Quantum Computational Advantage with String Order Parameters of One-Dimensional Symmetry-Protected Topological Order

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Nonlocal games with advantageous quantum strategies give arguably the most fundamental demonstration of the power of quantum resources over their classical counterparts. Recently, certain multiplayer generalizations of nonlocal games have been used to prove unconditional separations between limited computational complexity classes of shallow-depth circuits. Here, we show advantageous strategies for these nonlocal games for generic ground states of one-dimensional symmetry-protected topological orders (SPTOs), when a discrete invariant of a SPTO known as a twist phase is nontrivial and -1. Our construction demonstrates that sufficiently large string order parameters of such SPTOs are indicative of globally constrained correlations useful for the unconditional computational separation.

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Introduction.—Entanglement underlies nonclassical features of quantum mechanics. On one hand, local hidden variable models cannot produce nonlocal quantum correlations [1,2]. This idea is elegantly illustrated with nonlocal games [3,4], whereby players who implement strategies utilizing entangled resources can accomplish a distributed computational task without classical communication. Moreover, in Ref. [5], it was shown that local hidden variable models assisted by even a limited amount of classical communication fail to mimic Pauli-measurement outcomes on graph states [6]. On the other hand, contextuality [7–11], the degree to which locally incompatible measurements evade global explanation, is another nonclassical feature related to the hardness of computation and quantum advantage [12–19]. Combining these features, seminal works by Bravyi *et al.* [20] and others [21–25] compared certain many-body generalizations of nonlocal games assisted by limited classical communication to classical computation with bounded fan-in gates. This perspective is successful in proving unconditional exponential separations between limited computational complexity classes, demonstrating the power of shallow quantum circuits over their classical counterparts.

Advantageous quantum strategies for these multiplayer games possess two key properties: contextuality of the measurements performed and long-range entanglement accessible by arbitrarily distant players. Motivated by this key observation, we establish a general connection between the shared quantum resource and many-body entanglement ubiquitously present in ground states of quantum phases of matter called symmetry-protected topological order (SPTO) [26–29]. Namely, we show that local measurements that collectively resolve global measurements of symmetries and so-called twist phases [30] (an invariant of 1D SPTO phases with an Abelian symmetry group) give a desired state-dependent contextuality property. Furthermore, the string order parameter [31,32], a nonlocal order parameter of 1D SPTO related to the long-ranged order [33–37], gives the desired entanglement structure, which is known to be useful for measurement-based quantum computation (MBQC) [38–48].

Our Letter indicates that the aforementioned computational separation between shallow-depth classical and quantum circuits carries over to generic 1D SPTO ground states. This will be illustrated using various states in the 1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTO phase, such as the cluster state [49,50] and the Affleck-Kennedy-Tasaki-Lieb (AKLT) state [51]. It is intriguing to see how the stringlike correlations of 1D SPTO states have similar utility as the two-point correlations of the Greenberger-Horne-Zeilinger (GHZ) state, as the so-called GHZ paradox [52] has been a canonical example in nonlocal games and nonadaptive MBQC [12,53–56]. Our result assists to tighten an inherent connection between MBQC, contextuality, and group cohomology pursued in Refs. [57–62]. In comparison, however, our obstruction to a noncontextual description of the triangle game below arises directly from a cohomological signature of 1D SPTOs. Our results also complement studies of nonlocality in many-body systems [63–66]. Last but not least, as quantum simulation of various 1D SPTO states is of broad interest in experimental realizations [67–69], our construction may pave a way toward observation of quantum computational advantage using 1D SPTO and its string order parameter.

The triangle game.—We begin with a motivating example adapted from Refs. [5,20], as seen in Fig. 1. Consider a game where three players, indexed by $j \in \{0,1,2\}$, each receive a random input bit $x_j \in \{0,1\}$. Each fills a three-bit

