A p-ADIC WALDSPURGER FORMULA

YIFENG LIU, SHOUWU ZHANG, and WEI ZHANG

Abstract

In this article, we study p-adic torus periods for certain p-adic-valued functions on Shimura curves of classical origin. We prove a p-adic Waldspurger formula for these periods as a generalization of recent work of Bertolini, Darmon, and Prasanna. In pursuing such a formula, we construct a new anti-cyclotomic p-adic L-function of Rankin–Selberg type. At a character of positive weight, the p-adic L-function interpolates the central critical value of the complex Rankin–Selberg L-function. Its value at a finite-order character, which is outside the range of interpolation, essentially computes the corresponding p-adic torus period.

Contents

1. Introduction

The aim of this article is to generalize a recent formula of Bertolini, Darmon, and Prasanna in [1] which relates the p-adic logarithm of Heegner points in Abelian varieties parameterized by the modular curve $X_0(N)$ and certain p-adic L-values at a point outside its range of interpolation, for a prime p split in the imaginary quadratic field. The paper [1] works in the same setting as the Gross–Zagier formula (see [12]) under the Heegner hypothesis. Prior to the Bertolini–Darmon–Prasanna formula, Rubin [23] obtained a similar formula for elliptic curves with complex multipli-

DUKE MATHEMATICAL JOURNAL
Vol. 167, No. 4, © 2018 DOI 10.1215/00127094-2017-0045
Received 19 October 2014. Revision received 31 August 2017.
First published online 9 February 2018.
2010 *Mathematics Subject Classification*. Primary 11G40; Secondary 11F67, 11G18, 11J95.

cation, and after the Bertolini–Darmon–Prasanna formula, Brooks [4] also obtained a similar formula allowing the modular curve to be a rational Shimura curve.

Our formula is for the general case concerning Heegner points on Abelian varieties parameterized by Shimura curves over a totally real number field F, for a prime $\mathfrak{p}$ of F split in a CM field E. Even in the case in which $F = \mathbb{Q}$, our result is new since we remove all ramification restrictions from [1] and [4]. Moreover, we will place our formula in the setting of the Waldspurger formula (see [29], [30]) which compares the global torus periods of automorphic forms with products of global central L-values and local torus periods. More precisely, we will define the relevant p-adic L-function, introduce the notion of p-adic Maass functions and their torus periods, and compare them with products of p-adic L-values and local torus periods. For practical applications of our formula, one may need a formula for local torus periods of Gross–Prasad test vectors. Fortunately, this formula was worked out recently by Cai, Shu, and Tian [6].

To construct the p-adic L-function and prove our p-adic Waldspurger formula, we study the congruence relation for both global (torus) periods and local (torus) periods appearing in the complex Waldspurger formula. A key ingredient of our construction is the existence of action of the Lubin–Tate formal group on Shimura curves at the infinite level; this allows us to use p-adic Fourier analysis from [26].

In the rest of this section, we will sketch our construction and the proof for the formula in the case of elliptic curves over $\mathbb{Q}$. To be consistent with the notation in the main body of the article, we fix (1) an elliptic curve A over $\mathbb{Q}$, (2) an indefinite quaternion algebra B over $\mathbb{Q}$, and (3) an imaginary quadratic field E embedded into B.

As usual, put $\mathbb{A} = \mathbb{R} \times \widehat{\mathbb{Q}}$ as the ring of adèles of $\mathbb{Q}$, and put $\mathbb{A}_E := \mathbb{A} \otimes_{\mathbb{Q}} E$. By the modularity theorem, the elliptic curve A determines an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A})$. We assume that this representation has a nontrivial Jacquet–Langlands correspondence $\pi_{\mathbb{C}}$ to $B^\times$, uniquely realized on a subspace of $\mathcal{A}_{\mathbb{C}}(B^\times)$—the space of automorphic forms on $B^\times\backslash(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$.

1.1. Complex Waldspurger formula

First let us review the (complex) Waldspurger formula (see [29], [30]) for the cuspidal automorphic representation $\pi_{\mathbb{C}}$ of $(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$. Let $\chi\colon E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ be an automorphic character. Then we can form the (torus) period integrals

$$\mathcal{P}_{\mathbb{C}}(\phi, \chi^{\pm 1}) := \int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times} f(t)\chi^{\pm 1}(t)\,dt, \quad \phi \in \pi_{\mathbb{C}}. \tag{1.1}$$

Here we adopt the Haar measure such that the total volume of $E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times$ is 2. We consider these integrals as elements in the linear dual of representation spaces as follows:

$$\mathcal{P}_{\mathbb{C}}(\cdot, \chi) \in \mathrm{Hom}_{\mathbb{A}_E^\times}(\pi_{\mathbb{C}} \otimes \chi, \mathbb{C}), \qquad \mathcal{P}_{\mathbb{C}}(\cdot, \chi^{-1}) \in \mathrm{Hom}_{\mathbb{A}_E^\times}(\pi_{\mathbb{C}} \otimes \chi^{-1}, \mathbb{C}).$$

By a theorem of Saito and Tunnell (see [24], [28]), either both spaces have dimension 1 or they have dimension 0. Suppose that we are in the first case. Although we do not know how to construct a canonical basis in either space, we do know how to construct a canonical one in their tensor product. Namely, we have the element

$$\alpha = \prod_{v \leq \infty} \alpha_v \in \mathrm{Hom}_{\mathbb{A}_E^\times}(\pi_{\mathbb{C}} \otimes \chi, \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{A}_E^\times}(\pi_{\mathbb{C}} \otimes \chi^{-1}, \mathbb{C})$$

defined via the integration of local matrix coefficients

$$\alpha_v(\phi_1, \phi_2; \chi) := \frac{L(1, \eta_v) L(1, \Pi_v, \mathrm{Ad})}{\zeta_v(2) L(1/2, \Pi_v, \chi_v)} \int_{\mathbb{Q}_v^\times \backslash E_v^\times} \big(\pi_{\mathbb{C}}(t)\phi_1, \phi_2\big)_v \chi_v(t)\, dt, \tag{1.2}$$

where $\eta = \prod_v \eta_v$ is the quadratic character corresponding to the quadratic field extension $E/\mathbb{Q}$, and $(\cdot,\cdot) = \prod_v (\cdot,\cdot)_v$ is the bilinear Petersson inner product pairing on $\pi_{\mathbb{C}}$ defined by the Haar measure on $(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$ such that the total volume of $B^\times \mathbb{A}^\times \backslash (B \otimes_{\mathbb{Q}} \mathbb{A})^\times$ is 2. It was proved by Waldspurger [29, Section 3] that α is in fact a finite product for every pair of test vectors (ϕ_1, ϕ_2).

Thus, there is a unique constant $\mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi) \in \mathbb{C}$, depending only on $\pi_{\mathbb{C}}$ and χ, such that

$$\mathscr{P}_{\mathbb{C}}(\cdot, \chi) \cdot \mathscr{P}_{\mathbb{C}}(\cdot, \chi^{-1}) = \mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi) \cdot \alpha(\cdot, \cdot; \chi).$$

The Waldspurger formula gives an expression for $\mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi)$ in terms of the Rankin–Selberg central value $\Lambda(1/2, \Pi, \chi)$.

THEOREM 1.1.1 (Waldspurger)
We have

$$\mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi) = \frac{\Lambda_{\mathbb{Q}}(2)}{2\Lambda(1, \eta)\Lambda(1, \Pi, \mathrm{Ad})} \Lambda(1/2, \Pi, \chi).$$

In other words, for every pair of vectors $\phi_1, \phi_2 \in \pi_{\mathbb{C}}$, we have

$$\mathscr{P}_{\mathbb{C}}(\phi_1, \chi)\mathscr{P}_{\mathbb{C}}(\phi_2, \chi^{-1}) = \frac{\Lambda_{\mathbb{Q}}(2)\Lambda(1/2, \Pi, \chi)}{2\Lambda(1, \eta)\Lambda(1, \Pi, \mathrm{Ad})} \cdot \alpha(\phi_1, \phi_2; \chi).$$

Remark 1.1.2
In the above theorem, Λ stands for *complete* global L-functions, that is, those as products of local L-functions over *all* places. However, in the main body of the article, we use global L-functions that are products of local L-functions over non-Archimedean places, which will be denoted by L (except for $\zeta_F(s)$ with F a number field). For example, if $\chi_\infty(z) = (z/\bar{z})^{\pm k}$ with $k \geq 1$, then we have

$$\mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi) = \frac{k!(k-1)!}{(2\pi)^{2k-1}} \cdot \frac{\zeta_{\mathbb{Q}}(2)L(1/2, \Pi, \chi)}{2L(1,\eta)L(1,\Pi,\mathrm{Ad})}.$$

It is a simple computation using the formulas in, for instance, [21, Lemma 2.3].

Remark 1.1.3
Note that, unlike our unified choice of the Tamagawa measure on $\mathbb{A}^\times\backslash\mathbb{A}_E^\times$, in [29] and [30] the Haar measure in (1.1) has volume 1 on $E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times$, and the product Haar measure in (1.2) has volume $2\Lambda(1,\eta)$ on $E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times$. Therefore, the constant $\mathbf{\Lambda}(\pi_{\mathbb{C}}, \chi)$ in their formulas differs from ours by $4\Lambda(1,\eta)$.

1.2. *p-Adic Maass functions*

From now on, we fix a prime p and equip B with an isomorphism $B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$. For each (sufficiently small) open compact subgroup U of $(B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}})^\times$, the double quotient

$$B^\times\backslash(\mathbb{C} \setminus \mathbb{R}) \times (B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}})^\times / U \tag{1.3}$$

is the set of complex points of a Shimura curve X_U defined over $\mathbb{Q}$. The curve X_U is smooth over $\mathbb{Q}$, and it is proper if and only if B is division. We put $X = \varprojlim_U X_U$ as a scheme over $\mathbb{Q}$ with a right action of $(B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}})^\times$ under which $X_U = X/U$.

We say that a function $\phi\colon X(\mathbb{C}_p) \to \mathbb{C}_p$ is a *p-adic Maass function* on X if it is the pullback of some locally analytic function $X_U(\mathbb{C}_p) \to \mathbb{C}_p$ on X_U. Denote by $\mathcal{A}_{\mathbb{C}_p}(B^\times)$ the $\mathbb{C}_p$-vector space of all p-adic Maass functions on X. It is a representation of $(B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}})^\times$.

Denote by $\pi_{\mathbb{C}_p}$ the subspace of $\mathcal{A}_{\mathbb{C}_p}(B^\times)$ spanned by functions of the form

$$f^*\log_\omega\colon X(\mathbb{C}_p) \xrightarrow{f} A(\mathbb{C}_p) \xrightarrow{\log_\omega} \mathbb{C}_p,$$

where $f: X \to A$ is a nonconstant map, ω is a differential form on $A \otimes_{\mathbb{Q}} \mathbb{C}_p$, and $\log_\omega$ is the p-adic logarithm map (see, e.g., [3]). The subspace $\pi_{\mathbb{C}_p} \subset \mathcal{A}_{\mathbb{C}_p}(B^\times)$ is a subrepresentation of $(B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}})^\times$. Thus, on one hand we have a complex realization $\pi_{\mathbb{C}}$, and on the other hand we have a p-adic realization $\pi_{\mathbb{C}_p}$. They are related as follows. For every isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have a canonical isomorphism

$$\pi_{\mathbb{C}_p} \otimes_{\mathbb{C}_p,\iota} \mathbb{C} \xrightarrow{\sim} \pi_{\mathbb{C}}^{(2)}, \tag{1.4}$$

where $\pi_{\mathbb{C}}^{(2)} \subset \pi_{\mathbb{C}}$ is the subspace of weight 2 forms. It sends $f^*\log_\omega$ to $f^*\iota\omega$, which is well defined. The latter is a differential form on $X \otimes_{\mathbb{Q}} \mathbb{C}$ and, hence, induces an element in $\mathcal{A}_{\mathbb{C}}(B^\times)$.

1.3. p-Adic torus periods

From now on, we also fix an embedding $E \subset \mathbb{C}_p$ and an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. Then we have an induced isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}$. We assume that the isomorphism $B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$ is chosen such that the induced embedding $\mathbb{C} \simeq E \otimes_{\mathbb{Q}} \mathbb{R} \to B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$ is the standard one sending $x + iy$ to $\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$.

We proceed exactly as in the complex Waldspurger formula. Let $\chi\colon E^\times \widehat{\mathbb{Q}}^\times \backslash \widehat{E}^\times \to \mathbb{C}_p^\times$ be a finite-order character. Then parallel to (1.1), we can define the p-adic (torus) period integral as

$$\mathscr{P}_{\mathbb{C}_p}(\phi, \chi^{\pm 1}) := \int_{E^\times \widehat{\mathbb{Q}}^\times \backslash \widehat{E}^\times} \phi\big(\iota^{-1}[\pm i, t]\big) \chi^{\pm 1}(t)\, dt, \quad \phi \in \pi_{\mathbb{C}_p}. \tag{1.5}$$

Here we have used the double coset presentation (1.3) of $X(\mathbb{C})$ and adopt the Haar measure on $E^\times \widehat{\mathbb{Q}}^\times \backslash \widehat{E}^\times$ of total volume 2. Note that the above integrals are actually finite sums and, respectively, induce elements

$$\mathscr{P}_{\mathbb{C}_p}(\cdot, \chi^{\pm 1}) \in \mathrm{Hom}_{\widehat{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi^{\pm 1}, \mathbb{C}_p).$$

Similar to the complex case, both spaces $\mathrm{Hom}_{\widehat{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi^{\pm 1}, \mathbb{C}_p)$ have the same dimension—either 1 or 0. Suppose that they have dimension 1. Now we construct a basis of their tensor product. For $\phi_1, \phi_2 \in \pi_{\mathbb{C}_p}$, we define

$$\alpha'(\phi_1, \phi_2; \chi) = \prod_{v<\infty} \iota^{-1} \alpha_v(\iota\phi_1, \iota\phi_2; \iota\chi),$$

where α_v is the same as (1.2). Here, by abuse of notation, $\iota\phi$ denotes the image of ϕ under the map (1.4). Then α' is a basis of $\mathrm{Hom}_{\widehat{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi, \mathbb{C}_p) \otimes \mathrm{Hom}_{\widehat{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi^{-1}, \mathbb{C}_p)$. The invariant pairing $\prod_{v<\infty}(\cdot,\cdot)_v$ we use in the definition of α' is the one such that $\prod_{v<\infty}(\iota\phi_1, \iota\phi_2)_v$ is equal to the (bilinear) Petersson product of $\iota\phi_1$ and $\pi_{\mathbb{C}}(\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)_\infty)\iota\phi_2$.

Thus, there is a unique constant $\mathbf{L}(\pi_{\mathbb{C}_p}, \chi) \in \mathbb{C}_p$, depending only on $\pi_{\mathbb{C}_p}$ and χ, such that

$$\mathscr{P}_{\mathbb{C}_p}(\cdot, \chi) \cdot \mathscr{P}_{\mathbb{C}_p}(\cdot, \chi^{-1}) = \mathbf{L}(\pi_{\mathbb{C}_p}, \chi) \cdot \alpha'(\cdot, \cdot; \chi).$$

Our main objective is to give a formula for $\mathbf{L}(\pi_{\mathbb{C}_p}, \chi)$, which we call the *p-adic Waldspurger formula*, under the *only* assumption that p splits in E.

Thus, from now on we assume that p splits in E. Denote by $\mathfrak{P}$ the place of E induced by the default embedding $E \subset \mathbb{C}_p$, and denote by $\mathfrak{P}^{\mathrm{c}}$ the other one above p.

1.4. p-Adic characters

Put $G = E^\times \widehat{\mathbb{Q}}^\times \backslash \widehat{E}^\times$, which is a profinite group. Denote by $\hat{G}$ the continuous dual over $\mathbb{Q}_p$. In other words, for every complete (commutative) $\mathbb{Q}_p$-algebra R, $\hat{G}(R)$

is the set of all continuous characters from G to $R^\times$. Then $\hat{G}$ is represented by a (complete) $\mathbb{Q}_p$-algebra $\mathcal{D}(G)$. Thus, there is a universal character $\delta\colon G \to \mathcal{D}(G)^\times$ such that composing with δ induces a bijection

$$\operatorname{Hom}\bigl(\mathcal{D}(G), R\bigr) \simeq \hat{G}(R) \tag{1.6}$$

for every complete $\mathbb{Q}_p$-algebra R, where Hom is taken in the category of topological $\mathbb{Q}_p$-algebras.

The place $\mathfrak{P}$ induces an injective homomorphism $\mathbb{Z}_p^\times \hookrightarrow G$. We say that a character $\chi \in \hat{G}(R)$ has weight $w \in \mathbb{Z}$ if $\chi|_V$ is the wth power homomorphism for some subgroup $V \subset \mathbb{Z}_p^\times$ of finite index. For a character $\chi\colon G \to \mathbb{C}_p^\times$ of weight w, there is a standard way to attach an automorphic character $\chi^{(\iota)}\colon E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ under ι; in fact, $\chi^{(\iota)}$ is the unique automorphic character satisfying (1) $\chi^{(\iota)}|_{\widehat{E}^{\times,p}} = \iota \circ \chi|_{\widehat{E}^{\times,p}}$ and (2) $\chi^{(\iota)}_\infty(z) = (z/\bar{z})^w$ for $z \in \mathbb{C} \simeq E \otimes_\mathbb{Q} \mathbb{R}$.

A character $\chi \in \hat{G}(\mathbb{C}_p)$ induces a homomorphism $\mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \to \mathbb{C}_p$ via (1.6), and we denote its kernel by I_χ, which is a closed ideal of $\mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Put

$$\mathcal{D}(G;\pi_{\mathbb{C}_p}) = \mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \Big/ \bigcap_{\chi \in \Xi(\pi_{\mathbb{C}_p})} I_\chi,$$

where $\Xi(\pi_{\mathbb{C}_p})$ is the set of all χ such that $\dim \operatorname{Hom}_{\widehat{E}^{\times,p}}(\pi_{\mathbb{C}_p} \otimes \chi, \mathbb{C}_p) = 1$. In particular, elements in $\mathcal{D}(G;\pi_{\mathbb{C}_p})$ can be viewed as functions on $\Xi(\pi_{\mathbb{C}_p})$ valued in $\mathbb{C}_p$.

1.5. p-Adic Waldspurger formula

Our first theorem is about the existence of a p-adic L-function interpolating values $\mathbf{\Lambda}(\pi_\mathbb{C}, \chi^{(\iota)})$, which appeared in Theorem 1.1.1, for χ of *positive* weight.

THEOREM 1.5.1
There is a unique element $\mathcal{L}(\pi_{\mathbb{C}_p}) \in \mathcal{D}(G;\pi_{\mathbb{C}_p})$ such that, for every $\chi \in \Xi(\pi_{\mathbb{C}_p})$ of weight $k \geq 1$, we have

$$\iota\mathcal{L}(\pi_{\mathbb{C}_p})(\chi) = \mathbf{\Lambda}(\pi_\mathbb{C}, \chi^{(\iota)}) \cdot 2\pi^{2k-1} \cdot \frac{\epsilon(1/2, \psi, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})}{L(1/2, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})^2}.$$

Here, $\psi\colon \mathbb{Q}_p \to \mathbb{C}^\times$ is the standard additive character.

Remark 1.5.2
We have the following remarks concerning the above theorem.
(1) By the theorem of Saito and Tunnell, a character $\chi \in \hat{G}(\mathbb{C}_p)$ of an integer weight belongs to $\Xi(\pi_{\mathbb{C}_p})$ if and only if, for every finite place v of $\mathbb{Q}$ other than p, we have $\epsilon(1/2, \Pi_v, \chi^{(\iota)}_v) = \eta_v(-1)\epsilon(B_v)$, where $\epsilon(B_v)$ is the Hasse invariant.

(2) The uniqueness part is clear, since the subset of characters in $\Xi(\pi_{\mathbb{C}_p})$ of positive weight is dense in $\Xi(\pi_{\mathbb{C}_p})$.

Using this p-adic L-function, we can answer the question at the end of Section 1.3 about the ratio $\mathbf{L}(\pi_{\mathbb{C}_p}, \chi)$.

THEOREM 1.5.3
Let $\chi \in \Xi(\pi_{\mathbb{C}_p})$ be a finite-order character, that is, χ has weight 0. Then we have

$$\mathbf{L}(\pi_{\mathbb{C}_p}, \chi) = \mathscr{L}(\pi_{\mathbb{C}_p})(\chi) \cdot \iota^{-1}\Big(\frac{L(1/2, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})}\Big).$$

In other words, for every pair of vectors $\phi_1, \phi_2 \in \pi_{\mathbb{C}_p}$, we have

$$\mathscr{P}_{\mathbb{C}_p}(\phi_1, \chi)\mathscr{P}_{\mathbb{C}_p}(\phi_2, \chi^{-1}) = \mathscr{L}(\pi_{\mathbb{C}_p})(\chi) \cdot \iota^{-1}\Big(\frac{L(1/2, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \Pi_p \otimes \chi^{(\iota)}_{\mathfrak{P}^c})}\Big) \cdot \alpha'(\phi_1, \phi_2; \chi).$$

Theorems 1.5.1 and 1.5.3 follow from the more general context of Theorems 3.2.10 and 3.4.4. See Remark 3.4.5 for the reduction process.

1.6. Main ideas of the proofs

We now explain the main ideas of our proofs. The same ideas work for the general case as well. There are three major steps in the proofs of our main theorems:

(1) construct universal torus periods;
(2) construct universal matrix coefficient integrals;
(3) construct the p-adic L-function.

For (1), by a universal torus period, we mean an element in $\mathcal{D}(G; \pi_{\mathbb{C}_p})$ such that it specializes to Waldspurger periods at characters of positive weight. A key ingredient in our construction is a Mellin transform for forms on the Shimura curve with the infinite Iwahori level structure at p. This seems to be new and matches the philosophy that things look more canonical at the infinite level, which has appeared in some other works recently. The Mellin transform of a form f has two variables: the Shimura curve itself and the weight space. If we restrict the Shimura curve to an arbitrary open disk which reduces to a point on the special fiber, then we recover the (local) Mellin transform on the Lubin–Tate group from [26]. If we restrict to a classical point (a nonnegative integer, actually) on the weight space, then this recovers an iteration of the Atkin–Serre operator on the Shimura curve.

For (2), by a universal matrix coefficient integral, we mean again an element in $\mathcal{D}(G; \pi_{\mathbb{C}_p})$ such that it specializes to classical matrix coefficient integrals at characters of integral weight. In our construction, we need to choose suitable test vectors in the representation $\pi_{\mathbb{C}_p}$ and show that the classical matrix coefficient integrals form a rigid analytic family. Our key idea is to use the Kirillov model to deal with arbitrary ramification at p of $\pi_{\mathbb{C}_p}$ and characters in $\Xi(\pi_{\mathbb{C}_p})$ for the matrix coefficient integrals.

For (3), by the p-adic L-function, we mean an element in $\mathcal{D}(G;\pi_{\mathbb{C}_p})$ such that it specializes to complex special L-values appearing in the complex Waldspurger formula at characters χ of positive weight. The p-adic L-function is defined essentially as the ratio of a universal torus period to a universal matrix coefficient integral. The complex Waldspurger formula will imply that this ratio is independent of the choice of the test vectors. In order to show that we have enough universal matrix coefficient integrals whose nonvanishing loci cover the entire space, we use a classical result of Saito and Tunnel on the dichotomy of matrix coefficient integrals and some argument in rigid analytic geometry. In particular, we need our constructions in (1) and (2) to be applicable to sufficiently many test vectors.

Finally, to obtain the p-adic Waldspurger formula, that is, the special value formula for finite-order characters in terms of the p-adic logarithm of Heegner cycles, we use the multiplicity one property, a property from the global Mellin transform, and slight generalization of Coleman's work from Appendix A.

1.7. A glance at the general case

In the main body of the article, we will put ourselves in a more general context. Since it is a p-adic theory, we fix a CM number field E inside $\mathbb{C}_p$, with the maximal totally real subfield F. Let $\mathfrak{p}$ be the distinguished place of F induced by the inclusion $F \subset \mathbb{C}_p$. Recall that an Abelian variety A over F is of GL(2)-type if $M_A := \mathrm{End}(A) \otimes \mathbb{Q}$ is a field of the same degree as the dimension of A.

Given a modular Abelian variety A over F of GL(2)-type up to isogeny equipped with an embedding $M := M_A \hookrightarrow \mathbb{C}_p$, we will construct a p-adic L-function $\mathcal{L}(A)$ and prove a p-adic Waldspurger formula or, rather, a family of p-adic Waldspurger formulas for all relevant realizations of A via p-adic Maass functions. Note that A has a central character $\omega_A\colon F^\times\backslash\widehat{F}^\times \to M^\times$.

The space of all locally $F_\mathfrak{p}$-analytic and smooth-away-from-$\mathfrak{p}$ characters $\chi\colon E^\times\backslash \widehat{E}^\times \to K^\times$ with a complete field extension $K/MF_\mathfrak{p}$ such that $\omega_A \cdot \chi|_{F^\times\backslash\widehat{F}^\times} = 1$ can be organized into an ind-rigid analytic variety $\mathcal{E}$ over $MF_\mathfrak{p}$. It has a disjoint union decomposition $\mathcal{E} = \mathcal{E}_+ \coprod \mathcal{E}_-$ defined by a certain Rankin–Selberg ϵ-factor of A. We denote by $\mathcal{D}(A,K)$ the coordinate algebra of $\mathcal{E}_-\widehat{\otimes}_{MF_\mathfrak{p}} K$ for every complete field extension $K/MF_\mathfrak{p}$. In Theorem 3.2.10, we construct our p-adic L-function for A as an element

$$\mathcal{L}(A) \in (\mathrm{Lie}\, A \otimes_{F^M} \mathrm{Lie}\, A^\vee) \otimes_{F^M} \mathcal{D}(A, MF_\mathfrak{p}^{\mathrm{lt}}),$$

where $MF_\mathfrak{p}^{\mathrm{lt}}$ is the complete subfield of $\mathbb{C}_p$ generated by M, the maximal unramified extension of $F_\mathfrak{p}$, and the Lubin–Tate period (see Section 1.8); and $F^M := F \otimes_\mathbb{Q} M$, which maps to $MF_{\mathfrak{p}}^{\mathrm{lt}}$ naturally. Our p-adic L-function $\mathscr{L}(A)$ interpolates classical Rankin–Selberg central critical values for algebraic characters χ of positive weight with respect to an *arbitrary* comparison isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. In other words, we will *not* choose an Archimedean place of F as the theory should be entirely p-adic. In Theorem 3.4.4, we prove a p-adic Waldspurger formula computing p-adic torus periods of p-adic Maass functions coming from A in terms of special values of $\mathscr{L}(A)$ at finite-order characters.

1.8. Notation and conventions

The article is self-contained from now on in the sense that if readers would like to study the general case directly, they can start from here, and no other preliminaries will be used. Throughout the article, we fix a prime p and a CM number field $E \subset \mathbb{C}_p$ with F the maximal totally real subfield contained in E. Thus, we obtain a distinguished place $\mathfrak{p}$ (resp., $\mathfrak{P}$) of F (resp., E) above p. We introduce the following key (and only) assumption of the article.

Assumption 1.8.1

We assume that $\mathfrak{p}$ splits in E; in other words, $F_{\mathfrak{p}} = E_{\mathfrak{P}}$.

We introduce the following notation and conventions.

- Let g be the degree of F.
- Let $\mathrm{c} \in \mathrm{Gal}(E/F)$ be the (nontrivial) Galois involution.
- Denote by $\mathbb{A}$ (resp., $\mathbb{A}^{\infty}$) the ring of adèles (resp., finite adèles) of F. Put $\mathbb{A}_E = \mathbb{A} \otimes_F E$ and $\mathbb{A}_E^{\infty} = \mathbb{A}^{\infty} \otimes_F E$.
- Let $\eta = \prod \eta_v\colon F^{\times}\backslash\mathbb{A}^{\times} \to \{\pm 1\}$ be the quadratic character associated to E/F. In particular, we have the L-function $L(s,\eta) = \prod_{v<\infty} L(s,\eta_v)$.
- Let $\mathrm{d}_E \in \mathbb{Z}_{>0}$ be the absolute value of the discriminant of E.
- We denote by $O_{\mathfrak{p}}$ the ring of integers of $F_{\mathfrak{p}}$. We denote by $F_{\mathfrak{p}}^{\mathrm{nr}}$ (resp., $F_{\mathfrak{p}}^{\mathrm{ab}}$) the completion of the maximal unramified (resp., Abelian) extension of $F_{\mathfrak{p}}$ in $\mathbb{C}_p$ and by $O_{\mathfrak{p}}^{\mathrm{nr}}$ (resp., $O_{\mathfrak{p}}^{\mathrm{ab}}$) its ring of integers. Denote by κ the residue field of $O_{\mathfrak{p}}^{\mathrm{nr}}$, which is an algebraic closure of $\mathbb{F}_p$.
- Denote by $\overline{F^{\times}}$ (resp., $\overline{E^{\times}}$) the closure of $F^{\times}$ (resp., $E^{\times}$) in $\mathbb{A}^{\infty\times}$ (resp., $\mathbb{A}_E^{\infty\times}$). Put $O_{\mathfrak{p}}^{\mathrm{anti}} = O_{E_{\mathfrak{p}}}^{\times}/O_{\mathfrak{p}}^{\times}$.
- We write elements $t \in E_{\mathfrak{p}} = E \otimes_F F_{\mathfrak{p}}$ in the form $(t_{\bullet}, t_{\circ})$, where $t_{\bullet} \in F_{\mathfrak{p}}$ (resp., $t_{\circ} \in F_{\mathfrak{p}}$) is the component of t at $\mathfrak{P}^{\mathrm{c}}$ (resp., $\mathfrak{P}$). We fix the following embedding $E_{\mathfrak{p}} \to \mathrm{Mat}_2(F_{\mathfrak{p}})$ of $F_{\mathfrak{p}}$-algebras:

$$t \mapsto \begin{pmatrix} t_{\bullet} & \\ & t_{\circ} \end{pmatrix}. \tag{1.7}$$

- We adopt $\mathbb{N} = \{m \in \mathbb{Z} \mid m \geq 0\}$ and write elements in $A^{\oplus m}$ in columns for an object A (with a well-defined underlying set) in an Abelian category.
- Denote by J the two-by-two matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
- For $m \in \mathbb{Z}$, define the $\mathfrak{p}$-*Iwahori subgroup of level* m of $\mathrm{GL}_2(O_{\mathfrak{p}})$ to be

$$U_{\mathfrak{p},m} = \left\{ g \in \mathrm{GL}_2(O_{\mathfrak{p}}) \,\middle|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{p}^m \right\} \quad \text{if } m \geq 0,$$

$$U_{\mathfrak{p},m} = \left\{ g \in \mathrm{GL}_2(O_{\mathfrak{p}}) \,\middle|\, g \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod \mathfrak{p}^{-m} \right\} \quad \text{if } m < 0.$$

- We adopt the convention that the local or global Artin reciprocity maps send uniformizers to geometric Frobenii.
- If G is a reductive group over F, we always take the Tamagawa measure when we integrate on the adèlic group $G(\mathbb{A})$. In particular, the total volume of $E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times$ is 2.
- For a relative (formal) scheme X/S, we will simply write Ω_X^1 instead of $\Omega_{X/S}^1$ for the sheaf of relative differentials if the base is clear from the context. The tensor product of quasicoherent sheaves on X will simply be denoted as $\otimes$, instead of $\otimes_{\mathcal{O}_X}$, where $\mathcal{O}_X$ is the structure sheaf of X.
- Denote by $\widehat{\mathbf{G}}_m$ (resp., $\widehat{\mathbf{G}}_a$) the multiplicative (resp., additive) formal group. They have the coordinate T. We denote by $\widehat{\mathbf{G}}_m[p^\infty]$ the induced (formal) p-divisible group of $\widehat{\mathbf{G}}_m$.
- Denote by $\mathcal{LT}$ the Lubin–Tate $O_{\mathfrak{p}}$-formal group over $O_{\mathfrak{p}}^{\mathrm{nr}}$, which is unique up to isomorphism. We denote by $\mathcal{LT}[\mathfrak{p}^\infty]$ the induced (formal) $O_{\mathfrak{p}}$-divisible group of $\mathcal{LT}$ (see Section B.1 if not familiar with the terminology).
- Denote by $F_{\mathfrak{p}}^{\mathrm{lt}} \subset \mathbb{C}_p$ the complete field extension of $F_{\mathfrak{p}}^{\mathrm{nr}}$ generated by the "period" of the Lubin–Tate group $\mathcal{LT}$ (see [26, p. 460]). Its valuation is discrete only when $F_{\mathfrak{p}} = \mathbb{Q}_p$ by [26, Lemma 3.9], in which case $F_{\mathfrak{p}}^{\mathrm{lt}} = \mathbb{Q}_p^{\mathrm{nr}}$.
- In this article, we will only use basic knowledge about rigid analytic varieties over complete p-adic fields in the sense of Tate. Readers may use the book [2] for a reference. If $\mathcal{X}$ is an L-rigid analytic variety for some complete non-Archimedean field L, we denote by $\mathcal{O}(\mathcal{X}, K)$ the complete K-algebra of K-valued rigid analytic functions on $\mathcal{X}$ for every complete field extension K/L.

Definition 1.8.2 (Abelian Haar measure)
We fix the Haar measure dt_v on $F_v^\times\backslash E_v^\times$ for every place v of F determined by the following conditions:

- When v is Archimedean, the total volume of $F_v^\times\backslash E_v^\times \simeq \mathbb{R}^\times\backslash\mathbb{C}^\times$ is 1.
- When v is split, the volume of the maximal compact subgroup of $F_v^\times\backslash E_v^\times \simeq F_v^\times$ is 1.

- When v is nonsplit and unramified, the total volume is 1.
- When v is ramified, the total volume is 2.

Then the product measure $\prod_v dt_v$ equals the product of $2^{-g}\mathrm{d}_E^{-1/2}L(1,\eta)$ and the Tamagawa measure (cf. [30, Section 1.6]).

Notation 1.8.3
In the main part of the article, we will fix the choices of an additive character $\psi\colon F_\mathfrak{p} \to \mathbb{C}_p^\times$ of level 0 and a generator $\nu\colon \mathcal{LT} \to \widehat{\mathbf{G}}_m$ in the free $O_\mathfrak{p}$-module $\mathrm{Hom}(\mathcal{LT}, \widehat{\mathbf{G}}_m)$ of rank 1. Then there are unique isomorphisms

$$\nu_\pm\colon F_\mathfrak{p}/O_\mathfrak{p} \xrightarrow{\sim} \mathcal{LT}[\mathfrak{p}^\infty](\mathbb{C}_p) \tag{1.8}$$

such that the induced composite maps

$$\nu(\mathbb{C}_p)\circ\nu_\pm\colon F_\mathfrak{p}/O_\mathfrak{p} \to \widehat{\mathbf{G}}_m[p^\infty](\mathbb{C}_p) \subset \mathbb{C}_p^\times$$

coincide with $\psi^\pm$, respectively, where $\psi^+ = \psi$ and $\psi^- = \psi^{-1}$.

Definition 1.8.4
Recall from [30, Section 1.2.1] that a *coherent/incoherent totally definite quaternion algebra* over $\mathbb{A}$ is a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ such that the ramification set of $\mathbb{B}$, which is a finite set, contains all Archimedean places and has even/odd cardinality. For such $\mathbb{B}$, put $\mathbb{B}^\infty = \mathbb{B}\otimes_\mathbb{A}\mathbb{A}^\infty$.

An *E-embedding* of a totally definite quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ is an embedding

$$\mathsf{e} = \prod\nolimits_v' \mathsf{e}_v\colon \mathbb{A}_E^\infty = \prod\nolimits_{v<\infty}' E\otimes_F F_v \hookrightarrow \mathbb{B}^\infty \tag{1.9}$$

of $\mathbb{A}^\infty$-algebras. We say that $\mathbb{B}$ is *E-embeddable* if there exists an E-embedding of $\mathbb{B}$.

2. Arithmetic of quaternionic Shimura curves

In this section, we study some p-adic arithmetic properties of quaternionic Shimura curves over a totally real field. In Section 2.1, we start from the local theory of some p-adic Fourier analysis on Lubin–Tate groups, following the work of [26]. In Section 2.2, we study the Gauss–Manin connection and the Kodaira–Spencer isomorphism for quaternionic Shimura curves. This is followed by a discussion of universal convergent modular forms in Section 2.3. In particular, we prove Theorem 2.3.17, which is one of the most crucial technical results of the article. In Section 2.4, we prove some results involving comparisons with transcendental constructions under a given complex uniformization. Finally, Section 2.5 contains the proofs of the six claims in the previous sections, which require the auxiliary use of unitary Shimura curves.

2.1. *Fourier theory on Lubin–Tate groups*

We use [25] for some terminologies from non-Archimedean functional analysis. Let G be a topologically finitely generated Abelian locally $F_{\mathfrak{p}}$-analytic group—for example, $G = O_{\mathfrak{p}}$, which will be studied later. For a complete field K containing $F_{\mathfrak{p}}$, denote by $C(G, K)$ the locally convex K-vector space of locally ($F_{\mathfrak{p}}$-)analytic K-valued functions on G, and denote by $D(G, K)$ its strong dual (see Remark 2.1.1), which is a topological K-algebra with the multiplication given by convolution (see [26, Section 1]). We have a natural continuous injective homomorphism

$$\delta\colon G \to D(G, K)^{\times}$$

sending $g \in G$ to the Dirac distribution δ_g. Moreover, we have $D(G, K)\widehat{\otimes}_K K' \simeq D(G, K')$ for a complete field extension K'/K.

Remark 2.1.1

We briefly recall the notion of strong dual from [25]. Let V be a locally convex K-vector space, like $C(G, K)$ above. Denote by $\mathcal{L}(V, K)$ the K-vector space of continuous K-linear maps from V to K. For every bound subset B of V (i.e., for every open neighborhood $U \subset V$ of 0, there exists $a \in K$ such that $B \subset aU$) and an ideal I of O_K, the subset $\mathcal{L}(B, I) := \{f \in \mathcal{L}(V, K) \mid f(B) \subset I\}$ is a lattice in $\mathcal{L}(V, K)$. Then the strong dual of V is the (topological) K-vector space $\mathcal{L}(V, K)$ equipped with the topology defined by the family of lattices $\mathcal{L}(B, I)$ for all bounded subsets B of V and ideals I of O_K. When G is compact, there is a more explicit description of $D(G, K)$ on [26, p. 451].

Notation 2.1.2

Let $\mathcal{B}$ be the generic fiber of (the underlying formal scheme of) $\mathcal{LT}$, which is isomorphic to the open unit disk over $F_{\mathfrak{p}}^{\mathrm{nr}}$. We have a map

$$\alpha\colon \mathcal{B} \times_{\operatorname{Spf} F_{\mathfrak{p}}^{\mathrm{nr}}} \mathcal{B} \to \mathcal{B}$$

induced by the formal group law, and we have a map $O_{\mathfrak{p}} \times \mathcal{B} \to \mathcal{B}$ denoted by $(a, z) \mapsto a \cdot z$ coming from the $O_{\mathfrak{p}}$-action on $\mathcal{LT}$. Denote by $\mathcal{O}(\mathcal{B}, K)$ the set of all K-valued rigid analytic functions on $\mathcal{B}$, which is a topological K-algebra.

