Probabilistic Discrete Time Robust $\mathcal{H}_2$ Controller Design

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***Abstract*— Optimal $\mathcal{H}_2$ control theory is appealing, since it allows for optimizing a performance index frequently arising in practical situations. Moreover, in the state feedback case, the resulting closed loop system has an infinite gain margin and a phase margin of at least 60^o. However, these properties no longer hold in the output feedback case, where it is well known that there exist cases where the system is arbitrarily fragile. Motivated by this observation, since the early 1980's a large research effort has been devoted to the problem of designing robust $\mathcal{H}_2$ controllers. To this effect several relaxations of the original problem have been introduced, but all of these lead to conservative solutions. Surprisingly, the original problem remains, to date, still open. To address this issue, in this paper we present a randomization based algorithm that seeks to solve a relaxation of the original problem. Contrary to existing approaches, the performance of the resulting controller can be made—in a sense precisely defined in the paper—arbitrarily close to the optimal one. These results are illustrated with an academic example.**

I. Introduction

$\mathcal{H}_2$ control theory has been a mainstay of optimal control for the past four decades. Its success arises from the fact that it allows for optimizing a performance index frequently arising in practical situations, such as minimizing the RMS value of the output due to Gaussian white inputs. Moreover, in the state feedback case, the resulting closed loop system exhibits very good robustness properties. Unfortunately, these properties vanish in the output feedback case, where it is well known that there are cases where infinitesimal model perturbations can destabilize the system [1]. Starting in the 1980's several approaches were proposed to "robustify" LQG controllers. Initial efforts sought to recover closed-loop robustness properties through Loop Transfer Recovery (see for instance the tutorial [2]). This approach attempts to shape the closed loop transfer functions to guarantee robustness and acceptable performance, through proper weight selections in an $\mathcal{H}_2$ framework. These initial efforts were followed by "mixed" $\mathcal{H}_2/\mathcal{H}_\infty$ control, where the goal was to find a controller that minimized the closed loop $\mathcal{H}_2$ norm for the *nominal* plant, subject to an $\mathcal{H}_\infty$ constraint on the transfer function where the (dynamic) uncertainty entered the loop [3]–[6]. Thus, the resulting controllers guaranteed optimal performance for the nominal plant and robust stability against bounded dynamic perturbations. While these efforts represented substantial progress towards endowing $\mathcal{H}_2$ control with robustness, they suffered from the fact that performance for the actual plant could be arbitrarily bad, since only nominal performance was optimized. In addition, as shown in [7], optimal $\mathcal{H}_2/\mathcal{H}_\infty$ controllers are infinite dimensional, and thus optimization based methods seeking to approximate them can result in very high order controllers, necessitating some form of model reduction in order to be practically implementable. However, the original performance and robustness guarantees may be severely degraded by this approximation step.

As a first step to synthesize robust $\mathcal{H}_2$ controllers, starting in the mid 1990's, several approaches were proposed to analyze worst-case $\mathcal{H}_2$ performance in the presence of uncertainty. The paper [8] introduced bounds on the worst–case performance, under the assumption of non–causal, non–linear time varying model uncertainty. While these bounds lead to tractable synthesis problems, the uncertainty class considered is too broad, leading to conservative results in the case of LTI uncertainty. Time and frequency domain bounds on the worst-case $\mathcal{H}_2$ norm were proposed in [9], [10]. While these bounds work for the case of SISO systems, as shown in [11], they can be conservative by a factor of $\sqrt{\text{dimension of the input}}$. Finally, necessary and sufficient convex conditions for robust $\mathcal{H}_2$ performance under arbitrarily slowly time varying structured dynamic uncertainty (or LTI uncertainty with up to 2 uncertainty blocks) were derived in [12]. These conditions have the form of a frequency dependent LMI coupled with an integral constraint. Thus, while they can be used to assess performance once a controller has been found, as in the case with μ-synthesis, they are bilinear in the pair (controller/open-loop plant). In principle, a $D-K$ type iteration can be used to find a controller (albeit with the additional complexity of the integral condition), but such an approach can only be guaranteed to converge to a local minima. Moreover, the conditions there are only tight for (arbitrarily slow) time varying, non-causal, uncertainty and thus can be arbitrarily conservative for LTI, causal one.

