CONFIGURATION SETS WITH NONEMPTY INTERIOR

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Abstract. A theorem of Steinhaus states that if $E \subset \mathbb{R}^d$ has positive Lebesgue measure, then the difference set $E - E$ contains a neighborhood of 0. Similarly, if E merely has Hausdorff dimension $\dim_{\mathcal{H}}(E) > (d+1)/2$, a result of Mattila and Sjölin states that the distance set $\Delta(E) \subset \mathbb{R}$ contains an open interval. In this work, we study such results from a general viewpoint, replacing $E - E$ or $\Delta(E)$ with more general Φ-configurations for a class of $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$, and showing that, under suitable lower bounds on $\dim_{\mathcal{H}}(E)$ and a regularity assumption on the family of generalized Radon transforms associated with Φ, it follows that the set $\Delta_\Phi(E)$ of Φ-configurations in E has nonempty interior in $\mathbb{R}^k$. Further extensions hold for Φ-configurations generated by two sets, E and F, in spaces of possibly different dimensions and with suitable lower bounds on $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F)$.

1. Introduction

A classical theorem of Steinhaus [38] states that if $E \subset \mathbb{R}^d, d \geq 1$, with positive Lebesgue measure, $|E|_d > 0$, then the difference set $E - E \subset \mathbb{R}^d$ contains a neighborhood of the origin. $E - E$ can interpreted as the set of two-point configurations, $x - y$, of points of E modulo the translation group. A variant of this was obtained by Mattila and Sjölin [27] for *thin sets*, i.e., E with $|E|_d = 0$ but satisfying a lower bound on the Hausdorff dimension, $\dim_{\mathcal{H}}(E)$, in the context of the Falconer distance problem: if $\Delta(E)$ is the distance set of E, $\Delta(E) := \{|x - y| : x, y \in E\} \subset \mathbb{R}$, then if $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, it follows that $\Delta(E)$ contains an open interval. The purpose of the current paper is to generalize these results in two ways: to two-point configurations in E as measured by a general class of Φ-*configurations*, which can be nontranslation-invariant, and indeed not even in Euclidean space, and to allow *asymmetric* configurations between sets in different spaces, e.g., between points and lines or points and circles in $\mathbb{R}^2$, or lines and lines in $\mathbb{R}^3$. In the process, we shall establish non-empty interior results for some configuration sets for which previously it was not even known that the configuration space has positive Lebesgue measure.

In order to formulate these more general results, consider the models $E - E$ and $\Delta(E)$ as the images of $E \times E$ under the maps $(x, y) \to x - y$ and $(x, y) \to |x - y|$, resp. Now consider a C^∞ function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to R^k$, $k \leq d$, which is a defining function (vector-valued if $k > 1$) in the sense that the differential $D\Phi(x, y)$ has maximal rank everywhere. Thus, Φ is a submersion and hence for each $\vec{t} \in \mathbb{R}^k$, the level set

$$Z_{\vec{t}} := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \Phi(x,y) = \vec{t}\right\} \tag{1.1}$$

is a smooth, codimension k surface in $\mathbb{R}^d \times \mathbb{R}^d$, and the $Z_{\vec{t}}$ form a family of incidence relations on $\mathbb{R}^d$, indexed by $\vec{t}$. (For $k = 1$, the scalar will be denoted by t.) More generally, since many Φ of interest, such as those defining the generalized distance sets discussed below in Sec. 2, have points in the domain where they fail to be smooth, or have critical values in the codomain, it is useful to restrict the domain or codomain and consider $\Phi : X \times Y \to W$, where (for now) $X, Y \subset \mathbb{R}^d$ and $W \subset\subset R^k$.

If the assumption that $\mathrm{rank}(D\Phi\,(x,y)) = k$ is strengthened slightly to the condition (which is standard in integral geometry) that

$$\mathrm{rank}(D_x\Phi) = k \;\text{ and } \mathrm{rank}(D_y\Phi) = k \tag{1.2}$$

everywhere, then each of the two projections,

$$\pi_X,\, \pi_Y : Z_{\vec{t}} \to \mathbb{R}^d, \quad \pi_X(x,y) := x, \quad \pi_Y(x,y) := y,$$

are submersions, and $Z_{\vec{t}}$ is a *double fibration* in the sense of Gelfand (see [14]) and Helgason [16]. In particular, for each $\vec{t} \in W$ and $x \in X$,

$$Z^x_{\vec{t}} = \left\{y : (x,y) \in Z_{\vec{t}}\right\} \subset Y \tag{1.3}$$

is a smooth surface of codimension k. As in the paper [10] by Grafakos, Palsson and the first two authors, for a compact subset $E \subset \mathbb{R}^d$, we define the (two-point) Φ-*configuration set* of E as

$$\Delta_\Phi(E) = \left\{\Phi(x,y) : x, y \in E\right\} \subset \mathbb{R}^k. \tag{1.4}$$

Thus, if $k = d$ and $\Phi(x,y) = x - y$, then $\Delta_\Phi(E) = E - E$, while if $k = 1$ and $\Phi(x,y) = |x - y|$, $\Delta_\Phi(E)$ is the distance set of E [1]. We give further examples below.

Our goal is to find a threshold, $s_0 = s_0(\Phi)$, such that if $\dim_{\mathcal{H}}(E) > s_0$ then $\Delta_\Phi(E)$ has nonempty interior. Similar questions have been studied in, or are accessible to the methods of, a number of works, e.g., [11, 2, 5, 23, 24, 21]. (There is of course an extensive literature on the related Falconer distance problem [8], and its generalizations to configurations, where the question is what lower bound on $\dim_{\mathcal{H}}(E)$ ensures that $\Delta_\Phi(E)$ has positive Lebesque measure [41, 7, 6, 15].) We will show that a sufficient value of $s_0(\Phi)$ can be expressed in terms of d, k and α_Φ, the amount of smoothing on L^2-based Sobolev spaces satisfied by the family of generalized Radon transforms $\mathcal{R}_{\vec{t}}$ defined by Φ.

[1]In applications, one can easily localize away from singularities of Φ at degenerate configurations (at $x = y$ for this Φ).

There are a number of results in the configuration literature, such as Henriot-Łaba-Pramanik [17], Chan-Łaba-Pramanik [5], and Fraser-Guo-Pramanik [9], which make a Fourier dimension assumption on the set E. In some sense, the Sobolev regularity assumption we make here is analogous to a Fourier decay estimate, but on the family of configurations rather than on E, since in the translation-invariant case the Sobolev regularity of the $\mathcal{R}_{\vec{t}}$ corresponds to the uniform Fourier decay of the measures that are the convolution kernels of the $\mathcal{R}_{\vec{t}}$. This allows our condition on the set E to be solely in terms of the Hausdorff dimension.

The amount α_Φ of smoothing on $L^2_s(\mathbb{R}^d)$ is defined as follows. For each $\vec{t} \in W$, choose a compactly supported smooth density $d\rho_{\vec{t}}$ on $Z_{\vec{t}}$ (such as the Leray density induced by Φ), and define the generalized Radon transform

$$\mathcal{R}_{\vec{t}} f(x) = \int_{Z^x_{\vec{t}}} f(y). \tag{1.5}$$

Thus, the Schwartz kernel of $\mathcal{R}_{\vec{t}}$ is $d\rho_{\vec{t}}(x,y)$. It is known that each $\mathcal{R}_{\vec{t}}$ is a Fourier integral operator (FIO), of order $-(d-k)/2$ and associated to a canonical relation $C_{\vec{t}}$, where $C_{\vec{t}} = N^*Z'_{\vec{t}} \subset T^*\mathbb{R}^d \times T^*\mathbb{R}^d$, the conormal bundle of $Z_{\vec{t}}$; see Guillemin and Sternberg [13], and also Phong and Stein [33].