Definition 2.1.3 (Stable function)

A function $\phi \in \mathcal{O}(\mathcal{B}, K)$ is *stable* if

$$\sum_{z \in \operatorname{Ker}[\mathfrak{p}]} \phi\bigl(\alpha(z, \cdot)\bigr) = 0,$$

where $\mathrm{Ker}[\mathfrak{p}] \subset \mathcal{B}(K^{\mathrm{ac}})$ is the subset of z such that $\varpi \cdot z = 0$ for one and hence all uniformizers ϖ of $O_\mathfrak{p}$. We denoted by $\mathcal{O}(\mathcal{B}, K)^\heartsuit$ the subspace of $\mathcal{O}(\mathcal{B}, K)$ of stable functions.

From now on, we will assume that K contains $F_\mathfrak{p}^{\mathrm{lt}}$ (see Section 1.8). By [26, Theorems 2.3 and 3.6] (together with the remark after [26, Corollary 3.7]), we have a *Fourier transform*

$$\mathcal{F} : D(O_\mathfrak{p}, K) \xrightarrow{\sim} \mathcal{O}(\mathcal{B}, K),$$

which is an isomorphism of topological K-algebras, with respect to the homomorphism $\nu : \mathcal{LT} \to \widehat{\mathbf{G}}_m$ (Notation 1.8.3).

Remark 2.1.4
In fact, the pairing $(a, z) \mapsto \nu(a \cdot z)$ on $O_\mathfrak{p} \times \mathcal{B}$ identifies $\mathcal{B}$ as the rigid analytic space parameterizing locally analytic characters of $O_\mathfrak{p}^\times$; and the Fourier transform $\mathcal{F}$ is the unique isomorphism satisfying $\mathcal{F}(\delta_a)(z) = \nu(a \cdot z)$ for $z \in \mathcal{B}$. In particular, the topological K-vector space $\mathcal{O}(\mathcal{B}, K)$ is topologically generated by rigid analytic functions ν^a on $\mathcal{B}$ defined by $\nu^a(z) = \nu(a \cdot z)$, for $a \in O_\mathfrak{p}$. See [26] for more details.

Remark 2.1.5
We have an action of $O_\mathfrak{p}$ on $\mathcal{B}$ coming from the Lubin–Tate group and, hence, an action of $O_\mathfrak{p}$ on $D(O_\mathfrak{p}, K)$ via $\mathcal{F}$. More precisely, the action of $t \in O_\mathfrak{p}$ on $D(O_\mathfrak{p}, K)$ is given by the multiplication of the Dirac distribution δ_t.

We identify $D(O_\mathfrak{p}^\times, K)$ with the closed subspace of $D(O_\mathfrak{p}, K)$ consisting of distributions supported on $O_\mathfrak{p}^\times$.

LEMMA 2.1.6
We have the following.

(1) *The isomorphism $\mathcal{F}$ restricts to an isomorphism $\mathcal{F} : D(O_\mathfrak{p}^\times, K) \xrightarrow{\sim} \mathcal{O}(\mathcal{B}, K)^\heartsuit$ of topological K-vector spaces.*
(2) *The image of $\alpha^*|_{\mathcal{O}(\mathcal{B},K)^\heartsuit}$ is contained in $\mathcal{O}(\mathcal{B}, K)^\heartsuit \widehat{\otimes}_K \mathcal{O}(\mathcal{B}, K)^\heartsuit$.*

Proof
By [26, Section 3], for $z \in \mathcal{B}(K)$, we have a locally analytic character κ_z of $O_\mathfrak{p}$ such that $\kappa_z(a) = \nu(a \cdot z)$ for every $a \in O_\mathfrak{p}$, and $\lambda(\kappa_z) = \mathcal{F}(\lambda)(z)$ for $\lambda \in D(O_\mathfrak{p}, K)$. Moreover, the set of κ_z is dense in $C(O_\mathfrak{p}, K)$. Let e_1 (resp., e_0) be the characteristic function of $O_\mathfrak{p}^\times$ (resp., $O_\mathfrak{p} \backslash O_\mathfrak{p}^\times$), viewed as elements in $C(O_\mathfrak{p}, K)$.

For (1), we have the identity

$$\sum_{z\in \mathrm{Ker}[\mathfrak{p}]} \kappa_z f = (\# O_{\mathfrak{p}}/\mathfrak{p}) e_0 f.$$

Thus, if f is supported on $O_{\mathfrak{p}}\backslash O_{\mathfrak{p}}^{\times}$, then $f = e_0 f = (\# O_{\mathfrak{p}}/\mathfrak{p})^{-1}\sum_{z\in\mathrm{Ker}[\mathfrak{p}]}\kappa_z f$. By [26, Lemma 4.6.5], for $\phi \in \mathcal{O}(\mathcal{B}, K)^{\heartsuit}$, we have

$$\{\phi, f\} = \Big\{\phi, (\# O_{\mathfrak{p}}/\mathfrak{p})^{-1}\sum_{z\in\mathrm{Ker}[\mathfrak{p}]}\kappa_z f\Big\} = (\# O_{\mathfrak{p}}/\mathfrak{p})^{-1}\Big\{\sum_{z\in\mathrm{Ker}[\mathfrak{p}]}\phi\big(\alpha(z,\cdot)\big), f\Big\} = 0,$$

where $\{\phi, f\} = \mathcal{F}^{-1}(\phi)(f)$ and the same for the others. This means that $\mathcal{F}^{-1}\mathcal{O}(\mathcal{B}, K)^{\heartsuit} \subset D(O_{\mathfrak{p}}^{\times}, K)$. On the other hand, if $\phi \in \mathcal{F} D(O_{\mathfrak{p}}^{\times}, K)$, then for an arbitrary $f \in C(O_{\mathfrak{p}}, K)$, we have

$$(\# O_{\mathfrak{p}}/\mathfrak{p})^{-1}\Big\{\sum_{z\in\mathrm{Ker}[\mathfrak{p}]}\phi\big(\alpha(z,\cdot)\big), f\Big\} = \Big\{\phi, (\# O_{\mathfrak{p}}/\mathfrak{p})^{-1}\sum_{z\in\mathrm{Ker}[\mathfrak{p}]}\kappa_z f\Big\} = \{\phi, e_0 f\} = 0.$$

This means that $\mathcal{F} D(O_{\mathfrak{p}}^{\times}, K) \subset \mathcal{O}(\mathcal{B}, K)^{\heartsuit}$.

For (2), we consider the map $\alpha^!$, defined as the following composite map:

$$\begin{aligned} D(O_{\mathfrak{p}}, K) &\xrightarrow{\mathcal{F}} \mathcal{O}(\mathcal{B}, K) \xrightarrow{\alpha^*} \mathcal{O}(\mathcal{B}\times_{\mathrm{Spf}\, F_{\mathfrak{p}}^{\mathrm{nr}}}\mathcal{B}, K) \simeq \mathcal{O}(\mathcal{B}, K)\widehat{\otimes}_K\mathcal{O}(\mathcal{B}, K) \\ &\xrightarrow{\mathcal{F}^{-1}\widehat{\otimes}\mathcal{F}^{-1}} D(O_{\mathfrak{p}}, K)\widehat{\otimes}_K D(O_{\mathfrak{p}}, K) \to \big(C(O_{\mathfrak{p}}, K)\otimes_K C(O_{\mathfrak{p}}, K)\big)^{\vee}. \end{aligned}$$

In view of (1), it suffices to show that, for every $\lambda \in D(O_{\mathfrak{p}}, K)$ and $f_1, f_2 \in C(O_{\mathfrak{p}}, K)$, we have the formula

$$\alpha^!\lambda(f_1\otimes f_2) = \lambda(f_1 f_2). \tag{2.1}$$

For this, we may assume that $f_i = \kappa_{z_i}$ for some $z_i \in \mathcal{B}(K)$ $(i = 1, 2)$ as the image of $D(O_{\mathfrak{p}}, K)\widehat{\otimes}_K D(O_{\mathfrak{p}}, K)$ consists of continuous linear forms. Then we have, for $\lambda \in D(O_{\mathfrak{p}}, K)$,

$$\alpha^!\lambda(\kappa_{z_1}\otimes\kappa_{z_2}) = \big((\mathcal{F}^{-1}\otimes\mathcal{F}^{-1})\big(\alpha^*\mathcal{F}(\lambda)\big)\big)(\kappa_{z_1}\otimes\kappa_{z_2}) = \big(\alpha^*\mathcal{F}(\lambda)\big)(z_1, z_2)$$

by [26, Lemma 4.6.3]. But

$$\big(\alpha^*\mathcal{F}(\lambda)\big)(z_1, z_2) = \mathcal{F}(\lambda)\big(\alpha(z_1, z_2)\big) = \lambda(\kappa_{\alpha(z_1,z_2)}) = \lambda(\kappa_{z_1}\kappa_{z_2})$$

as $\kappa_{z_1}\kappa_{z_2} = \kappa_{\alpha(z_1,z_2)}$. Thus, (2.1) holds, and (2) follows. □

Remark 2.1.7
Lemma 2.1.6 implies that the function ν^a in Remark 2.1.4 is stable if and only if a

belongs to $O_{\mathfrak{p}}^{\times}$. Moreover, the topological K-vector space $\mathcal{O}(\mathcal{B},K)^{\heartsuit}$ is topologically generated by ν^a for $a\in O_{\mathfrak{p}}^{\times}$. The notion of stable functions is a local avatar of the notation of stabilization in the theory of p-adic modular/automorphic forms.

In a later argument, we will work on the compact Abelian locally $F_{\mathfrak{p}}$-analytic group $O_{\mathfrak{p}}^{\mathrm{anti}}$. Note that we have identified $O_{\mathfrak{p}}^{\mathrm{anti}}$ with $O_{\mathfrak{p}}^{\times}$ via $t\mapsto t/t^{\mathrm{c}}$. Thus, we have the following definition.

Definition 2.1.8 (Local Mellin transform)
We call the following composite map

$$\mathbf{M}_{\mathrm{loc}}\colon \mathcal{O}(\mathcal{B},K)^{\heartsuit}\to \mathcal{O}(\mathcal{B},K)^{\heartsuit}\widehat{\otimes}_K\mathcal{O}(\mathcal{B},K)^{\heartsuit} \xrightarrow{\mathrm{id}\widehat{\otimes}\mathcal{F}^{-1}} \mathcal{O}(\mathcal{B},K)^{\heartsuit}\widehat{\otimes}_{F_{\mathfrak{p}}} D(O_{\mathfrak{p}}^{\mathrm{anti}},F_{\mathfrak{p}}),$$

fulfilled by Lemma 2.1.6, the *local Mellin transform*.

Remark 2.1.9
In fact, the composite map

$$\mathbf{M}'\colon \mathcal{O}(\mathcal{B},K)\to \mathcal{O}(\mathcal{B},K)\widehat{\otimes}_K\mathcal{O}(\mathcal{B},K)\xrightarrow{\mathrm{id}\widehat{\otimes}\mathcal{F}^{-1}} \mathcal{O}(\mathcal{B},K)\widehat{\otimes}_{F_{\mathfrak{p}}} D(O_{\mathfrak{p}},F_{\mathfrak{p}})$$

is more like an analogue of the classical Mellin transform, as we may regard $\mathbf{M}'$ as a map sending a function on $\mathcal{B}$ valued in K to a function on $\mathcal{B}$ valued in (K-valued) distributions on the Lie group $O_{\mathfrak{p}}$. Recall that the classical Mellin transform M sends a function ϕ on $\mathbb{R}_+^{\times}$ to a function $\mathrm{M}(\phi)$ on $\mathbb{C}$. In fact, we may regard M as a map sending a function ϕ on $\mathbb{R}_+^{\times}$ valued in $\mathbb{C}$ to a function $x\mapsto \mathrm{M}(f(x\times\cdot))$ on $\mathbb{R}_+^{\times}$ valued in ($\mathbb{C}$-valued) distributions on the Lie group $\mathbf{G}_a(\mathbb{C})$. We have analogies between $\mathbb{R}_+^{\times}$ and $\mathcal{B}$—both are "spaces with Abelian group structure"—and between $\mathbf{G}_a(\mathbb{C})$ and $O_{\mathfrak{p}}$—both are commutative Lie groups. Moreover, the properties in [26, Lemma 4.6] are the analogues of those for the classical Mellin transform.

The continuous map $\mathbf{M}'$ is uniquely determined by the formula $\mathbf{M}'(\nu^a)=\nu^a\otimes\delta_a$ for $a\in O_{\mathfrak{p}}$, where ν^a is the function in Remark 2.1.4.

Notation 2.1.10
For every integer k, we have the character $\langle k\rangle\colon O_{\mathfrak{p}}^{\mathrm{anti}}\simeq O_{\mathfrak{p}}^{\times}\to O_{\mathfrak{p}}^{\times}\subset K$ sending t to $(t/t^{\mathrm{c}})^k$. It is an element in $C(O_{\mathfrak{p}}^{\mathrm{anti}},K)$.

LEMMA 2.1.11
For every integer N, the topological K-vector space $C(O_{\mathfrak{p}}^{\mathrm{anti}},K)$ is topologically generated by $\langle k\rangle$ for all $k\geq N$.

Proof
We may assume that $N = 0$, since for every $k \in \mathbb{Z}$, the function $\langle k \rangle$ is the limit of functions $\langle k' \rangle$ for $k' \gg 0$. By [26, Theorem 4.7], every function in $C(O_{\mathfrak{p}}, K)$ and, hence, $C(O_{\mathfrak{p}}^{\mathrm{anti}}, K)$ is the limit of finite linear combinations of polynomials on $O_{\mathfrak{p}}$. Thus, the lemma holds for $N = 0$ and then every N. □

Definition 2.1.12 (Lubin–Tate differential operator)
We define the *Lubin–Tate differential operator* Θ on $\mathcal{O}(\mathcal{B}, K)$ by the formula

$$\Theta\phi = \frac{d\phi}{\nu^* \frac{dT}{T}},$$

where we recall that ν is as in Notation 1.8.3 and T is the standard coordinate of $\widehat{\mathbf{G}}_m$.

Example 2.1.13
For $a \in O_{\mathfrak{p}}$, we have $\Theta\nu^a = a\nu^a$, where $\nu^a \in \mathcal{O}(\mathcal{B}, K)$ is the function in Remark 2.1.4.

The following lemma reveals the relation between the local Mellin transform and the Lubin–Tate differential operator.

LEMMA 2.1.14
Let $\phi \in \mathcal{O}(\mathcal{B}, K)^{\heartsuit}$ be a stable function. Then $\mathbf{M}_{\mathrm{loc}}(\phi)$ is the unique element in $\mathcal{O}(\mathcal{B}, K)^{\heartsuit} \widehat{\otimes}_{F_{\mathfrak{p}}} D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$ satisfying
(1) $\mathbf{M}_{\mathrm{loc}}(\phi)(\langle k \rangle) = \Theta^k \phi$ *for every* $k \geq 0$*; and*
(2) $\Theta \mathbf{M}_{\mathrm{loc}}(\phi)(\langle -1 \rangle) = \phi$.
Here $\langle k \rangle$ is introduced in Notation 2.1.10.

Proof
This follows from [26, Lemma 4.6.8] and Lemma 2.1.11. □

Definition 2.1.15 (Admissible function)
We say that a stable function $\phi \in \mathcal{O}(\mathcal{B}, K)^{\heartsuit}$ is *n-admissible* for some $n \in \mathbb{N}$ if

$$\phi\big(\alpha(\cdot, z)\big) = \nu(z)\phi$$

for every $z \in \mathrm{Ker}[\mathfrak{p}^n] \subset \mathcal{B}(K^{\mathrm{ac}})$.

LEMMA 2.1.16
Let $\phi \in \mathcal{O}(\mathcal{B}, K)^{\heartsuit}$ be an n-admissible stable function for some $n \geq 1$. Then $\mathcal{F}^{-1}(\phi)$ is supported on $1 + \mathfrak{p}^n$. In particular, we have

$$\mathbf{M}_{\mathrm{loc}}(\phi)\big(\langle k\rangle\big) = \mathbf{M}_{\mathrm{loc}}(\phi)\big(\chi\langle k\rangle\big)$$

for every $k \in \mathbb{Z}$ *and every (locally constant) character* $\chi\colon O_{\mathfrak{p}}^{\mathrm{anti}} \to K^{\times}$ *that is trivial on* $1+\mathfrak{p}^n$.

Proof
This again follows from [26, Lemma 4.6.5]. In fact, by a similar strategy to the proof of Lemma 2.1.6(1), it suffices to show that

$$\sum_{z \in \mathrm{Ker}[\mathfrak{p}^n]} \kappa_z(a)\nu(z)^{-1} = 0$$

for $a \in O_{\mathfrak{p}}^{\times} \backslash (1+\mathfrak{p}^n)$, where $\mathrm{Ker}[\mathfrak{p}^n] \subset \mathcal{B}(K^{\mathrm{ac}})$ is the subset of z such that $\varpi^n \cdot z = 0$ for one uniformizer and, hence, all uniformizers ϖ of $O_{\mathfrak{p}}$. This holds as $\kappa_z(a) = \nu(a \cdot z)$. □

Remark 2.1.17
Let $n \geq 1$ be an integer. Lemma 2.1.16 implies that the function ν^a in Remark 2.1.4 is n-admissible stable if and only if a belongs to $1+\mathfrak{p}^n$. Moreover, the topological K-vector space of n-admissible stable functions is topologically generated by ν^a for $a \in 1+\mathfrak{p}^n$.

2.2. *Shimura curves and Kodaira–Spencer isomorphism*

Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$ equipped with an isomorphism $\mathbb{B}_{\mathfrak{p}} \simeq \mathrm{Mat}_2(F_{\mathfrak{p}})$. Then we have the system of (noncompactified) Shimura curves $\{X(\mathbb{B})_U\}_U$ indexed by (sufficiently small) open compact subgroups U of $\mathbb{B}^{\infty\times}$ associated to $\mathbb{B}$ over Spec F (see, e.g., [30, Section 1.2.1]). More precisely, $X(\mathbb{B})_U$ is the scheme over Spec F, unique up to isomorphism, such that, for every embedding $\iota\colon F \hookrightarrow \mathbb{C}$, $X(\mathbb{B})_U \otimes_F \iota(F)$ is the canonical model of the complex Shimura curve

$$B(\iota)^{\times}\backslash \mathscr{H} \times \mathbb{B}^{\infty\times}/U$$

over the reflex field $\iota(F) \subset \mathbb{C}$, where $B(\iota)$ is a nearby quaternion algebra over F with respect to ι (see Definition 2.4.10 for more details).

As projective limits with affine transition morphisms exist in the category of schemes, we may put $X(\mathbb{B}) = \varprojlim_U X(\mathbb{B})_U$. We will simply write X_U and X if $\mathbb{B}$ is clear.

Notation 2.2.1
For an element $g \in \mathbb{B}^{\infty\times}$, we denote by $\mathrm{T}_g\colon X \to X$ the morphism induced by the right translation of g, known as the Hecke morphism.

Denote by $\mathfrak{U}$ the set of all open compact subgroups of $\mathbb{B}^{\infty\mathfrak{p}\times} = (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}^{\infty\mathfrak{p}})^\times$, which is a filtered partially ordered set under inclusion. For $U^\mathfrak{p} \in \mathfrak{U}$ and $m \in \mathbb{Z}$, put

$$X(m, U^\mathfrak{p}) = X_{U^\mathfrak{p} U_{\mathfrak{p},m}} \otimes_F F_\mathfrak{p}^{\mathrm{nr}},$$

where we recall that $U_{\mathfrak{p},m}$ is the $\mathfrak{p}$-Iwahori subgroup of level m as introduced in Section 1.8. Put

$$X(\pm\infty, U^\mathfrak{p}) := \varprojlim_{m\to\infty} X(\pm m, U^\mathfrak{p}).$$

For $m \in \mathbb{N} \cup \{\infty\}$, if we take the inverse limit over the partially ordered set $\mathfrak{U}$, then we obtain $F_\mathfrak{p}^{\mathrm{nr}}$-schemes

$$X(\pm m) = \varprojlim_{U^\mathfrak{p} \in \mathfrak{U}} X(\pm m, U^\mathfrak{p}).$$

We have successive surjective morphisms

$$X(\pm\infty) \to \cdots \to X(\pm 1) \to X(0),$$

which are equivariant under the Hecke actions of $\mathbb{B}^{\infty\mathfrak{p}\times}$. By the work of Carayol [7, Section 6], the $F_\mathfrak{p}^{\mathrm{nr}}$-scheme $X(0)$ admits a canonical smooth model (see [19, Definition 2.2] for its meaning) $\mathcal{X}$ over Spec $O_\mathfrak{p}^{\mathrm{nr}}$.

Remark 2.2.2
Strictly speaking, Carayol assumed that $F \neq \mathbb{Q}$. But when $F = \mathbb{Q}$, one may take $\mathcal{X}$ to be the model defined by modular interpretation using elliptic curves (resp., Abelian surfaces with quaternionic actions) when $\mathbb{B}^\infty$ is (resp., is not) the matrix algebra—this is well known.

We recall the construction in [7, Section 1.4] of an $O_\mathfrak{p}$-divisible group $\mathcal{G}$ on $\mathcal{X}$. We first introduce some notation.

Notation 2.2.3
For an integer $m \geq 1$, we write
(1) $U_{\mathfrak{p},m}^{\mathrm{pr}} := \{g \in U_{\mathfrak{p},0} \mid g \equiv 1 \mod \mathfrak{p}^m\}$ for the principal congruence subgroup of level $\mathfrak{p}^m$;
(2) $X(m)^{\mathrm{pr}} \to X(0)$ for the corresponding covering with respect to the subgroup $U_{\mathfrak{p},m}^{\mathrm{pr}}$; and
(3) $O_{E_\mathfrak{p},m}^\times := O_{E_\mathfrak{p}}^\times \cap U_{\mathfrak{p},m}^{\mathrm{pr}}$, where $E_\mathfrak{p}$ is an $F_\mathfrak{p}$-subalgebra of $\mathbb{B}_\mathfrak{p} \simeq \mathrm{Mat}_2(F_\mathfrak{p})$ via (1.7).

Consider the right action of $U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m}$ on $(\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}$ such that $v.g = g^{-1}v$ for $g \in U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m} \simeq \mathrm{GL}_2(O_{\mathfrak{p}}/\mathfrak{p}^m)$ and $v \in (\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}$. Then the quotient scheme

$$\big(X(m)^{\mathrm{pr}} \times (\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}\big)/(U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m})$$

defines a finite flat group scheme G_m over $X(0)$, with the obvious $O_{\mathfrak{p}}$-action. The inductive system $\{G_m\}_{m\geq 1}$ defines an $O_{\mathfrak{p}}$-divisible group G over $X(0)$ (which is, however, denoted by E_∞ in [7, Section 5]). In particular, over $X(+\infty)$ (resp., $X(-\infty)$), we have an exact sequence

$$0 \longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}} \longrightarrow G \longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}} \longrightarrow 0 \tag{2.2}$$

such that the second arrow is the inclusion into the first (resp., second) factor and the third arrow is the projection onto the second (resp., first) factor. By [7, Section 6.4], the $O_{\mathfrak{p}}$-divisible group G extends uniquely to an $O_{\mathfrak{p}}$-divisible group $\mathcal{G}$ of dimension 1 and height 2 over $\mathcal{X}$, together with an action by $\mathbb{B}^{\infty\mathfrak{p}\times}$ that is compatible with the Hecke action on the base.

Put $h = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. For $m \geq 1$, put $\mathcal{X}^{(m)} = \mathcal{X} \otimes_{O_{\mathfrak{p}}} O_{\mathfrak{p}}/\mathfrak{p}^m$ and $\mathcal{G}^{(m)} = \mathcal{G}|_{\mathcal{X}^{(m)}}$. We have the exact sequence

$$0 \longrightarrow \underline{\omega}_p^{\bullet(m)} \longrightarrow \mathcal{L}_p^{(m)} \longrightarrow (\underline{\omega}_p^{\circ(m)})^\vee \longrightarrow 0, \tag{2.3}$$

where

- $\mathcal{L}_p^{(m)}$ is the Dieudonné crystal of $\mathcal{G}^{(m)}$ evaluated at $\mathcal{X}^{(m)}$, which is a locally free sheaf of rank $2h$;
- $\underline{\omega}_p^{\bullet(m)}$ is the sheaf of invariant differentials of $\mathcal{G}^{(m)}/\mathcal{X}^{(m)}$, which is a locally free sheaf of rank 1; and
- $\underline{\omega}_p^{\circ(m)}$ is the sheaf of invariant differentials of $(\mathcal{G}^{(m)})^\vee/\mathcal{X}^{(m)}$, which is a locally free sheaf of rank $2h-1$.

They are equipped with actions of $O_{\mathfrak{p}}$ under which (2.3) is equivariant. The projective system of (2.3) for all $m \geq 1$ induces the following $O_{\mathfrak{p}}$-equivariant exact sequence

$$0 \longrightarrow \underline{\omega}_p^{\bullet} \longrightarrow \mathcal{L}_p \longrightarrow (\underline{\omega}_p^{\circ})^\vee \longrightarrow 0 \tag{2.4}$$

of locally free sheaves over $\widehat{\mathcal{X}}$, the formal completion of $\mathcal{X}$ along its special fiber. Let $\mathcal{L}$ (resp., $\underline{\omega}^{\circ\vee}$) be the maximal subsheaf of $\mathcal{L}_p$ (resp., $(\underline{\omega}_p^{\circ})^\vee$) where $O_{\mathfrak{p}}$ acts via the structure map. Then we have the $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant exact sequence

$$0 \longrightarrow \underline{\omega}^{\bullet} \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{\circ\vee} \longrightarrow 0, \tag{2.5}$$

where $\underline{\omega}^{\bullet} = \underline{\omega}_p^{\bullet}$. We call (2.5) the *formal Hodge exact sequence*.

We have the Gauss–Manin connection

$$\nabla_p \colon \mathcal{L}_p \to \mathcal{L}_p \otimes \Omega^1_{\widehat{\mathcal{X}}}, \tag{2.6}$$

for the Dieudonné crystal, which is equivariant under the Hecke action of $\mathbb{B}^{\infty\mathfrak{p}\times}$ and the action of $O_{\mathfrak{p}}$. Thus, it induces the *Gauss–Manin connection*

$$\nabla \colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{\widehat{\mathcal{X}}}, \tag{2.7}$$

which is equivariant under the Hecke action of $\mathbb{B}^{\infty\mathfrak{p}\times}$.

We have the following Lemma 2.2.4 and Proposition 2.2.6 whose proof will be given in Section 2.5.

LEMMA 2.2.4
The formal Hodge exact sequence (2.5) is algebraizable, that is, it is the formal completion of an exact sequence of locally free sheaves

$$0 \longrightarrow \underline{\omega}^{\bullet} \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{\circ\vee} \longrightarrow 0 \tag{2.8}$$

on $\mathcal{X}$. Here, by abuse of notation we adopt the same symbols for these quasicoherent sheaves. Moreover, the Gauss–Manin connection (2.7) is algebraizable.

We simply call (2.8) the *Hodge exact sequence*.

Remark 2.2.5
For $m \geq 1$, one may consider the right action of $U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m}$ on $(O_{\mathfrak{p}}/\mathfrak{p}^m)^{\oplus 2}$ such that $v.g = g^{-1}v$ for $g \in U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m} \simeq \mathrm{GL}_2(O_{\mathfrak{p}}/\mathfrak{p}^m)$ and $v \in (O_{\mathfrak{p}}/\mathfrak{p}^m)^{\oplus 2}$. Then the quotient scheme

$$\big(X(m)^{\mathrm{pr}} \times (O_{\mathfrak{p}}/\mathfrak{p}^m)^{\oplus 2}\big)/(U_{\mathfrak{p},0}/U^{\mathrm{pr}}_{\mathfrak{p},m})$$

defines an $O_{\mathfrak{p}}/\mathfrak{p}^m$-local system L_m on $X(0)$ of rank 2. Denote by L the $O_{\mathfrak{p}}$-local system over $X(0)$ defined by $(\mathrm{L}_m)_{m\geq 1}$. Then $\mathcal{O}_{X(0)} \otimes_{O_{\mathfrak{p}}} \mathrm{L}$ is canonically isomorphic to the restriction of $\mathcal{L}$ on the generic fiber $X(0)$. Moreover, the induced connection on $\mathcal{O}_{X(0)} \otimes_{O_{\mathfrak{p}}} \mathrm{L}$ coincides with the restriction of ∇ on $X(0)$, by the proof of Lemma 2.2.4.

PROPOSITION 2.2.6
The composite map

$$\underline{\omega}^{\bullet} \to \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega^1_{\mathcal{X}} \to \underline{\omega}^{\circ\vee} \otimes \Omega^1_{\mathcal{X}} \tag{2.9}$$

is an isomorphism of locally free sheaves on $\mathcal{X}$, where $\underline{\omega}^{\circ}$ is the dual sheaf of $\underline{\omega}^{\circ\vee}$.

Definition 2.2.7 (Kodaira–Spencer isomorphism)
We call the ($\mathbb{B}^{\infty p \times}$-equivariant) isomorphism

$$\mathrm{KS}\colon \underline{\omega}^{\bullet} \otimes \underline{\omega}^{\circ} \xrightarrow{\sim} \Omega^1_{\mathcal{X}}, \tag{2.10}$$

induced by the isomorphism (2.9), the *Kodaira–Spencer isomorphism*.

For $w \in \mathbb{N}$, put $\mathcal{L}^{[w]} = \mathrm{Sym}^w \mathcal{L} \otimes \mathrm{Sym}^w \mathcal{L}^\vee$. The Gauss–Manin connection $\nabla^\vee$ on the dual sheaf $\mathcal{L}^\vee$ and the original one ∇ induce a connection

$$\nabla^{[w]}\colon \mathcal{L}^{[w]} \to \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}}.$$

Define $\Theta^{[w]}$ to be the composite map

$$(\Omega^1_{\mathcal{X}})^{\otimes w} \xrightarrow{\mathrm{KS}^{-1}} (\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \to \mathcal{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}}, \tag{2.11}$$

where KS is the isomorphism (2.10).

Notation 2.2.8
Let $\mathcal{X}(0)$ be the (dense) open subscheme of $\mathcal{X}$ by removing all points on the special fiber where $\mathcal{G}$ is supersingular. For every integer $m \geq 1$, denote by $\mathcal{X}(m)$ the functor classifying $O_{\mathfrak{p}}/\mathfrak{p}^m$-equivariant *frames* over $\mathcal{X}(0)$, that is, homomorphisms $\mathcal{LT}[\mathfrak{p}^m] \to \mathcal{G}[\mathfrak{p}^m]$ and $\mathcal{G}[\mathfrak{p}^m] \to \mathfrak{p}^{-m}/O_{\mathfrak{p}}$ such that the sequence

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^m] \longrightarrow \mathcal{G}[\mathfrak{p}^m] \longrightarrow \mathfrak{p}^{-m}/O_{\mathfrak{p}} \longrightarrow 0$$

is exact.

Remark 2.2.9
The scheme $\mathcal{X}(m)$ is usually denoted by $\mathcal{X}(m)^{\mathrm{ord}}$ in the rest of the literature. But since we will only work with the ordinary locus, to reduce the burden of notation, we will omit the superscript.

For $m \in \mathbb{N}$, the functor $\mathcal{X}(m)$ is representable by a scheme that is finite étale over $\mathcal{X}(0)$, which we again denote by $\mathcal{X}(m)$. Note that the generic fiber of $\mathcal{X}(m)$ is canonically isomorphic to $X(m)$. Again as projective limits with affine transition morphisms exist in the category of schemes, we may put

$$\mathcal{X}(\infty) := \varprojlim_{m \to \infty} \mathcal{X}(m).$$

We define $\mathcal{G}$, $\nabla^{[w]}$, KS, $\Theta^{[w]}$, and the sequence (2.8) for $\mathcal{X}(m)$ ($m \in \mathbb{N} \cup \{\infty\}$) via restriction and denote them by the same notation. Over $\mathcal{X}(\infty)$, we have the *universal frame*

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^\infty] \xrightarrow{\varrho_\bullet^{\mathrm{univ}}} \mathcal{G} \xrightarrow{\varrho_\circ^{\mathrm{univ}}} F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0 \ . \tag{2.12}$$

By definition, there is an action of $O_{E_\mathfrak{p}}^\times$ on the morphism $\mathcal{X}(\infty) \to \mathcal{X}(0)$ such that the pullback of (2.12) along the action of $(t_\bullet, t_\circ) \in O_{E_\mathfrak{p}}^\times$ is the frame

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^\infty] \xrightarrow{t_\bullet^{-1}\circ\varrho_\bullet^{\mathrm{univ}}} \mathcal{G} \xrightarrow{\varrho_\circ^{\mathrm{univ}}\circ t_\circ^{-1}} F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0 \ . \tag{2.13}$$

This action is $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant. In what follows, we denote by

$$\Gamma_t \colon \mathcal{X}(\infty) \to \mathcal{X}(\infty) \tag{2.14}$$

the morphism induced by the action of $t \in O_{E_\mathfrak{p}}^\times$.

Definition 2.2.10
We define the *transition isomorphisms* to be

$$\Upsilon_\pm \colon X(\pm\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \xrightarrow{\sim} \mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}}$$

such that the pullbacks of (2.12) under $\Upsilon_\pm$ coincide with (2.2) in terms of the isomorphisms (1.8), respectively.

LEMMA 2.2.11
The Hecke morphism $\mathrm{T_J}$ *(Notation 2.2.1) descends to an (iso)morphism* $\mathrm{T_J} : X(+\infty) \to X(-\infty)$*, and the following diagram*

$$\begin{array}{ccc} X(+\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} & \xrightarrow{\mathrm{T_J}} & X(-\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \\ & \searrow^{\Upsilon_+} \quad {}^{\Upsilon_-}\swarrow & \\ & \mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} & \end{array}$$

commutes. Here, we regard $\mathrm{J} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *as an element in* $\mathbb{B}_\mathfrak{p}$ *via the fixed isomorphism* $\mathbb{B}_\mathfrak{p} \simeq \mathrm{Mat}_2(F_\mathfrak{p})$.

Moreover, the isomorphism Υ_+ *(resp.,* Υ_-*) is* $\mathbb{B}^{\infty\mathfrak{p}\times}$*-equivariant and* $O_{E_\mathfrak{p}}^\times$*-equivariant (resp.,* $O_{E_\mathfrak{p}}^\times$*-conjugate-equivariant).*

Proof
It is clear that the Hecke morphism $\mathrm{T_J}$ descends as the conjugation of J turns $U_{\mathfrak{p},m}$ to

$U_{\mathfrak{p},-m}$. The commutativity of the diagram follows from the fact that $\mathrm{T}_{\mathrm{J}}^{*}G|_{X(-\infty)}$ is isomorphic to $G|_{X(+\infty)}$.

The $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant property follows from the construction. By definition, Υ_{+} is $O_{E_{\mathfrak{p}}}^{\times}$-equivariant. The conjugate-equivariant property for Υ_{-} follows from the identity

$$\mathrm{J}\begin{pmatrix} t_{\bullet} & \\ & t_{\circ} \end{pmatrix}\mathrm{J}^{-1} = \begin{pmatrix} t_{\circ} & \\ & t_{\bullet} \end{pmatrix}$$

for every $t = (t_{\bullet}, t_{\circ}) \in O_{E_{\mathfrak{p}}}^{\times}$. □

2.3. *Universal convergent modular forms*

For $m \in \mathbb{N} \cup \{\infty\}$, denote by $\mathfrak{X}(m)$ the formal completion of $\mathcal{X}(m)$ along its special fiber. It is an affine formal scheme over $O_{\mathfrak{p}}^{\mathrm{nr}}$, equipped with an $O_{\mathfrak{p}}$-divisible group $\mathfrak{G}$ induced from $\mathcal{G}$. In particular, $\mathfrak{X}(\infty)$ is indeed the projective limit $\varprojlim_{m\to\infty} \mathfrak{X}(m)$ in the category of formal schemes over $O_{\mathfrak{p}}^{\mathrm{nr}}$.

The action of $O_{E_{\mathfrak{p}}}^{\times}$ (2.13) makes itself the Galois group of the $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant pro-étale Galois cover $\mathfrak{X}(\infty) \to \mathfrak{X}(0)$, in which $O_{E_{\mathfrak{p}},m}^{\times}$ (Notation 2.2.3) is the subgroup of $O_{E_{\mathfrak{p}}}^{\times}$ that fixes the subcover $\mathfrak{X}(m) \to \mathfrak{X}(0)$ for $m \in \mathbb{N}$. By abuse of notation, the formal completion of those quasicoherent sheaves on $\mathcal{X}(m)$ and their maps will be denoted by the same symbols.

The following lemma will be proved in Section 2.5.

LEMMA 2.3.1
There is a unique morphism $\Phi\colon \mathfrak{X}(0) \to \mathfrak{X}(0)$ lifting the Frobenius morphism on the special fiber of degree $\#O_{\mathfrak{p}}/\mathfrak{p}$ such that $\Phi^{*}\mathfrak{G} \simeq \mathfrak{G}/\mathfrak{G}^{0}[\mathfrak{p}]$, where $\mathfrak{G}^{0}$ is the formal part of $\mathfrak{G}$. In particular, Φ induces an endomorphism Φ^{*} on $\mathcal{L}$.

Moreover, we have a unique Φ^{*}-stable splitting

$$\mathcal{L} = \underline{\omega}^{\bullet} \oplus \mathcal{L}^{\circ} \tag{2.15}$$

with $\mathcal{L}^{\circ}$ an invertible quasicoherent formal sheaf on $\mathfrak{X}(0)$. In addition, $\mathcal{L}^{\circ}$ is horizontal with respect to the Gauss–Manin connection, that is, $\nabla\mathcal{L}^{\circ} \subset \mathcal{L}^{\circ} \otimes \Omega^{1}_{\mathfrak{X}(0)}$.

Remark 2.3.2
The splitting (2.15) is called *unit-root splitting*. It induces an isomorphism $\mathcal{L}^{\circ} \xrightarrow{\sim} \underline{\omega}^{\circ\vee}$. Dually, it induces a splitting $\mathcal{L}^{\vee} = \underline{\omega}^{\circ} \oplus \mathcal{L}^{\bullet}$ possessing similar properties as in Lemma 2.3.1, with an isomorphism $\mathcal{L}^{\bullet} \xrightarrow{\sim} \underline{\omega}^{\bullet\vee}$.

If we restrict the unit-root splitting in both Lemma 2.3.1 and Remark 2.3.2 to $\mathfrak{X}(m)$ for $m \in \mathbb{N} \cup \{\infty\}$, then we obtain a map

$$\theta_{\mathrm{ord}}^{[w]}\colon \mathcal{L}^{[w]} \to (\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \xrightarrow{\mathrm{KS}} (\Omega^1_{\mathfrak{X}(m)})^{\otimes w}$$

for all $w \in \mathbb{N}$, where KS is the (formal completion of the restriction of the) map (2.10).