Motivated by these difficulties, in this paper we propose an alternative approach to synthesize robust $\mathcal{H}_2$ controllers for LTI systems subject to LTI bounded dynamic uncertainty. Our main result is a tractable randomization based algorithm that leads to controllers achieving performance arbitrarily close, in a sense made clear in the paper, to the optimal one. This algorithm is obtained by first using the parameterization

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of all stabilizing controllers to recast the problem as a finite-dimensional robust optimization. Subsequently, this problem is solved using randomization based algorithms, leading to a controller that achieves, with probability arbitrarily close to 1, optimal performance.

II. Preliminaries

A. Notation

$\rho(M)$ spectral radius of the matrix M.

T_r Toeplitz matrix associated with the sequence $r_{i=0}^{(n-1)}$:

$$T_r = \begin{bmatrix} r_o & 0 & \dots & 0 \\ r_1 & r_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ r_{n-1} & \dots & r_1 & r_o \end{bmatrix}$$

$\mathcal{L}^\infty$ Lebesgue space of complex valued matrix functions which are essentially bounded on the unit circle, equipped with the norm:

$$\|G(\lambda)\|_\infty \doteq ess \sup_{\omega \in [0,2\pi)} \overline{\sigma}\left[G(e^{j\omega})\right]$$

where $\overline{\sigma}$ denotes the largest singular value.

$\mathcal{H}_\infty$ Subspace of functions in $\mathcal{L}^\infty$ with a bounded analytic continuation in $|\lambda| > 1$, equipped with the norm

$$\|G(\lambda)\|_\infty \doteq ess \sup_{|\lambda|>1} \overline{\sigma}\left[G(\lambda)\right]$$

$\mathcal{H}_{\infty,\rho}$ Subspace of $\mathcal{H}_\infty$ of functions analytic outside the disk of radius $0 < \rho < 1$, equipped with the norm

$$\|G(\lambda)\|_{\infty,\rho} \doteq ess \sup_{0\leq\theta\leq\pi} \overline{\sigma}\left[G(\rho e^{j\theta})\right]$$

$\mathcal{H}_2$ space of complex valued matrix functions $G(\lambda)$ with analytic continuation in $|\lambda| < 1$ and square integrable on the unit disk, equipped with the usual $\mathcal{H}_2$ norm:

$$\|G\|_2^2 \doteq \frac{1}{2\pi}\int_0^{2\pi} \text{Trace}\left[G(e^{j\omega})G(e^{j\omega})^*\right] d\omega,$$

$\mathcal{F}_\ell(M,K)$ Lower linear fractional transformation (LFT)

$$\mathcal{F}_\ell(M,K) \doteq M_{11}+M_{12}K\left(I-M_{22}K\right)^{-1}M_{21}$$

$\mathcal{F}_u(M,\Delta)$ Upper linear fractional transformation (LFT)

$$\mathcal{F}_u(M,\Delta) \doteq M_{22}+M_{21}\Delta\left(I-M_{11}\Delta\right)^{-1}M_{12}$$

B. Theoretical Background

Next we recall some theoretical results required to establish the main results of this paper.

Interpolation Theory:

Lemma 1: Given $r = (r_0, r_1, \dots, r_{n-1})$, and $s = (s_0, s_1, \dots, s_{n-1})$, there exists a $Q \in \mathcal{H}_{\infty,\rho}$, $\|Q\|_{\infty,\rho} < \gamma$ such that

$$s = T_q r,$$

if and only if $T_s^T R^2 T_s \prec \gamma^2 T_r^T R^2 T_r^T$, where $R = \text{diag}(1, \rho, \dots \rho^{(n-1)})$

Proof: This is a special case of Theorem 2 in [13]. ∎

Parameterization of All Suboptimal $\mathcal{H}_\infty$ Controllers: Consider the interconnection shown in Fig. 1, where the plant G has the following state-space realization:

$$G \equiv \left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right).$$

Assume that $\inf_{K \text{ stabilizing}} \|P_{\zeta_\infty w_\infty}(K)\|_\infty = \gamma^* < \gamma$, where $P_{\zeta_\infty w_\infty}(K)$ denotes the closed loop transfer function between the signals (w_∞, ζ_∞) corresponding to the controller K. It is well known that in this case, all suboptimal controllers that render $\|P_{\zeta_\infty w_\infty}(K)\|_\infty < \gamma$ can be parameterized in terms of a free parameter $Q \in \mathcal{H}_\infty$, $\|Q\|_\infty < \gamma$, as

$$K = \mathcal{F}_\ell(J_\infty, Q).$$

State space realizations of J_∞ can be found for instance in [14] for the continuous time case, and in [15] for its discrete time counterpart.