If $C_{\vec{t}_0}$ is a local canonical graph (see the more extended discussion in Sec. 3 below), then so are all the $C_{\vec{t}}$ for $\vec{t}$ close to $\vec{t}_0$, and, for $s \in \mathbb{R}$, $\mathcal{R}_{\vec{t}} : L^2_s \to L^2_{s+(d-k)/2}$ uniformly as $\vec{t}$ ranges over V. However, general canonical relations need not be local canonical graphs and there may be a loss of derivatives relative to this estimate; it is thus useful to describe the amount of smoothing both in terms of the absolute number, α_Φ, of derivatives that the $\mathcal{R}_{\vec{t}}$ add on L^2, and also in terms of the loss, β_Φ, relative to the optimal possible smoothing. Hence, for our first result, we state the regularity assumption on the $\mathcal{R}_{\vec{t}}$ as either of the equivalent conditions that

(i) there is an *absolute smoothing* α_Φ, $0 \le \alpha_\Phi \le (d-k)/2$ such that, for any $s \in \mathbb{R}$ one has

$$||\mathcal{R}_{\vec{t}}||_{L^2_s(\mathbb{R}^d)\to L^2_{s+\alpha_\Phi}(\mathbb{R}^d)} \le C_s, \tag{1.6}$$

with C_s bounded as $\vec{t}$ varies over W; or

(ii) for some *relative loss* β_Φ, $0 \le \beta_\Phi \le (d-k)/2$,

$$||\mathcal{R}_{\vec{t}}||_{L^2_s(\mathbb{R}^d)\to L^2_{s+(d-k)/2-\beta_\Phi}(\mathbb{R}^d)} \le C_s, \tag{1.7}$$

which holds iff (1.6) does, with $\beta_\Phi = (d-k)/2 - \alpha_\Phi$. Sharp values of α_Φ and β_Φ are known in a number of degenerate geometries, but in most for the examples below the canonical relations are nondegenerate, so that $\beta_\Phi = 0$. Our first result is the following.

Theorem 1.1. *Suppose that $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$ satisfies (1.2) and that the generalized Radon transforms $\mathcal{R}_{\vec{t}}$ in (1.5) satisfy (1.6), (1.7). Then, if $E \subset \mathbb{R}^d$ is a compact set with $\dim_{\mathcal{H}}(E) > d - \alpha_\Phi = \frac{d+k}{2} + \beta_\Phi$, it follows that $\Delta_\Phi(E)$ has nonempty interior.*

Corollary 1.2. *(Mattila-Sjölin [27]; Iosevich-Mourgoglou-Taylor [22]) Let $E \subset \mathbb{R}^d$ be a compact set and let $\Delta(E)$ be its distance set, $\Delta(E) = \{|x-y| : x, y \in E\}$. More generally, let $\Delta(E)$ be defined by the translation-invariant metric induced by a norm $||\cdot||$ on $\mathbb{R}^d$ whose unit sphere has strictly positive Gaussian curvature. Then, if $\dim_{\mathcal{H}}(E) > (d+1)/2$, it follows that $\Delta(E)$ contains an open interval.*

This follows from Thm. 1.1 by taking $k = 1$ and $\Phi(x, y) = ||x - y||$, so that $\Delta_\Phi(E) = \Delta(E)$. For $0 < t < \infty$, the generalized Radon transform $\mathcal{R}_t$ is the spherical mean operator for radius t with respect to $||\cdot||$; as is well- known (see, for example, [37]), this is an FIO of order $-(d-1)/2$ associated with a canonical graph, so that, on $L^2_s(\mathbb{R}^d)$, $\mathcal{R}_t$ is smoothing of order $\alpha_\Phi = (d-1)/2$ and $\beta_\Phi = 0$. Hence, in the same range as in [27], namely for $s > (d+1)/2$ (also the same range as in Falconer's original result for the distance set problem [8]), the distance set has nonempty interior, i.e., contains an open interval. (For certain specific thin sets below the $(d+1)/2$ threshold, it has been shown that the distance set has nonempty interior; see for example [35] by Simon and the third author of the current paper.)

Corollary 1.3. *Let (M, g) be a d-dimensional Riemannian manifold, $U \subset M$ an open set on which there are no conjugate points, and identify U with an open subset of $\mathbb{R}^d$. Setting $\Phi(x, y) = d_g(x, y)$, the Riemannian distance and, for $E \subset U$, the Riemannian distance set $\Delta^g(E) := \{d_g(x, y) : x, y \in U\}$, if $\dim_{\mathcal{H}}(E) > (d+1)/2$, then $\Delta^g(E)$ contains an open interval.*

This follows from Thm. 1.1 in the same way as for Cor. 1.2: the operators $\mathcal{R}_t$ are the Riemannian spherical means, which are still FIO of order $-(d-1)/2$ associated with canonical graphs [39], so that again $\alpha_\Phi = (d-1)/2$, $\beta_\Phi = 0$.

One can also prove a result for multi-parameter distance sets (cf. [20]):

Corollary 1.4. *Suppose $d = d_1 + \cdots + d_k$, with all $d_j > 1$, and write $x \in \mathbb{R}^d$ as $x = (x^1, \ldots, x^k)$ with $x^j \in \mathbb{R}^{d_j}$, $1 \le j \le k$. For $E \subset \mathbb{R}^d$ compact, define*

$$\Delta^{(k)}(E) = \left\{ \left(|x^1 - y^1|, \ldots, |x^k - y^k| \right) : x, y \in E \right\} \subset \mathbb{R}^k.$$

Then, if $\dim_{\mathcal{H}}(E) > d - \frac{1}{2}\min\{d_j - 1\} = \frac{1}{2}\max\{d - d_j + 1\}$, it follows that $\Delta^{(k)}(E)$ has nonempty interior in $\mathbb{R}^k$.

This follows by noting that, for $\vec{t} \in \mathbb{R}^k$, $t_j > 0$ for all j, the operator $\mathcal{R}_{\vec{t}}$ is convolution with a product of surface measures on spheres and thus has Fourier multiplier that decays uniformly as $(1+|\xi|)^{-\frac{1}{2}\min\{d_j-1\}}$ (and no better), so that Thm. 1.1 applies with the same k and $\alpha_\Phi = \frac{1}{2}\min\{d_j-1\}$.

Thm. 1.1 is subsumed in a more general version with two sets, whose proof is essentially the same. For $E,\ F \subset \mathbb{R}^d$, define the Φ-*configuration set of E and F* as

$$\Delta_\Phi(E,F) := \{\Phi(x,y) : x \in E,\ y \in F\} \subset \mathbb{R}^k. \tag{1.8}$$

Then we have

Theorem 1.5. *Suppose that $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$ satisfies (1.2) and that the generalized Radon transforms $\mathcal{R}_{\vec{t}}$ in (1.5) satisfy (1.6), (1.7). Then, if $E,\ F \subset \mathbb{R}^d$ are compact sets with $dim_{\mathcal{H}}(E)$ and $dim_{\mathcal{H}}(F)$ satisfying*

$$dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > 2d - 2\alpha_\Phi = d + k + 2\beta_\Phi, \tag{1.9}$$

it follows that $\Delta_\Phi(E,F)$ has nonempty interior.

The method for proving Thm. 1.5 is flexible and allows one to obtain the nonempty interior of Δ_Φ in asymmetric settings, where not only are E and F possibly different, but (i) Δ_Φ is determined by 'points' x and y which may belong to different spaces, of possibly different dimensions; and (ii) Φ can be manifold-valued.