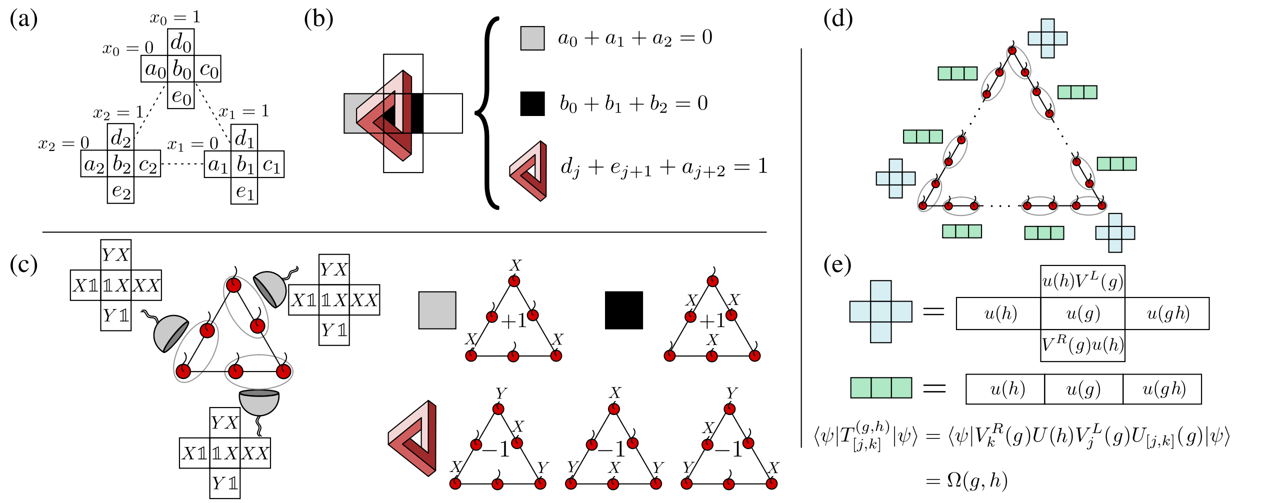

FIG. 1. Triangle game. (a) Players fill in the row or column of their table with a binary string if their input is 0 or 1, respectively. (b) The win conditions. Apart from the global even parity of the output, the dark and light shaded boxes denote that each entry in the row jointly have even parity. The Penrose triangle represents the condition that the top, bottom, and left entries of any clockwise ordering of the three players have odd parity. (c) Quantum strategy for the triangle game. For each pair of Pauli observables in the table, the left one corresponds to the qubit located at the corresponding corner of the triangle. Perfection of the strategy is ensured by five cluster state stabilizers, whose eigenvalues are ± 1 as shown. (d) Multiplayer triangle game (see [70]). Three arbitrary players, depicted at the corners of the triangle, measure the same Pauli observables as before on the $2n$-qubit cluster state and otherwise measure along the row. They still win the original game perfectly (up to inconsequential additional outputs by the other $n-3$ players). (e) Perfect quantum strategy on 1D SPTO fixed-point states is ensured when players measure on-site symmetry and boundary operators. The Penrose-triangle constraints in (c) manifest as a collective measurement of twisted string order parameters whose expectation value is an invariant of SPTOs, called a twist phase $\Omega(g,h)$, equal to -1.

string $\mathbf{y}_j \in \{0,1\}^3$ in the row or column of the table of Fig. 1(a) if $x_j = 0$ or 1, respectively. Suppose the players do not communicate and produce outputs dependent only on their given input, i.e., $\mathbf{y}_j = \mathbf{y}_j(x_j)$. The values recorded in the row or column of each table can be written $\mathbf{y}_j(0) = (a_j, b_j, c_j)$ and $\mathbf{y}_j(1) = (d_j, b_j, e_j)$. The players win the game whenever the full output string $(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \in \{0,1\}^9$ has even parity and

$$a_0 + a_1 + a_2 = 0, \tag{1a}$$

$$b_0 + b_1 + b_2 = 0, \tag{1b}$$

$$d_j + e_{j+1} + a_{j+2} = 1 \quad \forall \ j \in \{0,1,2\}. \tag{1c}$$

Notice that while Eq. (1b) must hold for all inputs, Eqs. (1a) and (1c) are input-dependent constraints. However, because summing Eqs. (1a)–(1c) gives $\sum_{j=0}^{2}(d_j + b_j + e_j) = 1$, the total output string for the input $\mathbf{x} = (1,1,1)$ cannot have even parity. This implies that the classical winning probability is bounded above by $\frac{7}{8}$, by failing on at least one of eight inputs.

