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Egalitarian Edge Orderings of Complete Graphs

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Abstract

For a consecutive ordering of the edges of a graph $G = (V, E)$, the point sum of a vertex is the sum of the indices of edges incident with that vertex. Motivated by questions of balancing accesses in data placements in the presence of popularity rankings, an edge ordering is egalitarian when all point sums are equal, and almost egalitarian when two point sums differ by at most 1. It is established herein that complete graphs on n vertices admit an egalitarian edge ordering when $n \equiv 1, 2, 3 \pmod 4$ and $n \notin \{3, 5\}$, or an almost egalitarian edge ordering when $n \equiv 0 \pmod 4$ and $n \neq 4$.

Keywords Egalitarian block labelling · Supermagic labelling · Complete graph · Difference sum

1 Introduction

As motivation, suppose that m data items are to be stored on n storage units. Because storage units may fail, each data item is to be placed on at least two storage units; because storage is not free, we want to place each data item on *exactly* two storage units, Of course, it may happen that two or more storage units fail simultaneously; when a data item is stored on these two storage units and no others, that data is lost. To minimize data loss, we may then insist that no two data items are stored on the same two storage units. Representing storage units as vertices V of a graph, each data item selects an unordered pair on V, an edge, as the storage units on which this data item is stored. When E is the set of edges chosen in this way, the mapping of data items to storage units is a (simple) graph $G = (V, E)$. Naturally one wants to store a large number of data items using few storage units, so dense graphs

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are of most interest here. Indeed the *complete graph* K_n with $m = \binom{n}{2}$ edges yields the most data items for a given number (n) of storage units. This is admittedly a model of data layout that is too simplified to have substantial practical relevance; see [1, 7, 11] and references therein for more general models.

When $G = (V, E)$ is used as a data layout, every data item is placed on two storage units; the number of data items stored on that unit is the degree of the corresponding vertex in V. In order to permit accesses to storage units to be balanced, one might ask that G be r-regular for some r. As remarked in [7], however, this does not take account of the differing popularities of the data items; if one is not careful, some storage unit may hold only frequently accessed data items, while another may hold only rarely accessed ones. Unfortunately, the popularities (long-term access frequencies) are at best known approximately. This led Dau and Milenkovic [7] to suggest ranking data items by popularity, and balancing the sum of the ranks of the data items on storage units. We describe one case of their general model, that of edge ordering of r-regular graphs. Let $G = (V, E)$ be an r-regular graph with n vertices and $m = \frac{1}{2}rn$ edges. Let $e_0, \ldots, e_{m-1}$ be a total ordering of the edges of E. For this edge ordering, the *point sum* σ_x of a vertex $x \in V$ is $\sum_{i:x \in e_i} i$. Forming a bijection from data items to edges so that the data item ranked i (with $0 \le i \le m-1$) maps to e_i, the point sum of x is the sum of ranks of the data items stored on storage unit x. Our goal is then to ensure that the point sums are as equal as possible; we minimize by choosing an appropriate edge ordering. (In fact, because the graph does not change, we are choosing a bijection from the data items to the edges, and the ordering is then inherited from the popularity ranks; but it is less cumbersome to focus directly on the edge ordering.)

The *difference sum* of an edge ordering for $G = (V, E)$ is $\max_{x \in V} \sigma_x - \min_{x \in V} \sigma_x$. The *difference sum* of G, $\mathsf{DiffSum}(G)$, is the smallest difference sum for any edge ordering of G. An edge ordering is *egalitarian* if it has difference sum equal to 0, and *almost egalitarian* if it has difference sum equal to 1.

Egalitarian edge labellings for graphs have been studied under the name *supermagic labellings* [9, 10]. Stewart [10] establishes the existence of supermagic labellings of K_n by an inductive method, settling the egalitarian cases examined in this paper. We provide different proofs for the egalitarian case and settle the almost egalitarian case completely. Our main motivation for developing different proofs is to address the generalization to Steiner systems [5, 6].