Robust $\mathcal{H}_2$ Analysis:

Consider the interconnection shown in Fig. 1, where the uncertainty Δ is of the form

$$\Delta \in \{\mathbf{\Delta} \doteq \text{diag}\left[\Delta_1, \dots, \Delta_n\right] : \Delta_i \in \mathcal{H}_\infty, \|\Delta\|_\infty \leq \gamma^{-1}\} \tag{1}$$

and define the set

$$\mathbf{X} = \{X \colon X = \text{diag}\left[x_1 I_{m_1} \dots x_n I_{m_n}\right], X = X^*\}$$

of scaling matrices that commute with the elements of $\mathbf{\Delta}$. Denote by $P(K) = \mathcal{F}_\ell(G,K)$ the closed loop plant, and by $P(\omega) \doteq P(K, e^{j\omega})$ its z-transform evaluated on the unit circle.

Theorem 1 ([12]) *Assume that the signal w is a scalar and that $\zeta \in \mathbb{R}^{n_z}$. If there exist a positive definite hermitian matrix $X(\omega) \in \mathbf{X}$ and a real transfer function $y(\omega) > 0$, such that*

$$P(\omega)^* \begin{bmatrix} \gamma^{-2}X(\omega) & 0 \\ 0 & I_{n_z \times n_z} \end{bmatrix} P(\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & y(\omega) \end{bmatrix} < 0 \tag{2}$$

holds for all $\omega \in [0, 2\pi)$, then

$$\|\mathcal{F}_u(P(K), \Delta\|_2^2 \leq \int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} \tag{3}$$

for all $\Delta \in \mathbf{\Delta}$ (not necessarily causal).

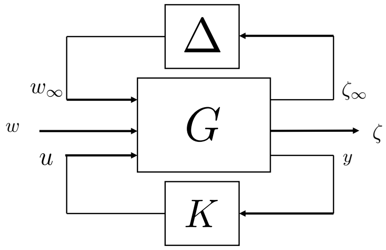

Fig. 1. Block diagram of the uncertain plant.

C. Problem Formulation

Consider an uncertainty structure of the form (1). The goal of this paper is to synthesize an internally stabilizing controller that robustly stabilizes the system and minimizes the worst-case $\mathcal{H}_2$ norm over Δ.

Problem 1 (Robust $\mathcal{H}_2$ Control Problem) *Find an internally stabilizing controller K that solves:*

$$K \doteq \underset{K\,stabilizing}{argmin}\left\{ \max_{\Delta\in\mathbf{\Delta}}(\|P_{\zeta w}(K,\Delta)\|_2) \right\} \quad (4)$$
$$\textit{where:}$$
$$P_{\zeta w}(K,\Delta) = \mathcal{F}_\ell(G(\Delta),K)$$
$$G(\Delta) = \mathcal{F}_u(G,\Delta).$$

For simplicity, we consider the case where $\mathbf{\Delta}$ is unstructured.

In the case of unstructured uncertainty, by using the parameterization of all suboptimal $\mathcal{H}_\infty$ controllers discussed in the previous section, this problem can be formally stated as:

Problem 2 (Robust $\mathcal{H}_2$ Control, unstructured $\mathbf{\Delta}$.) *Find the optimal value of performance measure μ and a parameter Q such that*

$$\mu \doteq \inf_{Q\in\mathcal{H}_\infty}\left\{ \max_{\Delta\in\mathbf{\Delta}}(\|P_{\zeta w}(Q,\Delta)\|_2) \textit{ subject to } \|Q\|_\infty < \gamma \right\}$$
$$\textit{where:}$$
$$P_{\zeta w}(Q,\Delta) = \mathcal{F}_\ell(G(\Delta),K)$$
$$G(\Delta) = \mathcal{F}_u(G,\Delta);\ K = \mathcal{F}_\ell(J,Q). \quad (5)$$

Remark 1: Note that if the problem above is feasible, then $\mu^* < \infty$ and the resulting controllers guarantee robust stability, since, for any Q, $\|Q\|_\infty < \gamma$, $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$.

III. Main Results

Problem 2 is a very challenging infinite dimensional robust optimization problem, since in principle both the variables Δ and Q are transfer functions with arbitrarily high order. Moreover, $P_{\zeta w}(Q,\Delta)$ is neither convex in Q, nor concave in Δ. To circumvent these two difficulties, we first introduce a convex relaxation of Problem 2 and then present a randomized algorithm for finding a solution to the convexified problem.