On the contrary, there is a perfect quantum strategy for this game. A quantum strategy for a nonlocal game is a tuple $(\rho, \mathcal{C}_x)$ consisting of a shared quantum state ρ and "contexts," sets of pairwise commuting local observables $\mathcal{C}_x = \{A_k(x)\}_k$ to be measured, for each $x \in \{0,1\}$. Let X, Y, and Z be the Pauli matrices and $\mathbb{1}$ be the identity matrix. Each player j holds qubits $2j$ and $2j+1$ from the 6-qubit 1D cluster state, $|\psi_{1\mathrm{DC}}\rangle = \prod_{k=0}^{5} CZ_{k,k+1}|+\rangle^{\otimes 6}$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $CZ = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z$ is the 2-qubit controlled-Z gate. $|\psi_{1\mathrm{DC}}\rangle$ is a stabilizer state, the joint $+1$ eigenstate of a commuting set of Pauli observables generated by $Z_k X_{k+1} Z_{k+2}$ for $k = 0, \ldots, 5$. Each player measures the 2-qubit Pauli observables from the horizontal or vertical contexts shown in Fig. 1(c) and records the outcomes in the table. The observables in each row and column multiply to the identity, constraining the measurement outcomes to form a string of even parity. Certain observables in each player's table collectively form stabilizers up to a sign, shown in Fig. 1(c), implying Eqs. (1a)–(1c) are satisfied. These stabilizers form an identity product [74], giving state-dependent contextuality.

Moreover, as described in Refs. [5,20], this game has a multiplayer generalization (see [70] for details). In each round of the game, three arbitrary players, labeled α, β, and γ, are given a bit from the input $\mathbf{x} \in \{0,1\}^3$ and each player outputs a three-bit string. The new win conditions are equivalent to Eqs. (1a)–(1c) up to the parity of a "correction string" given by the outputs of the other players. The corresponding quantum strategy utilizes a $2n$-qubit 1D cluster state and measurements in the same contexts, as depicted in Fig. 1(d). In this generalized n-player setting, quantum players can outperform even locally communicating classical players. For large enough n, a constant number of rounds of classical communication between players that are nearby with respect to the cycle cannot create the same global correlations attained in the quantum setting since distant players cannot communicate. These nonclassical, long-ranged correlations that persist for the quantum strategy in the large-n (i.e., thermodynamic) limit are naturally indicative of 1D symmetry-protected topological order.

Symmetry-protected topological order.—A 1D SPTO phase is topologically ordered in the presence of symmetry G, in that each ground state cannot be connected smoothly to a product state via symmetry-respecting perturbations. In the following, we focus on a global symmetry G that forms a finite group. The topological nature gives ground-state degeneracy dependent on boundary conditions. At the open boundary, there appear effective degrees of freedom that transform under a *projective* representation of G.

Algebraically, a projective representation of G is a collection of unitary matrices $\{V(g)\}_{g\in G}$ that obey the group multiplication law up to a G-dependent phase, i.e.,

$$V(g)V(h) = \omega(g,h)V(gh), \tag{2}$$

where $\omega(g,h) \in U(1)$ is called a 2-cocycle. Inequivalent projective representations, and thus 1D SPTO phases, are classified by a multiplicative group $H^2(G, U(1))$ called the second group cohomology [75], whose elements are equivalence classes of 2-cocycles, called cohomology classes, denoted $[\omega] \in H^2(G, U(1))$.

Symmetry twists.—The cohomological properties can be probed, even under periodic boundary conditions, by introducing artificial boundaries called symmetry twists [30,76]. Consider a system of n sites with global symmetry G carrying on-site representation $U(g) = u(g)^{\otimes n}$. Denote as $U_{[j,k]}(g) = \otimes_{j+1}^{k-1} u(g)$ a truncated symmetry operator acting only between sites j and k. Symmetry twists are low energy excitations that appear about sites j and k when $U_{[j,k]}(g)$ acts on the 1D SPTO ground state $|\psi\rangle$. In general, there are local operators $V_j^L(g)$ and $V_k^R(g)$, called "boundary operators," supported in the vicinity of sites j and k that annihilate the symmetry twists. Mathematically, this is realized by the trivial action of

$$S_{[j,k]}(g) = [V_j^L(g) \otimes V_k^R(g)]U_{[j,k]}(g) \tag{3}$$

on the state, i.e., $S_{[j,k]}(g)|\psi\rangle = |\psi\rangle$. The expectation value of $S_{[j,k]}(g)$ gives a string order parameter that characterizes the long-range order in the 1D SPTO phase [31,32]. For translationally invariant systems, one may drop the site dependence on $V_j^L(g)$ and $V_k^R(g)$.

