 40 (2020) 345–362.
[11] I. Gitler, I. López, On topological spin models and generalized $\Delta - Y$ transformations, Adv. Appl. Math. 32 (2004) 263–292.
[12] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.
[13] M.D. Hestenes, D.G. Higman, Rank 3 groups and strongly regular graphs, in: Proc. Symp. Applied Math., Amer. Math. Soc., Providence RI, 1971, pp. 141–160.
[14] F. Jaeger, On spin models, triply regular association schemes, and duality, J. Algebraic Comb. 4 (1995) 103–144.
[15] L. Lovász, Large Networks and Graph Limits, AMS Colloquium Publications, vol. 60, Amer. Math. Soc., 2012.
[16] W.J. Martin, H. Tanaka, Commutative association schemes, Eur. J. Comb. 30 (2009) 1497–1525.
[17] W.J. Martin, Scaffolds: a graph-based system for computations in Bose-Mesner algebras, preprint, https://arxiv.org/abs/2001.02346, January 2020.
[18] S. Penjić, A. Neumaier, A unified view of inequalities for distance-regular graphs. Part I, manuscript, https://www.mat.univie.ac.at/~neum/ms/uniDRG1.pdf, 2018.
[19] S. Penjić, A. Neumaier, A unified view of inequalities for distance-regular graphs. Part II, manuscript, https://www.mat.univie.ac.at/~neum/ms/uniDRG2.pdf, 2018.
[20] S. Reichard, Strongly regular graphs with the 7-vertex condition, J. Algebraic Comb. 41 (2015) 817–842.
[21] H. Suzuki, Imprimitive Q-polynomial association schemes, J. Algebraic Comb. 7 (1998) 165–180.
[22] H. Suzuki, Association schemes with multiple Q-polynomial structures, J. Algebraic Comb. 7 (1998) 181–196.
[23] P. Terwilliger, A characterization of P- and Q-polynomial association schemes, J. Comb. Theory, Ser. A 45 (1987) 8–26.

[24] P. Terwilliger, The subconstituent algebra of an association scheme (part I), J. Algebraic Comb. 1 (1992) 363–388.
[25] P. Terwilliger, Course lecture notes, Math 846 Algebraic Graph Theory, Spring term 2009, University of Wisconsin, http://www.math.wisc.edu/~terwilli/Htmlfiles/part2.pdf.
[26] K. Truemper, On the delta-wye reduction for planar graphs, J. Graph Theory 13 (1989) 141–148.
[27] D. Zhao, On eigenmatrices of polynomial association schemes, Lecture notes. G2R2@Novosibirsk, August 7, 2018.