A. A Finite-Dimensional Approximation

The first step towards solving Problem 2 is to restrict the closed loop poles to disk with radius $\rho < 1$ (effectively constraining the time constant of the closed loop system) and then show that under these conditions, it can be approximated, arbitrarily close, by a finite horizon problem. To this effect, we will constrain the uncertainty to $\Delta \in \mathbf{\Delta}_\rho \doteq \{\Delta \in \mathcal{H}_{\infty,\rho} \colon \|\Delta\|_{\infty,\rho} \le \gamma^{-1}\}$ and modify slightly (4) to:

$$K_\rho \doteq \underset{K\text{stabilizing}}{argmin}\left\{ \max_{\Delta\in\mathbf{\Delta}_\rho}(\|P_{\zeta w}(K,\Delta)\|_2 \right\}$$
$$\text{subject to } \ P_{\zeta w}(K,\Delta) \in \mathcal{H}_{\infty,\rho}. \quad (6)$$

Note that, by simply using the change of variable $z = \rho\hat{z}$ (or equivalently $A \to \frac{1}{\rho}A$, $B \to \frac{1}{\rho}B$) before performing the parameterization of all stabilizing controllers, the set of all controllers K_ρ such that $P_{\zeta w}(K_\rho) \in \mathcal{H}_{\infty,\rho}$, with $\|P_{\zeta w}(K_\rho)\|_{\infty,\rho} < \gamma$ can be parameterized as $K_\rho = \mathcal{F}_\ell(J_\infty, Q_\rho)$, with $Q_\rho \in \mathcal{H}_{\infty,\rho}$, $\|Q_\rho\|_{\infty,\rho} < \gamma$. Thus, (6) is equivalent to:

$$\inf_{Q\in\mathcal{H}_{\infty,\rho}} \left\{ \max_{\Delta\in\mathbf{\Delta}_\rho}(\|P_{\zeta w}(Q,\Delta)\|_2 \right\} \text{ subject to } \|Q\|_{\infty,\rho} < \gamma. \quad (7)$$

As we will show next, this problem can be approximated, arbitrarily close, by a finite dimensional one. In the sequel, for technical reasons we replace the constraint $\|Q\|_{\infty,\rho} < \gamma$ with $\|Q\|_{\infty,\rho} \le \gamma_\epsilon \doteq \gamma - \epsilon$, where $\epsilon > 0$ is arbitrarily small. In order to obtain a finite dimensional approximation of (7) we need the following preliminary result.

Lemma 2: Consider the closed loop system shown in Fig. 2 and let $P_{\zeta w}(\Delta, Q)$ denote the closed loop transfer function between the signals (w,ζ). Then, there exist some M that depends only on the problem data such that $\|P_{\zeta w}(\Delta,Q)\|_{\infty,\rho} \le M$, for all $\Delta \in \mathbf{\Delta}_\rho$ and all $Q \in \mathbf{Q}_\rho \doteq \{Q \in \mathcal{H}_{\infty,\rho}, \|Q\|_{\infty,\rho} \le \gamma_\epsilon\}$.

Proof: The proof is omitted to conserve space. ■

Using the Lemma above, we can now state the main result of this section.

Theorem 2: Let $m_k(\Delta,Q)$ denote the k^{th} Markov parameter of $P_{\zeta w}(\Delta,Q)$. Then, given $\delta > 0$, there exist some finite $N(\delta)$ that can be determined a-priori such that, for all $\Delta \in \mathbf{\Delta}_\rho$ and all $Q \in \mathbf{Q}_\rho$ we have that

$$\left|\|P_{\zeta,w}(\Delta,Q)\|_2 - \sum_{k=0}^{N-1}\|m_k(\Delta,Q)\|_2^2\right| \le \delta. \quad (8)$$

Proof: The proof is omitted to conserve space. ■

The theorem above allows for replacing the original problem (7) with the finite dimensional approximation:

$$\inf_{Q\in\mathbf{Q}_\rho}\left\{ \max_{\Delta\in\mathbf{\Delta}_\rho} \sum_{k=0}^{N-1}\|m_k(\Delta,Q)\|_2^2 \right\}. \quad (9)$$

Note that the problem above is still very challenging due to the non-convexity (concavity) of the objective and the fact that in principle Δ and Q are infinite dimensional. Next, we reformulate the problem to handle the non-convexity in Q and the fact that the set $\mathbf{Q}_\rho$ is infinite dimensional.