To make this precise, let W, X and Y be smooth manifolds, of dimensions $k \geq 1$ and $d_1 \geq d_2 \geq 1$, resp.; note that the notion of Hausdorff dimension for compact subsets of X or Y is well-defined. Let $\Phi : X \times Y \to W$ be a C^∞ function satisfying the double-fibration condition (1.2), at least away from a lower dimensional subvariety. If $E \subset X$, $F \subset Y$ are compact, define

$$\Delta_\Phi(E,F) := \{\Phi(x,y) : x \in E,\ y \in F\} \subset W. \tag{1.10}$$

Our goal is to find a lower bound on $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F)$ ensuring that $\Delta_\Phi(E,F)$ has nonempty interior in W. For $\vec{t} \in W$, the operators $\mathcal{R}_{\vec{t}}$ are now FIO of order

$$m := 0 + k/2 - (d_1+d_2)/4 = -(d_2-k)/2 - (d_1-d_2)/4,$$

where we have written the order on the right (a way that is standard for FIO between spaces of different dimensions) so as to the isolate the best possible smoothing order of the operator, namely $(d_2-k)/2$ (see Sec. 3 below). The analogue of (1.6), (1.7) is then

$$||\mathcal{R}_{\vec{t}}||_{L^2_s(Y)\to L^2_{s+\alpha_\Phi}(X)} = ||\mathcal{R}_{\vec{t}}||_{L^2_s(Y)\to L^2_{s+\frac{d_2-k}{2}-\beta_\Phi}(X)} \leq C_s, \ \text{ for all } \vec{t} \in W_0 \subset W, \tag{1.11}$$

for some $0 \leq \alpha_\Phi,\ \beta_\Phi \leq (d_2-k)/2$, related by $\alpha_\Phi = (d_2-k)/2 - \beta_\Phi$. When the $C_{\vec{t}}$ are nondegenerate canonical relations, $\alpha_\Phi = (d_2-k)/2$, $\beta_\Phi = 0$. In this generality, the following holds, and also implies Thm. 1.5 and thus Thm. 1.1.

Theorem 1.6. *Let W, X and Y be smooth manifolds, of dimensions $k \geq 1$ and $d_1 \geq d_2 \geq 1$, resp. Suppose that $\Phi : X \times Y \to W$ satisfies condition (1.2) and the $\mathcal{R}_{\vec{t}}$ in (1.5) satisfy (1.11) for some $0 \leq \alpha_\Phi,\ \beta_\Phi \leq (d_2 - k)/2$. If $E \subset X$, $F \subset Y$ are compact sets, then $\Delta_\Phi(E, F)$ defined in (1.10) has nonempty interior in W if*

$$dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d_1 + d_2 - 2\alpha_\Phi = d_1 + k + 2\beta_\Phi. \tag{1.12}$$

2. Examples

Before proving Thm. 1.6, we state a series of corollaries which follow by applying Thms. 1.5 or 1.6 to the relevant Φ, deferring of some of the details to Sec. 5.

2.1. Generalized distance sets.

A diverse class of examples with $k = 1$ arises when the x and y are points, subspaces or even submanifolds of some ambient $\mathbb{R}^d$ or Riemannian manifold (M, g). Suppose X and Y are smooth families of closed submanifolds (or possibly subvarieties) in $\mathbb{R}^d$ or (M, g), and let $\Phi : X \times Y \to \mathbb{R}$ be the standard distance between subsets,

$$\Phi(x, y) := \text{dist}(x, y) = \inf_{a\in x,\, b\in y} |a - b| \quad [\text{ or } \inf d_g(a, b)\,]. \tag{2.1}$$

In many cases, Φ is C^∞ on $X \times Y$, and (1.2) holds (with $k = 1$), away from a lower dimensional variety, corresponding to degenerate configurations. (Note that, even if $X = Y$, the function Φ is typically not a metric, since $\Phi(x, y) = 0$ does not imply that $x = y$, but only that $x \cap y \neq \emptyset$.) Then, for sets $E \subset X$, $F \subset Y$, let $\Delta_\Phi(E, F)$ be defined by (1.8). Applications of Thm. 1.6 to configuration sets of this type include the following:

Lines–points in $\mathbb{R}^2$ and hyperplanes-points in $\mathbb{R}^d$: Let $Y = \mathbb{R}^2$ and $X = M_{1,2}$, the space of all affine lines in $\mathbb{R}^2$. As with the Radon transform [16], it is convenient to parametrize $M_{1,2}$ as $\mathbb{S}^1 \times \mathbb{R}$, with $(\omega, s) \leftrightarrow \{y \in \mathbb{R}^2 : \omega \cdot y - s = 0\}$. (Since $(-\omega, -s)$ and (ω, s) correspond to the same line, there is a 2-1 redundancy, but this is harmless.) Then $\text{dist}((\omega, s), y) = |\omega \cdot y - s| =: \Phi((\omega, s), y)$, which is smooth away from the incidence relation Z_0. The question of interest is: for collections $E \subset M_{1,2}$ and $F \subset \mathbb{R}^2$ of lines and points, resp., what lower bounds on $\dim_{\mathcal{H}}(E)$ and $\dim_{\mathcal{H}}(F)$ ensure that $\Delta_\Phi(E, F)$ contains an interval? (Finite field Falconer-type problems for this geometry were studied in [40, 32, 4, 34, 3].)

The operators $\mathcal{R}_t$ are sums of translates by $\pm t$ of the Radon transform in the s variable so that the C_t are local canonical graphs and $\beta_\Phi = 0$. Similarly, for $d \geq 3$, let $Y = \mathbb{R}^d$ and $X = M_{d-1,d}$, the Grassmannian of all affine hyperplanes in $\mathbb{R}^d$. As for $d = 2$, we may parametrize $M_{d-1,d}$ as $\mathbb{S}^{d-1} \times \mathbb{R}$ and, with $k = 1$, take $\Phi((\omega, s), y) := \text{dist}((\omega, s), y) = |\omega \cdot y - s|$. Then the C_t are again local canonical graphs, so that $\beta_\Phi = 0$ and a consequence of Thm. 1.6 is

Corollary 2.1. *If $E \subset M_{d-1,d}$, $F \subset \mathbb{R}^d$ with $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d+1$, then $\Delta_\Phi(E,F)$ contains an interval.*

Circles–points in $\mathbb{R}^2$ and spheres-points in $\mathbb{R}^d$: For a family of non-equidimensional examples, again let $Y = \mathbb{R}^2$ and now let $X = S_2$ be the space of all circles in $\mathbb{R}^2$, which is 3-dimensional. We may parametrize S_2 by the center and radius of each circle, $S_2 = \{(a,r) : a \in \mathbb{R}^2,\ r > 0\}$. Then $\Phi((a,r),y) := |y-a| - r$ defines the circle-point relation; for each $t \in \mathbb{R}$, $Z_t^{\pm} = \{((a,r),y) : |y-a| = r \pm t\}$ is smooth away from the singular set at $r = \mp t$. The operator $\mathcal{R}_t \in I^{-\frac{1}{2}-\frac{1}{4}}(C_t)$ where C_t, which is the same canonical relation as for the forward solution of the wave equation (with r playing the role of time), but translated by $\pm t$ in the r variable. Since this is nondegenerate, $\beta_\Phi = 0$ and so it follows that if $E \subset S_2$, a set of circles, and $F \subset \mathbb{R}^2$, a set of points, satisfy $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > 4$, then Thm. 1.6 applies. This extends to higher dimensions:

Corollary 2.2. *If S_d is the $(d+1)$-dimensional space of all spheres in $\mathbb{R}^d$, and $E \subset S_d$, $F \subset \mathbb{R}^d$ satisfy $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d+2$, then $\Delta_\Phi(E,F) \subset \mathbb{R}$ has nonempty interior.*

Lines-points in $\mathbb{R}^d$, $d \geq 3$: Let $X = M_{1,d}$ be the $(2d-2)$-dimensional Grassmannian of all affine lines in $Y = \mathbb{R}^d$. For $L \in M_{1,d}$ and $y \in \mathbb{R}^3$, let $\Phi(L,y) = \text{dist}(L,y)$. One can show (see Sec. 5) that for $t > 0$, $\mathcal{R}_t$ is an FIO with a nondegenerate canonical relation and thus $\beta_\Phi = 0$. From Thm. 1.6 we obtain