We remark that the boundary operators are not universal (i.e., they vary for different states in the phase). Thus we focus on fixed-point boundary operators, which are defined with respect to the fixed-point state of the 1D SPTO phase obtained under renormalization group flow [29,77,78]. Hereafter, we redefine $V^L(g)$ and $V^R(g)$ to be the fixed-point boundary operators, which have local support of size m, typically two. In [70], we construct $V^L(g)$ and $V^R(g)$ explicitly from matrix-product state (MPS) representations of the fixed-point state. In the $[\omega]$-class 1D SPTO phase, operators $\{V^R(g)\}_{g\in G}$ and $\{V^L(g)\}_{g\in G}$ form projective representations of G residing in cohomology class $[\omega]$ and $[\omega^*]$, respectively. At the same site, they satisfy

$$V^R(g)V^R(h) = \omega(g,h)V^R(gh), \tag{4}$$

$$V^L(g)V^L(h) = \omega(g,h)^* V^L(gh), \tag{5}$$

$$V^R(g)V^L(h) = V^L(h)V^R(g), \tag{6}$$

$$V^R(g)V^L(g) = u(g)^{\otimes m}. \tag{7}$$

We prove Eqs. (4)–(7) in [70].

Twist phase and twisted string order parameter.—1D SPTO phases possess an invariant called a twist phase $\Omega(g,h) \in \mathrm{U}(1)$ [30] defined as

$$\Omega(g,h) = \frac{\omega(g,h)}{\omega(h,g)}. \tag{8}$$

For Abelian G, this object depends only on the cohomology class $[\omega]$. Conveniently, this phase is simply the overall phase accumulated upon commuting the projective representations of g and h through each other, i.e., $V(g)V(h) = \Omega(g,h)V(h)V(g)$.

In comparison to Eq. (3), it is convenient to define the "twisted" string order parameter as the expectation value of an operator

$$T_{[j,k]}^{(g,h)} = V_k^R(g)U(h)V_j^L(g)U_{[j,k]}(g). \tag{9}$$

By Eqs. (3)–(7), its expectation value on the fixed-point state is the twist phase

$$\langle\psi|T_{[j,k]}^{(g,h)}|\psi\rangle = \Omega(g,h). \tag{10}$$

See [70] for a proof of Eq. (10).

SPTO triangle game strategy from symmetry twists.—Now we present the main result of this Letter. We show that the measurement of twist phases for a particular class of 1D SPTO phases can be repurposed as a quantum strategy for the multiplayer triangle game. As on-site symmetries will always be measured over m sites, henceforth we redefine $u(g)$ to denote $u(g)^{\otimes m}$ for ease of the notation.

Lemma 1.—Consider a 1D SPTO ground state with a finite Abelian symmetry group G containing elements $g, h \in G$ such that the twist phase $\Omega(g,h) = -1$. There are two overlapping contexts of local observables by which the twisted string order parameter $T_{[j,k]}^{(g,h)}$ of Eq. (9) is composable.

Proof.—The operators appearing in $T^{(g,h)}_{[j,k]}$ can be organized in the following:

$$\begin{array}{|c|c|c|} \hline & u(h)V^L(g) & \\ \hline u(h) & u(g) & u(gh) \\ \hline & V^R(g)u(h) & \\ \hline \end{array} \; . \tag{11}$$

Since G is Abelian, all on-site symmetry operators in the row commute. By Eqs. (6) and (7), $u(g)$ commutes with $u(h)V^L(g)$ and $V^R(g)u(h)$. Finally, by Eqs. (4)–(6), $[u(h)V^L(g)][V^R(g)u(h)]{=}\Omega(g,h)^2 [V^R(g)u(h)][u(h)V^L(g)]$. Thus the operators in the column commute if and only if $\Omega(g,h) = \pm 1$. ■

Theorem 1.—Consider a 1D SPTO phase with a symmetry group G as described in Lemma 1. Any fixed-point ground state in the phase allows a perfect quantum strategy for the multiplayer triangle game.

Proof.—Suppose each player holds a block of m constituent particles from the 1D SPTO fixed point $|\psi\rangle$. Each player measures their block in the horizontal or vertical context of Eq. (11) if they are given input 0 or 1, respectively. By Eq. (7), the product of all operators in either context of Eq. (11) is $u(g^2h^2)$, so collectively the players measure global symmetry $U(g^2h^2)$ and the product of all outcomes is $+1$. Regardless of the input, each player measures $u(g)$ and thus they collectively measure global symmetry $U(g)$, implying Eq. (1b) is satisfied. For the input (0, 0, 0), each player measures $u(h)$, so collectively they measure global symmetry $U(h)$, implying Eq. (1a) is satisfied. Finally, for the input (1, 1, 0), player 0 measures $u(h)V^L(g)$, player 1 measures $V^R(g)u(h)$, and player 2 measures $u(h)$. Collectively they measure the three-site twisted string order parameter $T^{(g,h)}_{[0,1]}$ and the joint outcome is $\Omega(g,h) = -1$, by Eq. (10). Permutations of this argument show that Eq. (1c) is satisfied. The strategy for the multiplayer version follows accordingly. ■