Lemma 1 *An n-vertex, r-regular graph is egalitarian only if r is even or $n \equiv 2 \pmod 4$, and is almost egalitarian only if r is odd and $n \equiv 0, 1, 3 \pmod 4$.*

Proof Let G be an n-vertex, r-regular graph. Then G has $m = \frac{1}{2}rn$ edges. The sum of all n point sums is $2\sum_{i=0}^{m-1} i = m(m-1)$, so the average point sum is $\frac{1}{2}r[\frac{1}{2}rn - 1] = \frac{1}{4}r^2n - \frac{1}{2}r$. For an egalitarian edge ordering of G to exist, this must be an integer, and hence r is even *or* $n \equiv 2 \pmod 4$. When the average point sum is integral, if any point has point sum larger than the average, another must have point sum smaller than the average, and hence the difference sum must be at least 2 and the labelling is not almost egalitarian. When the average point sum is not integral (that is, when r is odd *and* $n \equiv 0, 1, 3 \pmod 4$), the difference sum must be at least 1. If it is 1, then $\frac{n}{2}$ vertices must have point sum equal to $\frac{1}{2}$ less than the average point sum, while the remaining $\frac{n}{2}$ vertices have point sum equal to $\frac{1}{2}$ more than the average. □

Of course, these arguments give necessary conditions for egalitarian and almost egalitarian edge orderings that are not sufficient for all regular graphs. For example, the difference sum of an n-vertex 1-regular graph is $\frac{n-2}{2}$, which can be almost egalitarian only when $n \in \{2, 4\}$. The average point sum of an n-vertex 2-regular graph (which has n edges) is $n - 1$. Because the number of unordered pairs from $\{0, \ldots, n-1\}$ having sum $n - 1$ is $\lfloor \frac{n}{2} \rfloor$, at most half the vertices can have point sum equal to the average, so the graph cannot be egalitarian or almost egalitarian.

Fortunately, we are concerned primarily with dense graphs. For the complete graph K_n, the average point sum is $\frac{1}{4}(n-1)(n+1)(n-2)$, which is an integer if and only if $n \equiv 1, 2, 3 \pmod 4$.

We require some basic graph-theoretic terminology. When $G = (V, E)$ and $H = (W, F)$ are graphs for which $W \subseteq V$ and $F \subseteq E$, H is a *subgraph* of G. When in addition $W = V$, the subgraph is *spanning*. When H is a spanning subgraph of G and H is d-regular, H is a *d-factor* of G. A partition of the edges of G in which each class forms a d-factor of G is a *d-factorization* of G.

Edge orderings for various purposes, and generalizations to block orderings of designs, have been extensively studied; an excellent overview is in [8]. Indeed, orderings to improve disk access performance have been studied in [2–4]. Those orderings impose a 'local' requirement to balance accesses, but here we are concerned with 'global' balance, as in [10].

2 Complete Graphs

We start by identifying exceptional cases.

Lemma 2 *No egalitarian or almost egalitarian edge ordering for K_n exists when $n \in \{3, 4, 5\}$.*

Proof

K_3 Without loss of generality, the edge ordering is $(\{0, 1\}, \{0, 2\}, \{1, 2\})$, having point sums 1, 2, and 3.

K_4 Suppose to the contrary that $e_0, \ldots, e_5$ is an almost egalitarian edge ordering on vertex set $\{0, 1, 2, 3\}$. Each point sum must be 7 or 8. Without loss of generality suppose that $e_5 = \{0, 1\}$. No other edge can contain both 0 and 1. Because each point sum is at least 7 and at most 8, without loss of generality $e_0 = \{0, 2\}$ and $e_3 = \{0, 3\}$, while 1 appears in e_1 and e_2. But then point 2 has point sum at most 6, which is a contradiction. (In fact, setting $e_1 = \{1, 3\}$, $e_2 = \{1, 2\}$ and $e_4 = \{2, 3\}$, point 2 has point sum 6, and the other three points each have point sum 8.)