Corollary 2.3. *If $E \subset M_{1,d}$ and $F \subset \mathbb{R}^d$ satisfy $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > 2d-1$, then $\Delta_\Phi(E,F)$ has nonempty interior.*

Lines-lines in $\mathbb{R}^d$, $d \geq 3$: Now let $X = Y = M_{1,d}$. For $L,\ L' \in M_{1,d}$, the distance $\text{dist}(L,L')$ as in (2.1) is positive if $L \cap L' = \emptyset$, and is a smooth function, satisfying the double fibration condition, away from the lower dimensional incidence variety $\{(L,L') :\ L \cap L' \neq \emptyset\}$. For $t > 0$, the $\mathcal{R}_t$ are FIO associated with canonical graphs, so that $\beta_\Phi = 0$, yielding

Corollary 2.4. *If $E,\ F \subset M_{1,d}$ are sets of lines with $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > 2d-1$, then $\Delta_\Phi(E,F)$ has nonempty interior.*

2.2. **Higher dimensional configuration sets.** Values of k greater than 1 arise when configurations are encoded by vector-valued (or manifold-valued) data.

Configurations determined by ensembles of quadratic forms: Let $Q_1, \dots, Q_k$ be quadratic forms on $\mathbb{R}^d$. Define the ensemble $\vec{Q} = (Q_1, \dots, Q_k)$ to be *nonsingular* if $c_1Q_1 + \cdots + c_kQ_k$ is nonsingular for all $\vec{c} = (c_1, \dots, c_k) \in \mathbb{R}^k \setminus \mathbf{0}$. The sharp restrictions on k and d in order that such ensembles to exist were found by Adams, Lax and Phillips [1]: there exists such a k-dimensional nonsingular family of quadratic forms on $\mathbb{R}^d$ iff $k \le \rho(d/2)+1$, where $\rho(\cdot)$ are Radon-Hurwitz numbers, defined by

Def. If n is a positive integer, write $n = (2l+1)\cdot 2^m$, the factorization of n into the product of an odd integer and an integral power of 2. Express m modulo 4 as $m = p + 4q$, $0 \le p \le 3$. Then the *Radon-Hurwitz number* of n is $\rho(n) := 2^p + 8q$. If n is a half-integer, then $\rho(n) := 0$.

Corollary 2.5. *If $\vec{Q}$ is a nonsingular family on $\mathbb{R}^d$, define $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^k$ by $\Phi(x,y) = \vec{Q}(x-y)$. Then if $E,\ F \subset \mathbb{R}^d$ are compact and $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > d+k$, it follows that*

$$\Delta_\Phi(E,F) = \left\{ \left(Q_1\left(x-y\right), \dots, Q_k\left(x-y\right)\right) \in \mathbb{R}^k : x \in E,\ y \in F \right\}$$

has nonempty interior in $\mathbb{R}^k$.

If d is odd, only $k=1$ is possible. For $d=2$, $\rho(d/2)+1 = 2$, and $Q_1(x) = x_1^2 - x_2^2$, $Q_2(x) = x_1 x_2$ is a nonsingular ensemble, but for this $\vec{Q}$ Thm. 1.5 is vacuous, since it requires $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > 4$. However, note that for $d=4$, one has $\rho(d/2)+1 = 2+1 = 3$, so that there exist nonsingular ensembles for which Cor. 2.5 yields a nontrivial result; e.g., with $k=3$, if $\dim_{\mathcal{H}}(E) > 7/2$ then $\Delta_\Phi(E,E)$ has nonempty interior.

Heisenberg circles and spheres: Let $\mathbb{H}^1$ be the 3-dimensional Heisenberg group, $\mathbb{H}^1 \simeq \mathbb{R}^2 \times \mathbb{R}$ with product

$$(x', x_3) \cdot (y', y_3) = \left(x' + y', x_3 + y_3 + \frac{1}{2}(x')^T J y'\right), \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Define $\Phi : \mathbb{H}^1 \times \mathbb{H}^1 \to \mathbb{R}^2$ by

$$\Phi(x,y) = \vec{t} = \left(|x' - y'|, x_3 - y_3 + \frac{1}{2}\left(x_1 y_2 - x_2 y_1\right) \right) \in \mathbb{R}^2, \tag{2.2}$$

where t_1 is the distance from the origin in the first two coordinates of $x \cdot y^{-1}$ and t_2 is the 'height' of $x \cdot y^{-1}$ above the 'plane of good directions', $\{x_3 = 0\}$. For each $\vec{t} \in \mathbb{R}_+ \times \mathbb{R}$, the corresponding $\mathcal{R}_{\vec{t}}$ is the generalized Radon transform which averages over group translates of the origin-centered circle of radius $t_1 \subset \{x_3 = 0\}$, translated in the central direction by t_2. Such operators (for $t_2 = 0$) have been studied by Nevo and Thangavelu [31] and Müller and Seeger [30]. Here, in the notation of Thm. 1.5, $d=3$, $k=2$, and it is known (see [30] and Sec. 5) that each $C_{\vec{t}}$ is a two-sided fold (or folding canonical relation in the sense of Melrose and Taylor [28]), so that $\beta_\Phi = 1/6$ by [28]. Hence, from Thm. 1.5 we obtain

Corollary 2.6. *If $E,\ F \subset \mathbb{H}^1$ with $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > 16/3$, then $\Delta_\Phi(E,F) \subset \mathbb{R}^2$ has nonempty interior.*

This generalizes to the more general setting of [30], with $\mathbb{H}^1$ replaced by a nondegenerate step-two nilpotent group G and $\mathbb{S}^1$ replaced by a hypersurface Σ with strictly positive Gaussian curvature in the step 1 space: Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a graded nilpotent Lie algebra, with $dim(\mathfrak{g}_1) = n$, $dim(\mathfrak{g}_2) = m$, and $G = exp(\mathfrak{g}) \simeq \mathbb{R}^{n+m}$ is the resulting connected, simply-connected Lie group, with projections $\pi_1 : G \to \mathfrak{g}_1 \simeq \mathbb{R}^n$ and $\pi_2 : G \to \mathfrak{g}_2 \simeq \mathbb{R}^m$. We assume that $\mathfrak{g}$ satisfies the nondegeneracy condition that the skew-linear form

$$B_\zeta(u,v) := \langle \zeta, [u,v] \rangle \text{ is nondegenerate on } \mathfrak{g}_1, \quad \forall \zeta \in \mathfrak{g}_2^* \setminus 0. \tag{2.3}$$

(In particular, n must be even.) Let $||\cdot||$ be a C^∞ norm on $\mathfrak{g}_1$ whose unit sphere Σ has everywhere positive Gaussian curvature, and define $\Phi : G \times G \to \mathbb{R}^{m+1}$ by

$$\Phi(x,y) = \vec{t} := \big(||\pi_1(x \cdot y^{-1})||, \pi_2(x \cdot y^{-1}) \big). \tag{2.4}$$

Then, one can show that the the canonical relations $C_{\vec{t}}$ of the $\mathcal{R}_{\vec{t}}$ are associated with two sided folds, so that again $\beta_\Phi = 1/6$, and the required lower bound on $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F)$ is $(n+m)+(m+1)+\frac{1}{3} = n+2m+\frac{4}{3}$, implying

Corollary 2.7. *If E, $F \subset G$ with $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > n+2m+\frac{4}{3}$, then $\Delta_\Phi(E,F)$ has nonempty interior.*

Configurations determined by a curve: Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a smooth curve. Writing $x = (x_1, x')$, etc., we assume γ is of the form $\gamma(\tau) = (\tau, g(\tau))$, $g = (g_2, \dots, g_d) : \mathbb{R} \to \mathbb{R}^{d-1}$. Define $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d-1}$ by

$$\Phi(x,y) = x' - y' - g(x_1 - y_1), \tag{2.5}$$

which measures the displacement of x off of $y + \gamma$. All of the operators $\mathcal{R}_{\vec{t}}$ are translates of $\mathcal{R}_{\mathbf{0}}$ and thus have the same structure and satisfy the same estimates. For example, if γ is the moment curve $\gamma(\tau) = (\tau, \tau^2, \dots, \tau^d)$, then $\alpha_\Phi = \frac{1}{d}$, and from Thm. 1.5 we obtain

Corollary 2.8. *If γ is the moment curve and compact sets E, $F \subset \mathbb{R}^d$ satisfy $dim_{\mathcal{H}}(E) + dim_{\mathcal{H}}(F) > 2d - \frac{2}{d}$, then $\Delta_\Phi(E,F)$ has nonempty interior.*

3. Background

We give a brief survey of the Fourier integral operator theory needed, referring to Hörmander [18, 19] for more background and further details.