Examples in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTO phase.—The simplest SPTO phase in which Theorem 1 holds is the nontrivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1D SPTO phase. The complete set of twist phases, given by $\Omega((a,b),(c,d)) = (-1)^{ad+bc}$ for $(a,b),(c,d) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, is identical to the Pauli algebra. We show how Theorem 1 encompasses the quantum strategy for the triangle game discussed above, and then extend Theorem 1 to generic states outside the fixed point. We illustrate these results using the 1D cluster and AKLT states, respectively. Both are known to be useful as 1D quantum logical wires in MBQC [50,79–82].

The 1D cluster state [49,50] is the fixed point of this phase. The on-site symmetry and boundary operators are

$$u[(a,b)] = X^a \otimes X^b, \tag{12a}$$

$$V^R[(a,b)] = Z^b X^a \otimes Z^a, \tag{12b}$$

$$V^L[(a,b)] = Z^b \otimes Z^a X^b, \tag{12c}$$

for $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Taking $g = (0,1)$ and $h = (1,0)$ in Eq. (11) gives the strategy presented in Fig. 1(c).

To study generic states beyond the fixed point, we will refer to the set of measurements to be performed as the protocol. The protocol corresponding to the contexts of Eq. (11) constructed with the fixed-point boundary operators will be referred to as the "fixed-point protocol." The quantum strategy formed by the fixed-point protocol implemented on arbitrary states in the phase no longer wins with unit probability, but extends Theorem 1 as follows.

Theorem 2.—Consider an arbitrary 1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTO ground state $|\phi\rangle$ and let $\langle\mathcal{S}\rangle = \min_{g\in G}\{\langle\phi|S_{[j,k]}(g)|\phi\rangle\}$ be the minimal value of any string order parameter constructed from the fixed-point boundary operators. The fixed-point protocol of Theorem 1 implemented on the state $|\phi\rangle$ yields an advantageous quantum strategy [i.e., $\mathrm{pr(win)} > 7/8$] whenever $1/3 < \langle\mathcal{S}\rangle \le 1$.

Proof.—In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTO phase $S_{[j,k]}(g)^2 = 1$ and each ground state $|\phi\rangle$ is symmetric. Denote by $\mathrm{pr}_\phi(\pm 1|O)$ the probability that the joint measurement outcome of a dichotomic observable O on $|\phi\rangle$ has parity ± 1. By definition, $\mathrm{pr}_\phi(\pm 1|O) = (1 \pm \langle\phi|O|\phi\rangle)/2$, so $\mathrm{pr}_\phi[+1|U(g)]\forall g$. Because $T^{(g,h)}_{[j,k]}{=}\Omega(g,h)U(h)S_{[j,k]}(g)$ by Eq. (7), $\mathrm{pr}_\phi(-1|T^{(g,h)}_{[j,k]}){=}\mathrm{pr}_\phi[+1|S_{[j,k]}(g)]{\ge}(1{+}\langle\mathcal{S}\rangle)/2$ when $\Omega(g,h) = -1$. Averaging over the eight possible inputs for the triangle game, we find $\mathrm{pr(win)}{=} \frac{5}{8}\mathrm{pr}_\phi[+1|U(g)]{+}\frac{3}{8}\mathrm{pr}_\phi(-1|T^{(g,h)}_{[j,k]}){\ge}(13{+}3\langle\mathcal{S}\rangle)/16$. Thus, $\mathrm{pr(win)} > 7/8$ whenever $1/3 < \langle\mathcal{S}\rangle \le 1$. ■