K_5 Suppose to the contrary that $e_0, \ldots, e_9$ is an egalitarian edge ordering on vertex set $\{0,1,2,3,4\}$. Each point sum must be 18. Without loss of generality suppose that $e_9 = \{0,1\}$. No other edge can contain both 0 and 1, so to obtain point sums 18 for both, without loss of generality one of the following cases must occur:

Case 1. $e_0 = \{0,2\}$, $e_1 = \{0,3\}$, $e_8 = \{0,4\}$, and 1 is in e_2, e_3, and e_4. Then 2 cannot appear in e_8 or e_9, so to achieve point sum 18 must appear in each of $\{e_5, e_6, e_7\}$. But then it appears in none of $\{e_2, e_3, e_4\}$.

Case 2. $e_0 = \{0,2\}$, $e_2 = \{0,3\}$, $e_7 = \{0,4\}$, and 1 is in e_1, e_3, and e_5. Then for 2 to achieve point sum 18, it must appear in each of $\{e_4, e_6, e_8\}$ and hence in none of $\{e_1, e_3, e_5\}$.

Case 3. $e_0 = \{0,2\}$, $e_4 = \{0,3\}$, $e_5 = \{0,4\}$, and 1 is in e_1, e_2, and e_6. Then for 2 to achieve point sum 18, it must appear in each of $\{e_3, e_7, e_8\}$ and hence in none of $\{e_1, e_2, e_6\}$.

In each case, a contradiction arises because the edge $\{1,2\}$ cannot appear in the ordering.

□

The edge ordering 34,02,13,01,04,12,23,24,14,03 for K_5 has point sums of 17, 18, and 19; this is the 'closest' to egalitarian for K_5.

Lemma 3 *There is an egalitarian edge ordering of K_n whenever $n \equiv 2 \pmod 4$.*

Proof Write $n = 4s+2$. We build the ordering of K_{4s+2} on vertex set $\mathbb{Z}_{4s+1} \cup \{\infty\}$. First we determine the initial sequence of $2s+1$ edges, by setting

$$e_i = \begin{cases} \{i+1, 4s-i\} & \text{if} \quad 0 \le i < s \\ \{\infty, 0\} & \text{if} \quad i = s \\ \{i, 4s+1-i\} & \text{if} \quad s < i \le 2s \end{cases}$$

We develop the initial run under addition to form the entire edge ordering as follows. For $j \in \mathbb{Z}_{4s+1}$ let $x \oplus j$ denote $(x+j) \bmod 4s+1$ when $x \in \mathbb{Z}_{4s+1}$, and ∞ when $x = \infty$. For $e = \{x, y\} \subset \mathbb{Z}_{4s+1} \cup \{\infty\}$, let $e \oplus j$ denote $\{x \oplus j, y \oplus j\}$. To extend the ordering, let $e_{j(2s+1)+i} = e_i \oplus j$ for $0 \le i \le 2s$ and $1 \le j \le 4s$. Define $F_j = \{e_{j(2s+1)+i} : 0 \le i \le 2s\}$ for $0 \le j \le 4s$. Then for each $0 \le j \le 4s$, F_j is a 1-factor, and $\{F_0, \ldots, F_{4s}\}$ is a 1-factorization.

Now we calculate point sums. For $x \in \mathbb{Z}_{4s+1} \cup \{\infty\}$, denote the point sum of x by σ_x. To compute σ_∞, observe that ∞ appears exactly in the edges $\{e_{j(2s+1)+s} : 0 \le j \le 4s\}$. So

$$\sigma_\infty = \sum_{j=0}^{4s} [j(2s+1)+s] = s(4s+1) + (2s+1)$$

$$\sum_{j=0}^{4s} j = s(4s+1) + (2s+1)(2s)(4s+1)$$

Thus $\sigma_\infty = s(4s+1)(4s+3)$. When $x \in \mathbb{Z}_{4s+1}$, x appears in exactly one edge of F_j

for each $0 \le j \le 4s$, so denote by $\ell_{x,j}$ the value for which $x \in e_{j(2s+1)+\ell_{x,j}}$. It follows that $\sigma_x = \sum_{j=0}^{4s}[j(2s+1) + \ell_{x,j}] = (2s+1)(4s+1)(2s) + \sum_{j=0}^{4s} \ell_{x,j}$. The additive development ensures that the multiset $\{\ell_{x,j} : 0 \le j \le 4s\}$ contains entry i twice for each $0 \le i \le 2s$ when $i \neq s$, and it contains s once. Hence $\sum_{j=0}^{4s} \ell_{x,j} = 2\sum_{i=0}^{2s} i - s = (2s+1)(2s) - s = (4s+1)s$. Consequently $\sigma_x = (2s+1)\,(4s+1)(2s) + (4s+1)s$, and so $\sigma_x = s(4s+1)(4s+3)$. Because all point sums are equal, the ordering is egalitarian. □