Let X be a smooth manifold of dimension d. The cotangent bundle T^*X is a symplectic manifold with respect to the *canonical two-form*, $\omega = \sum d\xi_j \wedge dx_j$ (with respect to any local coordinates). We denote the *zero-section* of T^*X, $\{\xi = 0\}$, by $\mathbf{0}$. A *conic Lagrangian submanifold* of T^*X is a smooth, conic (i.e., invariant under $(x,\xi) \to (x, \tau\xi)$ for $0 < \tau < \infty$) submanifold $\Lambda \subset T^*X \setminus \mathbf{0}$ of dimension $\dim(\Lambda) = d = \frac{1}{2}\dim(T^*X)$ such that $\omega|_\Lambda \equiv 0$.

Now let X and Y be smooth manifolds of dimensions d_1, d_2, resp. Then T^*X, T^*Y are each symplectic manifolds, with canonical two-forms we denote by ω_{T^*X}, ω_{T^*Y}, resp. Equip $T^*X \times T^*Y$ with the *difference symplectic form*, $\omega_{T^*X} - \omega_{T^*Y}$. For our purposes, a *canonical relation* will mean a submanifold, $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ (hence of dimension $d_1 + d_2$), which is conic Lagrangian with respect to $\omega_{T^*X} - \omega_{T^*Y}$.

For some $N \geq 1$, let $\phi : X \times Y \times (\mathbb{R}^N \setminus \mathbf{0}) \to \mathbb{R}$ be a smooth phase function which is positively homogeneous of degree 1 in $\theta \in \mathbb{R}^N$, i.e., $\phi(x, y, \tau\theta) = \tau \cdot \phi(x, y, \theta)$ for all $\tau \in \mathbb{R}_+$. Let Σ_ϕ be the *critical set* of ϕ in the θ variables,

$$\Sigma_\phi := \{(x, y, \theta) \in X \times Y \times (\mathbb{R}^N \setminus \mathbf{0}) : d_\theta \phi(x, y, \theta) = 0\},$$

and

$$C_\phi := \{(x, d_x\phi(x, y, \theta); y, -d_y\phi(x, y, \theta)) : (x, y, \theta) \in \Sigma_\phi\},$$

both of which are conic sets. If we impose the first order nondegeneracy conditions

$$d_x\phi(x, y, \theta) \neq 0 \text{ and } d_y\phi(x, y, \theta) \neq 0, \forall (x, y, \theta) \in \Sigma_\phi,$$

then $C_\phi \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$. If in addition one demands that

$$\operatorname{rank}[d_{x,y,\theta} d_\theta \phi(x, y, \theta)] = N, \; \forall \, (x, y, \theta) \in \Sigma_\phi,$$

then Σ_ϕ is smooth, $dim(\Sigma_\phi) = d_1 + d_2$, and the map

$$\Sigma_\phi \ni (x, y, \theta) \to (x, d_x\phi\,(x, y, \theta)\,; y, -\partial_y\phi\,(x, y, \theta)) \in C_\phi \tag{3.1}$$

is an immersion, whose image is an immersed canonical relation; the phase function ϕ is said to *parametrize* C_ϕ.

For a canonical relation $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ and $m \in \mathbb{R}$, one defines $I^m(X, Y; C) = I^m(C)$, the class of *Fourier integral operators* $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ of order m, as the collection of operators whose Schwartz kernels are locally finite sums of oscillatory integrals of the form

$$K(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(x, y, \theta)\, d\theta,$$

where $a(x, y, \theta)$ is a symbol of order $m - \frac{N}{2} + \frac{d_1+d_2}{4}$ and ϕ is a phase function as above, parametrizing some $C_\phi \subset C$.

The FIO relevant for this paper are the *generalized Radon transforms* $\mathcal{R}_{\vec{t}}$ determined by defining functions $\Phi : X \times Y \to \mathbb{R}^k$ satisfying (1.2). The Schwartz kernel of each $\mathcal{R}_{\vec{t}}$ is a smooth multiple of $\delta_k(\Phi(x, y) - \vec{t})$, where δ_k is the delta distribution on $\mathbb{R}^k$. From the Fourier inversion representation of δ_k, we see that $\mathcal{R}_{\vec{t}}$ has kernel

$$K_{\vec{t}}(x, y) = \int_{\mathbb{R}^k} e^{i(\Phi(x,y)-\vec{t})\cdot\theta}\, b(x, y) \cdot 1(\theta)\, d\theta,$$

where $b \in C_0^\infty$. Since the amplitude is a symbol of order 0, $\mathcal{R}_{\vec{t}}$ is an FIO of order $0 + \frac{k}{2} - \frac{d_1+d_2}{4} = -\frac{d_1+d_2-2k}{4}$ associated with the canonical relation parametrized as in (3.1) by $\phi(x,y,\theta) = (\Phi(x,y) - \vec{t}) \cdot \theta$, which is the twisted conormal bundle of the incidence relation $Z_{\vec{t}}$,

$$C_{\vec{t}} = N^* Z_{\vec{t}}' := \Big\{ \big(x, \sum_{j=1}^{k} d_x \Phi_j(x,y)\theta_j; y, -\sum_{j=1}^{k} d_x \Phi_j(x,y)\theta_j\big) : (x,y) \in Z_{\vec{t}},\, \theta \in \mathbb{R}^k \setminus \mathbf{0} \Big\}.$$

For W-valued defining functions Φ, as in Thm. 1.6, this discussion is modified slightly by introducing local coordinates on W.

For a general canonical relation, C, the natural projections $\pi_L : T^*X \times T^*Y \to T^*X$ and $\pi_R : T^*X \times T^*Y \to T^*Y$ restrict to C, and by abuse of notation we refer to the restricted maps with the same notation. One can show that, at any point $c_0 = (x_0, \xi_0; y_o, \eta_0) \in C$, one has $\operatorname{corank}(D\pi_L)(c_0) = \operatorname{corank}(D\pi_R)(c_0)$; we say that the canonical relation C is *nondegenerate* if this corank is zero at all points of C, i.e., if $D\pi_L$ and $D\pi_R$ are of maximal rank. If $\dim(X) = \dim(Y)$, then C is nondegenerate iff π_L, π_R are local diffeomorphisms, and then C is a *local canonical graph*, i.e., locally near any $c_0 \in C$ equal to the graph of a canonical transformation. If $\dim(X) = d_1 > d_2 = \dim(Y)$, then C is nondegenerate iff π_L is an immersion and π_R is a submersion. To describe the L^2-Sobolev estimates for FIO associated with C, it is convenient to normalize the order by considering $A \in I^{m - \frac{|d_1-d_2|}{4}}$. One has

Theorem 3.1. *Suppose that $\dim(X) = d_1$, $\dim(Y) = d_2$, $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ is a nondegenerate canonical relation, and $A \in I^{m - \frac{|d_1-d_2|}{4}}$ has a compactly supported Schwartz kernel. Then $A : L^2_s(Y) \to L^2_{s-m}(X)$ for all $s \in \mathbb{R}$.*

In particular, if C is a local canonical graph, then $A \in I^m \implies A : L^2_s \to L^2_{s-m}$; this is relevant to a number of the Corollaries above.