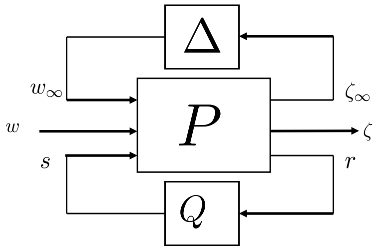

Fig. 2. Block diagram of the uncertain plant.

Let $P(\Delta) \doteq \mathcal{F}_u(P, \Delta)$ and define the following signals of length N

$$w = (w_0, w_1, \ldots, w_{N-1}), \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{N-1}),$$
$$r = (r_0, r_1, \ldots, r_{N-1}), s = (s_0, s_1, \ldots, s_{N-1}).$$

Choosing w to be Dirac delta function, the input-output relations in the interval $[0, N-1]$ can be written as

$$T_s = T_q T_r, \begin{cases} T_\zeta = T_{11} + T_{12} T_s \\ T_r = T_{21} + T_{22} T_s \end{cases}, \tag{10}$$

where T_{ij} and T_q are the toeplitz matrices formed by the first N Markov parameters of $P_{ij}(\Delta)$ and Q respectively. We remark that the signals r, s and ζ—and therefore the matrices T_r, T_s and T_ζ—depend on the uncertainty Δ, however, for the simplicity of notation, we have dropped the argument Δ. Using the IQC characterization of Q in terms of r, s given in Lemma 1 leads to the following finite dimensional problem

$$\begin{aligned} \widetilde{\mu} \doteq \inf_{s \in \mathbb{R}^n} \Big\{ & \max_{\Delta \in \mathbf{\Delta}_\rho} (\sum_{k=0}^{N-1} \|m_k\|_2^2) \text{ subject to:} \\ & {T_s}^T R^2 T_s \prec \gamma_\epsilon^2 {T_r}^T R^2 T_r, \forall \Delta \in \mathbf{\Delta}_\rho \Big\}. \end{aligned} \tag{11}$$

We note that since the disturbance w is selected to be an impulse, kth Markov parameters of $P_{\zeta w}(\Delta, Q)$, i.e. m_k is the same as ζ_k. Next, using (10) to eliminate T_r leads to:

$$\begin{aligned} & T_s^T (\gamma_\epsilon^{-2} R^2 - T_{22}^T R^2 T_{22}) T_s^T \prec \\ & T_{21}^T R^2 T_{21} + T_{21}^T R^2 T_{22} T_s + (T_{22} T_s)^T R^2 T_{21}. \end{aligned} \tag{12}$$

Finally, a Schur complement argument shows that (12) is equivalent to:

$$\begin{pmatrix} (\gamma_\epsilon^{-2} R^2 - T_{22}^T R^2 T_{22})^{-1} & T_s \\ {T_s}^T & F \end{pmatrix} \succeq 0, \tag{13}$$

where $F = T_{21}^T R^2 T_{21} + T_{21}^T R^2 T_{22} T_s + (T_{22} T_s)^T R^2 T_{21}$.

Remark 2: Note that by the choice of the parameterization, $\mathcal{F}_\ell(P(\Delta), Q)$ is internally stable, with its poles in $|z| \leq \rho < 1$ for all $\Delta \in \mathbf{\Delta}_\rho$ and $Q \in \mathcal{H}_{\infty,\rho}$, $\|Q\|_{\infty,\rho} < \gamma$. Thus, from a small gain argument, it follows that $\|T_{22}\|_{\infty,\rho} \leq \gamma^{-1}$, which implies (since $\gamma_\epsilon < \gamma$) that $(R^2 - \gamma_\epsilon^2 T_{22}^T R^2 T_{22}) \succ 0$. Hence (13) is well defined for all $\Delta \in \mathbf{\Delta}_\rho$.

From the derivations above, it follows that a finite dimensional approximation to Problem 2 is given by:

Problem 3 (Finite-dimensional Approximated Robust $\mathcal{H}_2$ Control Problem) *Find a sequence s_k^*, $k \in [0, N-1]$ that solves:*

$$s^* = \underset{s \in \mathbb{R}^N}{argmin} \max_{\Delta \in \mathbf{\Delta}_\rho} \sum_{k=0}^{N-1} \|m_k(\Delta, s)\|_2^2 \textit{ subject to } (13). \tag{14}$$

B. Sequential Randomized Algorithm for Relaxed Robust $\mathcal{H}_2$ Design

Since the set $\mathbf{\Delta}_\rho$ is uncountable, problem (14) is a semi-infinite optimization problem involving an infinite number of constraints. We note that, Problem 3 is not just a usual robust optimization problem. In fact, if one successfully finds a signal s^* as the solution of (14), it is still not possible to construct Q. The reason is that, the signal r depends on Δ and hence having s^*, there are infinite number of candidate signals r making it impossible to compute Q. In order to solve Problem 3, one needs to first find a pair of signals s and r robustly satisfying specifications in (11), and then use interpolation theory results such as [16] to reconstruct transfer function of the optimal parameter Q^*, and the optimal controller from $K^* = \mathcal{F}_\ell(J_\infty, Q^*)$. This brings further complexity to the picture and hence, direct use of other robust optimization techniques such the scenario approach [17] or other sample-based techniques such as [18], [19] is not possible.