The case when $n = 3$ is the only exception when $n \equiv 3 (\mathrm{mod}\ 4)$:

Lemma 4 *There is an egalitarian edge ordering of* K_n *whenever* $n \equiv 3(\mathrm{mod}\ 4)$ *and* $n \ge 7$.

Proof Suppose that $n \ge 7$ and write $n = 4s + 3$. We build the ordering of K_{4s+3} on vertex set $(\mathbb{Z}_{2s+1} \times \{0,1\}) \cup \{\infty\}$. First we determine an initial sequence of $4s + 3$ edges. When $n \equiv 3(\mathrm{mod}\ 8)$, write $n = 8m + 3$ and form the sequence

$$\begin{cases} e_i = \{(i+1,0),(4m-i,0)\} & \text{if} \quad 0 \le i < m \\ e_{m+i} = \{(i+1,1),(4m-i,1)\} & \text{if} \quad 0 \le i < m-1 \\ e_{2m-1} = \{(1,0),(4m,1)\} & \\ e_{2m} = \{(m,1),(3m+1,1)\} & \\ e_{2m+i} = \{(i+1,0),(4m-i,1)\} & \text{if} \quad 1 \le i < 2m \\ e_{4m} = \{\infty,(0,1)\} & \\ e_{4m+1} = \{(0,0),(0,1)\} & \\ e_{4m+2} = \{\infty,(0,0)\} & \\ e_{8m+2-i} = \{(i+1,1),(4m-i,0)\} & \text{if} \quad 2m \le i < 4m \\ e_{8m+2-i} = \{(i+1,0),(4m-i,1)\} & \text{if} \quad m \le i < 2m \\ e_{9m+2-i} = \{(i+1,0),(4m-i,0)\} & \text{if} \quad m \le i < 2m \end{cases}$$

When $n \equiv 7(\mathrm{mod}\ 8)$, write $n = 8m + 7$ and form the sequence

$$\begin{cases} e_i = \{(i+1,0),(4m+2-i,0)\} & \text{if} \quad 0 \le i < m \\ e_{m+i} = \{(i+1,1),(4m+2-i,1)\} & \text{if} \quad 0 \le i < m \\ e_{2m+i} = \{(i+1,0),(4m+2-i,1)\} & \text{if} \quad 0 \le i < 2m+1 \\ e_{4m+1} = \{\infty,(0,0)\} & \\ e_{4m+2} = \{(2m+1,1),(2m+2,1)\} & \\ e_{4m+3} = \{(0,0),(0,1)\} & \\ e_{4m+4} = \{(2m+1,0),(2m+2,0)\} & \\ e_{4m+5} = \{\infty,(0,1)\} & \\ e_{8m+6-i} = \{(i+1,1),(4m+2-i,0)\} & \text{if} \quad 2m+1 \le i < 4m+2 \\ e_{8m+6-i} = \{(i+1,0),(4m+2-i,1)\} & \text{if} \quad m \le i < 2m \\ e_{9m+6-i} = \{(i+1,0),(4m+2-i,0)\} & \text{if} \quad m \le i < 2m \end{cases}$$

We develop this initial run under addition to form the entire edge ordering as follows. For $j \in \mathbb{Z}_{2s+1}$ let $x \oplus j$ denote $((y+j) \bmod 2s+1, i)$ when $x = (y,i) \in \mathbb{Z}_{2s+1} \times \{0,1\}$, and ∞ when $x = \infty$. For $e = \{x,y\} \subset (\mathbb{Z}_{2s+1} \times \{0,1\}) \cup \{\infty\}$, let $e \oplus j$ denote $\{x \oplus j, y \oplus j\}$. To extend the ordering, let $e_{j(4s+3)+\ell} = e_\ell \oplus j$ for $0 \le \ell \le 4s+2$ and $1 \le j \le 2s$. Define $F_j = \{e_{j(4s+3)+i} : 0 \le i \le 4s+2\}$ for $0 \le j \le 2s$. Then for each $0 \le j \le 2s$, F_j is a 2-factor, and $\{F_0, \ldots, F_{2s}\}$ is a 2-factorization.