On the other hand, for Cors. 2.2 and 2.3, the canonical relations cannot be canonical graphs, since the dimensions of X and Y differ, but the canonical relations are nondegenerate. For Cor. 2.2, parametrized by the pair of phase functions $\phi^{\pm}((a,r), y; \theta) = (|y-a| - r \mp t)\theta$ on $\mathbb{R}^{d+1} \times \mathbb{R}^d \times (\mathbb{R} \setminus \mathbf{0})$, C_t is, away from $r \mp t = 0$, given by

$$C_t = \{(a, r, \theta\omega, \mp|\theta|; a + (r \pm t)\omega, \theta\omega) : (a,r) \in \mathbb{R}^{n+1}, \omega \in \mathbb{S}^{d-1}, \theta \in \mathbb{R} \setminus \mathbf{0}\}.$$

One sees by inspection that $D\pi_R$ has rank $2d$ everywhere and thus C_t is nondegenerate away from $\{r \mp t = 0\}$ (in fact, its natural extension across those points is also smooth and nondegenerate, but for our purposes we will not need that). Thus, when localized away from $r = \pm t$, the operators $\mathcal{R}_t$ are in $I^{-\frac{1}{2}-\frac{1}{4}}(C_t)$ and by Thm. 3.1 map $L^2_s(\mathbb{R}^d) \to L^2_{s+(1/2)}(\mathbb{R}^{d+1})$.

Returning to general C, if the corank of $D\pi_L$ (and thus that of $D\pi_R$) is $\le k$ at all points of C, then an $A \in I^m(C)$ maps no worse than $L^2_s(Y) \to L^2_{s-m-(k/2)}(X)$. However, for classes of C for which π_L and/or π_R degenerate in specific ways, the loss of derivatives is often less than $k/2$. The first and best known result of this type is the following, which we use in the analysis below of Cors. 2.6 and 2.7. Suppose $\dim(X) = \dim(Y) = d$ and at any degenerate points $c_0 \in C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$, both π_L and π_R have Whitney fold singularities. Such C were introduced by Melrose and Taylor [28] and called *folding canonical relations* (also called *two-sided folds* [12]).

Theorem 3.2. [28] *If C is a folding canonical relation and $A \in I^m(X, Y; C)$ with compactly supported Schwartz kernel, then $A : L^2_s(Y) \to L^2_{s-m-(1/6)}(X)$, $\forall s \in \mathbb{R}$.*

4. Proof of theorem 1.6

To begin the proof, recall[2] that if $E \subset \mathbb{R}^d$ is a compact set with $\dim_{\mathcal{H}}(E) > s$, then there exists a Frostman measure on E: a probability measure μ, supported on E and of finite s-energy:

$$\int_E \int_E |x-y|^{-s}\, d\mu(x)\, d\mu(y) < \infty,$$

or equivalently,

$$\int_E |\hat{\mu}(\xi)|^2 \cdot |\xi|^{s-d}\, d\xi < \infty. \tag{4.1}$$

Since $\mu \in \mathcal{E}'(\mathbb{R}^d)$, $\hat{\mu} \in C^\omega$ and thus it follows from (4.1) that $\mu \in L^2_{(s-d)/2}(\mathbb{R}^d)$. This last fact also holds in the more general setting of $E \subset X$, a compact subset of a d-dimensional manifold X with $\dim_{\mathcal{H}}(E) > s$.

Now, in the context of Thm. 1.6, suppose that $dim(X) = d_1$, $dim(Y) = d_2$, $E \subset X$, $F \subset Y$, with $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > d_1 + k + 2\beta_\Phi$. Then we can find s_1, s_2 such that $\dim_{\mathcal{H}}(E) > s_1$, $\dim_{\mathcal{H}}(F) > s_2$, also satisfying

$$s_1 + s_2 > d_1 + k + 2\beta_\Phi \tag{4.2}$$

If $\Phi : X \times Y \to W$ satisfies (1.2), then choices of Frostman measures μ_1, μ_2 on E, F, relative to s_1, s_2, resp., induce a *configuration measure* ν on W, defined by

$$\nu(A) = \int_E \mu_2\big(\{y \in F \,:\, \Phi(x,y) \in A\}\big)\, d\mu_1(x),$$

or equivalently, for $g \in C_0(W)$,

[2]We refer to the monographs of Mattila [25, 26] for background definitions and results.

$$\int_W g(\vec{t})\,d\nu = \int\int_{E\times F} g(\Phi(x,y))\,d\mu_1(x)\,d\mu_2(y).$$

We claim that ν is absolutely continuous (with respect to Lebesgue measure in any coordinate system on W), and its density is a continuous function, $\nu(\vec{t})$. First, note that one can formally write

$$\nu(\vec{t}) = \langle\, \mathcal{R}_{\vec{t}}\,\mu_2, \mu_1\,\rangle \tag{4.3}$$

with the pairing on the right hand between elements of Sobolev spaces on X. Since $\mu_1 \in L^2_{(s_1-d_1)/2}(X)$ and $\mu_2 \in L^2_{(s_2-d_2)/2}(Y)$, and the hypothesis (1.11) for Thm. 1.6 is that, for any $s \in \mathbb{R}$, $\mathcal{R}_{\vec{t}} : L^2_s(Y) \to L^2_{s+(d_2-k)/2-\beta_\Phi}(X)$ uniformly in $\vec{t}$, the sum of the Sobolev orders of the left- and right-side terms in (4.3) is

$$\frac{s_1-d_1}{2} + \frac{s_2-d_2}{2} + \frac{d_2-k}{2} - \beta_\Phi = \frac{1}{2}(s_1+s_2-d_1-k-2\beta_\Phi) > 0, \tag{4.4}$$

with the inequality due to (4.2). Thus, the integral representing (4.3) in terms of $\widehat{\mu_1}$, $\widehat{\mathcal{R}_{\vec{t}}\mu_2}$ is absolutely convergent by Cauchy-Schwarz, and by continuity of the integral it depends continuously on the parameter $\vec{t}$.

To make this rigorous we argue as follows, restricting the analysis to the case when $\Phi : X\times Y \to \mathbb{R}^k$; the proof extends to general W using local coordinates on W. For a $\chi \in C_0^\infty(\mathbb{R}^k)$ supported in a sufficiently small ball, $\chi \equiv 1$ near $\mathbf{0}$, and with $\int \chi\, d\vec{t} = 1$, set $\chi^\epsilon(\vec{t}) := \epsilon^{-k}\chi(\frac{\vec{t}}{\epsilon})$ the associated approximation to the identity, which converges to $\delta(\vec{t})$ weakly as $\epsilon \to 0^+$. Define $\mathcal{R}^\epsilon_{\vec{t}}$ to be the operator with Schwartz kernel $K^\epsilon_{\vec{t}}(x,y) := \chi_\epsilon(\Phi(x,y)-\vec{t})$. Then $\mathcal{R}^\epsilon_{\vec{t}}\,\mu_2 \in C^\infty(X)$ and depends smoothly on $\vec{t}$, and thus we can represent ν as the weak limit of absolutely continuous measures with smooth densities:

$$\nu = \lim_{\epsilon\to 0^+} \nu^\epsilon := \lim_{\epsilon\to 0^+} \langle\, \mathcal{R}^\epsilon_{\vec{t}}\,\mu_2, \mu_1\,\rangle.$$