Definition 2.3.3 (Atkin–Serre operator)
For $m \in \mathbb{N} \cup \{\infty\}$ and $w \in \mathbb{N}$, define the *Atkin–Serre operator* to be

$$\Theta_{\mathrm{ord}}^{[w]}\colon (\Omega^1_{\mathfrak{X}(m)})^{\otimes w} \xrightarrow{\Theta^{[w]}|_{\mathfrak{X}(m)}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathfrak{X}(m)} \xrightarrow{\theta_{\mathrm{ord}}^{[w]}} (\Omega^1_{\mathfrak{X}(m)})^{\otimes w+1},$$

where $\Theta^{[w]}$ is defined in (2.11). For $k \in \mathbb{N}$, define the *Atkin–Serre operator of degree* k to be

$$\Theta_{\mathrm{ord}}^{[w,k]} = \Theta_{\mathrm{ord}}^{[w+k-1]} \circ \cdots \circ \Theta_{\mathrm{ord}}^{[w]}\colon (\Omega^1_{\mathfrak{X}(m)})^{\otimes w} \to (\Omega^1_{\mathfrak{X}(m)})^{\otimes w+k}.$$

In what follows, w will always be clear from the text; hence, we will suppress w from notation. In other words, we simply write Θ_{ord} (resp., Θ_{ord}^k) instead of $\Theta_{\mathrm{ord}}^{[w]}$ (resp., $\Theta_{\mathrm{ord}}^{[w,k]}$) for all $w \in \mathbb{N}$.

By using Serre–Tate coordinates (Theorem B.1.1), the formal deformation space of the $O_{\mathfrak{p}}$-divisible group $\mathcal{LT}[\mathfrak{p}^\infty] \oplus F_{\mathfrak{p}}/O_{\mathfrak{p}}$ (over κ) is canonically isomorphic to $\mathcal{LT}$. Thus, we have the classifying morphism

$$c\colon \mathfrak{X}(\infty) \to \mathcal{LT}$$

of $O_{\mathfrak{p}}^{\mathrm{nr}}$-formal schemes. It induces a morphism

$$c_{/x}\colon \mathfrak{X}(\infty)_{/x} \to \mathcal{LT} \tag{2.16}$$

for every closed point $x \in \mathfrak{X}(\infty)(\kappa)$, where $\mathfrak{X}(\infty)_{/x}$ denotes the formal completion of $\mathfrak{X}(\infty)$ at x. The following Lemma 2.3.4 and Proposition 2.3.5 will be proved in Section 2.5.

LEMMA 2.3.4
The morphism $c_{/x}$ is an isomorphism for every x.

By the above lemma, we have, for every closed point $x \in \mathfrak{X}(\infty)(\kappa)$, a restriction map

$$\mathrm{res}_x\colon \mathcal{M}^0(\infty, K) \to \mathcal{O}(\mathcal{B}, K) \tag{2.17}$$

induced from $c_{/x}$ (see (2.16)).

PROPOSITION 2.3.5
There is a morphism $\beta\colon \mathcal{LT} \times_{\operatorname{Spf} O_{\mathfrak{p}}^{\mathrm{nr}}} \mathfrak{X}(\infty) \to \mathfrak{X}(\infty)$ *such that*

(1) *for every* $x \in \mathfrak{X}(\infty)(\kappa)$, β *preserves* $\mathfrak{X}(\infty)_{/x}$, *and the induced morphism*

$$\beta_{/x}\colon \mathcal{LT} \times_{\operatorname{Spf} O_{\mathfrak{p}}^{\mathrm{nr}}} \mathfrak{X}(\infty)_{/x} \to \mathfrak{X}(\infty)_{/x}$$

is simply the formal group law after identifying $\mathfrak{X}(\infty)_{/x}$ *with* $\mathcal{LT}$ *via* $c_{/x}$;

(2) *if we equip* $\mathcal{LT}$ *with the action of* $O_{E_{\mathfrak{p}}}^{\times} \times \mathbb{B}^{\infty\mathfrak{p}\times}$ *via the inflation* $O_{E_{\mathfrak{p}}}^{\times} \to O_{\mathfrak{p}}^{\times}$ *by* $t \mapsto t/t^{\mathrm{c}}$ *and trivially on the second factor, then* β *is* $O_{E_{\mathfrak{p}}}^{\times} \times \mathbb{B}^{\infty\mathfrak{p}\times}$*-equivariant;*

(3) *for every* $x \in F_{\mathfrak{p}}/O_{\mathfrak{p}}$, *the following diagrams*

$$\begin{array}{ccc} \mathfrak{X}(\infty)\widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{ab}} & \xrightarrow{\beta_{\nu_{\pm}(x)}} & \mathfrak{X}(\infty)\widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{ab}} \\ \downarrow \Upsilon_{\pm}^{-1} & & \downarrow \Upsilon_{\pm}^{-1} \\ X(\pm\infty) \otimes_{F_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{ab}} & \xrightarrow{\mathrm{T}_{\mathrm{n}^{\pm}(x)}} & X(\pm\infty) \otimes_{F_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{ab}} \end{array}$$

commute, where

$$\mathrm{n}^{+}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{n}^{-}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

respectively; and β_z *is the restriction of* β *to a point* z *of* $\mathcal{B}$.

In particular, $\mathcal{LT}$ *acts trivially on the special fiber of* $\mathfrak{X}(\infty)$, *and this* β *is unique.*

Definition 2.3.6
We call

$$\omega_{\nu} := c^{*}\nu^{*}\frac{dT}{T}$$

the *global Lubin–Tate differential*, where ν is the homomorphism in Notation 1.8.3. It is a nowhere-vanishing global differential form on $\mathfrak{X}(\infty)\widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} O_{\mathbb{C}_p}$, and in fact, it belongs to $\mathrm{H}^0(\mathfrak{X}(\infty), \Omega^1_{\mathfrak{X}(\infty)})\widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{lt}}$ by the definition of $F_{\mathfrak{p}}^{\mathrm{lt}}$.

Remark 2.3.7
The pullbacks $\Upsilon_{\pm}^{*}\omega_{\nu}$ depend only on ψ (or rather $\psi^{\pm}$), not on the choice of ν.

In the following definition, we generalize the notion of convergent modular forms first introduced by Katz [15] to Shimura curves.

Definition 2.3.8
Suppose that we have $m \in \mathbb{N} \cup \{\infty\}$, $w \in \mathbb{Z}$, and a complete field extension $K/F_{\mathfrak{p}}^{\mathrm{nr}}$.

(1) Define the space of (K-valued) *convergent modular forms of weight* w *and* $\mathfrak{p}$*-Iwahori level* m to be

$$\mathcal{M}^w(m, K) = \mathrm{H}^0\big(\mathfrak{X}(m), (\Omega^1_{\mathfrak{X}(m)})^{\otimes w}\big) \widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} K,$$

which is naturally a complete K-vector space.

(2) A convergent modular form of weight 0 is simply called a *convergent modular function*.

(3) Put $\mathcal{M}_\flat^w(m, K) = \bigcup_{U^{\mathfrak{p}} \in \mathfrak{U}} \mathcal{M}^w(m, K)^{U^{\mathfrak{p}}} \subset \mathcal{M}^w(m, K)$.

Remark 2.3.9
For $m \in \mathbb{N} \cup \{\infty\}$ and $w \in \mathbb{Z}$, the space $\mathcal{M}^w(m, K)$ has a natural action by $O_{E_{\mathfrak{p}}}^\times \times \mathbb{B}^{\infty\mathfrak{p}\times}$ under which $\mathcal{M}_\flat^w(m, K)$ is stable. Moreover, $\mathcal{M}^0(m, K)$ is the complete tensor product of the coordinate ring of $\mathfrak{X}(m)$ and the field K, and thus, $\mathcal{M}^w(m, K)$ is naturally a topological $\mathcal{M}^0(m, K)$-module.

In particular for $w \in \mathbb{N}$, we have the *Atkin–Serre operator*

$$\Theta_{\mathrm{ord}} \colon \mathcal{M}_\flat^w(m, K) \to \mathcal{M}_\flat^{w+1}(m, K) \tag{2.18}$$

induced from the corresponding operator of sheaves (Definition 2.3.3). The operator is $O_{E_{\mathfrak{p}}}^\times \times \mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant.

From now on, we suppose that K is a complete field extension of $F_{\mathfrak{p}}^{\mathrm{ab}}$. Then for every $w \in \mathbb{Z}$, the multiplication by ω_ν^w (Definition 2.3.6) induces a canonical $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant isomorphism $\mathcal{M}^0(\infty, K) \xrightarrow{\sim} \mathcal{M}^w(\infty, K)$.

Definition 2.3.10 (Stable convergent modular forms)
A convergent modular function $f \in \mathcal{M}^0(\infty, K)$ is *stable* if

$$\sum_{z \in \mathrm{Ker}[\mathfrak{p}] \subset \mathcal{B}(K^{\mathrm{ac}})} \beta_z^* f = 0,$$

where $\beta_z^* \colon \mathcal{M}^0(\infty, K) \to \mathcal{M}^0(\infty, K)$ is the map induced by β from Proposition 2.3.5. Denote by $\mathcal{M}^0(\infty, K)^\heartsuit$ the subspace of $\mathcal{M}^0(\infty, K)$ of stable convergent modular functions.

For $m \in \mathbb{N} \cup \{\infty\}$ and $w \in \mathbb{Z}$, put

$$\mathcal{M}_\flat^w(m, K)^\heartsuit = \mathcal{M}_\flat^w(m, K) \cap \mathcal{M}^0(\infty, K)^\heartsuit \cdot \omega_\nu^w \subset \mathcal{M}^w(m, K).$$

Remark 2.3.11
By Proposition 2.3.5, a convergent modular function f is stable if and only if $\mathrm{res}_x f$ from (2.17) is stable (Definition 2.1.3) for every $x \in \mathfrak{X}(\infty)(\kappa)$.

Remark 2.3.12
The space $\mathcal{M}_\flat^w(m, K)^\heartsuit$ does not depend on the choices of ψ or ν.

Definition 2.3.13 (Admissible convergent modular forms)
Let $n \geq 0$ be an integer. We say that a stable convergent modular function $f \in \mathcal{M}^0(\infty, K)^\heartsuit$ is *n-admissible* if $\beta_z^* f = \nu(z) f$ holds for all $z \in \mathrm{Ker}[\mathfrak{p}^n] \subset \mathcal{B}(K^{\mathrm{ac}})$. We say that $f \in \mathcal{M}_\flat^w(m, K)$ is an *n-admissible stable convergent modular form* if $f\omega_\nu^{-w}$ is an n-admissible stable convergent modular function.

Remark 2.3.14
By Proposition 2.3.5(1), a stable convergent modular function f is n-admissible if and only if $\mathrm{res}_x f$ from (2.17) is n-admissible (in the sense of Definition 2.1.15) for every $x \in \mathfrak{X}(\infty)(\kappa)$.

The following lemma is a comparison between the Atkin–Serre operator Θ_{ord} from (2.18) and the Lubin–Tate differential operator Θ (Definition 2.1.12).

LEMMA 2.3.15
For an element $f \in \mathcal{M}_\flat^w(m, K)$ for some $w, m \in \mathbb{N}$, we have

$$\mathrm{res}_x\big((\Theta_{\mathrm{ord}} f)\omega_\nu^{-w-1}\big) = \Theta\big(\mathrm{res}_x(f\omega_\nu^{-w})\big)$$

for every $x \in \mathfrak{X}(\infty)(\kappa)$.

Proof
It follows from Lemma 2.3.4, Theorem B.2.3, and the definition of Θ. □

Definition 2.3.16 (Universal convergent modular form)
A *universal convergent modular form* of depth $m \in \mathbb{N}$ and tame level $U^\mathfrak{p} \in \mathfrak{U}$ is an element

$$\mathbf{M} \in \mathcal{M}^0(\infty, K)\widehat{\otimes}_{F_\mathfrak{p}} D(O_\mathfrak{p}^{\mathrm{anti}}, F_\mathfrak{p})$$

such that $\mathbf{M}$ is $U^\mathfrak{p}$-invariant and

$$\Gamma_t^* \mathbf{M} = \delta_t^{-1} \cdot \mathbf{M} \tag{2.19}$$

for $t \in O_{E_{\mathfrak{p}},m}^{\times}$. Here, $\Gamma_t\colon \mathfrak{X}(\infty) \to \mathfrak{X}(\infty)$ is the formal completion of the morphism (2.14); and we regard δ_t as the Dirac distribution of the image of t under the quotient homomorphism $O_{E_{\mathfrak{p}}}^{\times} \to O_{\mathfrak{p}}^{\mathrm{anti}}$.

The next theorem produces universal convergent modular forms from a stable convergent modular form (of a fixed weight). In this way, the universal convergent modular forms can be regarded as p-adic families interpolating iterations of the Atkin–Serre operator.

THEOREM 2.3.17
Let $f \in \mathcal{M}_{\mathfrak{b}}^{w}(m,K)^{\heartsuit}$ be a stable convergent modular form for some $w, m \in \mathbb{N}$. Then there is a unique element

$$\mathbf{M}(f) \in \mathcal{M}^{0}(\infty,K)\widehat{\otimes}_{F_{\mathfrak{p}}} D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$$

such that, for every $k \in \mathbb{N}$,

$$\mathbf{M}(f)\big(\langle w+k\rangle\big) = (\Theta_{\mathrm{ord}}^{k} f)\omega_{\nu}^{-w-k}, \tag{2.20}$$

where Θ_{ord} is the map (2.18). Moreover, we have the following.

(1) *If f is fixed by $U^{\mathfrak{p}} \in \mathfrak{U}$, then so is $\mathbf{M}(f)$.*
(2) *$\mathbf{M}(f)$ is a universal convergent modular form of depth m (Definition 2.3.16).*
(3) *If $w \geq 1$, then we have*

$$\Theta_{\mathrm{ord}}\big(\mathbf{M}(f)\big(\langle w-1\rangle\big)\omega_{\nu}^{w-1}\big) = f.$$

(4) *Suppose that f is n-admissible (Definition 2.3.13). Then we have*

$$\mathbf{M}(f)\big(\langle k\rangle\big) = \mathbf{M}(f)\big(\chi\langle k\rangle\big)$$

for every $k \in \mathbb{Z}$ and every (locally constant) character $\chi\colon O_{\mathfrak{p}}^{\mathrm{anti}} \to K^{\times}$ that is trivial on $(1+\mathfrak{p}^{n})^{\times}$.

Proof
The uniqueness follows from Lemma 2.1.11. The morphism β in Proposition 2.3.5 induces a map

$$\beta^{*}\colon \mathcal{M}^{0}(\infty,K) \to \mathcal{M}^{0}(\infty,K)\widehat{\otimes}_{K}\mathcal{O}(\mathcal{B},K).$$

By Lemma 2.1.6(2) and Remark 2.3.11, it sends $\mathcal{M}^{0}(\infty,K)^{\heartsuit}$ into $\mathcal{M}^{0}(\infty,K)^{\heartsuit}\widehat{\otimes}_{K}$ $\mathcal{O}(\mathcal{B},K)^{\heartsuit}$. Thus, we may regard $\beta^{*}(f\omega_{\nu}^{-w})$ as an element in $\mathcal{M}^{0}(\infty,K)^{\heartsuit}\widehat{\otimes}_{F_{\mathfrak{p}}}$ $D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$ via the Fourier transform $\mathscr{F}$, since K contains $F_{\mathfrak{p}}^{\mathrm{lt}}$. Define a (continuous $F_{\mathfrak{p}}$-linear) translation map

$$\tau_w : D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}}) \to D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$$

such that $(\tau_w \phi)(g) = \phi(g \cdot \langle -w \rangle)$ for every $g \in C(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$. We take

$$\mathbf{M}(f) = \tau_w\big(\beta^*(f\omega_\nu^{-w})\big).$$

For the formula (2.20), it suffices to check it after applying res_x for every $x \in \mathfrak{X}(\infty)(\kappa)$. In fact, we have

$$\mathrm{res}_x \mathbf{M}(f)\big(\langle w+k\rangle\big) = \mathrm{res}_x\big(\beta^*(f\omega_\nu^{-w})\big)\big(\langle k\rangle\big) = \mathbf{M}_{\mathrm{loc}}\big(\mathrm{res}_x(f\omega_\nu^{-w})\big)\big(\langle k\rangle\big) \quad (2.21)$$

by Definition 2.1.8 and the definition of β. By Lemma 2.1.14, we have that (2.21) is equal to $\Theta^k(\mathrm{res}_x(f\omega_\nu^{-w}))$. Finally by Lemma 2.3.15, we have $\Theta^k(\mathrm{res}_x(f\omega_\nu^{-w})) = \mathrm{res}_x((\Theta_{\mathrm{ord}}^k f)\omega_\nu^{-w-k})$.

Property (1) follows from Proposition 2.3.5(2). Properties (3) and (4) follow from Lemmas 2.1.14 and 2.1.16, respectively. For property (2), we only need to show that (2.19) holds for $\mathbf{M} = \mathbf{M}(f)$ and $t \in O_{E_{\mathfrak{p}},m}^\times$. In fact, since f is fixed by $O_{E_{\mathfrak{p}},m}^\times$, we have $\mathbf{M}(f) = \mathbf{M}(\Gamma_t^* f)$, which equals $\delta_t \cdot \Gamma_t^* \mathbf{M}(f)$ by Proposition 2.3.5(2) and Remark 2.1.5. □

The following definition is suggested by the formula (2.21) in the proof of the above theorem.

Definition 2.3.18
We call $\mathbf{M}(f)$ in Theorem 2.3.17 the *global Mellin transform* of f.

2.4. *Comparison with Archimedean differential operators*

Now suppose that $\mathbb{B}$ is equipped with an E-embedding as in Definition 1.8.4 such that $\mathsf{e}_{\mathfrak{p}}$ coincides with (1.7) under the fixed isomorphism $\mathbb{B}_{\mathfrak{p}} \simeq \mathrm{Mat}_2(F_{\mathfrak{p}})$.

Definition 2.4.1 (CM-subscheme)
We define the *CM-subscheme* Y to be $X^{E^\times}$, the subscheme of X fixed by the action of $\mathsf{e}(E^\times)$ for e as in Definition 1.8.4. Define $Y^\pm$ to be the subschemes of Y such that $E^\times$ acts on the tangent space of points in $Y^\pm$ via the characters $t \mapsto (t/t^{\mathrm{c}})^{\pm 1}$, respectively. See also [30, Section 3.1.2].

In what follows, we will regard $Y^\pm$ as their base change to $F_{\mathfrak{p}}^{\mathrm{nr}}$; in particular, they are closed subschemes of $X \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\mathrm{nr}}$.

LEMMA 2.4.2
We have $Y = Y^+ \coprod Y^-$. Moreover, we have the following.

(1) *Both $Y^+(\mathbb{C}_p)$ and $Y^-(\mathbb{C}_p)$ are equipped with the natural profinite topology, isomorphic to $\overline{E^\times}\backslash\mathbb{A}_E^{\infty\times}$, and admit a transitive action of $\mathbb{A}_E^{\infty\times}$ via Hecke morphisms (see Section 1.8 for the notation $\overline{E^\times}$).*

(2) *The projection maps $X \otimes_{F_\mathfrak{p}} F_\mathfrak{p}^{\mathrm{nr}} \to X(\pm\infty)$ restrict to isomorphisms from $Y^\pm$ to their images, respectively. In particular, we may regard $Y^\pm$ as closed subschemes of $X(\pm\infty)$.*

(3) *The closed subschemes $Y^\pm(\infty) := \Upsilon_\pm Y^\pm$ of $\mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}}$ descend to closed subschemes of $\mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{nr}}$, where $\Upsilon_\pm$ are the transition isomorphisms in Definition 2.2.10.*

Proof

The decomposition follows directly from the definition. For the rest, we consider $Y^+(\mathbb{C}_p)$ without loss of generality.

Part (1) can be seen from the complex uniformization by choosing an arbitrary isomorphism $\mathbb{C}_p \simeq \mathbb{C}$. Part (2) follows from the fact that $\mathbb{A}_E^{\infty\times}$ does not contain any nontrivial unipotent element. Part (3) follows from the fact that $\mathrm{Gal}(F_\mathfrak{p}^{\mathrm{ab}}/F_\mathfrak{p}^{\mathrm{nr}})$ acts via local class field theory as the right multiplication of $O_\mathfrak{p}^\times$ on the double coset presentation $X(\infty)(\mathbb{C}_p)$ and, hence, preserves the subset $Y^+(\mathbb{C}_p)$. □

Notation 2.4.3

For $m \in \mathbb{N} \cap \{\infty\}$, denote by

(1) $Y^\pm(m)$ the image of $Y^\pm(\infty)$ in $\mathcal{X}(m) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{nr}}$,

(2) $\mathcal{Y}^\pm(m)$ the Zariski closure of $Y^\pm(m)$ in $\mathcal{X}(m)$, and

(3) $\mathfrak{Y}^\pm(m)$ the formal completion of $\mathcal{Y}^\pm(m)$ along the special fiber.

LEMMA 2.4.4

For $m \in \mathbb{N} \cup \{\infty\}$, we have $\mathcal{Y}^\pm(m)(\mathbb{C}_p) = \mathcal{Y}^\pm(m)(F_\mathfrak{p}^{\mathrm{nr}}) = \mathcal{Y}^\pm(m)(O_\mathfrak{p}^{\mathrm{nr}})$. Here, for an $O_\mathfrak{p}^{\mathrm{nr}}$-algebra R, $\mathcal{Y}^\pm(m)(R)$ are the sets of morphisms from Spec R *to $\mathcal{Y}^\pm(m)$ over* Spec $O_\mathfrak{p}^{\mathrm{nr}}$*, respectively.*

Proof

Without loss of generality, we only prove the case for $\mathcal{Y}^+(m)$. We first consider the case where $m = 0$. The first identity $\mathcal{Y}^\pm(0)(\mathbb{C}_p) = \mathcal{Y}^\pm(0)(F_\mathfrak{p}^{\mathrm{nr}})$ is well known from the class field theory. Take an element $x \in \mathcal{Y}^+(0)(F_\mathfrak{p}^{\mathrm{nr}})$. It induces a unique morphism $y\colon \operatorname{Spec} O_\mathfrak{p}^{\mathrm{nr}} \to \mathcal{X}$. Since y is fixed by $E^\times$, there are strict actions of $E^\times \cap O_{E_\mathfrak{p}}^\times$ and, hence, $O_{E_\mathfrak{p}}$ on the $O_\mathfrak{p}$-divisible group $\mathcal{G}_y$. Therefore, the reduction of $\mathcal{G}_y$ is ordinary. In other words, y factors through $\mathcal{X}(0)$. As $\mathcal{Y}^+(0)$ is defined as the Zariski closure of $Y^+(0)$ in $\mathcal{X}(0)$, we obtain an element $x' \in \mathcal{Y}^+(0)(O_\mathfrak{p}^{\mathrm{nr}})$ uniquely determined by x. Thus, we have $\mathcal{Y}^\pm(0)(F_\mathfrak{p}^{\mathrm{nr}}) = \mathcal{Y}^\pm(0)(O_\mathfrak{p}^{\mathrm{nr}})$.

The case for general m follows from the case for $m = 0$ and the following two facts: (1) $\mathcal{Y}^+(m)$ is a closed subscheme of $\mathcal{Y}^+(0) \times_{\mathcal{X}(0)} \mathcal{X}(m)$; (2) $\mathcal{X}(m) \to \mathcal{X}(0)$ is a finite étale morphism (resp., a projective limit of finite étale morphisms) for $m \in \mathbb{N}$ (resp., $m = \infty$). ☐

Notation 2.4.5
Let S be a scheme that is locally of finite type over $\operatorname{Spec}\mathbb{C}$. We denote by $\breve{S}$ the underlying real analytic space with the complex conjugation automorphism $\mathfrak{c}_S\colon \breve{S} \to \breve{S}$.

In what follows, we will sometimes deal with a complex scheme S that is of the form $\varprojlim_I S_i$, where I is a filtered partially ordered set and each S_i is a smooth complex scheme, with a sheaf $\mathcal{F}$ that is the restriction of a quasicoherent sheaf $\mathcal{F}_0$ on some S_0. Then we will write $\breve{S} = \{\breve{S}_i\}_{i\in I}$ for the projective system of the underlying real analytic spaces together with the complex conjugation $\mathfrak{c}_S$, and we will write $\breve{\mathcal{F}} = \{\breve{\mathcal{F}}_i\}_{i\geq 0}$ for the projective system of real analytification of the restricted sheaf $\mathcal{F}_i$ for $i \geq 0$. Moreover, we denote

$$\mathrm{H}^0(\breve{S}, \breve{\mathcal{F}}) := \varinjlim_{i\geq 0} \mathrm{H}^0(\breve{S}_i, \breve{\mathcal{F}}_i).$$

For an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, put $X_\iota = X \otimes_{F,\iota} \mathbb{C}$ and denote by

$$\mathfrak{c}_\iota\colon \breve{X}_\iota \to \breve{X}_\iota \tag{2.22}$$

the complex conjugation. Denote by $(\mathcal{L}_\iota, \nabla_\iota)$ the restriction of the pair $(\mathcal{L}, \nabla) \otimes_{O_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C}$ along $\pi_\iota\colon X_\iota \to \mathcal{X} \otimes_{O_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C}$, where $(\mathcal{L}, \nabla)$ appears in Lemma 2.2.4. Applying the same procedure to the sequence (2.8), we obtain the sequence

$$0 \longrightarrow \underline{\omega}_\iota^\bullet \longrightarrow \mathcal{L}_\iota \longrightarrow \underline{\omega}_\iota^{\circ\vee} \longrightarrow 0$$

of locally free sheaves on X_ι. Similarly, we have the Kodaira–Spencer isomorphism

$$\mathrm{KS}_\iota\colon \underline{\omega}_\iota^\bullet \otimes \underline{\omega}_\iota^\circ \xrightarrow{\sim} \Omega^1_{X_\iota} \tag{2.23}$$

induced by (2.10).

LEMMA 2.4.6
The natural map $\underline{\breve{\omega}}_\iota^\bullet \oplus \mathfrak{c}_\iota^* \underline{\breve{\omega}}_\iota^\bullet \to \breve{\mathcal{L}}_\iota$ *is an isomorphism of sheaves on the real analytic space* $\breve{X}$*. Moreover, we have* $\nabla_\iota(\mathfrak{c}_\iota^* \underline{\breve{\omega}}_\iota^\bullet) \subset (\mathfrak{c}_\iota^* \underline{\breve{\omega}}_\iota^\bullet) \otimes \breve{\Omega}^1_{X_\iota}$.

Proof
It follows from Lemma 2.4.12 later in this section. ☐

We have a remark similar to Remark 2.3.2, which together with Lemma 2.4.6 induces a map

$$\theta_\iota^{[w]}\colon \breve{\mathscr{L}}_\iota^{[w]} \to (\underline{\breve{\omega}}_\iota^{\bullet})^{\otimes w} \otimes (\underline{\breve{\omega}}_\iota^{\circ})^{\otimes w} \xrightarrow{\mathrm{KS}_\iota(2.23)} (\breve{\Omega}^1_{X_\iota})^{\otimes w} \tag{2.24}$$

for all $w \in \mathbb{N}$.

Definition 2.4.7
Similar to Definition 2.3.3, define the *Shimura–Maass operator* to be

$$\Theta_\iota^{[w]}\colon (\breve{\Omega}^1_{X_\iota})^{\otimes w} \xrightarrow{(2.11)} \breve{\mathscr{L}}^{[w]} \otimes \breve{\Omega}^1_{X_\iota} \xrightarrow{\theta_\iota^{[w]}(2.24)} (\breve{\Omega}^1_{X_\iota})^{\otimes w+1}.$$

For $k \in \mathbb{N}$, define the *Shimura–Maass operator of degree k* to be

$$\Theta_\iota^{[w,k]} = \Theta_\iota^{[w+k-1]} \circ \cdots \circ \Theta_\iota^{[w]}\colon (\breve{\Omega}^1_{X_\iota})^{\otimes w} \to (\breve{\Omega}^1_{X_\iota})^{\otimes w+k}.$$

As for Θ_{ord}, we will suppress w from the notation and write Θ_ι (resp., Θ_ι^k) for $\Theta_\iota^{[w]}$ (resp., $\Theta_\iota^{[w,k]}$). In particular, we have the map

$$\Theta_\iota\colon \mathrm{H}^0\big(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}\big) \to \mathrm{H}^0\big(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w+1}\big). \tag{2.25}$$

Notation 2.4.8
Put

$$X(m)_\iota = X(m) \otimes_{F_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C}, \quad m \in \mathbb{Z} \cup \{\pm\infty\},$$
$$Y^{\pm}(m)_\iota = Y^{\pm}(m) \otimes_{F_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C}, \quad m \in \mathbb{N} \cup \{\infty\}.$$

Let $F_{\mathfrak{p}}^{\mathrm{ab}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Take an element

$$f \in \mathrm{H}^0\big(X(m), (\Omega^1_{X(m)})^{\otimes w}\big) \otimes_F K$$

with $m \in \mathbb{Z} \cup \{\pm\infty\}$ and $w \in \mathbb{N}$. Then by the transition isomorphism and by the restriction to an ordinary locus, we have an element

$$f_{\mathrm{ord}} := \begin{cases} \Upsilon_{+*} f \in \mathcal{M}_\flat^w(m, K) & \text{for } m \geq 0, \\ \Upsilon_{-*} f \in \mathcal{M}_\flat^w(-m, K) & \text{for } m \leq 0. \end{cases} \tag{2.26}$$

By base change, f induces another element

$$f_\iota \in \mathrm{H}^0\big(X(m)_\iota, (\Omega^1_{X(m)_\iota})^{\otimes w}\big).$$

The following lemma shows that the Atkin–Serre operator (2.18) and the Shimura–Maass operator (2.25) coincide on CM points. Note that the operator Θ_ι descends along the projection map $X_\iota \to X(m)_\iota$.

LEMMA 2.4.9
Let the notation be as above. We have, for $k \in \mathbb{N}$,

$$\iota\Upsilon_{\pm}^{*}\big((\Theta_{\mathrm{ord}}^{k} f_{\mathrm{ord}})|_{\mathfrak{Y}^{\pm}(m)}\big) = (\Theta_{\iota}^{k} f_{\iota})|_{Y^{\pm}(m)_{\iota}}$$

as functions on $Y^{\pm}(m)_{\iota}$, regarded as closed subschemes of $X(\pm m)_{\iota}$ via the transition isomorphisms $\Upsilon_{\pm}$ in Definition 2.2.10, respectively.

Proof
Generally, once we restrict to stalks, we cannot apply differential operators anymore. Therefore, we need alternative descriptions of $\Theta_{\mathrm{ord}}^{[w,k]}$ and $\Theta_{\iota}^{[w,k]}$. (Here, we retrieve the original notation for clarity.)

We denote by ϑ^{w} the composite map

$$(\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \to \mathscr{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathscr{L}^{[w]} \otimes \Omega^{1}_{\mathfrak{X}(n)} \xrightarrow{\theta_{\mathrm{ord}}^{[w]}} (\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \otimes \Omega^{1}_{\mathfrak{X}(n)} \xrightarrow{\mathrm{KS}^{-1}} (\underline{\omega}^{\bullet})^{\otimes w+1} \otimes (\underline{\omega}^{\circ})^{\otimes w+1},$$

and by δ^{w} the composite map

$$\mathscr{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathscr{L}^{[w]} \otimes \Omega^{1}_{\mathfrak{X}(n)} \xrightarrow{\mathrm{KS}^{-1}} \mathscr{L}^{[w]} \otimes (\underline{\omega}^{\bullet} \otimes \underline{\omega}^{\circ}) \to \mathscr{L}^{[w+1]}.$$

Since we have $\nabla\mathscr{L}^{\circ} \subset \mathscr{L}^{\circ} \otimes \Omega^{1}_{\mathfrak{X}}$ by Lemma 2.3.1(2) and its dual version from Remark 2.3.2, the composition $\vartheta^{w+k-1} \circ \cdots \circ \vartheta^{w}$ coincides with the map

$$(\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \to \mathscr{L}^{[w]} \xrightarrow{\delta^{w+k-1} \circ \cdots \circ \delta^{w}} \mathscr{L}^{[w+k]} \xrightarrow{\theta_{\mathrm{ord}}^{[w+k]}} (\underline{\omega}^{\bullet})^{\otimes w+k} \otimes (\underline{\omega}^{\circ})^{\otimes w+k}.$$

Therefore, the map $\Theta_{\mathrm{ord}}^{[w,k]}$ coincides with the composite map

$$(\Omega^{1}_{\mathfrak{X}(n)})^{\otimes w} \xrightarrow{\mathrm{KS}^{-1}} (\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \to \mathscr{L}^{[w]} \xrightarrow{\delta^{w+k-1} \circ \cdots \circ \delta^{w}} \mathscr{L}^{[w+k]} \xrightarrow{\mathrm{KS} \circ \theta_{\mathrm{ord}}^{[w+k]}} (\Omega^{1}_{\mathfrak{X}(n)})^{\otimes w+k}.$$

The advantage of the above description is that θ_{ord} appears only at the end of the sequence of maps. Since we have $\nabla_{\iota}(\mathfrak{c}_{\iota}^{*}\underline{\breve{\omega}}_{\iota}^{\bullet}) \subset (\mathfrak{c}_{\iota}^{*}\underline{\breve{\omega}}_{\iota}^{\bullet}) \otimes \breve{\Omega}^{1}_{X_{\iota}}$ by Lemma 2.4.6, there is a similar description of $\Theta_{\iota}^{[w,k]}$ as above. Therefore, to prove the lemma, we only need to show that the splitting in Lemma 2.4.6 coincides with the restriction of the splitting

$$\mathscr{L} \otimes_{O_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C} = (\underline{\omega}^{\bullet} \otimes_{O_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C}) \oplus (\mathscr{L}^{\circ} \otimes_{O_{\mathfrak{p}}^{\mathrm{nr}},\iota} \mathbb{C})$$

on Y_{ι}^{+} and Y_{ι}^{-}.

Pick up an arbitrary point $y \in Y^+(m)_\iota(\mathbb{C}) \cup Y^-(m)_\iota(\mathbb{C})$. We have an action of $E^\times$ on both the splitting $\underline{\breve{\omega}}_\iota^\bullet|_y \oplus \mathfrak{c}_\iota^*\underline{\breve{\omega}}_\iota^\bullet|_y$ and $(\underline{\omega}^\bullet \otimes_{O_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}|_y) \oplus (\mathcal{L}^\circ \otimes_{O_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}|_y)$. By definition, $\underline{\breve{\omega}}_\iota^\bullet|_y$ and $\underline{\omega}^\bullet \otimes_{O_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}|_y$ coincide, which is one of the two complex eigenlines with respect to the $E^\times$-action. It follows that $\mathfrak{c}_\iota^*\underline{\breve{\omega}}_\iota^\bullet|_y$ and $\mathcal{L}^\circ \otimes_{O_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}|_y$ have to coincide as well, which contributes to the other complex eigenline. □

We now study the behavior of the Shimura–Maass operator under complex uniformization.

Definition 2.4.10 (ι-nearby data)
Let $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. An *ι-nearby data* for $\mathbb{B}$ consists of

- a quaternion algebra $B(\iota)$ over F such that $B(\iota)_v$ is definite for Archimedean places v other than $\iota|_F$,
- an isomorphism $B(\iota)_v \simeq \mathbb{B}_v$ for every finite place v other than $\mathfrak{p}$,
- an isomorphism $B(\iota)_\iota := B(\iota) \otimes_{F,\iota} \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$, and
- an embedding $\mathtt{e}(\iota)\colon E \hookrightarrow B(\iota)$ of F-algebras such that $\mathtt{e}(\iota)_v$ coincides with $\mathtt{e}_v$ under the isomorphism $B(\iota)_v \simeq \mathbb{B}_v$ for every finite place v other than $\mathfrak{p}$, and $\mathcal{H}^{E^\times} = \{\pm i\}$.

We now choose an ι-nearby data for $\mathbb{B}$. It induces a complex uniformization

$$X_\iota(\mathbb{C}) \simeq B(\iota)^\times \backslash \mathcal{H} \times \mathbb{B}^{\infty\times}/\overline{F^\times},$$

where $\mathcal{H} = \mathbb{C} \setminus \mathbb{R}$ denotes the union of Poincaré upper and lower half-planes. Let z be the standard coordinate on $\mathcal{H}$.

LEMMA 2.4.11
Denote by L_ι the $\mathbb{C}$-local system on X_ι defined by the quotient map

$$B(\iota)^\times \backslash \mathbb{C}^{\oplus 2} \times \mathcal{H} \times \mathbb{B}^{\infty\times}/\overline{F^\times} \to B(\iota)^\times \backslash \mathcal{H} \times \mathbb{B}^{\infty\times}/\overline{F^\times} \simeq X_\iota(\mathbb{C}),$$

where the action of $\gamma \in B(\iota)^\times$ is given by the formula

$$\gamma\big[(a_1,a_2)^t, z, g\big] = \big[\big((a_1,a_2)\iota(\gamma)^{-1}\big)^t, \iota(\gamma)(z), \gamma^\infty g\big].$$

Then we have a canonical isomorphism $\mathcal{L}_\iota \simeq \mathcal{O}_{X_\iota} \otimes_\mathbb{C} \mathrm{L}_\iota$ under which ∇_ι coincides with the induced connection on $\mathcal{O}_{X_\iota} \otimes_\mathbb{C} \mathrm{L}_\iota$.

Proof
It follows from the fact that L_ι is canonically isomorphic to the restriction of $\mathrm{L} \otimes_{O_\mathfrak{p},\iota} \mathbb{C}$ along the natural morphism π_ι, where L is the $O_\mathfrak{p}$-local system on $\mathcal{X}$ defined in Remark 2.2.5. □

The following lemma will be proved in Section 2.5.

LEMMA 2.4.12
Under the isomorphism $\mathscr{L}_\iota \simeq \mathscr{O}_{X_\iota} \otimes_{\mathbb{C}} \mathrm{L}_\iota$ in Lemma 2.4.11, the subsheaf $\underline{\omega}_\iota^\bullet$ is generated by the section $\omega_\iota^\bullet$ whose value at z is $(z,1)^t$.

The following lemma shows that our definition of Shimura–Maass operators coincides with the classical one.

LEMMA 2.4.13
For every $f \in \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ with some $w \in \mathbb{N}$, we have

$$\Theta_\iota f \otimes dz^{\otimes -w-1} = \Big(\frac{\partial}{\partial z} + \frac{2w}{z-\overline{z}}\Big) f \otimes dz^{\otimes -w}.$$

Proof
We may pass to the universal cover $\mathscr{H} \times \mathbb{B}^{\infty\times}/\overline{F^\times}$ and suppress the part $\mathbb{B}^{\infty\times}/\overline{F^\times}$ in what follows. Over $\mathscr{H}$, the sheaf $\mathscr{L}_\iota$ is trivialized as $\mathbb{C}^{\oplus 2}$, and the subsheaf $\underline{\omega}_\iota^\bullet$ is generated by the section $\omega_\iota^\bullet$ whose value at z is $(z,1)^t$ by Lemma 2.4.12. Dually, the sheaf $\mathscr{L}_\iota^\vee$ is trivialized as 2-dimensional complex row vectors, and the subsheaf $\underline{\omega}_\iota^\circ$ is generated by the section ω_ι° whose value at z is $(1,-z)$. Then we have $\mathrm{KS}(\omega_\iota^\bullet \otimes \omega_\iota^\circ) = dz$.