Now we compute the point sums. The additive development ensures that all points in $\{(x,i) \in \mathbb{Z}_{2s+1} \times \{0\}\}$ have the same point sum, and that all points in $\{(x,i) \in \mathbb{Z}_{2s+1} \times \{1\}\}$ have the same point sum. Each can be calculated using only the initial 2-factor. For $(x,i) \in \mathbb{Z}_{2s+1} \times \{0,1\}$, the point sum is

$$\left[\sum_{\ell=0}^{4s+2} \ell |e_\ell \cap \mathbb{Z}_{2s+1} \times \{i\}|\right] + 2\sum_{j=0}^{2s} j(4s+3)$$

Simplify using $2\sum_{j=0}^{2s} j(4s+3) = 2(4s+3)(2s+1)s$. For each $i \in \{0,1\}$, there are in total $4s+2$ occurrences of elements of $\mathbb{Z}_{2s+1} \times \{i\}$ in the edges of F_0. We show that $\left[\sum_{\ell=0}^{4s+2} \ell |e_\ell \cap \mathbb{Z}_{2s+1} \times \{i\}|\right] = (4s+2)(2s+1)$. To do this, first pair edges e_ℓ and $e_{4s+2-\ell}$ if and only if $|e_\ell \cap \mathbb{Z}_{2s+1} \times \{i\}| = |e_{4s+2-\ell} \cap \mathbb{Z}_{2s+1} \times \{i\}|$ and $\ell \ne 2s+1$. When e_ℓ and $e_{4s+2-\ell}$ are so paired, together they contribute $(4s+2)|e_\ell \cap \mathbb{Z}_{2s+1} \times \{i\}|$ to the sum; in these edges, each occurrence of an element of $\mathbb{Z}_{2s+1} \times \{i\}$ contributes, on average, $2s+1$ to the sum. We need only establish that the same average holds in edges that are not paired.

When $n = 8m+3$, edges not paired are

$$\{e_{2m-1}, e_{2m}, e_{4m}, e_{4m+1}, e_{4m+2}, e_{6m+2}, e_{6m+3}\},$$

which in total contain 4 elements from $\mathbb{Z}_{2s+1} \times \{0\}$ and 8 from $\mathbb{Z}_{2s+1} \times \{1\}$. Those in $\mathbb{Z}_{2s+1} \times \{0\}$ contribute $(2m-1)+(4m+1)+(4m+2)+(6m+2) = 4(2s+1)$, while those in $\mathbb{Z}_{2s+1} \times \{1\}$ contribute $(2m-1)+2(2m)+4m+(4m+1)+(6m+2)+2(6m+3) = 8(2s+1)$ to the sum. Hence on average each occurrence in an unpaired edge contributes $2s+1$, as required.

When $n = 8m + 7$, edges not paired are $\{e_{4m+1}, e_{4m+2}, e_{4m+3}, e_{4m+4}, e_{4m+5}\}$, which in total contain 4 elements from $\mathbb{Z}_{2s+1} \times \{0\}$ and 4 from $\mathbb{Z}_{2s+1} \times \{1\}$. Those in $\mathbb{Z}_{2s+1} \times \{0\}$ contribute $(4m + 1) + (4m + 3) + 2(4m + 4) = 4(2s + 1)$, while those in $\mathbb{Z}_{2s+1} \times \{1\}$ contribute $(4m + 5) + (4m + 3) + 2(4m + 2) = 4(2s + 1)$ to the sum. Hence on average each occurrence in an unpaired edge contributes $2s + 1$, as required.