Now, the operators $\mathcal{R}^\epsilon_{\vec{t}} \in I^{-\infty}(C_{\vec{t}})$, with symbols which converge in the Fréchet topology on the space of symbols as $\epsilon \to 0$ to the symbol of $\mathcal{R}_{\vec{t}}$. Since the singular limits $\mathcal{R}_{\vec{t}}$ satisfy (1.11), so do the $\mathcal{R}^\epsilon_{\vec{t}}$ uniformly in ϵ. Hence, $\nu(\vec{t})$, being the uniform limit of smooth functions of $\vec{t}$, is continuous. Furthermore, since $\epsilon^k \cdot \chi^\epsilon$ is bounded below by a constant times the characteristic function of the ball of radius ϵ in $\mathbb{R}^k$, with constant C_Φ uniform in $\vec{t}$ we have that

$$\nu\left(B\left(\vec{t},\epsilon\right)\right) := (\mu_1\times\mu_2)\left(\left\{(x,y) : \left|\Phi(x,y)-\vec{t}\right| < \epsilon\right\}\right) \leq C_\Phi \epsilon^k. \tag{4.5}$$

So far, we have shown that $\nu(\vec{t})$ is continuous, so that it is positive on an open set. Hence, $\Delta_\Phi(E,F)$ is open; to conclude the proof, we need to show that it is nonempty. However, this follows because, as a further consequence of the analysis above, it follows that $\Delta_\Phi(E,F)$ has positive k-dimensional Lebesque measure. In fact, since if $\{B(\vec{t}_j,\epsilon_j)\}$ is any cover of $\Delta_\Phi(E,F)$, we have

$$1=\mu_1(E)\cdot\mu_2(F)=(\mu_1\times\mu_2)(E\times F)\le\sum_j(\mu_1\times\mu_2)(\Phi^{-1}(B(\vec{t}_j,\epsilon_j))$$

$$=\sum_j\nu\left(B\left(\vec{t}_j,\epsilon_j\right)\right)\le C_\Phi\sum_j\epsilon_j^k$$

by (4.5), so that $\sum_j|B(\vec{t}_j,\epsilon_j)|_k\ge C'_\Phi$ is bounded below. Hence $\Delta_\Phi(E,F)$ has positive k-dimensional Lebesgue measure and is therefore nonempty; by the first part of the proof, it in fact has nonempty interior. Q.E.D.

5. Details of the examples and corollaries

This section contains calculations and additional details to show how some of the Corollaries in Sec. 2 follow from the Theorems.

5.1. **Mattila-Sjölin and generalizations (Cors. 1.2 and 1.3).** The results of Mattila-Sjölin [27] and Iosevich-Mourgoglou-Taylor [22] follow immediately from Thm. 1.1 as indicated in the discussion below Cor. 1.2: For $t>0$, the spherical mean operators $\mathcal{R}_t$ are FIO of order $-(d-1)/2$ associated to canonical relations which are (under the various assumptions) local canonical graphs and thus map $L^2_s\to L^2_{s+(d-1)/2}$; furthermore, by standard facts about the dependence on symbols and canonical relations, the operator norms (for fixed s) are uniform as t ranges over any compact subinterval of $(0,\infty)$. For the Riemannian setting of Cor. 1.3, one uses the fact that the same results hold within the conjugate locus [39].

Similarly, as indicated in the discussion above its statement, the analysis behind Cor. 2.1 concerning distances from points to lines in $\mathbb{R}^2$ or hyperplanes in $\mathbb{R}^3$ consists of standard facts about the L^2 regularity of the Radon transform.

5.2. **Spheres-points (Cor. 2.2).** Let $S_d=\{(a,r):a\in\mathbb{R}^d,\ r>0\}\subset\mathbb{R}^{d+1}$ denote the $(d+1)$-dimensional space of spheres in $\mathbb{R}^d$, and $Z_t^\pm$ be the smooth points of $\{((a,r),y)\in S_d\times\mathbb{R}^d:|y-a|=r\pm t\}$. For $t=0$, the twisted conormal bundle

$$C_0=N^*Z_0'\subset(T^*\mathbb{R}^{d+1}\setminus\mathbf{0})\times(T^*\mathbb{R}^d\setminus\mathbf{0})$$

is the same canonical relation as for the solution operator mapping the Cauchy data on $\mathbb{R}^d$ to the solution of the wave equation on $\mathbb{R}^{d+1}$, which is well-known to be nondegenerate (see, e.g., [29, 36]). For $t>0$, C_t is the union of two copies of C_0, translated by $\mp t$ in the r variable, and thus is also nondegenerate. By Thm. 3.1, if $A\in I^{m-\frac14}(C_t)$, then $A:L^2_s(\mathbb{R}^d)\to L^2_{s-m}(\mathbb{R}^{d+1})$, with norm that is bounded above as t ranges over a compact interval in $(0,\infty)$. Hence, (1.11) is satisfied with $\beta_\Phi=0$, and so Thm. 1.6 applies with $d_1=d+1$, $d_2=d$, $k=1$ and $\beta_\Phi=0$; thus, if $E\subset S_d$

and $F \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > (d+1) + 1 + 2 \cdot 0 = d+2$, then $\Delta_\Phi(E, F)$ has nonempty interior.

5.3. **Lines-points in $\mathbb{R}^d$ (Cor. 2.3).** We identify $M_{1,d}$, the Grassmannian of oriented affine lines in $\mathbb{R}^d$, with the tangent bundle $T\,\mathbb{S}^{d-1}$: $(\omega, v) \in T\,\mathbb{S}^{d-1}$ corresponds to the line $L_{\omega,v} := \{v + s\omega : s \in \mathbb{R}\}$. (Here, we identify $v \in T_\omega \mathbb{S}^{d-1}$ with the vector $v \in \mathbb{R}^d$, $v \perp \omega$.) Then define

$$\Phi((\omega, v), y) = \frac{1}{2} dist(y, L_{\omega,v})^2 = \frac{1}{2} dist(y - v, v^\perp)^2$$

$$= \frac{1}{2}\big(|y-v|^2 - (((y-v)\cdot\omega)\,\omega)^2\big) = \frac{1}{2}\big|\Pi_\omega^\perp(v-y)\big|^2,$$

where $\Pi_\omega^\perp$ denotes orthogonal projection onto $\omega^\perp$. Then

$$d\Phi'((\omega, v), y) = \big(-((y-v)\cdot\omega) i_\omega^*(y-v), \Pi_\omega^\perp(v-y);\, -\Pi_\omega^\perp(v-y)\big),$$

where i_ω denotes the inclusion of $T_\omega \mathbb{S}^{d-1} \hookleftarrow \mathbb{R}^d$ and i_ω^* its transpose. Note that $\Pi_\omega^\perp(v-y) \neq 0$ iff $y \notin \mathbb{R}\cdot\omega$. Thus, for $t > 0$,

$$Z_t = \{((\omega, v), y) \in M_{1,d} \times \mathbb{R}^d : dist(y, L_{\omega,v}) = t\} = \{\Phi = t^2\}$$

is smooth and satisfies (1.2) for

$$((\omega, v), y) \notin Z_t^{sing} := \{((\omega, v), y) : y \in L_{\omega,v}\},$$

since $\Pi_\omega^\perp(v-y) \neq 0$ at those points. As coordinates on the $(3d-3)$-dimensional $Z_t \setminus Z_t^{sing}$ we may use $y \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}$ and $v \in \{v \in T_\omega \mathbb{S}^{d-1} : |v| = t\}$. Letting $\theta \in \mathbb{R}\setminus 0$ be the additional cotangent variable on $C_t = N^*(Z_t \setminus Z_t^{sing})'$, the projection $\pi_R : C_t \to T^*\mathbb{R}^d$ is

$$(y, \omega, v, \theta) \to (y, -\theta \cdot \Pi_\omega^\perp(v-y)),$$

so that

$$rk(D\pi_R) = (d+1) + rk\Big(\frac{D(\Pi_\omega^\perp(v-y))}{D(\omega, v)}\Big) = (d+1) + (d-1) = 2d.$$

Thus, C_t is nondegenerate and Thm. 1.6, with $d_1 = 2d-2$, $d_2 = d$, $k = 1$ and $\beta_\Phi = 0$, implies that if $E \subset M_{1,3}$ and $F \subset \mathbb{R}^3$ are compact with

$$\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > (2d-2) + 1 + 0 = 2d-1,$$

then $\Delta_\Phi(E, F)$ has nonempty interior.