It is easy to see that

$$\Theta_\iota\big((\omega_\iota^\bullet)^{\otimes w} \otimes (\omega_\iota^\circ)^{\otimes w}\big) = \frac{2w}{z-\overline{z}}\big((\omega_\iota^\bullet)^{\otimes w} \otimes (\omega_\iota^\circ)^{\otimes w}\big) \otimes dz,$$

since $\mathfrak{c}_\iota^* \underline{\omega}_\iota^\bullet$ (resp., $\mathfrak{c}_\iota^* \underline{\omega}_\iota^\circ$) is generated by the section $(\overline{z},1)^t$ (resp., $(1,-\overline{z})$). The lemma follows. □

We now introduce the notion of automorphic forms.

Notation 2.4.14
For every $w \in \mathbb{Z}$, denote by $\mathcal{A}^{(2w)}(B(\iota)^\times)$ (resp., $\mathcal{A}^{(2w)}_{\mathrm{cusp}}(B(\iota)^\times)$) the space of real analytic (resp., cuspidal) automorphic forms on $B(\iota)^\times(\mathbb{A})$ of weight $2w$ at $\iota|_F$ and invariant under the action of $B(\iota)_v^\times$ at Archimedean places v other than $\iota|_F$.

The spaces $\mathcal{A}^{(2w)}(B(\iota)^\times)$ and $\mathcal{A}^{(2w)}_{\mathrm{cusp}}(B(\iota)^\times)$ are representations of $B(\iota)^\times(\mathbb{A})$ by the right translation R.

LEMMA 2.4.15
There is a natural $\mathbb{B}^{\infty\times}$-equivariant map

$$\phi_\iota\colon \mathrm{H}^0\big(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}\big) \to \mathcal{A}^{(2w)}\big(B(\iota)^\times\big)$$

such that, for $g_\iota \in B(\iota)_\iota^\times = \mathrm{GL}_2(\mathbb{R})$*,*

$$\phi_\iota(f)\big([g_\iota, 1]\big) j(g_\iota, i)^w = f\big(g_\iota(i)\big) \otimes dz^{\otimes -w},$$

where $j(g_\iota, i) = (\det g_\iota)^{-1} \cdot (ci + d)^2$ *is the square of the usual* j*-factor.*

Proof
This is the well-known dictionary between modular forms and automorphic forms. □

We denote by $\mathrm{H}^0_{\mathrm{cusp}}(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}) \subset \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ the inverse image of $\mathcal{A}^{(2w)}_{\mathrm{cusp}}(B(\iota)^\times)$ under ϕ_ι.

Definition 2.4.16
Define $\Delta_\pm$ to be the matrices

$$\frac{1}{4i}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$$

in $\mathfrak{gl}_{2,\mathbb{C}} = \mathrm{Mat}_2(\mathbb{C})$, respectively. For an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, define $\Delta_{\pm,\iota}$ to be the matrices $\Delta_\pm$ when regarded as elements in $\mathrm{Lie}_{\mathbb{C}}(B(\iota) \otimes_{F,\iota} \mathbb{C}) = \mathfrak{gl}_{2,\mathbb{C}}$, respectively. Finally, define $\Delta^k_{\pm,\iota} = \Delta_{\pm,\iota} \circ \cdots \circ \Delta_{\pm,\iota}$ to be the k-fold composition.

LEMMA 2.4.17
For every $f \in \mathrm{H}^0_{\mathrm{cusp}}(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ *and* $k \in \mathbb{N}$*, we have*

$$\phi_\iota(\Theta^k_\iota f) = \Delta^k_{+,\iota} \phi_\iota(f),$$

where ϕ_ι *is defined in Lemma 2.4.15.*

Proof
This follows from Lemma 2.4.13, together with [5, p. 130, p. 143, and Proposition 2.2.5 on p. 155]. □

2.5. Proofs of claims via unitary Shimura curves
In this section, we prove the six claims (Lemma 2.2.4, Proposition 2.2.6, Lemma 2.3.1, Lemma 2.3.4, Proposition 2.3.5, and Lemma 2.4.12) left from previous sections. Suppose that we are in the case of modular curves, that is, $F = \mathbb{Q}$ and $\mathbb{B}$ is unramified at every prime; then these statements except for Proposition 2.3.5 are clear. In fact, in this case,

- Lemma 2.2.4 follows from the fact that (2.5) is the formal completion of the Hodge sequence coming from the universal elliptic curve;
- Proposition 2.2.6 is well known;
- Lemma 2.3.1 is proved by Katz as [16, Theorem 1.11.27];
- Lemma 2.3.4 follows from Serre–Tate coordinates in [17];
- Proposition 2.3.5 again can be proved via Serre–Tate coordinates (one can adjust our proof below to the case of modular curves); and
- Lemma 2.4.12 is again well known.

The main idea is to use the existence of a universal family of elliptic curves with deformation theory. However, in the general case, $\mathcal{X}$ is not a moduli space; therefore, we have to use some auxiliary moduli space to deduce these statements. The reader may skip the rest of this section for the first reading.

Our strategy is to use the unitary Shimura curves considered by Carayol [7]. Thus, we will fix an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. In particular, $F_{\mathfrak{p}}^{\mathrm{nr}}$ is a subfield of $\mathbb{C}$. We also fix an ι-nearby data for $\mathbb{B}$ (Definition 2.4.10) and put $B = B(\iota)$ for short.

Note that when $F = \mathbb{Q}$ there is no need to change the Shimura data as $\mathcal{X}$ is already a moduli space. In order to unify the argument, we will choose to do so in this case as well. We will also assume that we are not in the case of classical modular curves (i.e., $F = \mathbb{Q}$ and $\mathbb{B}$ is unramified at every prime) where all these statements are known, as explained above.

Fix an element $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda > 0$, $-\lambda^2 \in \mathbb{N}$, p splits in $\mathbb{Q}(\lambda) \subset \mathbb{C}$, and $\mathbb{Q}(\lambda)$ is not contained in E. We have subfields $F(\lambda)$ and $E(\lambda)$ of $\mathbb{C}$, and we identify their completion inside $\mathbb{C} \simeq \mathbb{C}_p$ with $F_{\mathfrak{p}}$. In [7, Section 2] (see also [13, Section 2]), a reductive group G' over $\mathbb{Q}$ is defined as a subgroup of $\operatorname{Res}_{F/\mathbb{Q}}(B^\times \times_{F^\times} F(\lambda)^\times)$ (which itself is a subgroup of $\operatorname{Res}_{F(\lambda)/\mathbb{Q}}(B \otimes_F F(\lambda))^\times$) with "rational norms." In particular, we have

$$G'(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathrm{GL}_2(F_{\mathfrak{p}}) \times (\mathbb{B}_{\mathfrak{p}_2}^\times \times \cdots \times \mathbb{B}_{\mathfrak{p}_m}^\times),$$

where $\mathfrak{p}_2, \ldots, \mathfrak{p}_m$ are primes of F over p other than $\mathfrak{p}$. Put

$$G'^{\mathfrak{p}} = G'\Bigl(\Bigl(\prod_{q \neq p} \mathbb{Z}_q\Bigr) \otimes \mathbb{Q}\Bigr) \times (\mathbb{B}_{\mathfrak{p}_2}^\times \times \cdots \times \mathbb{B}_{\mathfrak{p}_m}^\times),$$

and let $\mathfrak{U}'$ be the set of all (sufficiently small) open compact subgroups $U'^{\mathfrak{p}}$ of $G'^{\mathfrak{p}}$. Then for each $U'^{\mathfrak{p}} \in \mathfrak{U}'$, there is a unitary Shimura curve $X'_{U'^{\mathfrak{p}}}$, smooth and projective over $\operatorname{Spec} F_{\mathfrak{p}}$, of the level structure $\mathbb{Z}_p^\times \times \mathrm{GL}_2(O_{\mathfrak{p}}) \times U'^{\mathfrak{p}}$. It has a canonical smooth model $\mathcal{X}'_{U'^{\mathfrak{p}}}$ over $\operatorname{Spec} O_{\mathfrak{p}}^{\mathrm{nr}}$ defined via a moduli problem (see [7, Section 6]). In particular, there is a universal Abelian variety $\pi\colon \mathcal{A}_{U'^{\mathfrak{p}}} \to \mathcal{X}'_{U'^{\mathfrak{p}}}$ with a specific p-divisible subgroup $\mathcal{G}'_{U'^{\mathfrak{p}}} := (\mathcal{A}_{U'^{\mathfrak{p}}}[p^\infty])_1^{2,1} \subset \mathcal{A}_{U'^{\mathfrak{p}}}[p^\infty]$ (it is denoted as $\mathbf{E}'_\infty$ in [7, Section 6]), which is an $O_{\mathfrak{p}}$-divisible group of dimension 1 and height 2. Here, for an object M with $O_{F(\lambda)} \otimes \mathbb{Z}_p$-action, we denote by $M^{2,1}$ the direct summand corresponding to the p-adic place $F(\lambda) \xrightarrow{\lambda \mapsto -\lambda} F(\lambda) \subset \mathbb{C}_p$. Then $(\mathcal{A}_{U'^{\mathfrak{p}}}[p^\infty])^{2,1}$ admits an action by $O_B \otimes_{O_F} O_{\mathfrak{p}} \simeq \mathrm{Mat}_2(O_{\mathfrak{p}})$. Put $e_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $(\mathcal{A}_{U'^{\mathfrak{p}}}[p^\infty])^{2,1}_1 = e_1(\mathcal{A}_{U'^{\mathfrak{p}}}[p^\infty])^{2,1}$. See [7, Section 2.6] for more details.

We let $\mathcal{X}'(0)_{U'^{\mathfrak{p}}}$ be the (dense) open subscheme of $\mathcal{X}'_{U'^{\mathfrak{p}}}$ by removing all points on the special fiber where $\mathcal{G}'$ is supersingular. For $n \in \mathbb{N}$, define $\mathcal{X}'(n)_{U'^{\mathfrak{p}}}$ to be the functor classifying $O_{\mathfrak{p}}$-equivariant extensions

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^n] \longrightarrow \mathcal{G}'[\mathfrak{p}^n] \longrightarrow \mathfrak{p}^{-n}/O_{\mathfrak{p}} \longrightarrow 0$$

of $\mathcal{G}'$ over $\mathcal{X}'(0)_{U'^{\mathfrak{p}}}$. The obvious map $\mathcal{X}'(n)_{U'^{\mathfrak{p}}} \to \mathcal{X}'(0)_{U'^{\mathfrak{p}}}$ is étale. Finally, put $\mathcal{X}'(\infty)_{U'^{\mathfrak{p}}} = \varprojlim_n \mathcal{X}'(n)_{U'^{\mathfrak{p}}}$.

The construction of Carayol amounts to saying that, for every sufficiently small $U^{\mathfrak{p}} \in \mathfrak{U}$ and a connected component $\mathcal{X}^\dagger_{U^{\mathfrak{p}}}$ of $\mathcal{X}_{U^{\mathfrak{p}}}$, there exists a member $U'^{\mathfrak{p}} \in \mathfrak{U}'$ such that

- $\mathcal{X}(n)^\dagger_{U^{\mathfrak{p}}} := \mathcal{X}^\dagger_{U^{\mathfrak{p}}} \times_{\mathcal{X}_{U^{\mathfrak{p}}}} \mathcal{X}(n)_{U^{\mathfrak{p}}}$ is isomorphic to the neutral connected component of $\mathcal{X}'(n)_{U'^{\mathfrak{p}}}$ for $n \in \mathbb{N} \cup \{\infty\}$; and
- under the above isomorphism, $\mathcal{G}_{U^{\mathfrak{p}}}|_{\mathcal{X}(n)^\dagger_{U^{\mathfrak{p}}}}$ is isomorphic to the restriction of $\mathcal{G}'_{U'^{\mathfrak{p}}}$ to (the neutral connected component of) $\mathcal{X}'(n)_{U'^{\mathfrak{p}}}$.

In what follows we may and will fix a sufficiently small subgroup $U^{\mathfrak{p}} \in \mathfrak{U}$, a connected component $\mathcal{X}^\dagger_{U^{\mathfrak{p}}}$ of $\mathcal{X}_{U^{\mathfrak{p}}}$, and a corresponding subgroup $U'^{\mathfrak{p}} \in \mathfrak{U}'$. To simplify notation, we will suppress $U^{\mathfrak{p}}$ and $U'^{\mathfrak{p}}$ and will regard $\mathcal{X}^\dagger$ as a connected component of $\mathcal{X}'$ as well.

Consider the Hodge exact sequence

$$0 \longrightarrow \pi_*\Omega^1_{\mathcal{A}/\mathcal{X}'} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}') \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{A}} \longrightarrow 0.$$

It has a direct summand

$$0 \longrightarrow (\pi_*\Omega^1_{\mathcal{A}/\mathcal{X}'})^{2,1}_1 \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}')^{2,1}_1 \longrightarrow (\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{A}})^{2,1}_1 \longrightarrow 0, \tag{2.27}$$

which is $O_{\mathfrak{p}}$-equivariant, where $(\cdot)^{2,1}_1$ is defined similarly as above. Here in (2.27), the three sheaves are locally constant of rank 1, $2h$, and $2h-1$, respectively, where $h = [F_{\mathfrak{p}} : \mathbb{Q}_p]$.

We introduce the following notation.

Notation 2.5.1
If M is a locally free sheaf on a scheme over Spec $O_{\mathfrak{p}}$ equipped with an $O_{\mathfrak{p}}$-action

$O_{\mathfrak{p}} \to \operatorname{End} M$, then we denote by $M^{O_{\mathfrak{p}}}$ the maximal subsheaf on which $O_{\mathfrak{p}}$ acts via the structure homomorphism.

In what follows, we denote the sequence (2.27) after applying $(\cdot)^{O_{\mathfrak{p}}}$ by

$$0 \longrightarrow \underline{\omega}'^{\bullet} \longrightarrow \mathcal{L}' \longrightarrow \underline{\omega}'^{\circ\vee} \longrightarrow 0. \tag{2.28}$$

Proof of Lemma 2.2.4
It suffices to consider the problem after the restriction to an arbitrarily chosen connected component $\mathcal{X}^{\dagger}$ of $\mathcal{X}$. By the definition of $\mathcal{G}'$ in [7, Section 5.4], we know that (2.27) is the Hodge exact sequence for $\mathcal{G}'$. Since $\mathcal{G}$ is isomorphic to $\mathcal{G}'$ on $\mathcal{X}^{\dagger}$, (2.27) is also the Hodge exact sequence for $\mathcal{G}$. Therefore, if we restrict (2.27) to $\mathcal{X}^{\dagger}$ and take formal completion, we recover the exact sequence (2.4) (restricted to $\widehat{\mathcal{X}^{\dagger}}$); and if we further apply the functor $(\cdot)^{O_{\mathfrak{p}}}$, then we recover the exact sequence (2.5). In other words, the formal completion of (2.28) coincides with (2.5), both restricted to $\widehat{\mathcal{X}^{\dagger}}$. This shows that (2.5) is algebraizable.

For the next assertion, we have the Gauss–Manin connection

$$\nabla'_{\mathcal{A}}\colon \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}') \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}') \otimes \Omega^1_{\mathcal{X}'}$$

and the induced connection

$$\nabla'_{p}\colon \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}')^{2,1}_{1} \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}')^{2,1}_{1} \otimes \Omega^1_{\mathcal{X}'}. \tag{2.29}$$

Since applying $(\cdot)^{2,1}_{1}$ commutes with the formation of the Gauss–Manin connection, we know that the formal completion of (2.29) coincides with ∇_p from (2.6) when restricted to $\widehat{\mathcal{X}^{\dagger}}$. Now applying the functor $(\cdot)^{O_{\mathfrak{p}}}$, we know that the formal completion of the induced connection

$$\nabla'\colon \mathcal{L}' \to \mathcal{L}' \otimes \Omega^1_{\mathcal{X}'} \tag{2.30}$$

coincides with ∇ from (2.7) when restricted to $\widehat{\mathcal{X}^{\dagger}}$. In other words, ∇ from (2.7) is algebraizable. Lemma 2.2.4 is proved. □

Proof of Proposition 2.2.6
Denote by

$$\mathrm{KS}'\colon \underline{\omega}'^{\bullet} \otimes \underline{\omega}'^{\circ} \to \Omega^1_{\mathcal{X}'} \tag{2.31}$$

the Kodaira–Spencer map induced from ∇' from (2.30), where $\underline{\omega}'^{\circ}$ is the dual sheaf of $\underline{\omega}'^{\circ\vee}$. We know from the proof of Lemma 2.2.4 that, when restricted to $\mathcal{X}^{\dagger}$, (2.28) coincides with (2.8) under which ∇' from (2.30) coincides with ∇ from (2.7). Therefore, under the previous identification, KS' coincides with KS from (2.10). Then Proposition 2.2.6 follows from the following analogous statement for $\mathcal{X}'$: (2.31) is an isomorphism.

The proof is similar to [9, Lemma 7], which essentially follows from the Grothendieck–Messing theory. Denote by $\mathcal{A}^\vee$ the dual Abelian variety of $\mathcal{A}$. Then $\underline{\omega}'^{\circ\vee}$ is canonically isomorphic to $(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1})^{O_\mathfrak{p}}$. We only need to show that, for every closed point $t\colon \operatorname{Spec} k(t) \to \mathcal{X}'$, the induced map

$$\underline{\omega}'^{\bullet} \otimes k(t) \to \big(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1}\big)^{O_\mathfrak{p}} \otimes \Omega^1_{\mathcal{X}'} \otimes k(t) \tag{2.32}$$

is surjective, where $\underline{\mathrm{Lie}}$ denotes the sheaf of tangent vectors.

Let $A/\operatorname{Spec} k(t)$ be the Abelian variety classified by t. Put $T = \operatorname{Spec} k(t)[\varepsilon]/(\varepsilon^2)$. The lifts A_ϕ of A (with other PEL structures) to T correspond to homomorphisms

$$\phi\colon t^*\underline{\omega}'^{\bullet} \to \big(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1}\big)^{O_\mathfrak{p}} \otimes k(t).$$

Since both sides are $k(t)$-vector spaces of dimension 1, we may choose a homomorphism ϕ that is surjective. Let $t_\phi\colon T \to \mathcal{X}'$ be the morphism that classifies A_ϕ/T. Compose the isomorphism $t_\phi^*\underline{\omega}'^{\bullet} \otimes k(t) \to t^*\underline{\omega}'^{\bullet}$ and the surjective map ϕ. By the isomorphism

$$\big(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1}\big)^{O_\mathfrak{p}} \otimes k(t) \simeq t_\phi^*\big(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1}\big)^{O_\mathfrak{p}} \otimes \Omega^1_{T/k(t)} \otimes k(t),$$

we obtain a surjective map

$$t_\phi^*\underline{\omega}'^{\bullet} \otimes k(t) \to t_\phi^*\big(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/\mathcal{X}')_1^{2,1}\big)^{O_\mathfrak{p}} \otimes \Omega^1_{T/k(t)} \otimes k(t),$$

which is the pullback of (2.32) under t_ϕ. Therefore, (2.32) is surjective. □

For $n \in \mathbb{N} \cup \{\infty\}$, denote by $\mathfrak{X}'(n)$ the formal completion of $\mathcal{X}'(n)$ along its special fiber, which is equipped with an $O_\mathfrak{p}$-divisible group $\mathfrak{G}'$ induced from $\mathcal{G}'$.

Proof of Lemma 2.3.1

By the proof of Lemma 2.2.4, it suffices to prove the same statement for $\mathfrak{X}'(0)$. The desired morphism $\Phi'\colon \mathfrak{X}'(0) \to \mathfrak{X}'(0)$ is constructed through the moduli interpretation of $\mathcal{X}'(0)$ by "dividing $\mathfrak{G}'^0[\mathfrak{p}]$," which lifts the Frobenius on the special fiber of degree $\#O_\mathfrak{p}/\mathfrak{p}$. The uniqueness of such Φ' is ensured by (the proof of) Lemma 2.3.4 and Theorem B.1.1.

The proof of the remaining part is similar to [16, Theorem 1.11.27]. The only modification we need is to show that the subsheaf $\mathcal{L}'^\circ$ of $\mathcal{L}'$ glues to a formal quasi-coherent sheaf. For this, we adopt the proof of [14, Theorem 4.1] in the case where

$\mathbb{Z}_p$ is replaced by $O_\mathfrak{p}$ and p is replaced by a uniformizer ϖ of F. The assumptions are satisfied because the Newton polygon of the underlying p-divisible group of $\mathfrak{G}'|_x$ for every $x \in \mathfrak{X}'(0)(\kappa)$ is the one starting with $(0,0)$, ending with $(2h,1)$, and having the unique breaking point at $(h,0)$. □

Remark 2.5.2
In fact, the induced map of Φ' constructed in the above proof on the coordinate ring is simply the operator Frob defined in [13, Definition 11.1].

Proof of Lemma 2.3.4
We only need to prove a similar statement for $\mathfrak{X}'(\infty)$. By the moduli interpretation of $\mathcal{X}'(\infty)$ and the Serre–Tate theorem on deformation of Abelian varieties, we have an isomorphism $\mathfrak{X}'(\infty)_{/x} \simeq \mathfrak{M}_x$, where $\mathfrak{M}_x$ is the formal scheme representing deformations of $\mathscr{G}'|_x$. By Theorem B.1.1, we know that $\mathfrak{M}_x$ is canonically isomorphic to $\mathscr{LT}$, and the induced isomorphism $\mathcal{X}'(\infty)_{/x} \simeq \mathscr{LT}$ is just $c_{/x}$ by definition. □

Proof of Proposition 2.3.5
Recall that we have a similarly defined formal scheme $\mathfrak{X}'(\infty)$ over $\operatorname{Spf} O_\mathfrak{p}^{\mathrm{nr}}$. The uniqueness of β is clear. Thus, by comparison, it suffices to construct the morphism $\beta' \colon \mathscr{LT} \times_{\operatorname{Spf} O_\mathfrak{p}^{\mathrm{nr}}} \mathfrak{X}'(\infty) \to \mathfrak{X}'(\infty)$ with similar properties to those in Proposition 2.3.5, since the action of $\mathscr{LT}$ is supposed to preserve the special fiber.

We use the moduli interpretation of $\mathfrak{X}'(\infty)$. For a scheme S over $\operatorname{Spec} O_\mathfrak{p}^{\mathrm{nr}}$ where p is locally nilpotent, $\mathfrak{X}'(\infty)(S)$ is the set of isomorphism classes of quintuples $(A, \iota, \theta, k^\mathfrak{p}, \kappa_\mathfrak{p})$, where $(A, \iota, \theta, k^\mathfrak{p})$ is the same data in [7, Section 5.2] but $k^\mathfrak{p}$ is an isomorphism instead of a class, and $\kappa_\mathfrak{p}$ is an exact sequence

$$0 \longrightarrow \mathscr{LT}[\mathfrak{p}^\infty] \longrightarrow (A_{p^\infty})_1^{2,1} \longrightarrow F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0.$$

On the other hand, $\mathscr{LT}(S)$ is the set of isomorphism classes of (G, κ_G) where κ_G is an exact sequence

$$0 \longrightarrow \mathscr{LT}[\mathfrak{p}^\infty] \longrightarrow G \longrightarrow F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0.$$

Using the group structure on $\mathscr{LT}$, we may add the above two exact sequences to a new one, denoted by $\alpha(\kappa_\mathfrak{p}, \kappa_G)$, which can be written as

$$0 \longrightarrow \mathscr{LT}[\mathfrak{p}^\infty] \longrightarrow \alpha\big((A_{p^\infty})_1^{2,1}, G\big) \longrightarrow F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0.$$

By the Serre–Tate theorem on deformation of Abelian varieties and the fact that étale level structures are determined on the special fiber, we canonically associate a quintuple $(A', \iota', \theta', k'^{\mathfrak{p}}, \kappa'_{\mathfrak{p}})$ with $\kappa'_{\mathfrak{p}} = \alpha(\kappa_{\mathfrak{p}}, \kappa_G)$. This defines the morphism β'. The properties of Proposition 2.3.5 for β' follow directly from the construction. □

Proof of Lemma 2.4.12
We define X'_ι similarly as the projective limit over all level structures over $\mathbb{C}$. Then we have the complex uniformization

$$X'_\iota(\mathbb{C}) \simeq G'(\mathbb{Q})\backslash \mathcal{H} \times G'(\mathbb{A}^\infty),$$

where $G'(\mathbb{Q})$ acts on $\mathcal{H}$ via the ι-component of $G'(\mathbb{R})$. We similarly define a $\mathbb{C}$-local system L'_ι on X'_ι via the quotient map

$$G'(\mathbb{Q})\backslash \mathbb{C}^{\oplus 2} \times \mathcal{H} \times G'(\mathbb{A}^\infty) \to G'(\mathbb{Q})\backslash \mathcal{H} \times G'(\mathbb{A}^\infty),$$

where the action of $\gamma \in G'(\mathbb{Q})$ is given by the formula

$$\gamma\big[(a_1, a_2)^t, z, g\big] = \big[\big((a_1, a_2)\iota(\gamma)^{-1}\big)^t, \iota(\gamma)(z), \gamma^\infty g\big],$$

where we regard $\iota(\gamma)$ as an element in $\mathrm{GL}_2(\mathbb{C})$ in the formula $(a_1, a_2)\iota(\gamma)^{-1}$. By the same reasoning as in Lemma 2.4.11, we have a canonical isomorphism $\mathcal{L}'_\iota \simeq \mathcal{O}_{X'_\iota} \otimes_{\mathbb{C}} \mathrm{L}'_\iota$. Here, we regard $\mathcal{L}'_\iota$ as the restriction of $\mathcal{L}'$ from (2.28) to X'_ι. By the comparison between (2.28) and (2.8) established in the proof of Lemma 2.2.4, it suffices to show that the subsheaf $\underline{\omega}'^{\bullet}_\iota$ is generated by the section $\omega'^{\bullet}_\iota$ whose value at z is $(z, 1)^t$.

However, the coherent sheaf $\mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{X}')$ is obtained from the local system

$$G'(\mathbb{Q})\backslash \mathbb{C}^{\oplus 2g} \times \mathcal{H} \times G'(\mathbb{A}^\infty) \to G'(\mathbb{Q})\backslash \mathcal{H} \times G'(\mathbb{A}^\infty),$$

where $G'(\mathbb{Q})$ acts on $\mathbb{C}^{\oplus 2g} = \mathbb{C}^{\oplus 2} \oplus \cdots \oplus \mathbb{C}^{\oplus 2}$ diagonally via all Archimedean places of F. From the Hodge homomorphism in the Shimura data of G', we see that the restriction of $\underline{\omega}'^{\bullet} \simeq (\pi_* \Omega^1_{\mathcal{A}/\mathcal{X}'})^{2,1}_1$ to X'_ι is generated by the section $\omega'^{\bullet}_\iota$ whose value at z is $(z, 1)^t$. This follows from the same computation for the case of modular curves. Therefore, Lemma 2.4.12 is proved. □

3. Statements of main theorems

In this section, we state our main theorems about p-adic L-functions and the p-adic Waldspurger formula for the general case. We start by recalling some background about representations of incoherent algebras and Abelian varieties of GL(2)-type in Section 3.1. In Section 3.2, we state the main theorem about p-adic L-functions in terms of Heegner cycles on Abelian varieties. In Section 3.3, we state the main theorem about the p-adic Waldspurger formula in terms of Heegner cycles on Abelian varieties. In Section 3.4, we provide an alternative formulation of our main theorems in terms of periods of p-adic Maass functions, in the same spirit as in Section 1, and deduce them from the previous formulation via Heegner cycles on Abelian varieties.

3.1. *Representations for incoherent quaternion algebras*

We recall some materials from [30, Section 3.2]. Let $\iota_1, \ldots, \iota_g$ be all Archimedean places of F. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$. As in Section 2.2, there is an associated projective system of Shimura curves $\{X_U = X(\mathbb{B})_U\}_U$ over F, and $X = \varprojlim_U X_U$. We recall the following definition from [30, Section 3.2.2].

Notation 3.1.1

Let L be a field embeddable into $\mathbb{C}$. Denote by $\mathcal{A}(\mathbb{B}^\times, L)$ the set of isomorphism classes of irreducible (admissible) representations Π of $\mathbb{B}^{\infty\times}$ over L such that, for some and hence all embeddings $L \hookrightarrow \mathbb{C}$, the Jacquet–Langlands transfer of $\Pi \otimes_L \mathbb{C}$ to $\mathrm{GL}_2(\mathbb{A}^\infty)$ is a finite direct sum of (finite components of) irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2.

Let A be an Abelian variety over F.

Notation 3.1.2

Recall from [30, Section 3.2.3] the following notation

$$\Pi(\mathbb{B})_A := \varinjlim_U \mathrm{Hom}_{\xi_U}(X_U^*, A),$$

where

- the colimit is taken over all open compact subgroups U of $\mathbb{B}^{\infty\times}$;
- X_U^* is the smooth compactification of X_U (which is simply X_U unless in the case of classical modular curves);
- ξ_U is the normalized Hodge class on X_U^* (see [30, Section 3.1.3]); and
- $\mathrm{Hom}_{\xi_U}(X_U^*, A)$ denotes the $\mathbb{Q}$-vector space of *modular parameterizations*, that is, the Abelian group of morphisms from X_U^* to A that send ξ_U to a torsion point, tensoring with $\mathbb{Q}$.

We simply write Π_A for $\Pi(\mathbb{B})_A$ if $\mathbb{B}$ is clear from the context.

If we denote by J_U the Jacobian of X_U^*, then ξ_U induces a morphism $X_U^* \to J_U$. Thus, $\mathrm{Hom}_{\xi_U}(X_U^*, A)$ is canonically identified with $\mathrm{Hom}^0(J_U, A) := \mathrm{Hom}(J_U, A) \otimes \mathbb{Q}$.

Put $M_A := \mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$. It is clear that both $\Pi(\mathbb{B})_A$ and M_A depend only on A up to isogeny.

Definition 3.1.3

We say that A can be *parameterized by* $\mathbb{B}$ if there is a nonconstant morphism from

$X = X(\mathbb{B})$ to A. Denote by $\mathrm{AV}^0(\mathbb{B})$ the set of simple Abelian varieties over F that can be parameterized by $\mathbb{B}$ up to isogeny.

The set $\mathrm{AV}^0(\mathbb{B})$ is stable under duality. Take an element $A \in \mathrm{AV}^0(\mathbb{B})$. Then Π_A is a nonzero rational irreducible representation of $\mathbb{B}^{\infty\times}$, which is an element in $\mathcal{A}(\mathbb{B}^\times, \mathbb{Q})$ (Notation 3.1.1). The assignment $A \mapsto \Pi_A$ induces a bijection between $\mathrm{AV}^0(\mathbb{B})$ and $\mathcal{A}(\mathbb{B}^\times, \mathbb{Q})$. Moreover, M_A is a field of degree equal to the dimension of A, and it acts on the representation Π_A. Denote by $A^\vee$ the dual Abelian variety (up to isogeny) of A, and we have $\Pi_{A^\vee}$ similarly. There is a canonical isomorphism $M_{A^\vee} \simeq M_A$ as in [30, Section 3.2.4].

Definition 3.1.4 (Canonical pairing, [30, Section 3.2.4])
We have a canonical pairing

$$(\cdot,\cdot)_A \colon \Pi_A \times \Pi_{A^\vee} \to M_A$$

induced by maps

$$(\cdot,\cdot)_U \colon \operatorname{Hom}^0(J_U, A) \times \operatorname{Hom}^0(J_U, A^\vee) \to M_A$$

defined by the assignment $(f_+, f_-) \mapsto \mathrm{vol}(X_U)^{-1} \circ f_+ \circ f_-^\vee \in \operatorname{End}^0(A) = M_A$ for all open compact subgroups U of $\mathbb{B}^{\infty\times}$.

Recall that an Abelian variety A (up to isogeny) over F is *of* GL(2)*-type* if M_A is a field of degree equal to the dimension of A. Let A be such an Abelian variety (up to isogeny), and denote by

$$\omega_A \colon F^\times \backslash \mathbb{A}^{\infty\times} \to M_A^\times$$

the central character associated to A. For a finite place v of F, choose a rational prime ℓ that does not divide v. We have a Galois representation $\rho_{A,v}$ of D_v, the decomposition group at v, on the ℓ-adic Tate module $\mathrm{V}_\ell(A)$ of A, which is a free module over $M_{A,\ell} := M_A \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ of rank 2. It is well known that the characteristic polynomial

$$P_v(T) = \det_{M_{A,\ell}}\bigl(1 - \mathrm{Frob}_v \bigm| \mathrm{V}_\ell(A)^{\mathrm{I}_v}\bigr)$$

belongs to $M_A[T]$ and is independent of ℓ, where $\mathrm{I}_v \subset \mathrm{D}_v$ is the inertia subgroup and $\mathrm{Frob}_v \in \mathrm{D}_v/\mathrm{I}_v$ is the geometric Frobenius.

Remark 3.1.5
We use ω to denote both differential forms and central characters, since both ways are standard. We hope this does not cause any confusion for readers.

Definition 3.1.6 (L-functions and ϵ-factors)
Let K be a field containing M_A.

(1) Define the local L-function of A as $L(s, \rho_{A,v}) := P_v(N_v^{-s-1/2})^{-1} \in M_A \otimes_{\mathbb{Q}} \mathbb{C}$. In a similar manner, we define the local adjoint L-function of A, which we denote as $L(s, \rho_{A,v}, \mathrm{Ad})$; in particular, $L(1, \rho_{A,v}, \mathrm{Ad}) \in M_A$.

(2) For a locally constant character $\chi_v : F_v^\times \to K^\times$, we have the twisted local L-function $L(s, \rho_{A,v} \otimes \chi_v) \in K \otimes_{\mathbb{Q}} \mathbb{C}$. If $\psi : F_v \to K^\times$ is a nontrivial additive character, then we have the ϵ-factor $\epsilon(1/2, \psi, \rho_{A,v} \otimes \chi_v)$.

(3) For a locally constant character $\chi_v : E_v^\times \to K^\times$ such that $\omega_{A,v} \cdot \chi_v|_{F_v^\times} = 1$, we have the local Rankin–Selberg L-function $L(s, \rho_{A,v}, \chi_v) \in K \otimes_{\mathbb{Q}} \mathbb{C}$ and the ϵ-factor $\epsilon(1/2, \rho_{A,v}, \chi_v)$. See Remark 3.1.7 for more details.

(4) Let $\iota : K \hookrightarrow \mathbb{C}$ be an embedding, which induces a homomorphism $\iota : K \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$ by abuse of notation. We define the global L-function of A (with respect to ι) to be

$$L(s, \rho_A^{(\iota)}) := \prod_{v<\infty} \iota L(s, \rho_{A,v}).$$

Similarly, we have the global version $L(s, \rho_A^{(\iota)}, \mathrm{Ad})$ and $L(s, \rho_A^{(\iota)}, \chi^{(\iota)})$ of other L-functions as well.

(5) We say that A is *automorphic* if $L(s, \rho_A^{(\iota)})$, for some and hence all ι, is (the finite component of) the L-function of an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$.

Remark 3.1.7
If v splits into two places v_1 and v_2 of E, then $L(s, \rho_{A,v}, \chi_v)$ is defined to be the product $L(s, \rho_{A,v_1} \otimes \chi_{v_1}) L(s, \rho_{A,v_2} \otimes \chi_{v_2})$. If v induces a single place w of E, then we define $L(s, \rho_{A,v}, \chi_v) := L(s, (\rho_{A,v}|_{\mathrm{D}_w}) \otimes \chi_v)$. By choosing a nontrivial additive character $\psi : F_v \to \mathbb{C}^\times$, we have the local Rankin–Selberg ϵ-factor $\epsilon(s, \psi, \rho_{A,v}, \chi_v)$. It is well known that $\epsilon(1/2, \psi_v, \rho_{A,v}, \chi_v)$ belongs to $\{\pm 1\}$ and does not depend on the choice of ψ. We denote its value by $\epsilon(1/2, \rho_{A,v}, \chi_v)$. The global L-functions $L(s, \rho_A^{(\iota)})$, $L(s, \rho_A^{(\iota)}, \mathrm{Ad})$, and $L(s, \rho_A^{(\iota)}, \chi^{(\iota)})$ are always absolutely convergent for $\mathrm{Re}\, s > 1$.

Remark 3.1.8
It is conjectured that every Abelian variety of GL(2)-type is automorphic. In particular, when $F = \mathbb{Q}$, every Abelian variety of GL(2)-type is parameterized by modular curves. This follows from Serre's modularity conjecture (for $\mathbb{Q}$) [22, Theorem 4.4], where the latter has been proved by Khare and Wintenberger [18].

3.2. p-Adic Rankin–Selberg L-functions for Abelian varieties of GL(2)*-type*

From now on, we fix an Abelian variety A of GL(2)-type over F up to isogeny that is automorphic (Definition 3.1.6(5)) and *equipped with an embedding* $M_A \subset \mathbb{C}_p$. For simplicity, in what follows, we put $M := M_A = M_{A^\vee}$ regarded as a subfield of $\mathbb{C}_p$, and we put $F^M = F \otimes_{\mathbb{Q}} M$, which is naturally equipped with a homomorphism to $\mathbb{C}_p$.

Notation 3.2.1

Denote by $\mathscr{B}(A)$ the (finite) set of isomorphism classes of totally definite incoherent quaternion algebras $\mathbb{B}$ over $\mathbb{A}$ that is E-embeddable (Definition 1.8.4) and such that A can be parameterized by $\mathbb{B}$ (Definition 3.1.3).

For each (representative) $\mathbb{B} \in \mathscr{B}(A)$, we fix an isomorphism $\mathbb{B}_{\mathfrak{p}} \simeq \mathrm{Mat}_2(F_{\mathfrak{p}})$ and an E-embedding (Definition 1.8.4) under which $\mathsf{e}_{\mathfrak{p}}$ coincides with (1.7). Then we have the F-scheme $X = X(\mathbb{B})$ and its closed subscheme $Y = Y^+ \coprod Y^-$ (Definition 2.4.1). We also fix an $\mathbb{A}_E^{\infty\times}$-equivariant isomorphism

$$\mathbf{c}\colon Y^+(\mathbb{C}_p) \xrightarrow{\sim} Y^-(\mathbb{C}_p), \tag{3.1}$$

which we call an *abstract conjugation* for $\mathbb{B}$.

Definition 3.2.2

Denote by $\mathfrak{V}$ the set of open compact subgroups of $\mathbb{A}_E^{\infty\mathfrak{p}\times}$, which is a filtered partially ordered set under inclusion. Let K be a complete field extension of $F_{\mathfrak{p}}$.

(1) A (K-valued) character

$$\chi\colon E^\times\backslash\mathbb{A}_E^{\infty\times} \to K^\times$$

is a *character of weight* $w \in \mathbb{Z}$ if the following hold.

- χ is invariant under some $V^{\mathfrak{p}} \in \mathfrak{V}$.
- There is an open compact subgroup $V_{\mathfrak{p}}$ of $E_{\mathfrak{p}}^\times$ such that $\chi(t) = (t_{\mathfrak{P}}/t_{\mathfrak{P}^{\mathtt{c}}})^w$ for $t \in V_{\mathfrak{p}}$.

We call $V^{\mathfrak{p}}$ the *tame level* of χ.

(2) For a character χ of weight w as above, we define two characters $\check{\chi}_{\mathfrak{P}}$ and $\check{\chi}_{\mathfrak{P}^{\mathtt{c}}}$ of $F_{\mathfrak{p}}^\times$ by the formulas $\check{\chi}_{\mathfrak{P}}(t) = t^{-w}\chi_{\mathfrak{P}}(t)$ and $\check{\chi}_{\mathfrak{P}^{\mathtt{c}}}(t) = t^w\chi_{\mathfrak{P}^{\mathtt{c}}}(t)$.