It follows that every point in $\mathbb{Z}_{2s+1} \times \{0, 1\}$ has the same point sum, $(4s + 2)(2s + 1) + (4s + 3)(4s + 2)s = (4s + 2)(4s + 1)(s + 1)$. Because the average point sum over all $n = 4s + 3$ points is also $(n - 1)(n + 1)(n - 2)/4 = (4s + 2)(4s + 1)(s + 1)$, point ∞ must also have the average point sum, and the edge ordering is egalitarian. □

Lemma 5 *There exist egalitarian edge orderings for K_9 and K_{13}.*

Proof For K_9, use the edge ordering

```
13 14 27 08 06 35 16 28 47 38 67 58 57 24 45 34 01 02
23 05 68 56 07 78 26 04 36 15 12 46 17 25 18 37 03 48
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For K_{13}, use the edge ordering

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2a 35 4a 06 9b 1c 38 16 8b 27 3c 68 12 6c 45 29 56 4c 0a 57 2b
7b 07 24 bc 13 49 9c 8a 05 79 15 3b 58 47 69 18 9a 0b 04 1a 19
08 02 78 3a 4b 7a 37 03 0c 7c 59 5a 6b 28 36 6a 01 ac 34 89 46
39 17 5b 2c 23 8c 14 48 09 5c 67 1b 26 ab 25
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□

Lemma 6 *There is an egalitarian edge ordering of K_n whenever $n \equiv 1 (\mathrm{mod}\ 4)$ and $n \geq 17$.*

Proof Suppose that $n \geq 17$ and write $n = 4s + 1$. We choose a set X of size $2s$ and build the ordering of K_{4s+1} on vertex set $(X \times \{0, 1\}) \cup \{\infty\}$. Form a K_{2s+1} on $X \cup \{\infty\}$ and let G be an arbitrary s-regular spanning subgraph of it. This K_{2s+1} has an egalitarian edge ordering $f_0, \ldots, f_{s(2s+1)-1}$, as follows. When $2s + 1 \equiv 3(\mathrm{mod}\ 4)$ apply Lemma 4; when $2s + 1 \in \{9, 13\}$, apply Lemma 5; otherwise proceed inductively. For $i \in \{0, 1\}$, let $\phi_i : X \cup \{\infty\} \mapsto (X \times \{i\}) \cup \{\infty\}$ so that $\phi_i(x) = (x, i)$ for $x \in X$ and $\phi_i(\infty) = \infty$, For $\{x, y\} \subset X \cup \{\infty\}$, $\phi_i(\{x, y\}) = \{\phi_i(x), \phi_i(y)\}$. Now we determine a consecutive interval in the ordering for K_{4s+1}. For $0 \leq i < s(2s + 1)$, if f_i is an edge of G, set $e_{2s^2+2i} = \phi_0(f_i)$ and $e_{2s^2+2i+1} = \phi_1(f_i)$; otherwise set $e_{2s^2+2i} = \phi_1(f_i)$ and $e_{2s^2+2i+1} = \phi_0(f_i)$. Within the consecutive run $\{e_{2s^2}, \ldots e_{6s^2+2s-1}\}$ one finds all edges of K_{4s+1} containing ∞; because ∞ has point sum $s(s + 1)(2s - 1)$ in the egalitarian ordering $f_0, \ldots, f_{s(2s+1)-1}$, it has point sum $2s^2(4s) + 4s(s + 1)(2s - 1) + 2s = s(4s + 2)(4s - 1)$, which is the desired average point sum. All points of $(X \times \{0, 1\})$ have the same point sum in the consecutive run, because (1) $f_0, \ldots, f_{s(2s+1)-1}$ is egalitarian, and (2) for each $x \in X$, exactly half of the edges containing x are placed with $(x, 0)$ before placement with $(x, 1)$, and the remaining half are placed in the opposite order.