In what follows, we propose a randomized algorithm for finding a solution to problem (14) which is feasible for the entire set of uncertainty $\mathbf{\Delta}_\rho$ except a subset having an arbitrary small probability measure. The randomized algorithm has a sequential nature and falls within the class of sequential probabilistic verification algorithms [20]. The algorithm has two main steps: verification and optimization.

Formally, we assume that Δ is a random variable and a probability measure $\mathbb{P}$ over the Borel $\sigma-$algebra of $\mathbf{\Delta}_\rho$ is given. More precisely—if Δ is unstructured—we assume that the first N Markov parameters of $\mathbf{\Delta}_\rho$ are uniformly distributed over their support set (see [21] for a precise description of this set). At each verification step k, we generate a new multi-sample $\mathbb{\Delta}_k$ with cardinality M_k from the set of uncertainty

$$\mathbb{\Delta}_k \doteq \{\Delta_k^{(1)}, \ldots, \Delta_k^{(M_k)}\} \in \mathbf{\Delta}_\rho^{M_k},$$

according to the measure $\mathbb{P}$, where $\mathbf{\Delta}_\rho^{M_k} \doteq \mathbf{\Delta}_\rho \times \mathbf{\Delta}_\rho \times \cdots \times \mathbf{\Delta}_\rho$ (M_k times). We remark that $\mathbf{\Delta}_\rho^{M_k}$ is the domain of random variables $\mathbb{\Delta}_k$. The generation of this multi-sample can be (approximately) done using the procedure described in [21]. The probability measure on $\mathbb{\Delta}_k$ is defined on this space, and it is denoted as $\mathbb{P}_\infty$.

Next, having a candidate solution Q_k and a candidate performance measure $\widetilde{\mu}_k$, we check if $\sum_{j=0}^{N-1} \|\zeta_j\|_2^2 \leq \widetilde{\mu}_k$ for all the extracted samples. If a violation is observed, the violating sample is declared as a violation certificate $\Delta_{\texttt{Viol}}$. In the optimization step, first having the violation certificate $\Delta_{\texttt{Viol}}$, we solve problem (14) only at the violated sample—instead of the entire set $\mathbf{\Delta}_\rho$. Having the signal s

as the solution of the optimization problem, the signal r in the interval $[0, N-1]$ and the first N Markov parameters of Q can be reconstructed from $r = T_{21} + T_{22}s$ and $q_k = (T_r)^{-1}s$. Then, having the truncated impulse response of Q_k, we compute the finite-dimensional Hankel matrix and use subspace identification technique [22], [23] to recover the candidate solution Q_k.

The two steps—verification and optimization—are performed till no violating sample is observed. The sequential algorithm is formally presented in Algorithm 1.

Algorithm 1 Sequential Randomized Algorithm

Input: $P(\Delta, Q)$, N, ε, β
Output: Q^*, $\widetilde{\mu}^*$
Initialization:
Set $k = 1$, $Q_1 = 0$, $\Delta = 0$, and let $\widetilde{\mu}_1 = \|\zeta\|_2^2$
Evolution:
Verification:

1) **Extract**

$$M_k = \left\lceil \frac{2.3 + 1.1 \ln k + \ln \frac{1}{\beta}}{\ln \frac{1}{1-\varepsilon}} \right\rceil \tag{15}$$

i.i.d samples $\mathbb{\Delta}_k \doteq \{\Delta_k^{(1)}, \ldots, \Delta_k^{(M_k)}\} \in \mathbf{\Delta}_\rho^{M_k}$.

2) **If** $\sum_{j=0}^{N-1} \|\zeta_j(\Delta_k^{(i)})\|_2^2 \leq \widetilde{\mu}_k$ for all $i = 1, \ldots, M_k$, **set** $Q^* = Q_{k-1}, \widetilde{\mu}^* = \widetilde{\mu}_{k-1}$ and **exit**, **otherwise** set $\Delta_{\mathtt{Viol}}$ as the first sample for which $\sum_{j=0}^{N-1} \|\zeta_j(\Delta_{\mathtt{Viol}})\|_2^2 > \widetilde{\mu}_k$.