(3) Suppose that K is contained in $\mathbb{C}_p$. Let χ be a K-valued character of weight w. Given an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we define the following local characters:

- $\chi_v^{(\iota)} = 1$ if $v \mid \infty$ but is not equal to $\iota|_F$;
- $\chi_v^{(\iota)}(z) = (z/\bar{z})^w$ for $v = \iota|_F$, where $z \in E \otimes_{F,\iota} \mathbb{R} \xrightarrow{\iota|_E} \mathbb{C}$;
- $\chi_v^{(\iota)} = \iota\chi_v$ for $v < \infty$ but $v \neq \mathfrak{p}$;

- $\chi_{\mathfrak{p}}^{(\iota)}(t) = \iota(\check{\chi}_{\mathfrak{P}}(t_\circ)\check{\chi}_{\mathfrak{P}^c}(t_\bullet))$ for $t \in E_{\mathfrak{p}}^\times$.

The product $\chi^{(\iota)} := \bigotimes_v \chi_v^{(\iota)} : \mathbb{A}_E^\times \to \mathbb{C}^\times$ is called the *ι-avatar* of χ.

(4) Suppose that K contains M. Denote by $\Xi(A, K)_w$ the set of all K-valued characters of weight w such that
   (a) $\omega_A \cdot \chi|_{\mathbb{A}^{\infty\times}} = 1$;
   (b) $\#\{v < \infty, v \neq \mathfrak{p} \mid \epsilon(1/2, \rho_{A,v}, \chi_v) = -1\} \equiv g - 1 \mod 2$.

   Put $\Xi(A, K) = \bigcup_{\mathbb{Z}} \Xi(A, K)_w$.

Remark 3.2.3
The character $\chi^{(\iota)}$ is automorphic, that is, it factors through $E^\times \backslash \mathbb{A}_E^\times$.

LEMMA 3.2.4
For a character $\chi \in \Xi(A, K)$, there is a unique element $\mathbb{B}_\chi \in \mathcal{B}(A)$ such that $\epsilon(1/2, \rho_{A,v}, \chi_v) = \chi_v(-1)\eta_v(-1)\epsilon(\mathbb{B}_{\chi,v})$ for every finite place $v \neq \mathfrak{p}$ of F.

Proof
The existence of such $\mathbb{B}_\chi$ follows from Definition 3.2.2(4.b). The uniqueness is clear, since $\mathbb{B}_\chi$ is unramified at $\mathfrak{p}$ and the $\epsilon(\mathbb{B}_{\chi,v})$'s are prescribed at all other places v. ☐

The following definition generalizes the discussion in [26, Section 1].

Definition 3.2.5 (Distribution algebra)
Let $K/F_{\mathfrak{p}}$ be a complete field extension that contains M.

(1) For a locally constant character $\omega : F^\times \backslash \mathbb{A}^{\infty\times} \to M^\times$, denote by $\mathcal{C}(\omega, K)$ the locally convex K-vector space of K-valued locally analytic functions f on the locally $F_{\mathfrak{p}}$-analytic group $E^\times \backslash \mathbb{A}_E^{\infty\times}$ satisfying that
   - f is invariant under translation by some $V^{\mathfrak{p}} \in \mathfrak{V}$;
   - $f(xt) = \omega(t)^{-1} f(x)$ for all $x \in E^\times \backslash \mathbb{A}_E^{\infty\times}$ and $t \in F^\times \backslash \mathbb{A}^{\infty\times}$.

   Let $\mathcal{D}(\omega, K)$ be the strong dual of $\mathcal{C}(\omega, K)$ as a topological K-algebra (see Remarks 2.1.1 and 3.2.6).
(2) Define $\mathcal{D}(A, K)$ to be the quotient K-algebra of $\mathcal{D}(\omega_A, K)$ divided by the closed ideal generated by elements that vanish on $\Xi(A, K) \subset \mathcal{C}(\omega_A, K)$.
(3) For $\mathbb{B} \in \mathcal{B}(A)$, define $\mathcal{D}(A, \mathbb{B}, K)$ to be the quotient K-algebra of $\mathcal{D}(\omega_A, K)$ divided by the closed ideal generated by elements that vanish on $\chi \in \Xi(A, K)$ with $\mathbb{B}_\chi \simeq \mathbb{B}$ for $\mathbb{B}_\chi$ as in Lemma 3.2.4.
(4) Define

$$\delta : E^\times \backslash \mathbb{A}_E^{\infty\times} \to \mathcal{D}(\omega_A, K)^\times \to \mathcal{D}(A, K)^\times \to \mathcal{D}(A, \mathbb{B}, K)^\times$$

   to be various continuous homomorphisms given by Dirac distributions.

Remark 3.2.6
The topological K-vector space $\mathcal{D}(\omega, K)$ is a commutative topological K-algebra with the multiplication given by convolution (see [26, Section 1]). For a complete field extension K'/K, we have $\mathcal{D}(\omega, K)\widehat{\otimes}_K K' \simeq \mathcal{D}(\omega, K')$. Moreover, if K is discretely valued, then $\mathcal{D}(\omega, K)$ may be written as a projective limit, indexed by tame levels $V^{\mathfrak{p}} \in \mathfrak{V}$, of nuclear Fréchet–Stein K-algebras with finite étale transition homomorphisms (see Remark 4.4.5) and, thus, complete. We have similar remarks for $\mathcal{D}(A, K)$ and $\mathcal{D}(A, \mathbb{B}, K)$.

Remark 3.2.7
Suppose that $F = M = \mathbb{Q}$ and that $\omega = \mathbf{1}$ is the trivial character. Fix a (sufficiently small) tame level $V^p \in \mathfrak{V}$. Define $\mathcal{C}(\mathbf{1}, \mathbb{Q}_p, V^p)$ similarly to Definition 3.2.5 by requiring that f be invariant under translation by $\mathbb{A}^{\infty\times} V^p$, and define $\mathcal{D}(\mathbf{1}, \mathbb{Q}_p, V^p)$ as the strong dual of $\mathcal{C}(\mathbf{1}, \mathbb{Q}_p, V^p)$ as a topological $\mathbb{Q}_p$-algebra. Then for every complete field extension $K/\mathbb{Q}_p$, there is a natural bijection between continuous characters $\mathcal{D}(\mathbf{1}, \mathbb{Q}_p, V^p) \to K^\times$ and continuous characters $E^\times \mathbb{A}^{\infty\times} \backslash \mathbb{A}_E^{\infty\times}/V^p \to K^\times$. In particular, $\mathcal{D}(\mathbf{1}, \mathbb{Q}_p, V^p)$ is isomorphic to the coordinate ring of a finite disjoint union of open unit disks over $\mathbb{Q}_p$ (compare with Section 2.1). See Remark 4.4.5 for an interpretation in the more general case.

For a representative $\mathbb{B} \in \mathcal{B}(A)$, put $\Omega_{X,Y^\pm} = \Omega_X^1|_{Y^\pm}$. For $t \in \overline{E^\times}\backslash \mathbb{A}_E^{\infty\times}$, there are canonical isomorphisms $\mathrm{T}_t^* \Omega_{X,Y^\pm} \simeq \Omega_{X,Y^\pm}$. Put

$$\omega_{\psi^\pm} = (\Upsilon_\pm^* \omega_\nu)|_{Y^\pm \otimes_{F_{\mathfrak{p}}^{\mathrm{nr}}} F_{\mathfrak{p}}^{\mathrm{lt}} F_{\mathfrak{p}}^{\mathrm{ab}}}, \tag{3.2}$$

where $\Upsilon_\pm$ are in Definition 2.2.10 and ω_ν is the global Lubin–Tate differential in Definition 2.3.6. Then $\omega_{\psi^\pm}$ are sections of $\Omega_{X,Y^\pm} \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^{\mathrm{lt}} F_{\mathfrak{p}}^{\mathrm{ab}}$, respectively, depending only on the additive character ψ (Remark 2.3.7).

Let $MF_{\mathfrak{p}}^{\mathrm{lt}} F_{\mathfrak{p}}^{\mathrm{ab}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Take a character $\chi \in \Xi(A, K)_k$ with $k \geq 0$, and take $\mathbb{B} = \mathbb{B}_\chi$. Define $\sigma_\chi^\pm$ to be the K-subspaces of $\mathrm{H}^0(Y^\pm, \Omega_{X,Y^\pm}^{\otimes -k}) \otimes_F K$ consisting of φ such that $\mathrm{T}_t^* \varphi = \chi(t)^{\pm 1} \varphi$, respectively. By Lemma 2.4.2(1), both σ_χ^+ and σ_χ^- have dimension 1. The abstract conjugation $\mathbf{c}$ from (3.1) induces an $\mathbb{A}_E^{\infty\times}$-invariant bilinear pairing

$$(\cdot,\cdot)_\chi \colon \sigma_\chi^+ \times \sigma_\chi^- \to K$$

by the formula $(\varphi_+, \varphi_-)_\chi = (\varphi_+ \otimes \omega_{\psi^+}^k) \cdot \mathbf{c}^*(\varphi_- \otimes \omega_{\psi^-}^k)$, where the right-hand side is a K-valued constant function on Y^+ and, hence, can be regarded as an element in K.

Put $A^+ = A$, $A^- = A^\vee$, and $\Pi^\pm = \Pi_{A^\pm} \in \mathcal{A}(\mathbb{B}_\chi^\times, \mathbb{Q})$. We have the canonical pairing $(\cdot,\cdot)_A \colon \Pi^+ \times \Pi^- \to M \subset \mathbb{C}_p$ (Definition 3.1.4).

LEMMA 3.2.8

Assume that $k \geq 1$. For every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have a unique $\mathbb{B}^{\infty\times} \times \mathbb{A}_E^{\infty\times}$-invariant bilinear pairing

$$(\cdot,\cdot)^{(\iota)}_{A,\chi}\colon (\Pi^+ \otimes_{FM} \sigma_\chi^+) \times (\Pi^- \otimes_{FM} \sigma_\chi^-) \to (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$$

such that, for every $f_\pm \in \Pi^\pm$, $\varphi_\pm \in \sigma_\chi^\pm$, and $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$, we have

$$\begin{aligned}&\big\langle \omega_+ \otimes \omega_-, (f_+ \otimes \varphi_+, f_- \otimes \varphi_-)^{(\iota)}_{A,\chi}\big\rangle \\ &\qquad = (\iota\varphi_+ \otimes \mathfrak{c}_\iota^* \iota\varphi_- \otimes \mu^k) \int_{X_\iota(\mathbb{C})} \frac{\Theta_\iota^{k-1} f_+^* \omega_+ \otimes \mathfrak{c}_\iota^* \Theta_\iota^{k-1} f_-^* \omega_-}{\mu^k}\, dx, \end{aligned} \tag{3.3}$$

where

- *$\langle\cdot,\cdot\rangle$ is the canonical pairing between $\mathrm{H}^0(A^+, \Omega^1_{A^+}) \otimes_{FM} \mathrm{H}^0(A^-, \Omega^1_{A^-})$ and $\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-$;*
- *μ is an arbitrary Hecke invariant hyperbolic metric on $X_\iota(\mathbb{C})$;*
- *$\mathfrak{c}_\iota$ is the complex conjugation on $\check{X}_\iota$ (2.22);*
- *$\iota\varphi_+ \otimes \mathfrak{c}_\iota^* \iota\varphi_- \otimes \mu^k$ is a constant function on $Y_\iota^+(\mathbb{C})$ and, hence, is viewed as a complex number;*
- *Θ_ι is the Shimura–Maass operator (Definition 2.4.7); and*
- *dx is the Tamagawa measure on $X_\iota(\mathbb{C})$.*

Moreover, there is a unique (nonzero) element $\mathbf{P}_\iota(A,\chi) \in (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$ such that

$$(\cdot,\cdot)^{(\iota)}_{A,\chi} = \mathbf{P}_\iota(A,\chi) \cdot \iota(\cdot,\cdot)_A \otimes \iota(\cdot,\cdot)_\chi.$$

Proof

For given $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$, the formula (3.3) defines a bilinear pairing

$$(\Pi^+ \otimes_{FM} \sigma_\chi^+) \times (\Pi^- \otimes_{FM} \sigma_\chi^-) \to \mathbb{C},$$

which is $\mathbb{B}^{\infty\times} \times \mathbb{A}_E^{\infty\times}$-invariant. By duality, all these pairings for different $\omega_\pm$ give rise to a nonzero pairing

$$(\cdot,\cdot)^{(\iota)}_{A,\chi}\colon (\Pi^+ \otimes_{FM} \sigma_\chi^+) \times (\Pi^- \otimes_{FM} \sigma_\chi^-) \to (\mathrm{Lie}\, A^+ \otimes_{\mathbb{Q}} \mathrm{Lie}\, A^-) \otimes_{F,\iota} \mathbb{C},$$

and it is easy to see that $(\cdot,\cdot)^{(\iota)}_{A,\chi}$ takes values in $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$. The existence of $\mathbf{P}_\iota(A,\chi)$ follows from the uniqueness of the Petersson inner product and the fact that $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$ is a $\mathbb{C}$-vector space of dimension 1. $\square$

Remark 3.2.9

The element $\mathbf{P}_\iota(A,\chi)$ can be viewed as a function on the set $\bigcup_{k\geq 1} \Xi(A,K)_k$ valued in the 1-dimensional $\mathbb{C}$-vector space $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$. It depends on the choices of $\mathbf{c}$ and ψ.

THEOREM 3.2.10
There is a unique element

$$\mathscr{L}(A) \in (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathfrak{D}(A, MF_{\mathfrak{p}}^{\mathrm{lt}})$$

such that, for every character $\chi \in \Xi(A, K)_k$ *with* $k \geq 1$ *and* $MF_{\mathfrak{p}}^{\mathrm{lt}} F_{\mathfrak{p}}^{\mathrm{ab}} \subset K \subset \mathbb{C}_p$ *a complete intermediate field and for every* $\iota \colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, *we have*

$$\iota \mathscr{L}(A)(\chi) = L(1/2, \rho_A^{(\iota)}, \chi^{(\iota)}) \cdot \frac{2^{g-1} \mathrm{d}_E^{1/2} \zeta_F(2) \mathbf{P}_\iota(A, \chi)}{L(1, \eta)^2 L(1, \rho_A^{(\iota)}, \mathrm{Ad})} \cdot \iota\Big(\frac{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2} \Big) \tag{3.4}$$

as an equality in $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM,\iota} \mathbb{C}$.

Remark 3.2.11
The element $\mathscr{L}(A)$ depends only on the choices of (1) an additive character ψ of $F_{\mathfrak{p}}$ of level 0 and (2) the abstract conjugation $\mathbf{c}$ from (3.1) for *each* (representative) $\mathbb{B} \in \mathscr{B}(A)$, in an elementary way. More precisely,

(1) if we change ψ to ψ_a for some $a \in O_{\mathfrak{p}}^\times$, where $\psi_a(x) = \psi(ax)$ for $x \in F_{\mathfrak{p}}$, then $\mathscr{L}(A)$ is multiplied by $\omega_{\pi_{\mathfrak{p}}}(a) \cdot \delta_a^2$, where a is regarded at the place $\mathfrak{P}^c$ in the Dirac distribution δ_a;

(2) if we write $\mathscr{L}(A) = \{\mathscr{L}(A, \mathbb{B})\}_{\mathbb{B} \in \mathscr{B}(A)}$ under the canonical isomorphism $\mathfrak{D}(A, K) \simeq \prod_{\mathbb{B} \in \mathscr{B}(A)} \mathfrak{D}(A, \mathbb{B}, K)$ (Remark 4.4.4) and change $\mathbf{c}$ (for $\mathbb{B}$) to $\mathbf{c}' = \mathrm{T}_t \circ \mathbf{c}$ for some $t \in \mathbb{A}_E^{\infty\times}$, then the component $\mathscr{L}(A, \mathbb{B})$ is multiplied by δ_t (Definition 3.2.5(4)).

3.3. *p-Adic Waldspurger formula*

Let K be a complete field extension of M. Consider an element $\chi \in \Xi(A, K)_0$. We take $\mathbb{B} = \mathbb{B}_\chi \in \mathscr{B}(A)$. Choose a CM point $P^+ \in Y^+(E^{\mathrm{ab}}) = Y^+(\mathbb{C}_p)$, and put $P^- = \mathbf{c}P^+$.

Definition 3.3.1
For every $f_\pm \in \Pi^\pm$, we define the *Heegner cycles* $P_\chi^\pm(f_\pm)$ on $A^\pm$ to be

$$P_\chi^\pm(f_\pm) = \int_{\overline{E^\times} \backslash \mathbb{A}_E^{\infty\times}} f_\pm(\mathrm{T}_t P^\pm) \otimes_M \chi(t)^{\pm 1}\, dt,$$

a finite sum in fact. Here, we recall that T_t is the Hecke morphism (Notation 2.2.1), and we adopt the Haar measure dt of total volume 2.

Suppose now that K contains $MF_{\mathfrak{p}}^{\mathrm{ab}}$. We have K-linear maps

$$\log_{A^\pm}\colon A^\pm(K)\otimes_M K \to \mathrm{Lie}\, A^\pm \otimes_{FM} K$$

given by p-adic logarithms on $A^\pm$ (see, e.g., [3]). As a functional on $\Pi^+\times\Pi^-$, the product $\log_{A^+} P_\chi^+(f_+)\cdot\log_{A^-} P_\chi^-(f_-)$ defines an element in the 1-dimensional K-vector space

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^+\otimes\chi, K)\otimes_K \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^-\otimes\chi^{-1}, K)\otimes_{FM}(\mathrm{Lie}\, A^+\otimes_{FM}\mathrm{Lie}\, A^-).$$

It depends on the choice of $\mathbf{c}$ but not on the choice of P^+.

THEOREM 3.3.2 (p-adic Waldspurger formula)
There exists a unique element

$$\alpha_\chi(\cdot,\cdot)\in \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^+\otimes\chi, K)\otimes_K \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^-\otimes\chi^{-1}, K)$$

such that, for every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\alpha_\chi(f_+,f_-)=\alpha^\natural(f_+,f_-;\chi^{(\iota)})$$

for every $f_\pm\in\Pi^\pm$, *where the right-hand side is the (normalized) matrix coefficient integral appearing in the complex Waldspurger formula (which will be recalled in Definition 4.1.4). Moreover, for a character* $\chi\in\Xi(A,K)_0$, *we have*

$$\log_{A^+} P_\chi^+(f_+)\cdot\log_{A^-} P_\chi^-(f_-)=\mathscr{L}(A)(\chi)\cdot\frac{L(1/2,\rho_{A,\mathfrak{p}}\otimes\chi_{\mathfrak{P}^c})^2}{\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\chi_{\mathfrak{P}^c})}\cdot\alpha_\chi(f_+,f_-)$$

for every $f_\pm\in\Pi^\pm$.

3.4. p-Adic Maass functions and alternative formulation

Let $\mathbb{B}$ be an arbitrary totally definite incoherent quaternion algebra over $\mathbb{A}$. As in Section 2.2, we have $X(\mathbb{B})=\varprojlim_U X(\mathbb{B})_U$ as the projective limit of Shimura curves associated to $\mathbb{B}$ over $\mathrm{Spec}\, F$. The following definition generalizes the one in Section 1.2.

Definition 3.4.1 (p-adic Maass function)
We say that a function $\phi\colon X(\mathbb{B})(\mathbb{C}_p)\to\mathbb{C}_p$ is a *p-adic Maass function* on $X(\mathbb{B})$ if it is the pullback of some locally analytic function $X(\mathbb{B})_U(\mathbb{C}_p)\to\mathbb{C}_p$. Denote by $\mathcal{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ the $\mathbb{C}_p$-vector space of all p-adic Maass functions on $X(\mathbb{B})$. It is a representation of $\mathbb{B}^{\infty\times}$.

We go back to the setting in Section 3.2, where we have fixed an Abelian variety A of GL(2)-type over F up to isogeny that is automorphic and equipped with an embedding $M = M_A \subset \mathbb{C}_p$. Denote by $\pi(\mathbb{B})_A^{\mathrm{rat}}$ the subspace of $\mathcal{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ spanned by functions of the form

$$f^* \log_\omega\colon X(\mathbb{B})(\mathbb{C}_p) \xrightarrow{f} A(\mathbb{C}_p) \xrightarrow{\log_\omega} \mathbb{C}_p,$$

where $f: X(\mathbb{B}) \to A$ is a nonconstant map, ω is a differential form on $A \otimes_{\mathbb{Q}} \mathbb{C}_p$, and $\log_\omega = \langle \log_A, \omega \rangle$. The subspace $\pi(\mathbb{B})_A^{\mathrm{rat}}$ is a subrepresentation of $\mathbb{B}^{\infty\times}$, which also receives an action of M by acting on A. Denote by $\pi(\mathbb{B})_A$ the subspace of $\pi(\mathbb{B})_A^{\mathrm{rat}}$ on which M acts via the default embedding $M \subset \mathbb{C}_p$, which is again a subrepresentation of $\mathbb{B}^{\infty\times}$.

LEMMA 3.4.2
Suppose that $\mathbb{B}$ belongs to $\mathcal{B}(A)$ (Notation 3.2.1). For every nonzero differential form $\omega \in \mathrm{H}^0(A, \Omega_A^1)$, the map

$$\varsigma_\omega\colon \Pi(\mathbb{B})_A \to \pi(\mathbb{B})_A$$

sending f to $f^* \log_\omega |_{X(\mathbb{B})(\mathbb{C}_p)}$ is $\mathbb{B}^{\infty\times}$-equivariant and M-linear, and the induced map $\Pi(\mathbb{B})_A \otimes_M \mathbb{C}_p \to \pi(\mathbb{B})_A$ is an isomorphism.

Proof
It follows directly from the definition that ς_ω is $\mathbb{B}^{\infty\times}$-equivariant and M-linear. To show the isomorphism, it suffices to show that $\pi(\mathbb{B})_A$ is a nonzero irreducible representation of $\mathbb{B}^{\infty\times}$. Since $\mathbb{B}$ belongs to $\mathcal{B}(A)$, the space $\pi(\mathbb{B})_A^{\mathrm{rat}}$ is nonzero and, hence, so is $\pi(\mathbb{B})_A$.

For the irreducibility, we choose an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. Consider the map $\pi(\mathbb{B})_A \otimes_{\mathbb{C}_p,\iota} \mathbb{C} \to \mathcal{A}_{\mathrm{cusp}}(B(\iota)^\times)$ sending $f^* \log_\omega$ to $\iota f^* \omega$ regarded as a weight 2 holomorphic cusp form on $B(\iota)^\times(\mathbb{A})$. The map is well defined, injective, and $\mathbb{B}^{\infty\times}$-equivariant. Its image coincides with the weight 2 subspace of the cuspidal automorphic representation of $B(\iota)^\times(\mathbb{A})$ determined by A and the embedding $\iota\colon M \subset \mathbb{C}$ (see [30, Theorem 3.3.2]). It follows that the image is irreducible as a representation of $\mathbb{B}^{\infty\times}$. Therefore, $\pi(\mathbb{B})_A$ itself is an irreducible representation of $\mathbb{B}^{\infty\times}$. □

From now on, we fix a representative $\mathbb{B}$ in $\mathcal{B}(A)$, and we will prove a p-adic Waldspurger formula for p-adic Maass functions on $X := X(\mathbb{B})$ contained in $\pi(\mathbb{B})_A$. Take two nonzero differential forms $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$. By Lemma 3.4.2, we have isomorphisms

$$\varsigma_{\omega_\pm}\colon \Pi(\mathbb{B})_{A^\pm} \otimes_M \mathbb{C}_p \xrightarrow{\sim} \pi(\mathbb{B})_{A^\pm}.$$

Let $\chi \in \Xi(A, \mathbb{C}_p)_0$ be a character such that $\mathbb{B}_\chi \simeq \mathbb{B}$. For $\phi_\pm \in \pi(\mathbb{B})_{A^\pm}$, we put

$$\alpha_\chi^{\omega_+,\omega_-}(\phi_+,\phi_-) = \alpha_\chi(\varsigma_{\omega_+}^{-1}\phi_+, \varsigma_{\omega_-}^{-1}\phi_-) \in \mathbb{C}_p,$$

where $\alpha_\chi(\cdot,\cdot)$ is the pairing in Theorem 3.3.2. Then $\alpha_\chi^{\omega_+,\omega_-}(\cdot,\cdot)$ is a basis of the 1-dimensional space

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\big(\pi(\mathbb{B})_{A^+} \otimes \chi, \mathbb{C}_p\big) \otimes_{\mathbb{C}_p} \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}\big(\pi(\mathbb{B})_{A^-} \otimes \chi^{-1}, \mathbb{C}_p\big). \tag{3.5}$$

Globally, we have the following definition. Choose a CM point $P^+ \in Y^+(E^{\mathrm{ab}}) = Y^+(\mathbb{C}_p)$, and put $P^- = \mathbf{c}P^+$ as in Section 3.3.

Definition 3.4.3
For $\phi_\pm \in \pi(\mathbb{B})_{A^\pm}$ and $\chi \in \Xi(A, \mathbb{C}_p)_0$, we define the *$p$-adic torus period* to be

$$\mathscr{P}_{\mathbb{C}_p}(\phi_\pm, \chi^{\pm 1}) := \int_{\overline{E^\times}\backslash \mathbb{A}_E^{\infty\times}} \phi_\pm(\mathrm{T}_t P^\pm) \cdot \chi(t)^{\pm 1}\, dt,$$

where the Haar measure dt has total volume 2 as in Definition 3.3.1.

The above integrals are, in fact, finite sums valued in $\mathbb{C}_p$. The product $\mathscr{P}_{\mathbb{C}_p}(\cdot,\chi)\cdot \mathscr{P}_{\mathbb{C}_p}(\cdot,\chi^{-1})$ defines another element in (3.5), which depends on the choice of $\mathbf{c}$ but not on the choice of P^+. In particular, it is proportional to $\alpha_\chi^{\omega_+,\omega_-}(\cdot,\cdot)$.

The following theorem is the p-adic Waldspurger formula for p-adic Maass functions. Recall that we have the p-adic L-function $\mathscr{L}(A)$ from Theorem 3.2.10. Put $\pi = \pi(\mathbb{B})_A$ as an irreducible subrepresentation of $\mathcal{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$.

THEOREM 3.4.4 (p-adic Waldspurger formula for p-adic Maass functions)
Put

$$\mathscr{L}_{\omega_+,\omega_-}(\pi) = \big\langle \omega_+ \otimes \omega_-, \mathscr{L}(A)\big\rangle,$$

regarded as an element in $\mathscr{D}(A, \mathbb{C}_p)$. Then for a character $\chi \in \Xi(A, \mathbb{C}_p)_0$, we have

$$\mathscr{P}_{\mathbb{C}_p}(\phi_+,\chi)\mathscr{P}_{\mathbb{C}_p}(\phi_-,\chi^{-1}) = \mathscr{L}_{\omega_+,\omega_-}(\pi)\cdot \frac{L(1/2, \pi_{\mathfrak{p}} \otimes \chi_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \pi_{\mathfrak{p}} \otimes \chi_{\mathfrak{P}^c})} \cdot \alpha_\chi^{\omega_+,\omega_-}(\phi_+,\phi_-)$$

for every $\phi_\pm \in \pi(\mathbb{B})_{A^\pm}$.

Proof
It follows from Theorem 3.3.2, after pairing with $\omega_+ \otimes \omega_-$. ☐

Remark 3.4.5
In this remark, we explain how to deduce Theorems 1.5.1 and 1.5.3. We take A to be an elliptic curve over $\mathbb{Q}$. In particular, we have that $F = M = \mathbb{Q}$, $A^+ = A^- = A$ and that $\omega_A = \mathbf{1}$ is the trivial character. We also *fix* an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. We have the indefinite quaternion algebra B over $\mathbb{Q}$. Take $\mathbb{B} \in \mathcal{B}(A)$ such that $\mathbb{B}^\infty \simeq B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$. So we may identify B with $B(\iota)$ in the ι-nearby data for $\mathbb{B}$ (Definition 2.4.10). Moreover, we take $\psi\colon \mathbb{Q}_p \to \mathbb{C}_p^\times$ to be the additive character such that $\iota \circ \psi$ is the standard one. We choose the abstract conjugation $\mathbf{c}$ from (3.1) such that $\mathbf{c} \otimes_{\mathbb{C}_p,\iota} \mathbb{C}$ coincides with the restriction of the complex conjugation on $\check{X}_\iota$. We also note that $\mathcal{D}(G)$ is simply $\mathcal{D}(\mathbf{1}, \mathbb{C}_p)$; and $\mathcal{D}(G; \pi_{\mathbb{C}_p})$ is simply $\mathcal{D}(A, \mathbb{B}, \mathbb{C}_p)$ (Definition 3.2.5).

(1) We first deduce Theorem 1.5.1. Take the p-adic L-function $\mathcal{L}(A)$ as in Theorem 3.2.10, regarded as an element in $(\mathrm{Lie}\, A)^{\otimes 2} \otimes_{\mathbb{Q}} \mathcal{D}(A, \mathbb{C}_p)$. Take a basis ω of $\mathrm{H}^0(A, \Omega^1_A)$. Then there is a unique element $\mathbf{P}_\omega \in \mathbb{C}_p^\times$ such that $\iota(\mathbf{P}_\omega^{-1}(f_1, f_2)_A)$ is equal to the (bilinear) Petersson inner product of $\phi_\iota(f_1^*\omega)$ and $\mathrm{R}(\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)_\infty)\phi_\iota(f_2^*\omega)$ for every $f_1, f_2 \in \Pi_A$. Now we define $\mathcal{L}(\pi_{\mathbb{C}_p})$ to be the image of

$$\mathbf{P}_\omega \cdot \iota^{-1}\big(\mathrm{d}_E^{-1/2} L(1,\eta)\big) \cdot \big\langle \omega \otimes \omega, \mathcal{L}(A)\big\rangle$$

under the canonical projection $\mathcal{D}(A, \mathbb{C}_p) \to \mathcal{D}(A, \mathbb{B}, \mathbb{C}_p) = \mathcal{D}(G; \pi_{\mathbb{C}_p})$. It is clear that $\mathcal{L}(\pi_{\mathbb{C}_p})$ does not depend on the choice of ω and, hence, is well defined. Then Theorem 1.5.1 follows from Theorem 3.2.10, Remark 1.1.2, and Lemma 3.4.6 below (with $r = 2$).

(2) Now we deduce Theorem 1.5.3. In Definition 3.4.3, we choose P^+ such that $\iota P^+ = [+i, 1]$, and thus, $\iota P^- = [-i, 1]$. Then $\mathcal{P}_{\mathbb{C}_p}(\cdot, \chi^{\pm 1})$ in Definition 3.4.3 coincide with those in (1.5). Therefore, Theorem 1.5.3 follows from Theorem 3.4.4.

LEMMA 3.4.6
Let π be the discrete series representation of weight $r \geq 2$ of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character. Fix a nonzero $\mathrm{GL}_2(\mathbb{R})$-equivariant bilinear pairing $(\cdot,\cdot)\colon \pi \times \pi \to \mathbb{C}$. Let $f_+ \in \pi$ be a generator of weight r, that is, the Archimedean component of holomorphic modular forms of weight r. Put $f_- = \pi(\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)) f_+$, which is a generator of weight $-r$. Then we have

$$\frac{(\Delta_+^k f_+, \Delta_-^k f_-)}{(f_+, f_-)} = \frac{k!(k+r-1)!}{4^k (r-1)!},$$

where $\Delta_\pm$ are as in Definition 2.4.16.

Proof
It is well known that $(f_+, f_-) \neq 0$. Put

$$X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} = 2i\,\Delta_\pm, \qquad H = -i\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Recall from [5, p. 157, (2.39)] the Casimir element

$$\Delta = -\frac{1}{4}(2X_+X_- + 2X_-X_+ + H^2).$$

It acts on π by the scalar

$$\lambda_r := \frac{r}{2}\Big(1 - \frac{r}{2}\Big).$$

We say that a vector $g \in \pi$ has weight μ if $Hg = \mu g$. For such g, we have

$$X_-X_+g = -\Big(\frac{H^2 + 2H}{4} + \Delta\Big)g = -\Big(\frac{\mu^2 + 2\mu}{4} + \lambda_r\Big)g.$$

Now for each $k \geq 0$, the vector $X_+^k f_+$ is of weight $r + 2k$. Therefore, we have for $k \geq 1$ the formula

$$X_-X_+^k f_+ = X_-X_+(X_+^{k-1} f_+) = -\Big(\frac{(r + 2k - 2)(r + 2k)}{4} + \lambda_r\Big)X_+^{k-1} f_+.$$

We prove the identity

$$\frac{(X_+^k f_+, X_-^k f_-)}{(f_+, f_-)} = \frac{k!(k + r - 1)!}{(r - 1)!}$$

by induction on $k \geq 0$. The case $k = 0$ is trivial. Suppose that we know this for $k - 1$. Then we have

$$\begin{aligned}(X_+^k f_+, X_-^k f_-) &= -(X_-X_+^k f_+, X_-^{k-1} f_-)\\ &= \Big(\frac{(r + 2k - 2)(r + 2k)}{4} + \lambda_r\Big) \cdot (X_+^{k-1} f_+, X_-^{k-1} f_-)\\ &= k(k + r - 1) \cdot (X_+^{k-1} f_+, X_-^{k-1} f_-).\end{aligned}$$

The lemma follows as $X_\pm = 2i\Delta_\pm$. □

4. Proofs of main theorems

This section is dedicated to the proofs of Theorems 3.2.10 and 3.3.2. In Section 4.1, we construct the distribution interpolating matrix coefficient integrals appearing in the complex Waldspurger formula. We construct the universal torus period in Section 4.2, which is a crucial construction toward the p-adic L-function. In Section 4.3, we study the relation between universal torus periods and classical torus periods, based on which we complete the proofs of our main theorems in Section 4.4.

4.1. Distribution of matrix coefficient integrals

Recall that we have fixed an Abelian variety A of GL(2)-type over F up to isogeny that is automorphic, *equipped with an embedding* $M = M_A \subset \mathbb{C}_p$, as in Section 3.2. Let $K/MF_{\mathfrak{p}}$ be a complete field extension. Take a representative $\mathbb{B}$ in $\mathscr{B}(A)$. As in Section 3.2, we fix an isomorphism $\mathbb{B}_{\mathfrak{p}} \simeq \mathrm{Mat}_2(F_{\mathfrak{p}})$ and an E-embedding under which $\mathsf{e}_{\mathfrak{p}}$ coincides with (1.7). Recall that we put $\Pi^{\pm} = \Pi_{A^{\pm}} = \Pi(\mathbb{B})_{A^{\pm}}$ (Notation 3.1.2).

Definition 4.1.1 (Stable/admissible vector)
We say that elements $f_{\pm}$ in $\Pi^{\pm} \otimes_M K$ or $\Pi^{\pm}_{\mathfrak{p}} \otimes_M K$ are *stable vectors* if, respectively,

(1) $f_{\pm}$ are fixed by $\mathrm{N}^{\pm}(O_{\mathfrak{p}})$;
(2) $f_{\pm}$ satisfy the relation

$$\sum_{g \in \mathrm{N}^{\pm}(\mathfrak{p}^{-1})/\mathrm{N}^{\pm}(O_{\mathfrak{p}})} \Pi^{\pm}_{\mathfrak{p}}(g) f_{\pm} = 0.$$

We denote by $(\Pi^{\pm})^{\heartsuit}_K$ the subsets of $\Pi^{\pm} \otimes_M K$ consisting of stable vectors, respectively. We denote by $(\Pi^{\pm}_{\mathfrak{p}})^{\heartsuit}_K$ the subsets of $\Pi^{\pm}_{\mathfrak{p}} \otimes_M K$ consisting of stable vectors, respectively.

For $n \in \mathbb{N}$, we say that stable vectors $f_{\pm}$ in $(\Pi^{\pm})^{\heartsuit}_K$ or $(\Pi^{\pm}_{\mathfrak{p}})^{\heartsuit}_K$ are *n-admissible* if, respectively,

$$\Pi^{\pm}_{\mathfrak{p}}\big(\mathrm{n}^{\pm}(x)\big) f_{\pm} = \psi^{\pm}(x) f_{\pm}$$

for every $x \in \mathfrak{p}^{-n}/O_{\mathfrak{p}}$, where $\mathrm{n}^{\pm}(x)$ are the same as in Proposition 2.3.5.

Remark 4.1.2
If we realize $\Pi^{\pm}_{\mathfrak{p}}$ in their Kirillov models with respect to the pair $(\mathrm{N}^{+}, \psi^{\pm})$, then $f_{\pm\mathfrak{p}}$ belong to $(\Pi^{\pm}_{\mathfrak{p}})^{\heartsuit}_K$ if and only if $f_{+\mathfrak{p}}$ (resp., $\Pi^{-}_{\mathfrak{p}}(\mathrm{J}) f_{-\mathfrak{p}}$) is supported on $O^{\times}_{\mathfrak{p}}$, and they are n-admissible if and only if $f_{+\mathfrak{p}}$ (resp., $\Pi^{-}_{\mathfrak{p}}(\mathrm{J}) f_{-\mathfrak{p}}$) is supported on $(1+\mathfrak{p}^n)^{\times}$.

Definition 4.1.3
Let $w \in \mathbb{Z}$, let $n \in \mathbb{N}$ be integers, and let $\omega\colon F^{\times}\backslash\mathbb{A}^{\infty\times} \to M^{\times}$ be a locally constant character. We say that a K-valued character $\chi\colon E^{\times}\backslash\mathbb{A}^{\infty\times}_E \to K^{\times}$ of weight w is *of central type ω and depth n* if

- $\omega \cdot \chi|_{\mathbb{A}^{\infty\times}} = 1$; and
- $\chi_{\mathfrak{P}^c}(t) = t^{-w}$ for all $t \in (1+\mathfrak{p}^n)^{\times}$.

We denote by $\Xi(\omega, K)^n_w$ the set of all K-valued characters of weight w, central type ω, and depth n. Moreover, put $\Xi(\omega, K)^n = \bigcup_{\mathbb{Z}} \Xi(\omega, K)^n_w$.

We recall the definition of the classical (normalized) matrix coefficient integral. Suppose that K is contained in $\mathbb{C}_p$. We take a character $\chi \in \Xi(\omega_A, K)$. Let $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. Morally speaking, the integral should be defined as

$$\alpha^{\natural}(f_+, f_-; \chi^{(\iota)}) \text{ "=" } \int_{\mathbb{A}^{\infty\times}\backslash\mathbb{A}_E^{\infty\times}} \iota\big(\Pi(t) f_+, f_-\big)_A \cdot \chi^{(\iota)}(t)\, dt.$$

However, it is not absolutely convergent, so we need regularization recalled as follows.