Theorem 2 extends the quantum advantage to more realistic states that are neither fixed-point states nor Pauli stabilizer states. The AKLT state [51] is a spin-1 antiferromagnetic state that is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTO ground state of a two-body interacting Hamiltonian (as opposed to the three-body interactions of the 1D cluster state). Over two spin 1's, we introduce $|\tilde{e}\rangle$ as the singlet and $\{|\tilde{x}\rangle, |\tilde{y}\rangle, |\tilde{z}\rangle\}$ as a Cartesian basis of the triplet (see [70] for mathematical definitions). Using $\mu \in \{z,x,y\}$ to denote group elements $\{(0,1),(1,0),(1,1)\}$, respectively, the on-site symmetry and fixed-point boundary operators are

$$u(\mu) = \exp{(i\pi S^\mu)}^{\otimes 2}, \tag{13a}$$

$$V^R(\mu) = |\tilde{e}\rangle\langle\tilde{\mu}| + |\tilde{\mu}\rangle\langle\tilde{e}| + i \sum_{\nu,\gamma\in\{x,y,z\}} \epsilon_{\mu\nu\gamma}|\tilde{\nu}\rangle\langle\tilde{\gamma}|, \tag{13b}$$

$$V^L(\mu) = |\tilde{e}\rangle\langle\tilde{\mu}| + |\tilde{\mu}\rangle\langle\tilde{e}| - i \sum_{\nu,\gamma\in\{x,y,z\}} \epsilon_{\mu\nu\gamma}|\tilde{\nu}\rangle\langle\tilde{\gamma}|. \tag{13c}$$

The string order parameter formed by these operators [as per Eq. (3)] consists of dichotomic operators, in contrast to the conventional one based on spin-1 operators [31]. Note, however, that Eqs. (13a)–(13c) are equivalent to Eqs. (12a)–(12c) under a local isometry $|\tilde{e}\rangle\langle + + | + |\tilde{z}\rangle\langle - + | + |\tilde{x}\rangle\langle + - | + i|\tilde{y}\rangle\langle - - |$, as the fixed point is the 1D cluster state. For $g = z$ and $h = x$, an exact MPS calculation gives $\langle \mathcal{S}\rangle \geq \frac{4}{9}(\sqrt{\frac{2}{3}} + \frac{2}{3})^2 \approx 0.978$ and by Theorem 2, $\mathrm{pr}(\mathrm{win}) \geq \frac{13}{16} + \frac{1}{12}(\sqrt{\frac{2}{3}} + \frac{2}{3})^2 \approx 0.996$ (see [70] for details). Thus quantum advantage persists at the AKLT point.

Quantum computational advantage.—In Ref. [20], an exponential quantum speed-up was shown for a problem equivalent to a 2D multiplayer generalization of our triangle problem where players are situated on an $N \times N$ grid (elaborated in [70]). In this 2D setting, quantum players outperform nonlocally communicating classical players. Indeed, a constant number of rounds of classical communication between a constant number of arbitrarily distant players on the grid still leaves at least one cycle of locally communicating players in the large-N limit. This advantage can be rephrased in the language of circuit complexity. Classical Boolean circuits consisting of nonlocal gates with bounded fan-in require at least logarithmic depth (i.e., logarithmically many rounds of communication) to ensure a solution to the problem with arbitrarily high probability. On the other hand, it is possible to prepare generic SPTO ground states in constant depth when the symmetry G is disregarded [83,84]. Theorems 1 and 2 present a substantial extension regarding the required capability of a quantum device.

Corollary.—Consider a relation problem whereby players situated on an $N \times N$ 2D grid are tasked to play the multiplayer triangle game on an arbitrary cycle in the grid. A quantum device that can prepare a 1D SPTO ground state in constant time with string order parameters greater than $1/3$ on the arbitrary cycles and perform the fixed-point protocol of Theorem 2 solves the problem with probability greater than $7/8$ on all inputs, which any classical circuit with gates of fan-in at most K and depth less than $\log_K(N)$ cannot do.A precise statement and proof of Corollary 1 is given in [70].

Conclusion and outlook.—We have shown how to harness contextuality and the string order parameter of generic 1D SPTO ground states to construct advantageous quantum strategies for a nonlocal game that thwarts all classical strategies (even with assistance of limited long-range communication). Our approach, to be supplemented with a follow-up paper [85], contributes to unify recent insight about unconditional quantum advantage. For example, the magic-square game in [23] also admits general 1D SPTO strategies, and similar complexity-theoretic results using the GHZ state [24] can be understood using Kennedy-Tasaki duality maps [86,87]. Our relation of the string order parameter to robustness of the advantage may be applicable to robust self-testing [88–92] for fixed-point SPTO states. The use of SPTO is welcome in scalable and robust experimental demonstrations, as these ground states can be realized as the unique ground states of two-body Hamiltonians in contrast to the GHZ state. Broadly, our Letter is timely to promote the value of quantum simulation to prepare and detect 1D SPTOs for potential quantum advantage.

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