Optimization:

1) Solve (14) formulated at $\Delta = \Delta_{\mathtt{Viol}}$ and set $\widetilde{\mu}_k = \sum_{j=0}^{N-1} \|m_j\|_2^2$
2) Having s as the solution of (14), set $r = T_{21} + T_{22}s$ and compute the truncated impulse response of Q_k as $q_k = (T_r)^{-1}s$.
3) Find transfer function of Q_k from its truncated impulse response q_k
4) Set $k = k + 1$

We now explain Algorithm 1 in more details. The algorithm is initialized with $Q = 0$ and we set $\Delta = 0$ to compute a nominal performance index $\widetilde{\mu}_1$. In the verification step, having the candidate solution and performance measure Q_k and $\widetilde{\mu}_k$, we check if $\sum_{j=0}^{N-1} \|\zeta_j(\Delta_k^{(i)})\|_2^2 \leq \widetilde{\mu}_k,\ \forall i \in \{1, \ldots, M_k\}$. If there is no violation, Algorithm 1 is terminated successfully and the candidate solution Q_{k-1} is declared as a probabilistic robust solution to Problem 3. Otherwise, we set $\Delta_{\mathtt{Viol}}$ as the first sample for which $\sum_{j=0}^{N-1} \|\zeta_j(\Delta_{\mathtt{Viol}})\|_2^2 > \widetilde{\mu}_k$. The violated sample is used in the optimization step to construct a *more robust* solution. To this end, we solve the optimization problem (14) formulated only at $\Delta_{\mathtt{Viol}}$ instead of the entire uncertainty set $\mathbf{\Delta}_\rho$. Having s as the solution of (14), we compute $r = T_{21} + T_{22}s$ and $q_k = (T_r)^{-1}s$. Next, the transfer function Q_k can be either constructed by formulating the Hankel matrix of q_k and using subspace identification techniques [22], [23] or using interpolation results such as [16]. We now present a Theorem quantifying the properties of the solution Q^*.

Theorem 3: Given probabilistic parameters $\varepsilon, \beta \in (0, 1)$, suppose Algorithm 1 is terminated at iteration k, then the following inequality holds

$$\mathbb{P}_\infty \left\{ \mathbb{\Delta} \in \mathbf{\Delta}_\rho^{M_k} : \mathbb{P}\Big\{ \Delta \in \mathbf{\Delta}_\rho : \sum_{j=0}^{N-1} \|\zeta_j(\Delta)\|_2^2 > \widetilde{\mu}^* \Big\} \leq \varepsilon \right\} \geq 1 - \beta. \tag{16}$$

Proof: The proof is omitted to conserve space. ■

IV. Numerical Simulation

We check the effectiveness of the proposed algorithm in Section III, through numerical simulation. We note that although Algorithm 1 is presented for unstructured uncertainty, it can be easily applied to problems involving structured uncertainty. To this end, we consider an uncertain second order system of the form

$$G(s) = \frac{b w_n^2}{s^2 + 2\eta w_n s + w_n^2}, \tag{17}$$

where $b = 13$ is a constant, and η and w_n are uncertain damping ratio and natural frequency respectively. We assume that damping ratio η and natural frequency w_n are uncertain and vary by 10% from their nominal values $\overline{\eta} = 0.01$ and $\overline{w}_n = 1.25 \times 10^4$ rad/sec

$$\begin{aligned} w_n &= \overline{w}_n(1 + p_w \delta_w) \\ \eta &= \overline{\eta}(1 + p_\eta \delta_\eta), \end{aligned}$$

where $p_\eta = p_w = 0.1$ are the maximum relative uncertainties, and δ_η and δ_w are norm-bounded perturbations

$$\|\delta_\eta\|_\infty \leq 1,\ \|\delta_w\|_\infty \leq 1.$$

In order to *pull out* the uncertain parameters, system (17) is first written in state-space form

$$\begin{aligned} \dot{x}_1 &= w_n x_2, \\ \dot{x}_2 &= w_n(-x_1 - 2\eta x_2 + u), \\ y &= b x_1, \end{aligned} \tag{18}$$

and uncertain parameters η and w_n are represented in LFT form as

$$\eta = \mathcal{F}_u(M_\eta, \delta_\eta),\ w_n = \mathcal{F}_u(M_w, \delta_w),$$

where $M_\eta = \begin{pmatrix} 0 & \overline{\eta} \\ p_\eta & \overline{\eta} \end{pmatrix}$ and $M_w = \begin{pmatrix} 0 & \overline{w}_n \\ p_w & \overline{w}_n \end{pmatrix}$. We further obtain the perturbed system in the form of