Definition 4.1.4 (Regularized matrix coefficient integral)
Take an arbitrary decomposition $\iota(\cdot,\cdot)_A = \prod_{v<\infty} (\cdot,\cdot)_{\iota,v}$, where $(\cdot,\cdot)_{\iota,v}\colon \Pi_v^+ \times \Pi_v^- \to \mathbb{C}$ is a $\mathbb{B}_v^\times$-invariant bilinear pairing. For $f_\pm = \bigotimes_{v<\infty} f_{\pm v}$ such that $(f_{+v}, f_{-v})_{\iota,v} = 1$ for all but finitely many v's, we put

$$\alpha(f_{+v}, f_{-v}; \chi_v^{(\iota)}) = \int_{F_v^\times\backslash E_v^\times} \big(\Pi_v(t) f_{+v}, f_{-v}\big)_{\iota,v} \chi_v^{(\iota)}(t)\, dt;$$

$$\alpha^{\natural}(f_{+v}, f_{-v}; \chi_v^{(\iota)}) = \Big(\frac{\zeta_{F_v}(2) L(1/2, \rho_{A,v}^{(\iota)}, \chi_v^{(\iota)})}{L(1,\eta_v) L(1, \rho_{A,v}^{(\iota)}, \mathrm{Ad})}\Big)^{-1} \alpha(f_{+v}, f_{-v}; \chi_v^{(\iota)}).$$

Here, dt is the measure on $F_v^\times\backslash E_v^\times$ given in Section 1.8, and $\rho_{A,v}^{(\iota)}$ is the corresponding admissible complex representation of $\mathbb{B}_v^\times$ via ι. Then by [29, Section 3] we have $\alpha^{\natural}(f_{+v}, f_{-v}; \chi_v^{(\iota)}) = 1$ for all but finitely many v's, and the product

$$\alpha^{\natural}(f_+, f_-; \chi^{(\iota)}) := \prod_{v<\infty} \alpha^{\natural}(f_{+v}, f_{-v}; \chi_v^{(\iota)})$$

is well defined. We extend the functional $\alpha^{\natural}(\cdot,\cdot; \chi^{(\iota)})$ to all f_+, f_- by linearity.

Remark 4.1.5
The functional $\alpha^{\natural}(\cdot,\cdot; \chi^{(\iota)})$ does not depend on the choice of the decomposition of $\iota(\cdot,\cdot)_A$.

The following proposition is our main result, whose proof will be given at the end of this section. Note that, since $\Xi(\omega, K)^n$ is a subset of $\mathscr{C}(\omega, K)$, we have a natural pairing $\mathscr{D}(\omega, K) \times \Xi(\omega, K)^n \to K$.

PROPOSITION 4.1.6
Let $MF_{\mathfrak{p}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Let $f_\pm \in (\Pi^\pm)_K^\heartsuit$ be two n-admissible stable vectors for some (common) $n \in \mathbb{N}$. Then there is a unique element

$\mathcal{Q}(f_+, f_-) \in \mathcal{D}(\omega_A, K)$ *such that, for all K-valued characters $\chi \in \Xi(\omega_A, K)^n$ of central type ω_A and depth n and for $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have*

$$\iota\mathcal{Q}(f_+, f_-)(\chi) = \iota\Big(\frac{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}\Big) \cdot \alpha^\natural(f_+, f_-; \chi^{(\iota)}).$$

Definition 4.1.7
The element $\mathcal{Q}(f_+, f_-)$ is called the *(K-valued) local period distribution.*

Before giving the proof, we make a convenient choice of a decomposition of $(\cdot,\cdot)_A$. Realize the representations $\Pi^\pm_\mathfrak{p}$ in their Kirillov models as in Remark 4.1.2. We may assume that $f_\pm = \bigotimes f_{\pm v}$, with $f_{\pm v} \in \Pi^\pm_v \otimes_M K$, are decomposable and are fixed by some (common) sufficiently small open compact subgroup $V^\mathfrak{p} \in \mathfrak{V}$. Choose a decomposition $(\cdot,\cdot)_A = \prod_{v<\infty} (\cdot,\cdot)_v$ such that

(1) $(f_{+v}, f_{-v})_v = 1$ for all but finitely many v's;
(2) $(f'_{+v}, f'_{-v})_v \in K$ for all $f'_{\pm v} \in \Pi^\pm_v \otimes_M K$;
(3) for $f'_{\pm\mathfrak{p}} \in \Pi^\pm_\mathfrak{p} \otimes_M K$ that are compactly supported on $F^\times_\mathfrak{p}$,

$$(f'_{+\mathfrak{p}}, f'_{-\mathfrak{p}})_\mathfrak{p} = \int_{F^\times_\mathfrak{p}} f'_{+\mathfrak{p}}(a) f'_{-\mathfrak{p}}(a)\, da,$$

where da is the Haar measure on $F^\times_\mathfrak{p}$ such that the volume of $O^\times_\mathfrak{p}$ is 1.

We need two lemmas for the proof of Proposition 4.1.6. For simplicity, write $\omega = \omega_A$. For each finite place $v \neq \mathfrak{p}$ and an open compact subgroup V_v of $E^\times_v$, let $\mathcal{D}(\omega_v, K, V_v)$ be the quotient K-algebra of $D(E^\times_v/V_v, K)$ divided by the closed ideal generated by $\{\omega_v(t)\delta_t - 1 \mid t \in F^\times_v\}$. Put

$$\mathcal{D}(\omega_v, K) = \varprojlim_{V_v} \mathcal{D}(\omega_v, K, V_v),$$

where the limit runs over all V_v's. Let $\mathcal{D}(\omega_\mathfrak{p}, K)$ be the quotient of $D(E^\times_\mathfrak{p}, K)$ by the closed ideal generated by $\{\omega_\mathfrak{p}(t)\delta_t - 1 \mid t \in F^\times_\mathfrak{p}\}$. For every finite place v, we have a natural homomorphism $\mathcal{D}(\omega_v, K) \to \mathcal{D}(\omega, K)$.

LEMMA 4.1.8
Let $v \neq \mathfrak{p}$ be a finite place of F.
(1) *There exists a unique element*

$$\mathcal{L}^{-1}(\rho_{A,v}) \in \mathcal{D}(\omega_v, MF_\mathfrak{p})$$

such that, for every locally constant character $\chi_v\colon E^\times_v \to K^\times$ satisfying $\omega_v \cdot \chi_v|_{F^\times_v} = 1$, we have

$$\mathcal{L}^{-1}(\rho_{A,v})(\chi_v) = L(1/2, \rho_{A,v}, \chi_v)^{-1}.$$

(2) *For $f_{\pm v} \in \Pi_v^{\pm} \otimes_M K$, there exists a unique element*

$$\mathcal{Q}(f_{+v}, f_{-v}) \in \mathcal{D}(\omega_v, K)$$

such that, for every locally constant character $\chi_v \colon E_v^\times \to K^\times$ satisfying $\omega_v \cdot \chi_v|_{F_v^\times} = 1$ and for $\iota \colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have

$$\iota\mathcal{Q}(f_{+v}, f_{-v})(\chi_v) = \alpha^\natural(f_{+v}, f_{-v}; \chi_v^{(\iota)}).$$

Proof

The uniqueness is clear. In the following proof, we suppress v from the notation, and we will use the subscript ι for all changing of coefficients of representations via ι.

To prove (1), we first consider the following situation. Let $\tilde{F}$ be either F or E, and let $\tilde{\Pi}$ be an irreducible admissible M-representation of $\mathrm{GL}_2(\tilde{F})$. We claim that there is a (unique) element $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) \in D_\flat(\tilde{F}^\times, MF_\mathfrak{p})$, where

$$D_\flat(\tilde{F}^\times, K) := \varprojlim_V D(\tilde{F}^\times/V, K)$$

with V running over all open compact subgroups of $\tilde{F}^\times$ such that, for every locally constant character $\chi \colon \tilde{F}^\times \to K^\times$ and $\iota \colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi})(\chi) = L(1/2, \tilde{\Pi}_\iota \otimes \chi_\iota)^{-1}.$$

In fact, for a locally constant character $\mu \colon \tilde{F}^\times \to M^\times$, define $\mathcal{L}_{\tilde{F}}^{-1}(\mu) \in D_\flat(\tilde{F}^\times, MF_\mathfrak{p})$ by the formula

$$\mathcal{L}_{\tilde{F}}^{-1}(\mu)(h) = 1 - \int_{O_{\tilde{F}}^\times} \mu(\tilde{\varpi}a)h(\tilde{\varpi}a)\,da$$

for $h \in \varinjlim_V C(\tilde{F}^\times/V, MF_\mathfrak{p})$. Here, $\tilde{\varpi}$ is an arbitrary uniformizer of $\tilde{F}$, and da is the Haar measure on $O_{\tilde{F}}^\times$ with total volume 1. Then we have three cases.

- If $\tilde{\Pi}$ is supercuspidal, put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = 1$.
- If $\tilde{\Pi}$ is the unique irreducible subrepresentation of the unnormalized parabolic induction of $(\mu, \mu|\cdot|^{-2})$ for a character $\mu \colon \tilde{F}^\times \to M^\times$, then we put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = \mathcal{L}_{\tilde{F}}^{-1}(\mu)$.
- If $\tilde{\Pi}$ is the irreducible unnormalized parabolic induction of $(\mu^1, \mu^2|\cdot|^{-1})$ for a pair of characters $\mu^i \colon \tilde{F}^\times \to M^\times$ $(i = 1, 2)$, then we put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = \mathcal{L}_{\tilde{F}}^{-1}(\mu^1) \cdot \mathcal{L}_{\tilde{F}}^{-1}(\mu^2)$.

Here, we adopt the unnormalized induction in order to track the rationality properties.

Go back to (1). First, assume that E/F is nonsplit. Then we define $\mathcal{L}^{-1}(\rho_A)$ to be the image of $\mathcal{L}_E^{-1}(\Pi_E)$ in $\mathcal{D}(\omega, MF_\mathfrak{p})$, where Π_E is the base change of Π to

$\mathrm{GL}_2(E)$, which depends only on ρ_A. Second, assume that $E = F_\bullet \times F_\circ$ is split, where $F_\bullet = F_\circ = F$. Then we define $\mathscr{L}^{-1}(\rho_A)$ to be the image of $\mathscr{L}^{-1}_{F_\bullet}(\Pi) \otimes \mathscr{L}^{-1}_{F_\circ}(\Pi)$ in $\mathscr{D}(\omega, MF_{\mathfrak{p}})$.

Now we consider (2). First, assume that E/F is nonsplit. Then the torus $F^\times \backslash E^\times$ is compact; hence, the matrix coefficient $\Phi_{f_+,f_-}(g) := (\Pi^+(g)f_+, f_-)$ is finite under $E^\times$-translation. We may assume that the restriction $\Phi_{f_+,f_-}|_{E^\times} = \sum_i a_i \chi_i$ is a finite K-linear combination of K-valued (locally constant) characters χ_i of $E^\times$ such that $\omega \cdot \chi_i|_{F^\times} = 1$. Assigning to every locally constant function h on $E^\times$ satisfying $\omega(t)h(at) = h(a)$ for all $a \in E^\times$ and $t \in F^\times$ the integral

$$\sum_i a_i \int_{F^\times \backslash E^\times} \chi_i(t) h(t)\, dt,$$

which is a finite sum, defines an element $\alpha(f_+, f_-)$ in $\mathscr{D}(\omega, K)$. Put

$$\mathscr{Q}(f_+, f_-) = \Big(\frac{\zeta_F(2)}{L(1, \rho_A, \mathrm{Ad})L(1,\eta)}\Big)^{-1} \mathscr{L}^{-1}(\rho_A)\alpha(f_+, f_-).$$

Second, assume that $E = F_\bullet \times F_\circ$ is split. We may suppose that the embedding $E \to \mathrm{Mat}_2(F)$ is given by

$$(t_\bullet, t_\circ) \mapsto \begin{pmatrix} t_\bullet & \\ & t_\circ \end{pmatrix}$$

for $t_\bullet, t_\circ \in F$. Moreover, a character χ of $E^\times$ is given by a pair $(\chi_\bullet, \chi_\circ)$ of characters of $F^\times$ such that $\chi((t_\bullet, t_\circ)) = \chi_\bullet(t_\bullet)\chi_\circ(t_\circ)$.

Now we realize $\Pi^\pm$ in their Kirillov models with respect to (nontrivial) additive characters $\psi^\pm \colon F \to \mathbb{C}^\times$ of conductor 0, respectively, where $\psi^- = (\psi^+)^{-1}$. Moreover, we may assume, for $f_\pm \in \Pi^\pm \otimes_M K$ that are compactly supported on $F^\times$, that

$$(f_+, f_-) = \int_{F^\times} f_+(a) f_-(a)\, da,$$

where da is the Haar measure on $F^\times$ such that the volume of $O_F^\times$ is c for some $c \in M$. We have the formula

$$\begin{aligned}\alpha^\natural(f_+, f_-; \chi_\iota) &= \Big(\frac{\zeta_F(2)L(1/2, \rho_A^{(\iota)}, \chi_\iota)}{L(1,\eta)L(1, \rho_A^{(\iota)}, \mathrm{Ad})}\Big)^{-1} \\ &\quad \times \int_{F^\times} \iota f_+(a) \cdot \chi_{\bullet\iota}(a)\, da \int_{F^\times} \iota f_-(b) \cdot \chi_{\bullet\iota}(b^{-1})\, db \\ &= \Big(\frac{\zeta_F(2)}{L(1,\eta)L(1, \rho_A^{(\iota)}, \mathrm{Ad})}\Big)^{-1} Z(\iota f_+, \chi_{\bullet\iota}) Z(\iota f_-, \chi_{\bullet\iota}^{-1}), \end{aligned} \tag{4.1}$$

where

$$Z(\iota f_{\pm}, \chi_{\bullet\iota}^{\pm 1}) = L(1/2, \Pi_{\iota}^{\pm} \otimes \chi_{\bullet\iota}^{\pm 1})^{-1} \int_{F^{\times}} \iota f_{\pm}(a) \cdot \chi_{\bullet\iota}^{\pm 1}(a)\, da.$$

Note that the above integrals are simply local zeta integrals. To conclude, it suffices to show that there exist elements $\mathcal{Z}(f_{\pm}) \in D_{\flat}(F^{\times}, K)$ such that, for every locally constant character $\chi\colon F^{\times} \to K^{\times}$ and every isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have $\iota\mathcal{Z}(f_{\pm})(\chi) = Z(\iota f_{\pm}, \chi_{\iota}^{\pm 1})$, respectively. Without loss of generality, we only construct $\mathcal{Z}(f_{+})$.

By enlarging M if necessary to include $l^{1/2}$, where l is the cardinality of the residue field of F, there is a subspace $\Pi^{+,c}$ of Π^{+} such that $\Pi^{+,c} \otimes_M K$ is the subspace of $\Pi^{+} \otimes_M K$ of functions that are compactly supported on $F^{\times}$. For $f_{+} \in \Pi^{+,c} \otimes_M K$, we may define $\mathcal{Z}(f_{+})$ such that, for every locally constant function h on $F^{\times}$,

$$\mathcal{Z}(f_{+})(h) = \mathcal{L}_F^{-1}(\Pi^{+})(h) \times \int_{F^{\times}} f_{+}(a)h(a)\, da.$$

Therefore, we may conclude the proof if $\dim \Pi^{+}/\Pi^{+,c} = 0$. There are two cases remaining.

First, Π^{+} is a special representation, that is, $\dim \Pi^{+}/\Pi^{+,c} = 1$. We may choose a representative $f_{+} = \mu(a) \cdot \mathrm{ch}_{O_F \backslash \{0\}}(a)$ for some character $\mu\colon F^{\times} \to M^{\times}$. Then $Z(\iota f_{+}, \chi_{\iota}) = c$ (resp., 0) if $\mu \cdot \chi$ is unramified (resp., otherwise). Therefore, we may define $\mathcal{Z}(f_{+})$ such that

$$\mathcal{Z}(f_{+})(h) = \int_{O_F^{\times}} \mu(a)h(a)\, da$$

for every locally constant function h on $F^{\times}$.

Second, Π^{+} is a principal series, that is, $\dim \Pi^{+}/\Pi^{+,c} = 2$. There are two possibilities. In the first case, we may choose representatives $f_{+}^{i} = \mu^{i}(a) \cdot \mathrm{ch}_{O_F \backslash \{0\}}(a)$ for two *different* characters $\mu^1, \mu^2\colon F^{\times} \to M^{\times}$. Without loss of generality, we consider f_{+}^{1}. Then $Z(\iota f_{+}^{1}, \chi_{\iota}) = L(1/2, \mu_{\iota}^{2} \cdot \chi_{\iota})^{-1}$ (resp., 0) if $\mu^1 \cdot \chi$ is unramified (resp., otherwise). Therefore, we may define $\mathcal{Z}(f_{+}^{1})$ such that

$$\mathcal{Z}(f_{+}^{1})(h) = \mathcal{L}_F^{-1}(\mu^1)(h) \times \int_{O_F^{\times}} \mu(a)h(a)\, da$$

for every locally constant function h on $F^{\times}$. In the second case, we may choose representatives $f_{+}^{1} = \mu(a) \cdot \mathrm{ch}_{O_F \backslash \{0\}}(a)$ and $f_{+}^{2} = (1 - \log_l |a|)\mu(a) \cdot \mathrm{ch}_{O_F \backslash \{0\}}(a)$ for some character $\mu\colon F^{\times} \to M^{\times}$. The function f_{+}^{1} has been treated above. For f_{+}^{2}, we have $Z(\iota f_{+}^{2}, \chi_{\iota}) = c$ (resp., 0) if $\mu \cdot \chi$ is unramified (resp., otherwise). Therefore, we may define $\mathcal{Z}(f_{+})$ such that

$$\mathcal{Z}(f_+)(h) = \int_{O_F^\times} \mu(a)h(a)\,da$$

for every locally constant function h on $F^\times$. □

LEMMA 4.1.9
Let $f_{\pm\mathfrak{p}} \in (\Pi_\mathfrak{p}^\pm)_K^\heartsuit$ be two n-admissible stable vectors. There exists a unique element

$$\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}) \in \mathcal{D}(\omega_\mathfrak{p}, K)$$

with the following property: for every character $\chi_\mathfrak{p}\colon E_\mathfrak{p}^\times \to K^\times$ satisfying $\omega_\mathfrak{p} \cdot \chi_\mathfrak{p}|_{F_\mathfrak{p}^\times} = 1$ and $\chi_{\mathfrak{P}^c}(t) = t^{-w}$ for $t \in (1+\mathfrak{p}^n)^\times$ and some $w \in \mathbb{Z}$ and for $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have

$$\iota\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}})(\chi_\mathfrak{p}) = \iota\Big(\frac{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}\Big)\alpha^\natural(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}; \chi_\mathfrak{p}^{(\iota)}).$$

Here, $\check{\chi}$ is defined similarly as in Definition 3.2.2(2). Moreover, there are n-admissible stable vectors $f_{\pm\mathfrak{p}} \in (\Pi_\mathfrak{p}^\pm)_K^\heartsuit$ such that $\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}})(\chi_\mathfrak{p}) \neq 0$ for every such $\chi_\mathfrak{p}$.

Proof
The uniqueness of $\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}})$ is clear, as those characters $\chi_\mathfrak{p}$ in the statement span a dense subspace of $\mathcal{C}(\omega_\mathfrak{p}, K)$ by Lemma 2.1.11.

For the existence of $\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}})$, first note that the formula (4.1) also works for $v = \mathfrak{p}$. Moreover, we have the functional equation

$$Z(\iota f_{-\mathfrak{p}}, \chi_{\mathfrak{P}^c}^{(\iota)-1}) = \iota\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c}) \cdot Z\big(\iota\big(\Pi_\mathfrak{p}^-(\mathrm{J}) f_{-\mathfrak{p}}\big), \chi_{\mathfrak{P}^c}^{(\iota)}\big).$$

By Remark 4.1.2, we only need to show that, for $f \in \Pi_\mathfrak{p}^+ \otimes_M K$ that is supported on $(1+\mathfrak{p}^n)^\times$, there exists $\mathcal{Q}'(f) \in \mathcal{D}(\omega_\mathfrak{p}, K)$ such that, for $\chi_\mathfrak{p}$ as in the statement and ι,

$$\iota\mathcal{Q}'(f)(\chi_\mathfrak{p}) = \int_{O_\mathfrak{p}^\times} \iota f(a) \cdot \chi_{\mathfrak{P}^c}^{(\iota)}(a)\,da.$$

Then we may set

$$\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}) = \Big(\frac{\zeta_{F_\mathfrak{p}}(2)}{L(1, \rho_{A,\mathfrak{p}}, \mathrm{Ad})L(1, \eta_\mathfrak{p})}\Big)^{-1} \mathcal{Q}'(f_{+\mathfrak{p}})\mathcal{Q}'\big(\Pi_\mathfrak{p}^-(\mathrm{J}) f_{-\mathfrak{p}}\big).$$

For the existence of $\mathcal{Q}'(f)$, since $\chi_{\mathfrak{P}^c}^{(\iota)}$ restricts to the trivial character on $(1+\mathfrak{p}^n)^\times$, we have

$$\int_{O_\mathfrak{p}^\times} \iota f(a) \cdot \chi_{\mathfrak{P}^c}^{(\iota)}(a)\,da = \iota \int_{O_\mathfrak{p}^\times} f(a)\,da.$$

We may put

$$\mathcal{Q}'(f) = \int_{O_{\mathfrak{p}}^{\times}} f(a)\, da \in K, \tag{4.2}$$

which is a constant (depending only on f). The last part of the lemma follows from (4.2). □

Proof of Proposition 4.1.6
Let $f_{\pm} \in (\Pi^{\pm})_K^{\heartsuit}$ be two n-admissible stable vectors. It is clear that $\mathcal{Q}(f_{+v}, f_{-v})$ constructed in Lemma 4.1.8 is equal to 1 for almost all v's. Therefore, we may simply define $\mathcal{Q}(f_+, f_-)$ to be the image of

$$\mathcal{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}) \otimes \bigotimes_{v \neq \mathfrak{p}} \mathcal{Q}(f_{+v}, f_{-v})$$

in $\mathcal{D}(\omega_A, K)$. □

4.2. Universal torus periods
Let $\mathbb{B}$ be as in the previous section. As in Section 3.3, we choose a CM point $P^+ \in Y^+(E^{\mathrm{ab}})$ and put $P^- = \mathbf{c}P^+$. By Lemma 2.4.2, we regard $P^{\pm}$ as points in $X(\pm\infty)(F_{\mathfrak{p}}^{\mathrm{ab}})$, respectively. By the same lemma, the morphism Γ_t from (2.14) preserves $\mathfrak{Y}^{\pm}(\infty)$ for $t \in O_{E_{\mathfrak{p}}}^{\times}$, respectively.

Recall that, for $m \in \mathbb{N} \cup \{\infty\}$, we have the closed formal subscheme $\mathfrak{Y}^{\pm}(m)$ of $\mathfrak{X}(m)$ as in Section 2.4. For a complete field extension $K/F_{\mathfrak{p}}^{\mathrm{nr}}$, put

$$\mathcal{N}^{\pm}(m, K) = \mathrm{H}^0\big(\mathfrak{Y}^{\pm}(m), \mathcal{O}_{\mathfrak{Y}^{\pm}(m)}\big) \widehat{\otimes}_{O_{\mathfrak{p}}^{\mathrm{nr}}} K.$$

LEMMA 4.2.1
Suppose that K is a complete field extension of $F_{\mathfrak{p}}^{\mathrm{ab}}$. Then the respective maps from $\mathcal{N}^{\pm}(\infty, K)$ to the K-algebra of continuous K-valued functions on $\overline{E^{\times}}\backslash\mathbb{A}_E^{\infty\times}$ that send $f \in \mathcal{N}^{\pm}(\infty, K)$ to the functions $x \mapsto f(\Upsilon_{\pm}\mathrm{T}_x P^{\pm})$ are isomorphisms. We recall that $\Upsilon_{\pm}$ are in Definition 2.2.10. Moreover, the induced actions of $t \in O_{E_{\mathfrak{p}}}^{\times}$ on $\mathcal{N}^{\pm}(\infty, K)$ are, respectively, given by

$$(\Gamma_t^* f)(x) = \begin{cases} f(xt) & \text{for } f \in \mathcal{N}^+(\infty, K), \\ f(xt^{\mathrm{c}}) & \text{for } f \in \mathcal{N}^-(\infty, K), \end{cases}$$

for $x \in \overline{E^{\times}}\backslash\mathbb{A}_E^{\infty\times}$.

Proof
The isomorphism follows from Lemmas 2.4.2 and 2.4.4. The action is a consequence of Lemma 2.2.11. □

Notation 4.2.2
Consider a locally constant character

$$\omega\colon F^\times\backslash\mathbb{A}^{\infty\times}\to M^\times.$$

Let K be a complete field extension of $MF_\mathfrak{p}$. For every $V^\mathfrak{p}\in\mathfrak{V}$ on which ω is trivial, denote by $\mathscr{D}(\omega,K,V^\mathfrak{p})$ the quotient K-algebra of $D(\overline{E^\times}\backslash\mathbb{A}_E^{\infty\times}/V^\mathfrak{p},K)$ divided by the closed ideal generated by $\{\omega(t)\delta_t-1\mid t\in\mathbb{A}^{\infty\times}\}$.

Then by some standard facts from functional analysis (see [27, Propositions 2.11 and 2.12]) and Remark 2.1.1, we have a canonical isomorphism

$$\mathscr{D}(\omega,K)\simeq\varprojlim_{V^\mathfrak{p}\in\mathfrak{V}}\mathscr{D}(\omega,K,V^\mathfrak{p})$$

of topological K-algebras, where the former one is in Definition 3.2.5(1). The (unique) continuous homomorphism $D(O_{E_\mathfrak{p}}^\times,K)\to D(\overline{E^\times}\backslash\mathbb{A}_E^{\infty\times}/V^\mathfrak{p},K)$ sending δ_t to $\omega_\mathfrak{p}(t_\circ)\delta_t$ for $t=(t_\bullet,t_\circ)\in O_{E_\mathfrak{p}}^\times$ descends to a continuous homomorphism $\mathbf{w}\colon D(O_\mathfrak{p}^{\mathrm{anti}},K)\to\mathscr{D}(\omega,K,V^\mathfrak{p})$ of K-algebras, which is compatible with respect to the change of $V^\mathfrak{p}$. In other words, we have a homomorphism

$$\mathbf{w}\colon D(O_\mathfrak{p}^{\mathrm{anti}},K)\to\mathscr{D}(\omega,K). \tag{4.3}$$

Definition 4.2.3 (Universal character)
We respectively define the $\pm$-*universal character* to be

$$\chi_{\mathrm{univ}}^\pm\colon\overline{E^\times}\backslash\mathbb{A}_E^{\infty\times}\xrightarrow{\delta^{\pm1}}\mathscr{D}(\omega,MF_\mathfrak{p})^\times,$$

where δ is defined in Definition 3.2.5(4).

The universal characters depend on ω. Since we will always take $\omega=\omega_A$, we suppress it from notation.

LEMMA 4.2.4
The universal characters $\chi_{\mathrm{univ}}^\pm$ are elements in $\mathscr{N}^\pm(\infty,F_\mathfrak{p}^{\mathrm{ab}})\widehat{\otimes}_{F_\mathfrak{p}^{\mathrm{ab}}}\mathscr{D}(\omega,MF_\mathfrak{p}^{\mathrm{ab}})$ satisfying

$$\Gamma_t^*\chi_{\mathrm{univ}}^\pm=\delta_t\cdot\chi_{\mathrm{univ}}^\pm,$$

respectively, for $t\in O_{E_\mathfrak{p},m}^\times$ if ω is trivial on $F_\mathfrak{p}^\times\cap O_{E_\mathfrak{p},m}^\times$.

Proof
It follows from the definition, Lemma 4.2.1, and the observation that conjugation and inversion coincide on $O_\mathfrak{p}^{\mathrm{anti}}$. ☐

Suppose that K is a complete field extension of $MF_{\mathfrak{p}}^{\mathrm{lt}}F_{\mathfrak{p}}^{\mathrm{ab}}$. Given a stable convergent modular form $f \in \mathcal{M}_{\flat}^{w}(m,K)^{\heartsuit}$ for some $w,m \in \mathbb{N}$ (Definition 2.3.10), we have the global Mellin transform $\mathbf{M}(f)$ by Theorem 2.3.17, and by (4.3),

$$\mathbf{wM}(f) \in \mathcal{M}_{\flat}^{w}(m,K)^{\heartsuit}\widehat{\otimes}_K\mathcal{D}(\omega,K).$$

By restriction, we obtain elements

$$\mathbf{wM}(f)|_{\mathfrak{Y}^{\pm}(\infty)} \in \mathcal{N}^{\pm}(\infty,K)\widehat{\otimes}_K\mathcal{D}(\omega,K).$$

By Theorem 2.3.17(2) and Lemma 4.2.4, the product $(\mathbf{wM}(f)|_{\mathfrak{Y}^{\pm}(\infty)})\cdot\chi_{\mathrm{univ}}^{\pm}$ descends to an element in $\mathcal{N}^{\pm}(m,K)\widehat{\otimes}_K\mathcal{D}(\omega,K)$ if ω is trivial on $F_{\mathfrak{p}}^{\times}\cap O_{E_{\mathfrak{p}},m}^{\times}$.

For every $V^{\mathfrak{p}} \in \mathfrak{V}$ under which f is invariant, we regard $(\mathbf{wM}(f)|_{\mathfrak{Y}^{\pm}(\infty)})\cdot\chi_{\mathrm{univ}}^{\pm}$ as elements in $\mathcal{N}^{\pm}(m,K)\widehat{\otimes}_K\mathcal{D}(\omega,K,V^{\mathfrak{p}})$, respectively. They are invariant under the action of $V^{\mathfrak{p}}$ on $\mathcal{N}^{\pm}(m,K)$.

Definition 4.2.5 (Universal torus period)
We define the *universal torus periods* of f to be the elements

$$\mathcal{P}_{\omega}^{\pm}(f) := \frac{2}{|E^{\times}\backslash\mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O_{E_{\mathfrak{p}},m}^{\times}|}\sum_{E^{\times}\backslash\mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O_{E_{\mathfrak{p}},m}^{\times}}\big((\mathbf{wM}(f)|_{\mathfrak{Y}^{\pm}(\infty)})\cdot\chi_{\mathrm{univ}}^{\pm}\big)(t)$$

in $\mathcal{D}(\omega,K,V^{\mathfrak{p}})$.

Remark 4.2.6
We add the factor 2 in the above definition in order to be consistent with the Tamagawa measure we chose in the complex Waldspurger formula recalled in Section 1.1.

By construction, the elements $\mathcal{P}_{\omega}^{\pm}(f)$ are independent of m and are compatible with respect to the change of $V^{\mathfrak{p}}$. Therefore, they are elements in $\mathcal{D}(\omega,K)$. In fact, for a character $\chi \in \Xi(\omega,K)$, Lemma 4.2.1 allows us to write

$$\mathcal{P}_{\omega}^{\pm}(f)(\chi) = \int_{\overline{E^{\times}}\backslash\mathbb{A}_E^{\infty\times}}\mathbf{wM}\big((f_{\pm}^{*}\omega_{\pm})_{\mathrm{ord}}\big)(\chi)(\Upsilon_{\pm}\mathrm{T}_t P^{\pm})\cdot\chi(t)^{\pm 1}\,dt.$$

4.3. *Interpolation of universal torus periods*

We keep the setting from the previous section. Let $MF_{\mathfrak{p}}^{\mathrm{lt}}F_{\mathfrak{p}}^{\mathrm{ab}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field.

By Definition 4.1.1, elements $f^{\pm} \in \Pi^{\pm}\otimes_F K$ can be realized as K-linear combinations of morphisms from $X_{U^{\mathfrak{p}}U_{\mathfrak{p},\pm m}}$ to A, respectively, for some (common) $U^{\mathfrak{p}} \in \mathfrak{U}$ and $m \in \mathbb{N}$. We now assume this.

Take differential forms $\omega_{\pm} \in \mathrm{H}^0(A^{\pm},\Omega_{A^{\pm}}^1)$. Using the notation in (2.26), we have convergent modular forms $(f_{\pm}^{*}\omega_{\pm})_{\mathrm{ord}} \in \mathcal{M}_{\flat}^{2}(\infty,K)$. Then $(f_{\pm}^{*}\omega_{\pm})_{\mathrm{ord}}$ are stable (in the sense of Definition 2.3.10) if and only if $f^{\pm}$ are stable (in the sense of

Definition 4.1.1), respectively. By Proposition 2.3.5(3), $(f_{\pm}^{*}\omega_{\pm})_{\mathrm{ord}}$ are n-admissible (in the sense of Definition 2.3.13) if and only if $f^{\pm}$ are n-admissible (in the sense of Definition 4.1.1), respectively.

Notation 4.3.1
For stable vectors $f^{\pm} \in (\Pi^{\pm})_{K}^{\heartsuit}$, define the elements

$$\mathcal{P}_{\mathrm{univ}}^{\pm}(f_{\pm}) \in \mathrm{Lie}\, A^{\pm} \otimes_{F^M} \mathcal{D}(\omega_A, K)$$

by the formulas

$$\big\langle \omega_{\pm}, \mathcal{P}_{\mathrm{univ}}^{\pm}(f_{\pm})\big\rangle = \mathcal{P}_{\omega_A}^{\pm}\big((f_{\pm}^{*}\omega_{\pm})_{\mathrm{ord}}\big).$$

In this section, we study the relation between

$$\iota\mathcal{P}_{\mathrm{univ}}^{+}(f_{+})(\chi)\cdot\iota\mathcal{P}_{\mathrm{univ}}^{-}(f_{-})(\chi) \in (\mathrm{Lie}\, A^{+} \otimes_{F^M} \mathrm{Lie}\, A^{-}) \otimes_{F^M,\iota} \mathbb{C}$$

for a given isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, with classical torus periods, for $f^{\pm}$ as above and a character $\chi \in \Xi(\omega_A, K)_k^n$ of weight $k \geq 1$ and depth n (Definition 4.1.3). For this purpose, we choose an ι-nearby data for $\mathbb{B}$ (Definition 2.4.10). In particular, we have

$$Y_{\iota}^{\pm}(\mathbb{C}) = \overline{E^{\times}}\backslash\{\pm i\} \times \mathbb{A}_E^{\infty\times} \subset X_{\iota}(\mathbb{C}).$$

Choose elements $t_{\pm} \in \mathbb{A}_E^{\infty\times}$ such that $\iota P^{\pm}$ are represented by $[\pm i, t_{\pm}]$, respectively. Define $\zeta_{\iota}^{\pm} \in \mathbb{C}^{\times}$ such that

$$dz\big([\pm i, t_{\pm}]\big) = \zeta_{\iota}^{\pm} \cdot \iota\omega_{\psi^{\pm}}|_{P^{\pm}}, \tag{4.4}$$

where $\omega_{\psi^{\pm}}$ are defined in (3.2). We also introduce matrices $j_{\iota}^{\pm} = \left(\begin{smallmatrix} 1 & \\ & \pm 1\end{smallmatrix}\right)$ in $\mathrm{Mat}_2(\mathbb{R}) = B(\iota) \otimes_{F,\iota} \mathbb{R}$.

Notation 4.3.2
For a cusp form $\Phi \in \mathcal{A}_{\mathrm{cusp}}(B(\iota)^{\times})$ with central character $\iota \circ \omega_A^{\pm 1}$, we respectively define

$$\mathcal{P}_{\mathbb{C}}(\Phi, \chi^{(\iota)\pm 1}) := \int_{E^{\times}\mathbb{A}^{\times}\backslash\mathbb{A}_E^{\times}} \Phi(t)\chi^{(\iota)}(t)^{\pm 1}\, dt$$

to be the complex torus periods appearing in the complex Waldspurger formula.

LEMMA 4.3.3
Let the notation be as above. We have

$$\begin{aligned}&\iota\big\langle\omega_+,\mathscr{P}^+_{\mathrm{univ}}(f_+)(\chi)\big\rangle\cdot\iota\big\langle\omega_-,\mathscr{P}^-_{\mathrm{univ}}(f_-)(\chi)\big\rangle\\&\quad=(\zeta^+_\iota\zeta^-_\iota)^k\cdot\chi^{(\iota)}(t_+^{-1}t_-)\\&\qquad\times\mathscr{P}_{\mathbb{C}}\big(\Delta^{k-1}_{+,\iota}\big(\mathrm{R}(j^+_\iota)\phi_\iota(f^*_+\omega_+)\big),\chi^{(\iota)+1}\big)\mathscr{P}_{\mathbb{C}}\big(\Delta^{k-1}_{-,\iota}\big(\mathrm{R}(j^-_\iota)\phi_\iota(f^*_-\omega_-)\big),\chi^{(\iota)-1}\big),\end{aligned}$$

where ϕ_ι is defined in Lemma 2.4.15 and $\chi^{(\iota)}$ is the ι-avatar of χ as in Definition 3.2.2(3).

Proof
Take $V^{\mathfrak{p}}\in\mathfrak{V}$ under which $f_\pm$ and χ are invariant. By Theorem 2.3.17 and Definition 4.2.5, we have

$$\begin{aligned}&\big\langle\omega_\pm,\mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi)\big\rangle\\&\quad=\frac{2}{|E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|}\\&\qquad\times\sum_{E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}}\Theta^{k-1}_{\mathrm{ord}}(f^*_\pm\omega_\pm)_{\mathrm{ord}}(\Upsilon_\pm\mathrm{T}_tP^\pm)\cdot\omega_\nu^{-k}(\Upsilon_\pm\mathrm{T}_tP^\pm)\cdot\chi^{\pm1}(t)\\&\quad=\frac{2}{|E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|}\\&\qquad\times\sum_{E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}}\Upsilon^*_\pm\Theta^{k-1}_{\mathrm{ord}}(f^*_\pm\omega_\pm)_{\mathrm{ord}}(\mathrm{T}_tP^\pm)\cdot\omega^{-k}_{\psi\pm}(\mathrm{T}_tP^\pm)\cdot\chi^{\pm1}(t)\end{aligned}$$

for some sufficiently large $m\geq n$. By (4.4) and Lemma 2.4.9, we have

$$\begin{aligned}&\iota\big\langle\omega_\pm,\mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi)\big\rangle\\&\quad=\frac{2(\zeta^\pm_\iota)^k}{|E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|}\\&\qquad\times\sum_{E^\times\backslash\mathbb{A}^{\infty\times}_E/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}}\Theta^{k-1}_\iota f^*_\pm\omega_\pm(\mathrm{T}_tP^\pm)\,dz\big([\pm i,t_\pm t]\big)^{-k}\cdot\chi^{(\iota)}(t)^{\pm1}\\&\quad=(\zeta^\pm_\iota)^k\int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}^\times_E}\mathrm{R}(j^\pm_\iota)\phi_\iota(\Theta^{k-1}_\iota f^*_\pm\omega_\pm)(t_\pm t)\cdot\chi^{(\iota)}(t)^{\pm1}\,dt\\&\quad=(\zeta^\pm_\iota)^k\cdot\chi^{(\iota)}(t^{\mp1}_\pm)\int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}^\times_E}\mathrm{R}(j^\pm_\iota)\phi_\iota(\Theta^{k-1}_\iota f^*_\pm\omega_\pm)(t)\cdot\chi^{(\iota)}(t)^{\pm1}\,dt,\end{aligned}$$

which by Lemma 2.4.17 equals

$$(\zeta_\iota^{\pm})^k \cdot \chi^{(\iota)}(t_\pm^{\mp 1}) \int_{E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times} \mathrm{R}(j_\iota^{\pm})\big(\Delta_{+,\iota}^{k-1}\phi_\iota(f_\pm^*\omega_\pm)\big)(t)\cdot \chi^{(\iota)}(t)^{\pm 1}\,dt$$

$$= (\zeta_\iota^{\pm})^k \cdot \chi^{(\iota)}(t_\pm^{\mp 1}) \int_{E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times} \Delta_{\pm,\iota}^{k-1}\big(\mathrm{R}(j_\iota^{\pm})\phi_\iota(f_\pm^*\omega_\pm)\big)(t)\cdot \chi^{(\iota)}(t)^{\pm 1}\,dt.$$

This completes the proof. □

PROPOSITION 4.3.4
Given n-admissible stable vectors $f_\pm \in (\Pi^\pm)_K^\heartsuit$ and a character $\chi \in \Xi(\omega_A, K)_k^n$ of weight $k \geq 1$ and depth n, we have

$$\begin{aligned}
&\iota\mathscr{P}_{\mathrm{univ}}^{+}(f_+)(\chi)\cdot \iota\mathscr{P}_{\mathrm{univ}}^{-}(f_-)(\chi)\\
&\quad = \iota\mathscr{Q}(f_+, f_-)(\chi)\\
&\qquad \times L(1/2, \rho_A^{(\iota)}, \chi^{(\iota)})\cdot \frac{2^{g-1}\mathrm{d}_E^{1/2}\zeta_F(2)\mathbf{P}_\iota(A,\chi)}{L(1,\eta)^2 L(1,\rho_A^{(\iota)},\mathrm{Ad})}\cdot \iota\Big(\frac{\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})}{L(1/2,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})^2}\Big),
\end{aligned}$$

as an equality in $(\mathrm{Lie}\,A^+ \otimes_{F^M} \mathrm{Lie}\,A^-) \otimes_{F^M,\iota} \mathbb{C}$.