5.4. **Lines-lines (Cor. 2.4).** Using the same notation as for lines-points, we now consider pairs of lines in $\mathbb{R}^d$, say $L = L_{\omega,v}$, $L' := L_{\omega',v'}$ with (ω, v), $(\omega', v') \in T\,\mathbb{S}^{d-1}$. Then, parametrizing L' by $s' \to v' + s'\omega'$, we have

$$dist(L, L')^2 = \inf_{y' \in L'} dist(y', L)^2 = \inf_{y' \in L'} \left|\Pi_\omega^\perp \left(v - (v' + s'\omega')\right)\right|^2.$$

A calculation shows that the critical point of the quadratic function of s' in the last expression is $s' = (v - v') \cdot \omega'$, and thus

$$\Phi((\omega, v), (\omega', v')) := \frac{1}{2} dist(L, L')^2 = \frac{1}{2}\left|\Pi_\omega^\perp \Pi_{\omega'}^\perp (v - v')\right|^2$$

$$= \left|v - v' - ((v - v') \cdot \omega')\omega' - ((v - v') \cdot \omega)\omega + ((v - v') \cdot \omega')(\omega \cdot \omega')\omega\right|^2.$$

By a slightly more complicated calculation than in the lines-points case, one sees that this satisfies the double fibration condition away from a lower- dimensional singular set, and the canonical relations C_t are nondegenerate. Hence by Thm. 1.6, again with $d_1 = 2d - 2$, $k = 1$, $\beta_\Phi = 0$, it follows that if E, $F \subset M_{1,d}$ are compact sets of lines with $\dim_\mathcal{H}(E) + \dim_\mathcal{H}(F) > 2d - 1$, then $\Delta_\Phi(E, F)$ has nonempty interior.

5.5. **Ensembles of quadratic forms (Cor. 2.5).** Writing each $Q_j(x) = x^T A_j x$ with A_j $(d \times d)$ symmetric, the nonsingularity of $\vec{Q}$ implies that all notrivial linear combinations of the A_j are nonsingular, which implies that at all $x - y \neq 0$ and $\vec{t} \in \mathbb{R}^k$ with $t_j \neq 0$ for all j, the gradients of the $Q_j(x - y) - t_j$ are linearly independent, so that $\Phi(x, y) = \vec{Q}(x - y)$ does in fact satisfy (1.2). Furthermore, $\Sigma_{\vec{t}} := \{x \in \mathbb{R}^d : Q_1(x) = \cdots = Q_k(x) = 0\}$ is a smooth codimension k surface, and the operator $\mathcal{R}_{\vec{t}}$ is convolution with a smooth multiple of surface measure on $\Sigma_{\vec{t}}$.

5.6. **Heisenberg spheres (Cors. 2.6 and 2.7).** For simplicity, we only treat the case of the Heisenberg group $\mathbb{H}^1$, with the proof for general nondegenerate step two groups being similar. Rather than the original $\Phi : \mathbb{H}^1 \times \mathbb{H}^1 \to \mathbb{R}^2$ in (2.2), for simplicity we square the first component and work with

$$\Phi(x, y) = \vec{t} = \left(|x' - y'|^2, x_3 - y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right) \in \mathbb{R}^2. \tag{5.1}$$

Evaluating $D\Phi' = [D_x\Phi, -D_y\Phi]$, one sees that $D_x\Phi$ and $D_y\Phi$ have rank 2 away from the lower dimensional set $Z^{sing} := \{x' = y'\}$, so that for $\vec{t} \in \mathbb{R}_+ \times \mathbb{R}$, $Z_{\vec{t}} := \{(x, y) : \Phi(x, y) = \vec{t}\}$ is smooth, codimension $k = 2$ in $\mathbb{H}^1 \times \mathbb{H}^1$. On $Z_{\vec{t}}$ we can use $x \in \mathbb{H}^1$ and $\omega = (x' - y')/\sqrt{t_1} \in \mathbb{S}^1$ as coordinates. The phase function for $\mathcal{R}_{\vec{t}}$ is

$$\phi_{\vec{t}}(x, y, \theta) = (|x' - y'|^2 - t_1)\theta_1 + \left(x_3 - y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right)\theta_2,$$

and on the resulting 6-dimensional canonical relation $C_{\vec{t}}$ we can use x, ω and $\theta \in \mathbb{R}^2 \setminus \mathbf{0}$ as coordinates. With respect to these,

$$\pi_L(x, \omega, \theta) = \big(x; 2\sqrt{t_1}\theta_1\omega + (1/2)\theta_2 J(\omega), \theta_2\big),$$

where the cotangent variables have been split into $\xi = (\xi', \xi_3)$ and J is the standard 2×2 symplectic matrix above (2.2). From this we see that

$$rk(D\pi_L) = 4 + rk\Big(\frac{D\xi'}{D(\omega, \theta_1)}\Big),$$

and this $= 6$ where $\theta_1 \neq 0$. Furthermore, at the hypersurface $\{\theta_1\} \subset C_{\vec{t}}$, the kernel of $D\pi_L$ is spanned by a vector with a nonzero $\partial/\partial\theta_1$ component, so that $ker(D\pi_L)$ is transverse to $\{\theta_1 = 0\}$ and thus π_L has a Whitney fold singularity at the points where it is not a local canonical graph. By symmetry, the same is true for π_R, and thus $C_{\vec{t}}$ is a folding canonical relation, so that by Thm. 3.2, there is a loss of $1/6$ derivatives on L^2-based Sobolev spaces. Hence, Thm. 1.6 applies with $d_1 = 3$, $k = 2$ and $\beta_\Phi = 1/6$. Hence, for E, $F \subset \mathbb{H}^1$ with $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) > 3 + 2 + 2 \cdot (1/6) = 16/3$, $\Delta_\Phi(E, F)$ has nonempty interior.

6. Final comment

We observe that the threshold in Thm. 1.1 cannot in general be lowered. Returning to the setting of Steinhaus' theorem, one has $k = d$, $\Phi(x, y) = x - y$, and for each $\vec{t} \in \mathbb{R}^d$ the operator $\mathcal{R}_{\vec{t}}$ is just translation by $\vec{t}$, which is an FIO of order 0 associated with a canonical graph. Hence $\beta_\Phi = 0$ and $\frac{d+k}{2} + \beta_\Phi = d$. The resulting sufficient lower bound in Thm. 1.1 is then $\dim_{\mathcal{H}}(E) > d$, so that the theorem is vacuous in this case, and does not imply Steinhaus' result. However, the threshold $\frac{d+k}{2} + \beta_\Phi$ cannot be lowered in this case: Falconer's original counterexamples related to the distance problem (see [8, Thm. 2.4]) can be modified to show that for any $s < d$ there is an $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) = s$ and $\mathrm{int}(E - E) = \emptyset$, showing that the range of $\dim_{\mathcal{H}}(E)$ in Thm. 1.1 cannot *in general* be lowered below the endpoint $\frac{d+k}{2} + \beta_\Phi$. Of course, this leaves open the possibility of improvement for other, specific Φ.

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