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \hline y_w \\ z_w \\ y_\eta \\ \hline y \end{array}\right] = \left[\begin{array}{cc|cccc|c} 0 & \overline{w}_n & 0 & p_w & 0 & & 0 \\ -\overline{w}_n & -2\overline{\eta}\overline{w}_n & p_w & 0 & -2\overline{w}p_\eta & & \overline{w}_n \\ \hline -\overline{w}_n & -2\overline{\eta}\overline{w}_n & 0 & 0 & -2\overline{w}p_\eta & & \overline{w}_n \\ 0 & \overline{w}_n & 0 & 0 & 0 & & 0 \\ 0 & \overline{\eta} & 0 & 0 & 0 & & 0 \\ \hline b & 0 & 0 & 0 & 0 & & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline u_w \\ v_w \\ u_\eta \\ \hline u \end{array}\right]$$

Defining $\Delta = \text{diag}(\delta_w, \delta_w, \delta_\eta)$, the uncertain system (17) can be written as $G \doteq \mathcal{F}_u(\Lambda, \Delta)$. In order to design controller K, as shown in Fig. 3, we augment the plant with weighting function $W_p = \frac{s+6000}{1.2s+480}$. The augmented system is then discretized using *Tustin's* method with sampling

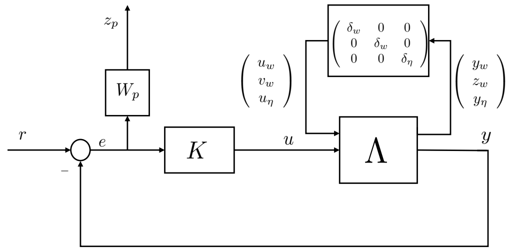

Fig. 3. The augmented plant with performance specifications.

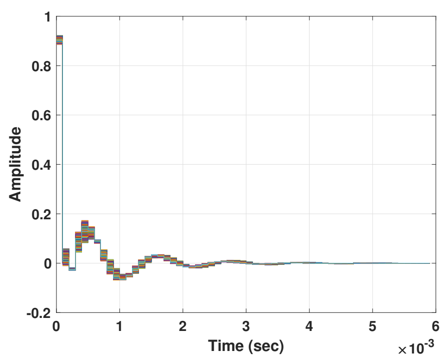

Fig. 4. Impulse response of 1000 uncertain closed loop systems.

time $T_s = 10^{-4}$. Letting $\zeta_\infty \doteq (y_w, z_w, y_\eta)^T$, $w_\infty \doteq (u_w, v_w, u_\eta)^T$, $w \doteq r$, and $\zeta \doteq z_p$ the uncertain system becomes in the form defined in Problem 1.

For simulation purpose, we select $\rho = 0.97$, $\delta = 0.01$, $\varepsilon = 0.01$, and $\beta = 10^{-9}$ which leads to $N(\delta) \geq 129.94$. We set $N(\delta) = 130$ and used Algorithm 1 to solve the formulated problem. Algorithm terminates with the robust optimal solution after 14 iterations by checking the robustness of the solution for 2572 number of random samples (M_k at the last iteration) extracted from the set of uncertainty. The probability measure is selected to be uniform. Figure 4 shows the truncated impulse response of $P_{\zeta w}$ for 1000 random uncertain closed loop systems. We further check the robustness of the computed performance measure $\widetilde{\mu}^* = 0.8945$ by performing *a-posteriori* Monte Carlo simulation using $100,000$ samples. Only 53 out of $100,000$ samples led to a performance measure greater than 0.8945 resulting in the empirical violation of 5.3×10^{-4} which is much smaller than the selected accuracy level ε.

V. Concluding Remarks

In this paper, we study the problem of synthesizing robust $\mathcal{H}_2$ controllers for LTI systems subject to LTI bounded dynamic uncertainty. To address this complex problem, we propose a computationally efficient randomized algorithm that provides controllers with performance arbitrarily close to the optimal one. To develop this algorithm, results on stabilizing controller parameterization are used to obtain a finite dimensional robust optimization problem that is solved using a randomized approach.

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