Proof
It suffices to show the equality after pairing with $\omega_+ \otimes \omega_-$ for an arbitrary pair of differential forms $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$.

By the complex Waldspurger formula (see [29] and [30, Theorem 1.4.2]) and Proposition 4.1.6, we have

$$\begin{aligned}
&\mathscr{P}_\mathbb{C}\big(\Delta_{+,\iota}^{k-1}\big(\mathrm{R}(j_\iota^{+})\phi_\iota(f_+^*\omega_+)\big), \chi^{(\iota)+1}\big)\mathscr{P}_\mathbb{C}\big(\Delta_{-,\iota}^{k-1}\big(\mathrm{R}(j_\iota^{-})\phi_\iota(f_-^*\omega_-)\big), \chi^{(\iota)-1}\big)\\
&\quad = C_\iota \frac{\zeta_F(2)L(1/2,\rho_A^{(\iota)},\chi^{(\iota)})}{L(1,\rho_A^{(\iota)},\mathrm{Ad})L(1,\eta)}\frac{2}{2^{-g}\mathrm{d}_E^{-1/2}L(1,\eta)}\cdot \iota\Big(\frac{\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})}{L(1/2,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})^2}\Big)\\
&\qquad \cdot \iota\mathscr{Q}(f_+, f_-)(\chi),
\end{aligned}$$

where C_ι is the complex constant such that

$$\big(\Delta_{+,\iota}^{k-1}\big(\mathrm{R}(j_\iota^{+})\phi_\iota(f_+^*\omega_+)\big), \Delta_{-,\iota}^{k-1}\big(\mathrm{R}(j_\iota^{-})\phi_\iota(f_-^*\omega_-)\big)\big)_{\mathrm{Pet}} = C_\iota \cdot \iota(f_+, f_-)_A$$

holds for all f_+ and f_-. Here $(\cdot,\cdot)_{\mathrm{Pet}}$ is the bilinear Petersson inner product pairing. By Lemma 4.3.3, the proposition is reduced to the formula

$$\big\langle \omega_+ \otimes \omega_-, \mathbf{P}_\iota(A,\chi)\big\rangle = C_\iota \cdot (\zeta_\iota^+\zeta_\iota^-)^k \cdot \chi^{(\iota)}(t_+^{-1}t_-). \tag{4.5}$$

By Lemma 3.2.8, it suffices to show that

$$\frac{\iota\varphi_+ \otimes \mathfrak{c}_\iota^*\iota\varphi_- \otimes \mu^k}{\iota(\varphi_+,\varphi_-)_\chi}\int_{X_\iota(\mathbb{C})}\frac{\Theta_\iota^{k-1}f_+^*\omega_+ \otimes \mathfrak{c}_\iota^*\Theta_\iota^{k-1}f_-^*\omega_-}{\mu^k}\,dx$$
$$= (\zeta_\iota^+\zeta_\iota^-)^k \cdot \chi^{(\iota)}(t_+^{-1}t_-)\cdot\big(\Delta_{+,\iota}^{k-1}\big(\mathrm{R}(j_\iota^+)\phi_\iota(f_+^*\omega_+)\big), \Delta_{-,\iota}^{k-1}\big(\mathrm{R}(j_\iota^-)\phi_\iota(f_-^*\omega_-)\big)\big)_{\mathrm{Pet}}$$

for some choice of $\varphi_\pm \in \sigma_\chi^\pm(\mathbb{C}_p)$. We take elements $\varphi_\pm$ such that $\iota\varphi_\pm(P^\pm) = dz([\pm i, t_\pm])^k$, respectively. Then $\iota(\varphi_+,\varphi_-)_\chi = (\zeta_\iota^+\zeta_\iota^-)^{-k}$ by (4.4). Now we take μ to be the standard invariant hyperbolic metric on $\check{X}_\iota = B(\iota)^\times\backslash\mathscr{H}\times\mathbb{B}^{\infty\times}/\overline{F^\times}$. Then $\iota\varphi_+ \otimes \mathfrak{c}_\iota^*\iota\varphi_- \otimes \mu^k$ is the constant $\chi^{(\iota)}(t_+^{-1}t_-)$, and

$$\int_{X_\iota(\mathbb{C})}\frac{\Theta_\iota^{k-1}f_+^*\omega_+ \otimes \mathfrak{c}_\iota^*\Theta_\iota^{k-1}f_-^*\omega_-}{\mu^k}\,dx$$
$$= \big(\Delta_{+,\iota}^{k-1}\big(\mathrm{R}(j_\iota^+)\phi_\iota(f_+^*\omega_+)\big), \Delta_{-,\iota}^{k-1}\big(\mathrm{R}(j_\iota^-)\phi_\iota(f_-^*\omega_-)\big)\big)_{\mathrm{Pet}}.$$

Thus, (4.5) holds, and the proposition follows. □

The proposition has the following corollary.

COROLLARY 4.3.5
For $\chi \in \Xi(\omega_A, K)_k^n$ with $k \geq 1$, the ratio

$$\frac{\mathscr{P}_{\mathrm{univ}}^+(f_+)(\chi)\mathscr{P}_{\mathrm{univ}}^-(f_-)(\chi)}{\mathscr{Q}(f_+,f_-)(\chi)} \in (\mathrm{Lie}\,A^+ \otimes_{FM} \mathrm{Lie}\,A^-)\otimes_{FM} K,$$

if the denominator is nonzero, is independent of the choice of n-admissible stable vectors $f_\pm \in (\Pi^\pm)_K^\heartsuit$. Moreover, for $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have

$$\iota\Big(\frac{\mathscr{P}_{\mathrm{univ}}^+(f_+)(\chi)\mathscr{P}_{\mathrm{univ}}^-(f_-)(\chi)}{\mathscr{Q}(f_+,f_-)(\chi)}\Big)$$
$$= L(1/2, \rho_A^{(\iota)}, \chi^{(\iota)}) \cdot \frac{2^{g-1}\mathrm{d}_E^{1/2}\zeta_F(2)\mathbf{P}_\iota(A,\chi)}{L(1,\eta)^2 L(1,\rho_A^{(\iota)},\mathrm{Ad})}\cdot\iota\Big(\frac{\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})}{L(1/2,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})^2}\Big).$$

PROPOSITION 4.3.6
For n-admissible stable vectors $f_\pm \in (\Pi^\pm)_K^\heartsuit$ and a character $\chi \in \Xi(\omega_A, K)_0^n$ of weight 0 and depth n, we have

$$\mathscr{P}_{\mathrm{univ}}^\pm(f_\pm)(\chi) = \log_{A^\pm} P_\chi^\pm(f_\pm).$$

Proof
We may choose a tame level $U^\mathfrak{p} \in \mathfrak{U}$ that fixes both f_+ and f_- and such that χ is fixed by $U^\mathfrak{p} \cap \mathbb{A}_E^{\infty\times}$. We may realize $f_\pm$ as K-linear combinations of morphisms from

$X_{U^{\mathfrak{p}}U_{\mathfrak{p},\pm m}}$ to $A^{\pm}$, respectively, for some sufficiently large integer $m \geq n$. By linearity, we may assume that $f_{\pm}$ are just morphisms from $X_{U^{\mathfrak{p}}U_{\mathfrak{p},\pm m}}$ to $A^{\pm}$, respectively.

For $\omega_{\pm} \in \mathrm{H}^0(A^{\pm}, \Omega^1_{A^{\pm}})$, we have by Theorems 2.3.17(3) and 2.3.17(4) that

$$d\mathbf{M}\big((f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}\big)(\chi|_{O^{\times}_{E_{\mathfrak{P}^c}}}) = \Theta_{\mathrm{ord}}\mathbf{M}\big((f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}\big)(\chi|_{O^{\times}_{E_{\mathfrak{P}^c}}}) = (f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}.$$

On the other hand, by Proposition A.0.1, we know that $(f_{\pm}^* \log_{\omega_{\pm}})_{\mathrm{ord}}$ are Coleman integrals of $(f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}$ on (the generic fiber of) $\mathfrak{X}(m, U^{\mathfrak{p}})$, respectively. Therefore, we have

$$\mathbf{M}\big((f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}\big)(\chi|_{O^{\times}_{E_{\mathfrak{P}^c}}}) = (f_{\pm}^* \log_{\omega_{\pm}})_{\mathrm{ord}} \tag{4.6}$$

on $\mathfrak{X}(m, U^{\mathfrak{p}})$, since both of them are Coleman integrals of $f_{\pm}^*\omega_{\pm}$ on $\mathfrak{X}(m, U^{\mathfrak{p}})$ that belong to $\mathcal{M}^0_{\flat}(m, K)^{\heartsuit}$, respectively. By Definition 3.3.1, we have

$$\begin{aligned}
\log_{\omega_{\pm}} P^{\pm}_{\chi}(f_{\pm}) &= \int_{\overline{E^{\times}}\backslash \mathbb{A}_E^{\infty\times}} \log_{\omega_{\pm}} f_{\pm}(\mathrm{T}_t P^{\pm}) \cdot \chi(t)^{\pm 1}\, dt \\
&= \int_{\overline{E^{\times}}\backslash \mathbb{A}_E^{\infty\times}} f_{\pm}^* \log_{\omega_{\pm}}(\mathrm{T}_t P^{\pm}) \cdot \chi(t)^{\pm 1}\, dt \\
&= \int_{\overline{E^{\times}}\backslash \mathbb{A}_E^{\infty\times}} (f_{\pm}^* \log_{\omega_{\pm}})_{\mathrm{ord}}(\Upsilon_{\pm}\mathrm{T}_t P^{\pm}) \cdot \chi(t)^{\pm 1}\, dt,
\end{aligned}$$

which by (4.6) is equal to

$$\begin{aligned}
&\int_{\overline{E^{\times}}\backslash \mathbb{A}_E^{\infty\times}} \mathbf{M}\big((f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}\big)(\chi|_{O^{\times}_{E_{\mathfrak{P}^c}}})(\Upsilon_{\pm}\mathrm{T}_t P^{\pm}) \cdot \chi(t)^{\pm 1}\, dt \\
&\quad = \int_{\overline{E^{\times}}\backslash \mathbb{A}_E^{\infty\times}} \mathbf{wM}\big((f_{\pm}^*\omega_{\pm})_{\mathrm{ord}}\big)(\chi)(\Upsilon_{\pm}\mathrm{T}_t P^{\pm}) \cdot \chi(t)^{\pm 1}\, dt,
\end{aligned}$$

respectively. Then the proposition follows from Remark 4.2.6. □

4.4. *Proofs of main theorems*

Let K be a complete field extension of $MF_{\mathfrak{p}}$. For $V^{\mathfrak{p}} \in \mathfrak{V}$, denote by $\mathscr{C}(\omega, K, V^{\mathfrak{p}})$ the (closed) subspace of $\mathscr{C}(\omega, K)$ (Definition 3.2.5) of functions that are invariant under the right translation of $V^{\mathfrak{p}}$. It is also a closed subspace of $C(E^{\times}\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}, K)$. The strong dual of $\mathscr{C}(\omega, K, V^{\mathfrak{p}})$ is canonically isomorphic to $\mathscr{D}(\omega, K, V^{\mathfrak{p}})$ (Notation 4.2.2).

We consider totally definite (not necessarily incoherent) quaternion algebras $\mathbb{B}$ over $\mathbb{A}$ such that, for a finite place v of F, $\epsilon(\mathbb{B}_v) = 1$ if v is split in E *or* the Galois representation $\rho_{A,v}$ corresponds to a principal series.

For such (a representative in the isomorphism class of) $\mathbb{B}$, we choose an E-embedding as (1.9), which is possible. We define representations

$$\Pi(\mathbb{B})^{\mathrm{tame}}_{A^\pm} = \bigotimes\nolimits'_M \Pi_{v,A^\pm},$$

where the restricted tensor products (over M) are taken over all finite places $v \neq \mathfrak{p}$ of F and $\Pi_{v,A^\pm}$ are M-representations of $\mathbb{B}_v^\times$ determined by $\rho_{A^\pm,v}$, respectively. In particular, if $\mathbb{B}$ is incoherent (i.e., $\mathbb{B} \in \mathcal{B}(A)$ in Notation 3.2.1), then $\Pi(\mathbb{B})^{\mathrm{tame}}_{A^\pm}$ are isomorphic to the away-from-$\mathfrak{p}$ components of $\Pi(\mathbb{B})_{A^\pm}$ (Notation 3.1.2), respectively.

Notation 4.4.1
Let $\mathcal{I}_+(\omega_A, K, V^\mathfrak{p})$ be the closed ideal of $\mathcal{D}(\omega_A, K, V^\mathfrak{p})$ generated by

$$\left\{\mathcal{Q}(f_+, f_-) \;\middle|\; f_\pm \in \left(\Pi(\mathbb{B})^{\mathrm{tame}}_{A^\pm}\right)^{V^\mathfrak{p}} \otimes_M K, \epsilon(\mathbb{B}) = +1\right\},$$

and let $\mathcal{I}_-(\omega_A, K, V^\mathfrak{p})$ be the closed ideal of $\mathcal{D}(\omega_A, K, V^\mathfrak{p})$ generated by

$$\left\{\mathcal{Q}(f_+, f_-) \;\middle|\; f_\pm \in \left(\Pi(\mathbb{B})^{\mathrm{tame}}_{A^\pm}\right)^{V^\mathfrak{p}} \otimes_M K, \epsilon(\mathbb{B}) = -1\right\},$$

where $\mathcal{Q}(f_+, f_-)$ is defined as the product of those elements $\mathcal{Q}(f_{+v}, f_{-v})$ in Lemma 4.1.8(2).

Let $\mathcal{C}_+(\omega_A, K, V^\mathfrak{p})$ (resp., $\mathcal{C}_-(\omega_A, K, V^\mathfrak{p})$) be the subspace of $\mathcal{C}(\omega_A, K, V^\mathfrak{p})$ consisting of functions lying in the kernel of every element in $\mathcal{I}_-(\omega_A, K, V^\mathfrak{p})$ (resp., $\mathcal{I}_+(\omega_A, K, V^\mathfrak{p})$). Put $\Xi(A, K, V^\mathfrak{p}) = \Xi(A, K) \cap \mathcal{C}(\omega_A, K, V^\mathfrak{p})$ and $\Xi(\omega_A, K, V^\mathfrak{p}) = \Xi(\omega_A, K) \cap \mathcal{C}(\omega_A, K, V^\mathfrak{p})$, where $\Xi(A, K)$ and $\Xi(\omega_A, K)$ are introduced in Definition 3.2.5.

Remark 4.4.2
The ideals $\mathcal{I}_\pm(\omega_A, K, V^\mathfrak{p})$ are topologically finitely generated. The subspaces $\mathcal{C}_\pm(\omega_A, K, V^\mathfrak{p})$ are closed in $\mathcal{C}(\omega_A, K, V^\mathfrak{p})$.

The following lemma concerns some algebraic properties of the objects introduced above.

LEMMA 4.4.3
Suppose that $V^\mathfrak{p} \in \mathfrak{V}$ is sufficiently small. We have
(1) $\mathcal{I}_+(\omega_A, K, V^\mathfrak{p}) \cap \mathcal{I}_-(\omega_A, K, V^\mathfrak{p}) = 0$;
(2) $\mathcal{I}_+(\omega_A, K, V^\mathfrak{p}) + \mathcal{I}_-(\omega_A, K, V^\mathfrak{p}) = \mathcal{D}(\omega_A, K, V^\mathfrak{p})$;
(3) $\mathcal{C}(\omega_A, K, V^\mathfrak{p}) = \mathcal{C}_+(\omega_A, K, V^\mathfrak{p}) \oplus \mathcal{C}_-(\omega_A, K, V^\mathfrak{p})$;
(4) *the subset $\Xi(A, K, V^\mathfrak{p})$ is contained in and generates a dense subspace of $\mathcal{C}_-(\omega_A, K, V^\mathfrak{p})$;*

(5) $\mathcal{J}_+(\omega_A, K, V^{\mathfrak{p}})$ *is the closed ideal generated by elements that vanish on* $\Xi(A, K, V^{\mathfrak{p}})$.

Proof
We first realize that $\Xi(\omega_A, K, V^{\mathfrak{p}})$ generates a dense subspace of $\mathcal{C}(\omega_A, K, V^{\mathfrak{p}})$. Thus, (1) follows from the dichotomy theorem of Saito and Tunnell (see [28] and [24]). For (2), assume the converse, and suppose that $\mathcal{J}_+(\omega_A, K, V^{\mathfrak{p}}) + \mathcal{J}_-(\omega_A, K, V^{\mathfrak{p}})$ is contained in a (closed) maximal ideal with residue field K'. Then all local period distributions $\mathcal{Q}(f_+, f_-)$ will vanish on the character

$$E^\times\backslash\mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}} \xrightarrow{\delta} \mathcal{D}(\omega_A, K, V^{\mathfrak{p}}) \to K',$$

which contradicts the theorem of Saito and Tunnell. Part (3) is a direct consequence of (1) and (2). It is clear that $\Xi(A, K, V^{\mathfrak{p}})$ is contained in $\mathcal{C}_-(\omega_A, K, V^{\mathfrak{p}})$ and, by the theorem of Saito and Tunnell, $\Xi(\omega_A, K, V^{\mathfrak{p}}) \setminus \Xi(A, K, V^{\mathfrak{p}}) \subset \mathcal{C}_+(\omega_A, K, V^{\mathfrak{p}})$, which together imply (4). Finally, (5) follows from (4). □

Remark 4.4.4
If we put $\mathcal{D}(A, K, V^{\mathfrak{p}}) = \mathcal{D}(\omega_A, K, V^{\mathfrak{p}})/\mathcal{J}_+(\omega_A, K, V^{\mathfrak{p}})$, then we obtain a canonical isomorphism

$$\mathcal{D}(A, K) \xrightarrow{\sim} \varprojlim_{V^{\mathfrak{p}} \in \mathfrak{V}} \mathcal{D}(A, K, V^{\mathfrak{p}}).$$

Moreover, we have $\mathcal{D}(A, K) \simeq \prod_{\mathbb{B}\in\mathcal{B}(A)} \mathcal{D}(A, \mathbb{B}, K)$ (Definition 3.2.5). We have $\mathcal{D}(A, K, V^{\mathfrak{p}})\widehat{\otimes}_K K' \simeq \mathcal{D}(A, K', V^{\mathfrak{p}})$ and $\mathcal{D}(A, K)\widehat{\otimes}_K K' \simeq \mathcal{D}(A, K')$ for a complete field extension K'/K.

Remark 4.4.5
In fact, for sufficiently small $V^{\mathfrak{p}} \in \mathfrak{V}$, the morphism $\mathbf{w}$ from (4.3) is injective with the quotient which is a finite étale K-algebra. We also have $D(O_{\mathfrak{p}}^{\mathrm{anti}}, K) \cap \mathcal{J}_+(\omega_A, K, V^{\mathfrak{p}}) = \{0\}$. Thus, if K is discretely valued, then $\mathcal{D}(A, K, V^{\mathfrak{p}})$ is a (commutative) nuclear Fréchet–Stein K-algebra (defined, e.g., in [10, Definition 1.2.10]). Moreover, it is not hard to see that the transition homomorphism $\mathcal{D}(A, K, V'^{\mathfrak{p}}) \to \mathcal{D}(A, K, V^{\mathfrak{p}})$ is finite étale for $V'^{\mathfrak{p}} \subset V^{\mathfrak{p}}$. The rigid analytic variety $\mathcal{E}_-(V^{\mathfrak{p}})$ associated to $\mathcal{D}(A, MF_{\mathfrak{p}}, V^{\mathfrak{p}})$ is a smooth rigid curve over $MF_{\mathfrak{p}}$, which may be regarded as an eigencurve for the group $\mathrm{U}(1)_{E/F}$ of tame level $V^{\mathfrak{p}}$, twisted by (the cyclotomic character) ω_A and cut off by the condition that $\epsilon(1/2, \rho_A, \cdot) = -1$. The ind-rigid analytic variety $\mathcal{E}_-$ mentioned in Section 1.7 is actually $\varinjlim_{V^{\mathfrak{p}}\in\mathfrak{V}} \mathcal{E}_-(V^{\mathfrak{p}})$.

Proof of Theorem 3.2.10
For the existence, note that the union $\bigcup_{k\geq 1} \Xi(\omega_A, \mathbb{C}_p)_k^0$ already spans a dense sub-

space of $\mathscr{C}(\omega_A, \mathbb{C}_p)$ by Lemma 2.1.11. By Corollary 4.3.5, Lemma 4.4.3, and (the nonvanishing part of) Lemma 4.1.9, the collection of ratios

$$\frac{\mathscr{P}^+_{\mathrm{univ}}(f_+)\mathscr{P}^-_{\mathrm{univ}}(f_-)}{\mathscr{Q}(f_+, f_-)}$$

for $f_\pm$ running over $(\Pi(\mathbb{B})_{A^\pm})_K^\heartsuit$ with $\epsilon(\mathbb{B}) = -1$ defines an element

$$\mathscr{L}(A) \in (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathscr{D}(A, \mathbb{C}_p).$$

It actually belongs to $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathscr{D}(A, MF_{\mathfrak{p}}^{\mathrm{lt}})$ by the lemma below. We need to show that the element

$$\mathscr{L}(A) \in (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathscr{D}(A, MF_{\mathfrak{p}}^{\mathrm{lt}})$$

introduced in Definition 4.4.7 satisfies (3.4). However, this follows from Corollary 4.3.5.

The uniqueness follows from the fact that $\bigcup_{k\ge 1} \Xi(\omega_A, \mathbb{C}_p)_k$ is dense in $\mathscr{C}(\omega_A, \mathbb{C}_p)$, which we already used in the construction of $\mathscr{L}(A)$. □

LEMMA 4.4.6
The element $\mathscr{L}(A)$ *belongs to* $(\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathscr{D}(A, MF_{\mathfrak{p}}^{\mathrm{lt}})$.

Proof
Note that, in the definition of $\mathscr{L}(A)$, we only need to consider $f_\pm \in (\Pi(\mathbb{B})_{A^\pm})^\heartsuit_{MF_{\mathfrak{p}}^{\mathrm{lt}}}$ such that both f_+ and $\Pi(\mathbb{B})_{A^-}(\mathrm{J})f_-$ are invariant under $O^\times_{\mathfrak{P}^c}$. Then the lemma follows if we can show that, for every $\chi \in \bigcup_{k\ge 1} \Xi(A, \mathbb{C}_p)^0_k$ and $\sigma \in \mathrm{Gal}(\mathbb{C}_p / MF_{\mathfrak{p}}^{\mathrm{lt}})$, we have

$$\sigma\langle \omega_\pm, \mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi)\rangle = \langle \omega_\pm, \mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\sigma\circ\chi)\rangle. \tag{4.7}$$

Without loss of generality, we consider the one for f^+. As in the proof of Lemma 4.3.3, we have the equality

$$\begin{aligned}&\langle \omega_+, \mathscr{P}^+_{\mathrm{univ}}(f_+)(\chi)\rangle\\ &\quad = C \sum_{E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}} O^\times_{E_{\mathfrak{p}}}} \Theta^{k-1}_{\mathrm{ord}}(f_+^*\omega_+)_{\mathrm{ord}}(\Upsilon_+ \mathrm{T}_t P^+)\cdot \omega_\nu^{-k}(\Upsilon_+\mathrm{T}_t P^+)\cdot \chi(t),\end{aligned}$$

where C is a positive rational constant. However, the product $\Theta^{k-1}_{\mathrm{ord}}(f_+^*\omega_+)_{\mathrm{ord}}\cdot\omega_\nu^{-k}$ is naturally an element in $\mathscr{M}^0_\flat(\infty, MF_{\mathfrak{p}}^{\mathrm{lt}})$ (Definition 2.3.8), and χ can be viewed as an element in $\mathscr{N}^+(\infty, \mathbb{C}_p)$ (Section 4.2). Thus, (4.7) holds, and the lemma follows. □

Definition 4.4.7 (p-adic L-function)
We call the element

$$\mathcal{L}(A) \in (\mathrm{Lie}\, A^+ \otimes_{FM} \mathrm{Lie}\, A^-) \otimes_{FM} \mathcal{D}(A, MF_{\mathfrak{p}}^{\mathrm{lt}})$$

in the proof of Theorem 3.2.10 the *anti-cyclotomic p-adic L-function* attached to A.

Proof of Theorem 3.3.2
It follows from Propositions 4.3.6 and 4.1.6. □

Appendices

A. Compatibility of logarithm and Coleman integral

In this appendix, we generalize a result of Coleman [8] about the compatibility of the p-adic logarithm and Coleman integral. This result will only be used in the proof of Proposition 4.3.6.

Let F be a local field contained in $\mathbb{C}_p$ with ring of integers O_F and residue field k. Let X be a quasiprojective scheme over F, and let $U \subset X^{\mathrm{rig}}$ be an affinoid domain with good reduction. We say a closed rigid analytic 1-form ω on U is *Frobenius proper* if there exist a Frobenius endomorphism ϕ of U and a polynomial $P(X)$ over $\mathbb{C}_p$ such that $P(\phi^*)\omega$ is the differential of a rigid analytic function on U and such that no root of $P(X)$ is a root of unity. Therefore, by [8, Theorem 2.1], there exists a locally analytic function f_ω on $U(\mathbb{C}_p)$, unique up to an additive constant on each geometric connected component, such that

- $df_\omega = \omega$;
- $P(\phi^*) f_\omega$ is rigid analytic.

Such f_ω is known as a *Coleman integral* of ω on U, which is independent of the choice of P (see [8, Corollary 2.1(b)]).

PROPOSITION A.0.1
Let X and U be as above. Let A be an Abelian variety over F which has either totally degenerate reduction or potentially good reduction. Then for a morphism $f: X \to A$ and a differential form $\omega \in \Omega^1(A/F)$, the form $f^*\omega|_U$ is Frobenius proper, which admits $f^* \log_\omega |_U$ as a Coleman integral, where $\log_\omega : A(\mathbb{C}_p) \to \mathbb{C}_p$ is the p-adic logarithm associated to ω.

Proof
We may assume that X is projective. Replacing F by a finite extension, we may assume that A has good reduction or split totally degenerate reduction (i.e., the connected neutral component $\mathcal{A}_s^\circ$ of the special fiber $\mathcal{A}_s$ of the Néron model $\mathcal{A}$ of A is

isomorphic to $\mathbf{G}_{m,k}^d$, where d is the dimension of A). The first case follows from [8, Theorem 2.8 and Proposition 2.2].

Now we consider the second case. Denote by $\mathcal{A}_\eta^\circ$ the analytic domain of A^{rig} of points whose reduction is in $\mathcal{A}_s^\circ$. By a well-known result of uniformization from [20, Section 6], we have $A^{\mathrm{rig}} \simeq (\mathbf{G}_{m,F}^{\mathrm{rig}})^d/\Lambda$ for a lattice $\Lambda \subset \mathbf{G}_{m,F}^d(F)$. Moreover, $\mathcal{A}_\eta^\circ$ is isomorphic to $\operatorname{Sp} F\langle T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}\rangle$, the rigid analytic multitorus of multiradius 1.

Choose an admissible covering $\mathcal{U}$ of X^{rig} containing U, which determines a formal model $X_{\mathcal{U}}$ of X over O_F. Since X is projective, we may assume that $X_{\mathcal{U}}$ is algebraic. Let Z be the nonsmooth locus of $X_{\mathcal{U}}$ over O_F. The set of closed points of X whose reduction is not in Z forms an analytic domain W of X^{rig}. Since U has good reduction, we have $U \subset W$. By the Néron mapping property, the morphism f extends uniquely to a morphism $X_{\mathcal{U}} - Z \to \mathcal{A}$, which induces a morphism $f'\colon U \to A^{\mathrm{rig}}$. Without loss of generality, we assume that $f'(U)$ is contained in $\mathcal{A}_\eta^\circ$. By [8, Proposition 2.2], we only need to show that $\omega|_{\mathcal{A}_\eta^\circ}$ is Frobenius proper and $\log_\omega|_{\mathcal{A}_\eta^\circ}$ is a Coleman integral of it.

In fact, we have

$$\left\{\omega|_{\mathcal{A}_\eta^\circ} \;\middle|\; \omega \in \Omega^1(A/F)\right\} = \operatorname{Span}_F\left\{\frac{dT_1}{T_1}, \dots, \frac{dT_d}{T_d}\right\}.$$

By linearity, we may assume $\omega^\circ := \omega|_{\mathcal{A}_\eta^\circ} = \frac{dT_1}{T_1}$. We choose the Frobenius endomorphism on $\mathcal{A}_\eta^\circ$ to be given by $\phi((T_1, \dots, T_d)) = (T_1^q, \dots, T_d^q)$, where $q = |k|$. We have that $P(\phi^*)\omega^\circ = 0$ for $P(X) = X - q$. On the other hand, the p-adic logarithm log on $\operatorname{Sp} F\langle T_1, T_1^{-1}\rangle$ is also killed by $P(\phi^*)$. Therefore, the function $(\log, 1, \dots, 1)$ on $\operatorname{Sp} F\langle T_1, T_1^{-1}\rangle \times \cdots \times \operatorname{Sp} F\langle T_d, T_d^{-1}\rangle \simeq \mathcal{A}_\eta^\circ$ is a Coleman integral of ω°, which coincides with the restriction of $\log_\omega$. ☐

B. Serre–Tate local moduli for $\mathcal{O}$-divisible groups (following N. Katz)

In this appendix, we describe the Kodaira–Spencer isomorphism for ordinary $\mathcal{O}$-divisible groups in terms of their Serre–Tate coordinates, generalizing a classical result of Katz [17, Theorem 3.7.1] which is for ordinary p-divisible groups. Only Theorems B.1.1 and B.2.3 will be used in the main part of the article. Some notation in this appendix may be different from that in Section 1.8.

B.1. $\mathcal{O}$-Divisible groups and Serre–Tate coordinates

Let F be a finite field extension of $\mathbb{Q}_p$, where p is a rational prime. Denote by $\breve{F}$ the completion of a maximal unramified extension of F. The ring of integers of F (resp., $\breve{F}$) is denoted by $\mathcal{O}$ (resp., $\breve{\mathcal{O}}$). Let k be the residue field of $\breve{\mathcal{O}}$, which is an algebraic closure of $\mathbb{F}_p$. For a p-divisible group G over $\operatorname{Spec} R$, we denote by $\Omega(G/R)$ the

R-module of invariant differentials of G over R, which is the dual R-module of the tangent space $\mathrm{Lie}(G/R)$ at the identity.

Let S be an $\breve{\mathcal{O}}$-scheme. Recall that an $\mathcal{O}$-divisible group over S is a p-divisible group G over S with an action by $\mathcal{O}$ such that the induced action of $\mathcal{O}$ on the sheaf $\underline{\mathrm{Lie}}(G/S)$ coincides with the natural action as an $\mathcal{O}_S$-module (hence, an $\mathcal{O}$-module). Denote by $\mathsf{BT}_S^{\mathcal{O}}$ the category of $\mathcal{O}$-divisible groups over S, which is an Abelian category. We omit the superscript $\mathcal{O}$ if it is $\mathbb{Z}_p$. The height h of G, as a p-divisible group, must be divisible by $[F:\mathbb{Q}_p]$. We define the *$\mathcal{O}$-height* of G to be $[F:\mathbb{Q}_p]^{-1}h$. An $\mathcal{O}$-divisible group G is connected (resp., étale) if its underlying p-divisible group is. We denote by $\mathcal{LT}$ the Lubin–Tate $\mathcal{O}$-formal group over $\operatorname{Spec}\breve{\mathcal{O}}$, which is unique up to isomorphism. We use the same notation for its base change to S.

For an $\mathcal{O}$-divisible group G over S, there exists an $\mathcal{O}$-formal group G^0 over S, unique up to isomorphism, such that its associated p-divisible group $G^0[p^\infty]$ is the maximal connected subgroup of G. In particular, $G^0[p^\infty]$ is an $\mathcal{O}$-divisible group. We define the *$\mathcal{O}$-Cartier dual* of G to be

$$G^D := \varinjlim_n \underline{\mathrm{Hom}}_{\mathcal{O}}\big(G[p^n], \mathcal{LT}[p^n]\big)$$

as in [11]. An $\mathcal{O}$-divisible group G is *ordinary* if $(G^0[p^\infty])^D$ is étale. Denote by $\mathrm{T}_p G = \varprojlim_n G[p^n]$ the Tate module functor. Denote by $\mathsf{Nilp}_{\breve{\mathcal{O}}}$ the category of $\breve{\mathcal{O}}$-schemes on which p is locally nilpotent.

THEOREM B.1.1 (Serre–Tate coordinates)
Let $\mathbf{G}$ be an ordinary $\mathcal{O}$-divisible group over k. Consider the moduli functor $\mathfrak{M}_{\mathbf{G}}$ on $\mathsf{Nilp}_{\breve{\mathcal{O}}}$ such that, for every $\breve{\mathcal{O}}$-scheme S on which p is locally nilpotent, $\mathfrak{M}_{\mathbf{G}}(S)$ is the set of isomorphism classes of pairs (G,φ), where G is an object in $\mathsf{BT}_S^{\mathcal{O}}$ and $\varphi\colon G\times_S(S\otimes_{\breve{\mathcal{O}}}k)\to \mathbf{G}\times_{\operatorname{Spec}k}(S\otimes_{\breve{\mathcal{O}}}k)$ is an isomorphism. Then $\mathfrak{M}_{\mathbf{G}}$ is canonically pro-represented by the $\breve{\mathcal{O}}$-formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$.

In particular, for every Artinian local $\breve{\mathcal{O}}$-algebra R with the maximal ideal $\mathfrak{m}_R$ and G/R a deformation of $\mathbf{G}$, we have a pairing

$$q(G/R;\cdot,\cdot)\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT}(R)=1+\mathfrak{m}_R.$$

It satisfies the following.

(1) *For every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$ and $\alpha_D\in\mathrm{T}_p\mathbf{G}^D(k)$, we have*

$$q(G/R;\alpha,\alpha_D)=q(G^D/R;\alpha_D,\alpha).$$

(2) *Suppose that we have another ordinary $\mathcal{O}$-divisible group $\mathbf{H}$ over k and its deformation H over R. Let $\mathbf{f}\colon\mathbf{G}\to\mathbf{H}$ be a homomorphism, and let $\mathbf{f}^D$ be its dual. Then $\mathbf{f}$ lifts to a (unique) homomorphism $f\colon G\to H$ if and only if*

$$q(G/R;\alpha,\mathbf{f}^D\beta_D)=q(H/R;\mathbf{f}\alpha,\beta_D)$$

for every $\alpha\in \mathrm{T}_p\mathbf{G}(k)$ *and* $\beta_D\in \mathrm{T}_p\mathbf{H}^D(k)$.

By abuse of notation, we will use $\mathfrak{M}_{\mathbf{G}}$ to denote the formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$. The proof of the theorem follows exactly that of [17, Theorem 2.1].

Proof

The fact that $\mathfrak{M}_G$ is pro-presentable is well known. Now we determine the representing formal scheme.

Since $\mathbf{G}$ is ordinary, we have a canonical isomorphism

$$\mathbf{G}\simeq \mathbf{G}^0[p^\infty]\times \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O}.$$

By the definition of $\mathcal{O}$-Cartier duality, we have a morphism

$$e_{p^n}:\mathbf{G}[p^n]\times\mathbf{G}^D[p^n]\to\mathcal{LT}[p^n].$$

The restriction of the first factor to $\mathbf{G}^0[p^n]$ gives rise to an isomorphism

$$\mathbf{G}^0[p^n]\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}}\big(\mathbf{G}^D[p^n](k),\mathcal{LT}[p^n]\big)$$

of group schemes over k preserving $\mathcal{O}$-actions. Passing to the limit, we obtain an isomorphism of $\mathcal{O}$-formal groups over k

$$\mathbf{G}^0\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT}\big),$$

which induces a pairing

$$E_{\mathbf{G}}:\mathbf{G}^0\times\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT}.$$

Let G/R be a deformation of $\mathbf{G}$. Then we have an extension

$$0\longrightarrow G^0[p^\infty]\longrightarrow G\longrightarrow \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}F/\mathcal{O}\longrightarrow 0 \tag{B.1}$$

of $\mathcal{O}$-divisible groups. We have pairings

$$E_{G,p^n}:G^0[p^n]\times\mathbf{G}^D[p^n]\to\mathcal{LT}[p^n],$$
$$E_G:G^0\times\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT},$$

which lift e_{p^n} and $E_{\mathbf{G}}$, respectively.

Similar to the p-divisible group case, the extension (B.1) is obtained from the extension

$$0 \longrightarrow \mathrm{T}_p\mathbf{G}(k) \longrightarrow \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F \longrightarrow \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O} \longrightarrow 0$$

by pushing out along a unique $\mathcal{O}$-linear homomorphism

$$\varphi_{G/R}\colon \mathrm{T}_p\mathbf{G}(k)\to G^0(R).$$

The homomorphism $\varphi_{G/R}$ may be recovered from (B.1) in the way described in [17, p. 151]. It is the composite

$$\mathrm{T}_p\mathbf{G}(k)\to \mathrm{T}_p\mathbf{G}[p^n](k)\xrightarrow{\langle p^n\rangle} G^0(R)$$

for every $n\geq 1$ such that $\mathfrak{m}_R^{n+1}=0$. Therefore, from G/R, we obtain a pairing

$$q(G/R;\cdot,\cdot)=E_G(R)\circ(\varphi_{G/R},\mathrm{id})\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k)\to \mathcal{LT}(R)=1+\mathfrak{m}_R.$$

This shows that the functor $\mathfrak{M}_{\mathbf{G}}$ is canonically pro-represented by the $\breve{\mathcal{O}}$-formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$.