It remains to specify the ordering for the remaining $4s^2$ edges, those in $\{\{(x,0),(y,1)\} : x,y \in X\}$. These form a complete bipartite graph $K_{2s,2s}$. Let L_0 and L_1 be two orthogonal latin squares of side $2s$, each with symbols indexed by X and with rows and columns indexed by $\{0,\ldots,2s-1\}$. (These exist because $2s \geq 8$.) Then set $e_{2si+j} = \{(L_0(i,j),0),(L_1(i,j),1)\}$ and $e_{6s^2+2s+2si+j} = \{(L_0(i+s,j),0),(L_1(i+s,j),1)\}$ for $0 \leq i < s$ and $0 \leq j < 2s$. Within these $4s^2$ edges, each point of $(X \times \{0,1\})$ has the same point sum. Hence $e_0,\ldots,e_{8s^2+2s-1}$ is an egalitarian edge ordering. $\square$

The following treats the remaining case, paralleling Lemma 3 closely.

Lemma 7 *There is an almost egalitarian edge ordering of K_n whenever $n \equiv 0 (\mathrm{mod}\ 4)$ and $n \geq 8$.*

Proof Write $n = 4s+4$. We build the ordering of K_{4s+4} on vertex set $\mathbb{Z}_{4s+3} \cup \{\infty\}$. First we determine an initial sequence of $2s+2$ edges. Begin by assigning edges $\{\{i+1, 4s+2-i\} : 0 \leq i \leq 2s, i \neq 2s-1\}$ distinct labels from $\{e_0,\ldots,e_{s-1}\} \cup \{e_{s+2},\ldots,e_{2s+1}\}$ (an arbitrary bijection is fixed throughout). Then set $e_s = \{2s, 2s+3\}$ and $e_{s+1} = \{\infty, 0\}$.

For $j \in \mathbb{Z}_{4s+3}$ let $x \oplus j$ denote $(x+j) \bmod 4s+3$ when $x \in \mathbb{Z}_{4s+3}$, and ∞ when $x = \infty$. For $e = \{x,y\} \subset \mathbb{Z}_{4s+3} \cup \{\infty\}$, let $e \oplus j$ denote $\{x \oplus j, y \oplus j\}$. To extend the ordering, let $e_{j(2s+2)+i} = e_i \oplus j$ for $0 \leq i \leq 2s+1$ with $i \notin \{s, s+1\}$ and $1 \leq j \leq 4s+2$. When $1 \leq j \leq 4s+2$, set $e_{j(2s+2)+s} = e_s \oplus j$ and $e_{j(2s+2)+s+1} = e_{s+1} \oplus j$ when j is even, and set $e_{j(2s+2)+s} = e_{s+1} \oplus j$ and $e_{j(2s+2)+s+1} = e_s \oplus j$ when j is odd. Define $F_j = \{e_{j(2s+2)+i} : 0 \leq i \leq 2s+1\}$ for $0 \leq j \leq 4s+2$. Then for each $0 \leq j \leq 4s+2$, F_j is a 1-factor, and $\{F_0,\ldots,F_{4s+2}\}$ is a 1-factorization.

Each element of $\{0,\ldots,4s+2\}$ appears exactly twice in the edges $\{e_{j(2s+2)+i} : 0 \leq j \leq 4s+2\}$ whenever $0 \leq i \leq 2s+1$ and $i \notin \{s, s+1\}$, and exactly three times in the edges $\{e_{j(2s+2)+s}, e_{j(2s+2)+s+1} : 0 \leq j \leq 4s+2\}$. Indeed, among the edges $\{e_{j(2s+2)+s} : 0 \leq j \leq 4s+2\}$, each of the *low points* $\{2j+1 : 0 \leq j \leq 2s, j \neq s\}$, $2s$, and $2s+2$, appear exactly twice while each of the remaining points (other than ∞) appears exactly once. Routine calculation then establishes that each of the $2s+2$ low points has point sum $\frac{1}{2}[(4s+3)(4s+5)(2s+1)-1]$, while each of the remaining $2s+2$ points has point sum $\frac{1}{2}[(4s+3)(4s+5)(2s+1)+1]$. Hence the edge ordering is almost egalitarian. $\square$

Combining these results, we have established:

Theorem 1 *An egalitarian edge ordering of K_n exists if and only if $n \equiv 1,2,3 (\mathrm{mod}\ 4)$ and $n \notin \{3,5\}$. An almost egalitarian edge ordering of K_n exists if and only if $n \equiv 0 (\mathrm{mod}\ 4)$ and $n \neq 4$.*

Generalizations to Steiner triple systems [5] and to Steiner systems in general [6] have also been considered.

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Declaration

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