For (2), if the given homomorphism $\mathbf{f}\colon \mathbf{G}\to\mathbf{H}$ can be lifted to $f: G\to H$, then we must have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT}[p^\infty]\big) & \longrightarrow & G & \longrightarrow & \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \circ\mathrm{T}_p\mathbf{f}^D(k)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{T}_p\mathbf{f}(k)\otimes_{\mathcal{O}}F/\mathcal{O}} & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}[p^\infty]\big) & \longrightarrow & H & \longrightarrow & \mathrm{T}_p\mathbf{H}(k)\otimes_{\mathcal{O}} F/\mathcal{O} & \longrightarrow & 0.
\end{array}$$

Conversely, if we may fill f in the above diagram, then $\mathbf{f}$ lifts.

The existence of the middle arrow is equivalent to the pushout of the top extension by the left arrow being isomorphic to the pullback of the lower extension by the right arrow. The above-mentioned pushout is an element in

$$\mathrm{Ext}_{\mathsf{BT}_R^{\mathcal{O}}}\big(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O},\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}[p^\infty]\big)\big),$$

which is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}(R)\big)$$

by the bilinear pairing

$$(\alpha,\beta_D)\mapsto q(G/R;\alpha,\mathbf{f}^D\beta_D).$$

Similarly, the above-mentioned pullback is an element in

$$\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}(R)\big)$$

defined by the bilinear pairing

$$(\alpha, \beta_D) \mapsto q(H/R; \mathbf{f}\alpha, \beta_D).$$

It remains to prove (1). Choose n such that $\mathfrak{m}_R^{n+1} = 0$. Then both $G^0(R)$ and $(G^D)^0(R)$ are annihilated by p^n. Denote by $\alpha(n)$ the image of α under the canonical projection $\mathrm{T}_p\mathbf{G}(k) \to G[p^n](k)$ and similarly for $\alpha_D(n)$. By construction, we have $\varphi_{G/R}(\alpha) = \langle p^n \rangle \alpha(n) \in G^0(R)$ and $\varphi_{G^D/R}(\alpha_D) = \langle p^n \rangle \alpha_D(n) \in (G^D)^0(R)$. Therefore, we have

$$q(G/R; \alpha, \alpha_D) = E_{G,p^n}\big(\langle p^n \rangle \alpha(n), \alpha_D(n)\big).$$

Similarly, we have $q(G^D/R; \alpha_D, \alpha) = E_{G^D,p^n}(\langle p^n \rangle \alpha_D(n), \alpha(n))$.

The remaining argument is formal, and one only needs to replace $\widehat{\mathbf{G}}_m$ (resp., Abelian varieties) by $\mathcal{LT}$ (resp., $\mathcal{O}$-divisible groups) in the proof of [17, Theorem 2.1]. In particular, we have the following. Given an integer $n \geq 1$ and elements $x \in G^0[p^n](R)$ and $y \in \mathbf{G}^D[p^n](k)$, there exist an Artinian local ring R' that is finite and flat over R and a point $Y \in G^D[p^n](R')$ lifting y. For every such R' and Y, we have the equality $E_{G,p^n}(x, y) = e_{p^n}(x, Y)$ inside $\mathcal{LT}(R')$. ☐

B.2. Main theorem

We fix an ordinary $\mathcal{O}$-divisible group $\mathbf{G}$ over k. Denote by $\mathfrak{R}$ the coordinate ring of $\mathfrak{M}_{\mathbf{G}}$, which is a complete $\breve{\mathcal{O}}$-algebra. We have the universal pairing

$$q\colon \mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k) \to \mathcal{LT}(\mathfrak{R}) \subset \mathfrak{R}^\times.$$

Therefore, we may regard $q(\alpha, \alpha_D)$ as a regular function on $\mathfrak{M}_{\mathbf{G}}$. For each $\mathcal{O}$-linear form $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$, denote by $D(\ell)$ the translation-invariant continuous derivation of $\mathfrak{R}$ given by

$$\mathbf{D}(\ell) q(\alpha, \alpha_D) = \ell(\alpha \otimes \alpha_D) \cdot q(\alpha, \alpha_D).$$

By abuse of notation, we also denote by $\mathbf{D}(\ell)$ the corresponding map $\Omega_{\mathfrak{R}/\breve{\mathcal{O}}} \to \mathfrak{R}$. Denote by $\mathfrak{G}$ the universal $\mathcal{O}$-divisible group over $\mathfrak{M}_{\mathbf{G}}$. We choose a logarithm $\log\colon \mathcal{LT} \to \widehat{\mathbf{G}}_a$ over $\breve{\mathcal{O}} \otimes \mathbb{Q}$ such that $\omega_0 := \log^* dT$ is a generator of the free rank 1 $\breve{\mathcal{O}}$-module $\Omega(\mathcal{LT}/\breve{\mathcal{O}})$.

Let R be as in Theorem B.1.1, and let G/R be a deformation of $\mathbf{G}$. We have the canonical isomorphism of $\mathcal{O}$-modules

$$\lambda_G\colon \mathrm{T}_p\mathbf{G}^D(k) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{BT}_R^{\mathcal{O}}}\big(G^0[p^\infty], \mathcal{LT}[p^\infty]\big).$$

Define the $\mathcal{O}$-linear map $\omega_G\colon \mathrm{T}_p\mathbf{G}^D(k) \to \Omega(G/R)$ by the formula

$$\omega_G(\alpha_D) = \lambda_G(\alpha_D)^* \omega_0 \in \Omega(G^0/R) = \Omega(G/R).$$

Let $L_G\colon \operatorname{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O}) \to \operatorname{Lie}(G/R)$ be the unique $\mathcal{O}$-linear map such that

$$\omega_G(\alpha_D)\cdot L_G(\alpha_D^\vee) = \alpha_D\cdot\alpha_D^\vee \in \mathcal{O}.$$

In fact, the R-linear extensions

$$\omega_G\colon \mathrm{T}_p\mathbf{G}^D(k)\otimes_{\mathcal{O}} R \to \Omega(G/R)$$

and

$$L_G\colon \operatorname{Hom}_{\mathcal{O}}\big(\mathrm{T}_p\mathbf{G}^D(k),R\big) \to \operatorname{Lie}(G/R)$$

are isomorphisms. Similarly, we have an isomorphism

$$\lambda_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k) = \mathrm{T}_p\mathbf{G}^{\text{ét}}(k) = \mathrm{T}_pG^{\text{ét}}(R) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{BT}_R}\big((G^{\text{ét}})^\vee, \widehat{\mathbf{G}}_m[p^\infty]\big),$$

which induces an isomorphism

$$\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathbb{Z}_p} R \xrightarrow{\sim} \Omega\big((G^{\text{ét}})^\vee/R\big)$$

by pulling back the differential form $\frac{dT}{T}$ on $\widehat{\mathbf{G}}_m$. It further induces an isomorphism

$$\omega_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} R = \big(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathbb{Z}_p} R\big)_{\mathcal{O}} \xrightarrow{\sim} \Omega\big((G^{\text{ét}})^\vee/R\big)_{\mathcal{O}}.$$

Here, the subscript $\mathcal{O}$ denotes the maximal flat quotient on which $\mathcal{O}$ acts via the structure map. By construction, we have the following lemma of functoriality.

LEMMA B.2.1
Let $\mathbf{f}\colon \mathbf{G}\to\mathbf{H}$ be as in Theorem B.1.1, and let $f\colon G\to H$ be a homomorphism lifting $\mathbf{f}$. Then the following hold.

(1) *We have $((f^{\text{ét}})^\vee)^*(\omega_{G^\vee}(\alpha)) = \omega_{H^\vee}(\mathbf{f}\alpha)$ for every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$, where $f^{\text{ét}}\colon G^{\text{ét}}\to H^{\text{ét}}$ is the induced homomorphism on the étale quotient.*

(2) *We have $f_*(L_H(\alpha_D^\vee)) = L_G(\alpha_D^\vee\circ\mathbf{f}^D)$ for every $\alpha_D^\vee\in\operatorname{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$.*

Denote by $\mathrm{D}(\mathbf{G})$ the (contravariant) Dieudonné crystal of $\mathbf{G}$. We have the exact sequence

$$0 \longrightarrow \Omega(\mathfrak{G}^\vee/\mathfrak{R}) \longrightarrow \mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}} \longrightarrow \operatorname{Lie}(\mathfrak{G}/\mathfrak{R}) \longrightarrow 0$$

and the *Gauss–Manin connection*

$$\nabla\colon \mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}} \to \mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\otimes_{\mathfrak{R}}\Omega_{\mathfrak{R}/\breve{\mathcal{O}}}.$$

They together define the *(universal) Kodaira–Spencer map*

$$\mathrm{KS}\colon \Omega(\mathfrak{G}^\vee/\mathfrak{R}) \to \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) \otimes_{\mathfrak{R}} \Omega_{\mathfrak{R}/\breve{\mathcal{O}}},$$

which factors through the quotient $\Omega(\mathfrak{G}^\vee/\mathfrak{R}) \to \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$. The following lemma is immediate.

LEMMA B.2.2
The natural map $\Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} \to \Omega((\mathfrak{G}^{\text{ét}})^\vee/\mathfrak{R})_{\mathcal{O}}$ is an isomorphism.

In particular, we may regard $\omega_{\mathfrak{G}^\vee}$ as a map from $\mathrm{T}_p\mathbf{G}(k)$ to $\Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$. The following result on the compatibility of the Kodaira–Spencer map and the Serre–Tate coordinate is the main theorem of this appendix.

THEOREM B.2.3
We have the equality

$$\omega_{\mathfrak{G}}(\alpha_D)\cdot \mathrm{KS}\big(\omega_{\mathfrak{G}^\vee}(\alpha)\big) = d\log\big(q(\alpha,\alpha_D)\big)$$

in $\Omega_{\mathfrak{R}/\breve{\mathcal{O}}}$ for every $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ and $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$.

Note that the definition of $\omega_{\mathfrak{G}}$, but not $\omega_{\mathfrak{G}^\vee}$, depends on the choice of log, which is compatible with the right-hand side.

B.3. Frobenius
Denote by σ the Frobenius automorphism of $\breve{\mathcal{O}}$ such that $\mathcal{O} = \breve{\mathcal{O}}^{\sigma=1}$. Put $X^\sigma = X \otimes_{\breve{\mathcal{O}},\sigma} \breve{\mathcal{O}}$ for every $\breve{\mathcal{O}}$-(formal) scheme X, let $\Sigma_X\colon X^\sigma \to X$ be the natural projection, and let $\mathsf{F}_X\colon X \to X^\sigma$ be the relative Frobenius morphism which is $\breve{\mathcal{O}}$-linear. We omit the subscript X if it is $\mathfrak{M}_{\mathbf{G}}$.

LEMMA B.3.1
We have the following results.
(1) *There is a natural isomorphism*

$$\mathfrak{M}_{\mathbf{G}}^\sigma \xrightarrow{\sim} \mathfrak{M}_{\mathbf{G}^\sigma}$$

under which the regular function $q(\sigma(\alpha),\sigma(\alpha_D))$ is mapped to $\Sigma^* q(\alpha,\alpha_D)$.
(2) *Under the map $\Sigma_{(G^{\text{ét}})^\vee}\colon (G^{\sigma\text{ét}})^\vee \simeq ((G^{\text{ét}})^\vee)^\sigma \to (G^{\text{ét}})^\vee$, we have*

$$\Sigma^*_{(G^{\text{ét}})^\vee}\omega_{G^\vee}(\alpha) = \omega_{(G^\sigma)^\vee}(\sigma\alpha)$$

for every $\alpha \in \mathrm{T}_p\mathbf{G}(k)$.
(3) *Under the map $\mathsf{F}_G\colon G \to G^\sigma$, we have*

$$\mathsf{F}_{G*}L_G(\alpha_D^\vee) = L_{G^\sigma}(\alpha_D^\vee \circ \sigma^{-1})$$

for every $\alpha_D^\vee \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$.

Proof
The proof is the same as that for [17, Lemmas 4.1.1 and 4.1.1.1]. ☐

From now on, we choose a uniformizer ϖ of F, which gives rise to an isomorphism $\mathcal{LT}^\sigma \simeq \mathcal{LT}$. In particular, we may identify $(G^D)^\sigma$ and $(G^\sigma)^D$. For a deformation G/R of $\mathbf{G}$, we denote by G'/R the quotient of G by the subgroup $G^0[\varpi]$. The induced projection map

$$\mathcal{F}_G : G \to G'$$

lifts the relative Frobenius morphism

$$\mathsf{F}_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}^\sigma.$$

Define the *Verschiebung* to be

$$\mathsf{V}_{\mathbf{G}} = (\mathsf{F}_{\mathbf{G}^D})^D : \mathbf{G}^\sigma \simeq \mathbf{G}^{D\sigma D} \to \mathbf{G}.$$

Note that the isomorphism depends on ϖ.

LEMMA B.3.2
For $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ *and* $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$*, we have formulas*
(1) $\mathsf{F}_{\mathbf{G}}(\alpha) = \sigma\alpha$ *and* $\mathsf{V}_{\mathbf{G}}(\sigma\alpha) = \varpi\alpha_D$*; and*
(2) $q(G'/R; \sigma\alpha, \sigma\alpha_D) = \varpi.q(G/R; \alpha, \alpha_D)$.

Proof
The proof is the same as that of [17, Lemma 4.1.2], with $\mathsf{V}_{\mathbf{G}} \circ \mathsf{F}_{\mathbf{G}} = \varpi$. ☐

LEMMA B.3.3
For $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ *and* $\alpha_D^\vee \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$*, we have formulas*
(1) $((\mathcal{F}_G^{\text{ét}})^\vee)^*\omega_{G^\vee}(\alpha) = \omega_{G'^\vee}(\sigma\alpha)$*; and*
(2) $\mathcal{F}_{G*}L_G(\alpha_D^\vee) = \varpi L_{G'}(\alpha_D^\vee \circ \sigma^{-1})$.

Proof
It follows from Lemmas B.2.1 and B.3.2. ☐

If we apply the construction to the universal object $\mathfrak{G}$, we obtain a formal deformation $\mathfrak{G}'/\mathfrak{R}$ of G^σ. Its classifying map is the unique morphism

$$\Phi\colon \mathfrak{M}_{\mathbf{G}} \to \mathfrak{M}_{\mathbf{G}^\sigma} \xrightarrow{\sim} \mathfrak{M}_{\mathbf{G}}^\sigma$$

such that $\Phi^*\mathfrak{G}^\sigma \simeq \mathfrak{G}'$. Therefore, we may regard $\mathcal{F}_{\mathfrak{G}}$ as a morphism

$$\mathcal{F}_{\mathfrak{G}}\colon \mathfrak{G} \to \Phi^*\mathfrak{G}^\sigma$$

of $\mathcal{O}$-divisible groups over $\mathfrak{M}_{\mathbf{G}}$. Taking the dual, we have

$$\mathcal{F}_{\mathfrak{G}}^\vee\colon \Phi^*\mathfrak{G}^{\sigma\vee} \simeq (\Phi^*\mathfrak{G}^\sigma)^\vee \to \mathfrak{G}^\vee.$$

LEMMA B.3.4

We have the following statements.

(1) *The map* $\omega_{\mathfrak{G}^\vee}\colon \mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathfrak{R} \to \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$ *induces an isomorphism*

$$\mathrm{T}_p\mathbf{G}(k) \xrightarrow{\sim} \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}^1 := \left\{\omega \in \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} \;\middle|\; (\mathcal{F}_{\mathfrak{G}}^\vee)^*\omega = \Phi^*\Sigma_{\mathfrak{G}^\vee}^*\omega\right\}$$

of $\mathcal{O}$*-modules.*

(2) *The map* $L_{\mathfrak{G}}\colon \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathfrak{R}) \to \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})$ *induces an isomorphism*

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}}\left(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O}\right)\\ &\qquad \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0 := \left\{\delta \in \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) \;\middle|\; \mathcal{F}_{\mathfrak{G}*}\delta = \varpi\,\Phi^*\mathsf{F}_{\mathfrak{G}*}\delta\right\}\end{aligned}$$

of $\mathcal{O}$*-modules.*

Proof

It can be proved in the same way as [17, Corollary 4.1.5] by using Lemmas B.3.1 and B.3.3. ☐

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} & \xrightarrow{a} & \left(\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\right)_{\mathcal{O}} & \xrightarrow{b} & \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle (\mathcal{F}_{\mathfrak{G}}^\vee)^*} & & \big\downarrow{\scriptstyle \mathrm{D}(\mathsf{F}_{\mathbf{G}}^\vee)} & & \big\downarrow{\scriptstyle \mathcal{F}_{\mathfrak{G}*}} & & \\
0 & \longrightarrow & \Omega(\mathfrak{G}'^\vee/\mathfrak{R})_{\mathcal{O}} & \longrightarrow & \left(\mathrm{D}(\mathbf{G}^{\sigma\vee})_{\mathfrak{R}}\right)_{\mathcal{O}} & \longrightarrow & \mathrm{Lie}(\mathfrak{G}'/\mathfrak{R}) & \longrightarrow & 0\\
 & & \big\uparrow{\scriptstyle \Phi^*\circ\Sigma_{\mathfrak{G}^\vee}^*} & & \big\uparrow{\scriptstyle \mathrm{D}(\Sigma_{\mathbf{G}^\vee})} & & \big\uparrow{\scriptstyle \Phi^*\circ\mathsf{F}_{\mathfrak{G}*}} & & \\
0 & \longrightarrow & \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} & \longrightarrow & \left(\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\right)_{\mathcal{O}} & \longrightarrow & \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) & \longrightarrow & 0
\end{array} \tag{B.2}$$

For $k \in \mathbb{Z}$, we define $\mathcal{O}$-modules

$$\begin{aligned}\mathrm{D}(\mathbf{G}^\vee)^k_{\mathfrak{R}} &= \{\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \mid \mathrm{D}(\mathsf{F}^\vee_{\mathbf{G}})\xi = \varpi^{1-k}\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi\} \\ &= \{\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \mid \mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi = \varpi^k\xi\}.\end{aligned}$$

LEMMA B.3.5

The maps $\omega_{\mathfrak{G}^\vee}$ and a in (B.2) together induce an isomorphism

$$a_1 : \mathrm{T}_p\mathbf{G}(k) \xrightarrow{\sim} \mathrm{D}(\mathbf{G}^\vee)^1_{\mathfrak{R}}$$

of $\mathcal{O}$-modules. The maps $L_{\mathfrak{G}}$ and b in (B.2) together induce an isomorphism

$$b_0 : \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$$

of $\mathcal{O}$-modules.

Proof

For the first part, by a similar argument to that in [17, Lemma 4.2.1], we know that $b(\xi) = 0$ for $\xi \in \mathrm{D}(\mathbf{G}^\vee)^1_{\mathfrak{R}}$, that is, ξ is in the image of a. The conclusion then follows from Lemma B.3.4(1).

For the second part, it is easy to see that $\mathrm{Im}(a) \cap \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{G}} = \{0\}$ by choosing an $\mathcal{O}$-basis of $\mathrm{T}_p\mathbf{G}(k)$. Therefore, b restricts to an injective map $\mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}} \to \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0$. We only need to show that this map is also surjective. For every $\delta \in \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0$, choose an element $\xi_0 \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$. Put $\xi_{n+1} = \mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi_n$ for $n \geq 0$. Then $b(\xi_n) = \delta$, and $\{\xi_n\}$ converge to an element $\xi \in \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}}$. □

LEMMA B.3.6

For every $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$, the action of $D(\ell)$ under the Gauss–Manin connection on $(\mathrm{D}(G^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ satisfies the formula

$$D(\ell)\big(\nabla\big(\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{G^\vee})\xi\big)\big) = \varpi\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\big(D(\ell)(\nabla\xi)\big)$$

for every $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$.

Proof

It is proved in the same way as [17, Lemma 4.3.3]. □

LEMMA B.3.7

If $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ satisfies $\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi = \lambda\xi$ for some $\lambda \in \breve{\mathcal{O}}$, then for every $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$, the element $D(\ell)(\nabla\xi) \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ satisfies

$$\varpi\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\big(D(\ell)(\nabla\xi)\big) = \lambda D(\ell)(\nabla\xi).$$

Proof
It follows immediately from Lemma B.3.6. □

PROPOSITION B.3.8
For $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ *and* $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$*, there exists a unique character* $Q(\alpha, \alpha_D)$ *of* $\mathfrak{M}_{\mathbf{G}}$ *such that*

$$\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}\big(\omega_{\mathfrak{G}^\vee}(\alpha)\big) = d \log Q(\alpha, \alpha_D).$$

Proof
Let $\{\alpha_i\}$ (resp., $\{\alpha_{D,j}\}$) be an $\mathcal{O}$-basis of $\mathrm{T}_p\mathbf{G}(k)$ (resp., $\mathrm{T}_p\mathbf{G}^D(k)$). Let $\{\ell_{i,j}\}$ be the basis of $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$ dual to $\{\alpha_i \otimes \alpha_{D,j}\}$. Then for every element $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$, we have

$$\nabla\xi = \sum_{i,j} D(\ell_{i,j})(\nabla\xi) \otimes d \log q(\alpha_i, \alpha_{D,j}).$$

In particular, for $\xi = \omega_{\mathfrak{G}^\vee}(\alpha)$, we have

$$\nabla\omega_{\mathfrak{G}^\vee}(\alpha) = \sum_{i,j} D(\ell_{i,j})\big(\nabla\omega_{\mathfrak{G}^\vee}(\alpha)\big) \otimes d \log q(\alpha_i, \alpha_{D,j}).$$

By Lemmas B.3.4 and B.3.7, $\nabla\omega_{\mathfrak{G}^\vee}(\alpha) \in \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}}$. Therefore, there exist unique elements $\alpha^\vee_{D,i,j} \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$ such that

$$\nabla\omega_{\mathfrak{G}^\vee}(\alpha) = b_0^{-1}(\alpha^\vee_{D,i,j})$$

for every i, j. By definition,

$$\mathrm{KS}\big(\omega_{\mathfrak{G}^\vee}(\alpha)\big) = \sum_{i,j} L_{\mathfrak{G}}(\alpha^\vee_{D,i,j}) \otimes d \log q(\alpha_i, \alpha_{D,j}),$$

and

$$\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}\big(\omega_{\mathfrak{G}^\vee}(\alpha)\big) = d \log\Big(\prod_{i,j} q(\alpha_i, \alpha_{D,j})^{\alpha_D \cdot \alpha^\vee_{D,i,j}}\Big).$$ □

The above proposition has the following two corollaries.

COROLLARY B.3.9
For elements $\alpha \in \mathrm{T}_p\mathbf{G}(k)$*,* $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$*, and* $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$*, we have that* $D(\ell)(\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)))$ *is a constant in* $\mathcal{O}$.

COROLLARY B.3.10

Suppose that, for every integer $n \geq 1$, we can find a homomorphism $f_n: \mathfrak{R} \to \breve{\mathcal{O}}/p^n$ such that

$$f_n\big(D(\ell)\big(\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}\big(\omega_{\mathfrak{G}^\vee}(\alpha)\big)\big)\big) = \ell(\alpha \otimes \alpha_D)$$

holds in $\breve{\mathcal{O}}/p^n$. Then $Q = q$, and Theorem B.2.3 follows.

The condition of this corollary is fulfilled by Theorem B.4.2. Therefore, we have reduced Theorem B.2.3 to Theorem B.4.2 in the next section.

B.4. Infinitesimal computation

Let R be an (Artinian) local $\breve{\mathcal{O}}$-algebra with the maximal ideal $\mathfrak{m}_R$ satisfying $\mathfrak{m}_R^{n+1} = 0$. Let G/R be the canonical deformation of $\mathbf{G}$. Let $\tilde{G}$ be a deformation of G to $\tilde{R} := R[\varepsilon]/(\varepsilon^2)$, which gives rise to a map $\partial: \Omega(G^\vee/R) \to \mathrm{Lie}(G/R)$. Note that the target $\mathrm{Lie}(G/R)$ may be identified with $\mathrm{Ker}(G^0(\tilde{R}) \to G^0(R))$.

LEMMA B.4.1

The reduction map $\mathrm{T}_p G(R) \to \mathrm{T}_p \mathbf{G}(k)$ is an isomorphism.

Proof

It follows from the same argument as in [17, Lemma 6.1]. □

In particular, we may define maps $\lambda_{G^\vee}: \mathrm{T}_p\mathbf{G}(k) \to \mathrm{Hom}_{\mathsf{BT}_R}(G^\vee, \widehat{\mathbf{G}}_m[p^\infty])$ and

$$\omega_{G^\vee}: \mathrm{T}_p\mathbf{G}(k) \to \Omega(G^\vee/R). \tag{B.3}$$

THEOREM B.4.2

The Serre–Tate coordinate for $\tilde{G}/\tilde{R}$ satisfies

$$q(\tilde{G}/\tilde{R}; \alpha, \alpha_D) = 1 + \varepsilon\omega_G(\alpha_D) \cdot \partial\big(\omega_{G^\vee}(\alpha)\big).$$

LEMMA B.4.3

For $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$ and $\alpha \in \mathrm{Ker}(G^0(\tilde{R}) \to G^0(R)) = \mathrm{Lie}(G/R)$, we have

$$E_G(\alpha, \alpha_D) = 1 + \varepsilon\omega_G(\alpha_D)\alpha.$$

Proof

By functoriality, we only need to prove the lemma for the universal object $\mathfrak{G}/\mathfrak{R}$. By definition,

$$1 + \varepsilon\omega_{\mathfrak{G}}(\alpha_D)\alpha = 1 + \varepsilon\big(\lambda_{\mathfrak{G}}(\alpha_D)_*\alpha \cdot \omega_0\big) \in \mathcal{LT}(\tilde{R}).$$

We also have

$$\lambda_{\mathfrak{G}}(\alpha_D)_*\alpha \cdot \omega_{\mathcal{LT}} = \big(\log \circ \lambda_{\mathfrak{G}}(\alpha_D)\big)_*\alpha \cdot dT$$

in $\mathfrak{R}[p^{-1}]$. Therefore, we have the equality

$$E_{\mathfrak{G}}(\alpha,\alpha_D) = 1 + \varepsilon\omega_{\mathfrak{G}}(\alpha_D)\alpha$$

in $\operatorname{Ker}(\mathcal{LT}(\tilde{\mathfrak{R}}[p^{-1}]) \to \mathcal{LT}(\mathfrak{R}[p^{-1}]))$. □

For an integer $N > n$, denote by α_N the image of α in $G[p^N](R)$. Let $\tilde{\alpha}_N \in \tilde{G}(\tilde{R})$ be an arbitrary lifting of α_N. Then

$$p^N\tilde{\alpha}_N \in \operatorname{Ker}\big(\tilde{G}(\tilde{R}) \to G(R)\big) = \operatorname{Ker}\big(\tilde{G}^0(\tilde{R}) \to G^0(R)\big) \simeq \operatorname{Lie}(G/R).$$

This process defines a map $\varphi_G \colon \mathrm{T}_pG(R) \to \operatorname{Lie}(G/R)$.

PROPOSITION B.4.4
We have $\partial\omega_{G^\vee}(\alpha) = \varphi_G(\alpha)$ *for every* $\alpha \in \mathrm{T}_pG(R)$.

Assuming the above proposition, we prove Theorem B.4.2.

Proof of Theorem B.4.2
It is clear that $G^0[p^\infty] \otimes_R \tilde{R}$ is the unique, up to isomorphism, deformation of $G^0[p^\infty]$ to $\tilde{R}$. Then the deformation $\tilde{G}$ corresponds to the extension

$$0 \longrightarrow G^0[p^\infty] \otimes_R \tilde{R} \longrightarrow \tilde{G} \longrightarrow \mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} F/\mathcal{O} \longrightarrow 0.$$

In particular, we may identify $\tilde{G}^0$ with $G^0 \otimes_R \tilde{R}$. We have

$$\begin{aligned}\operatorname{Ker}\big(\tilde{G}(\tilde{R}) \to G(R)\big) &= \operatorname{Ker}\big(\tilde{G}^0(\tilde{R}) \to G^0(R)\big) = \operatorname{Ker}\big(G^0(\tilde{R}) \to G^0(R)\big)\\ &= \operatorname{Lie}(G^0/R).\end{aligned}$$

For $D \in \operatorname{Lie}(G^0/R)$, we have

$$E_{\tilde{G}}(D,\cdot) = E_G(D,\cdot) \colon \mathrm{T}_p\mathbf{G}^D(k) \simeq \mathrm{T}_p\big(\mathbf{G}^0[p^\infty]\big)^D(k) \to \mathcal{LT}(\tilde{R}),$$

where in the pairing $E_{\tilde{G}}$ (resp., E_G), we view D as an element in $\tilde{G}^0(\tilde{R})$ (resp., $G^0(\tilde{R})$). For $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$, we have

$$E_G(D,\alpha_D) = 1 + \varepsilon\omega_G(\omega_\alpha)D$$

by definition. Therefore, Theorem B.4.2 follows from Proposition B.4.4 and the construction of q. □

The rest of the appendix is devoted to the proof of Proposition B.4.4. We will reduce it to certain statements from [17] about Abelian varieties. It is an interesting problem to find a proof purely using $\mathcal{O}$-divisible groups.

Recall that ordinary $\mathcal{O}$-divisible groups over k are classified by their dimension and $\mathcal{O}$-height. Let $\mathbf{G}_{r,s}$ be an $\mathcal{O}$-divisible group of dimension r and $\mathcal{O}$-height $r+s$ with $r \geq 0$ and $r+s>0$.

Proof of Proposition B.4.4
Choose a totally real number field E^+ such that $F \simeq E^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and choose an imaginary quadratic field K in which $p = \mathfrak{p}^+\mathfrak{p}^-$ splits. Put $E = E^+ \otimes_{\mathbb{Q}} K$. Suppose that $\tau_1, \tau_2, \dots, \tau_h$ are all complex embeddings of E^+. Consider the data $(\mathbf{A}_{r,s}, \theta, i)$ where

- $\mathbf{A}_{r,s}$ is an Abelian variety over k;
- $\theta\colon \mathbf{A}_{r,s} \to \mathbf{A}_{r,s}^\vee$ is a prime-to-p polarization;
- $i\colon O_E \to \operatorname{End}_k \mathbf{A}_{r,s}$ is an O_E-action which sends the complex conjugation on O_E to the Rosati involution and such that, in the induced decomposition

$$\mathbf{A}_{r,s}[p^\infty] = \mathbf{A}_{r,s}[p^\infty]^+ \oplus \mathbf{A}_{r,s}[p^\infty]^-$$

  of the $O_E \otimes \mathbb{Z}_p$-module $\mathbf{A}_{r,s}[p^\infty]$, the summand $\mathbf{A}_{r,s}[p^\infty]^+$ is isomorphic to $\mathbf{G}_{r,s}$ as an $\mathcal{O}$-divisible group.

It is clear that the polarization θ induces an isomorphism $\mathbf{A}_{r,s}[p^\infty]^+ \xrightarrow{\sim} (\mathbf{A}_{r,s}[p^\infty]^-)^\vee$. By the Serre–Tate theorem, $\mathfrak{M}_{\mathbf{G}_{r,s}}$ also parameterizes deformation of the triple $(\mathbf{A}_{r,s}, \theta, i)$. In what follows, we fix r, s and suppress them from notation. Let R be as in Theorem B.1.1, let A/R be the canonical deformation of $\mathbf{A}/k$, and let $\tilde{A}$ be a deformation of A to $\tilde{R}$ such that $\tilde{G} \simeq \tilde{A}[p^\infty]^+$.

There is a similar map (B.3) for A, and we have $\omega_{G^\vee}(\alpha) = \omega_{A^\vee}(\alpha)$ for $\alpha \in \mathrm{T}_p G(R) \subset \mathrm{T}_p A(R)$, where we view $\Omega(G^\vee/R)$ as a submodule of $\mathrm{H}^0(A^\vee, \Omega^1_{A^\vee/R})$. Moreover, the map $\varphi_G\colon \mathrm{T}_p G(R) \to \operatorname{Lie}(G/R)$ can be extended in the same way to a map $\varphi_A\colon \mathrm{T}_p A(R) \to \operatorname{Lie}(A/R)$. Then Proposition B.4.4 follows from [17, Lemma 5.4 and Section 6.5], where the argument uses normalized cocycles and does *not* require A to be ordinary in the usual sense. □

Acknowledgments. We would like to thank Daniel Disegni for his careful reading of the draft version and useful comments. We also thank the anonymous referees for their careful reading and helpful comments which, in particular, led us to the new introduction focusing on the case of rational elliptic curves.

Liu's work was partially supported by National Science Foundation (NSF) grant DMS-1302000. S. Zhang's work was partially supported by NSF grants DMS-

0970100 and DMS-1065839. W. Zhang's work was partially supported by NSF grants DMS-1301848 and DMS-1601144, and by a Sloan research fellowship.

References

[1] M. BERTOLINI, H. DARMON, and K. PRASANNA, *Generalized Heegner cycles and p-adic Rankin L-series*, with an appendix by B. Conrad, Duke Math. J. **162** (2013), 1033–1148. MR 3053566. DOI 10.1215/00127094-2142056. *(743, 744)*

[2] S. BOSCH, U. GÜNTZER, and R. REMMERT, *Non-Archimedean Analysis*, Grundlehren Math. Wiss. **261**, Springer, Berlin, 1984. MR 0746961. DOI 10.1007/978-3-642-52229-1. *(752)*

[3] N. BOURBAKI, *Lie Groups and Lie Algebras*, Chapters 1–3, reprint of the 1975 edition, Elem. Math. (Berlin), Springer, Berlin, 1989. MR 0979493. *(746, 793)*

[4] E. H. BROOKS, *Generalized Heegner cycles, Shimura curves, and special values of p-adic L-functions*, Ph.D. dissertation, University of Michigan, Ann Arbor, Michigan, 2013. *(744)*

[5] D. BUMP, *Automorphic Forms and Representations*, Cambridge Stud. Adv. Math. **55**, Cambridge Univ. Press, Cambridge, 1997. MR 1431508. DOI 10.1017/CBO9780511609572. *(778, 797)*

[6] L. CAI, J. SHU, and Y. TIAN, *Explicit Gross-Zagier and Waldspurger formulae*, Algebra Number Theory **8** (2014), 2523–2572. MR 3298547. DOI 10.2140/ant.2014.8.2523. *(744)*

[7] H. CARAYOL, *Sur la mauvaise réduction des courbes de Shimura*, Compos. Math. **59** (1986), 151–230. MR 0860139. *(760, 761, 779, 780, 781, 783)*

[8] R. F. COLEMAN, *Torsion points on curves and p-adic abelian integrals*, Ann. of Math. (2) **121** (1985), 111–168. MR 0782557. DOI 10.2307/1971194. *(816, 817)*

[9] F. DIAMOND and R. TAYLOR, *Nonoptimal levels of mod l modular representations*, Invent. Math. **115** (1994), 435–462. MR 1262939. DOI 10.1007/BF01231768. *(782)*

[10] M. EMERTON, *Locally analytic vectors in representations of locally p-adic analytic groups*, Mem. Amer. Math. Soc. **248** (2017), no. 1175. MR 3685952. *(814)*

[11] G. FALTINGS, *Group schemes with strict $\mathcal{O}$-action*, Mosc. Math. J. **2** (2002), 249–279. MR 1944507. *(818)*

[12] B. H. GROSS and D. B. ZAGIER, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320. MR 0833192. DOI 10.1007/BF01388809. *(743)*

[13] P. L. KASSAEI, *$\mathcal{P}$-adic modular forms over Shimura curves over totally real fields*, Compos. Math. **140** (2004), 359–395. MR 2027194. DOI 10.1112/S0010437X03000150. *(779, 783)*

[14] N. M. KATZ, "Travaux de Dwork" in *Séminaire Bourbaki 1971/1972* nos. 400–417, Lecture Notes in Math. **317**, Springer, Berlin, 1973, 167–200. MR 0498577. *(782)*

[15] ———, *Higher congruences between modular forms*, Ann. of Math. (2) **101** (1975), 332–367. MR 0417059. DOI 10.2307/1970994. *(767)*

[16] ———, *p-adic L-functions for CM fields*, Invent. Math. **49** (1978), 199–297. MR 0513095. DOI 10.1007/BF01390187. *(779, 782)*

[17] ———, "Serre–Tate local moduli" in *Algebraic Surfaces (Orsay, 1976–78)*, Lecture Notes in Math. **868**, Springer, Berlin, 1981, 138–202. MR 0638600. *(779, 817, 819, 820, 821, 824, 825, 826, 828, 830)*

[18] C. KHARE and J.-P. WINTENBERGER, *Serre's modularity conjecture, I*, Invent. Math. **178** (2009), 485–504; *II*, 505–586. MR 2551763. DOI 10.1007/s00222-009-0205-7. MR 2551764. DOI 10.1007/s00222-009-0206-6. *(787)*

[19] J. S. MILNE, "The points on a Shimura variety modulo a prime of good reduction" in *The Zeta Functions of Picard Modular Surfaces*, Univ. Montréal, Montreal, 1992, 151–253. MR 1155229. *(760)*

[20] D. MUMFORD, *An analytic construction of degenerating abelian varieties over complete rings*, Compos. Math. **24** (1972), 239–272. MR 0352106. *(817)*

[21] A. MURASE and H.-A. NARITA, *Fourier expansion of Arakawa lifting, II: Relation with central L-values*, Internat. J. Math. **27** (2016), 1650001. MR 3454680. DOI 10.1142/S0129167X16500014. *(746)*

[22] K. A. RIBET, "Abelian varieties over $\mathbf{Q}$ and modular forms" in *Algebra and Topology 1992 (Taejŏn)*, Korea Adv. Inst. Sci. Tech., Taejŏn, 1992, 53–79. MR 1212980. *(787)*

[23] K. RUBIN, *p-adic L-functions and rational points on elliptic curves with complex multiplication*, Invent. Math. **107** (1992), 323–350. MR 1144427. DOI 10.1007/BF01231893. *(743)*

[24] H. SAITO, *On Tunnell's formula for characters of* $\mathrm{GL}(2)$, Compos. Math. **85** (1993), 99–108. MR 1199206. *(745, 814)*

[25] P. SCHNEIDER, *Nonarchimedean Functional Analysis*, Springer Monogr. Math., Springer, Berlin, 2002. MR 1869547. DOI 10.1007/978-3-662-04728-6. *(754)*

[26] P. SCHNEIDER and J. TEITELBAUM, *p-adic Fourier theory*, Doc. Math. **6** (2001), 447–481. MR 1871671. *(744, 749, 752, 753, 754, 755, 756, 757, 758, 759, 789, 790)*

[27] J. L. TAYLOR, *Notes on locally convex topological vector spaces*, preprint, https://www.math.utah.edu/~taylor/LCS.pdf, 1995. *(806)*

[28] J. B. TUNNELL, *Local ϵ-factors and characters of* $\mathrm{GL}(2)$, Amer. J. Math. **105** (1983), 1277–1307. MR 0721997. DOI 10.2307/2374441. *(745, 814)*

[29] J.-L. WALDSPURGER, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compos. Math. **54** (1985), 173–242. MR 0783511. *(744, 745, 746, 799, 810)*

[30] X. YUAN, S.-W. ZHANG, and W. ZHANG, *The Gross–Zagier Formula on Shimura Curves*, Ann. of Math. Stud. **184**, Princeton Univ. Press, Princeton, 2013. MR 3237437. *(744, 746, 753, 759, 771, 785, 786, 794, 810)*

Liu
Department of Mathematics, Northwestern University, Evanston, Illinois, USA;
liuyf@math.northwestern.edu

Zhang
Department of Mathematics, Princeton University, Princeton, New Jersey, USA;
shouwu@math.princeton.edu

Zhang
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; weizhang@mit.edu