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Dissipativity learning control (DLC): Theoretical foundations of input–output data-driven model-free control

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Data-driven, model-free control strategies leverage statistical or learning techniques to design controllers based on data instead of dynamic models. We have previously introduced the dissipativity learning control (DLC) method, where the dissipativity property is learned from the input–output trajectories of a system, based on which L^2-optimal P/PI/PID controller synthesis is performed. In this work, we analyze the statistical conditions on dissipativity learning that enable control performance guarantees, and establish theoretical results on performance under nominal conditions as well as in the presence of statistical errors. The implementation of DLC is further formalized and is illustrated on a two-phase chemical reactor, along with a comparison to model identification-based LQG control.

1. Introduction

Data-driven techniques are playing an increasing role in control due to their potential of easing the derivation and maintenance of dynamic models [1]. Based on the different roles of data, we can classify data-driven control approaches into two categories: *model-based* and *model-free*. In data-driven model-based control, data is mainly used for the identification of a dynamic model. This ranges from the classical approaches of transfer function or linear state–space model construction [2], parameter estimation in adaptive control [3], neural networks [4] to more recent Koopman operator approaches [5]. In these model-based approaches, difficulties of establishing models that accurately describe the system dynamics are often encountered. To relieve the complexity of full system identification, "identification for control" has been pursued, which seeks a model that is sufficient for the resulting control performance [6,7].

In contrast, in data-driven model-free control, one directly seeks to learn from data some *essential control-relevant information*, which can be *much simpler than a dynamic model* but has a *more direct relation to the resulting control performance*. This idea dates back to the traditional PID tuning approaches based on the time and frequency constants on response curves [8]. More generic frameworks such as iterative feedback tuning (IFT) [9] and virtual reference feedback tuning (VRFT) [10] have been developed for linear systems, and the iterative learning control (ILC) approach [11] has been proposed for repetitive control tasks. Preliminary explorations on the theoretical foundations and potential of data-driven model-free control based on behavioral approaches have also been made for linear systems [12–14]. In recent years, approximate dynamic programming (ADP) and reinforcement learning (RL) have gained more popularity [15–17]. These approaches learn the optimal control policy and cost as the essential control-relevant information. However, rooted in Bellman's optimality principle, RL and ADP intrinsically depend on state–space information, either explicitly using full state measurements or implicitly by augmenting observable outputs with their memories, and their application on nonlinear systems is usually limited to small-scale ones with relatively simple dynamics.

We have recently developed [18,19] an *input–output data-driven model-free control* framework, named *dissipativity learning control* (DLC). The DLC framework is built upon the dissipative theory in classical nonlinear control [20–23], where dissipativity is used as a characterization of the input–output behavior of systems and as a basis for controller synthesis, and is also motivated by the works on dissipativity-based control [24–27] where the dissipativity property is obtained through a rigorous thermodynamic analysis. Different from these works on model-based dissipative control, in DLC, the dissipativity property is considered as the essential control-relevant information to be *learned from data* in the form of input–output trajectories under excitations. Such a dissipativity learning approach avoids the restrictions of thermodynamic analysis, and lends itself to a generic data-driven model-free control method applicable to nonlinear systems. We note that in parallel to our works, a series of papers in literature have addressed the determination and verification of input–output properties from trajectory data [28,29] including dissipativity [30–32], and have discussed the conditions and sampling strategies involved in these procedures [33,34].

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Despite the well-established dissipativity-based control theory in a model-based setting and the developments in dissipativity learning approaches, formal guarantees on the control performance of DLC are lacking. In this work, we aim at providing such a theoretical support with a more formalized DLC framework. Specifically, the following key results are established:

- If free of statistical errors, DLC yields the optimal dissipative output-feedback control law that is in a certain nearly L^2-optimal sense defined on a certain neighborhood of the origin;
- The errors resulting from data sampling and statistical inference of the dissipativity property cause an error in the learning result in terms of an upper bound of the L^2-gain from the exogenous disturbances to the inputs and outputs;
- Under small errors in dissipativity learning, the perturbation on the resulting upper bound of L^2-gain is also small, so that nearly L^2-optimal control performance is still achievable.

We introduce a series of novel definitions referring to key objects relevant to dissipativity learning in a generic nonlinear setting, such as admissible inputs (Definition 1), effective reachable domain (Definition 6), effective supply rate (Definition 7), and effective (dual) dissipativity set (Definitions 8 and 9). These concepts lend themselves to a different approach to establish dissipativity learning compared to the existing ones based on iterative experiment design [28] or persistent excitation conditions [32] for linear systems. Based on these definitions, two assumptions are made allowing us to establish the above results, namely the existence of closed-loop invariance on the effective reachable domain (Assumption 1) and a dense sampling oracle on the effective dual dissipativity set (Assumption 2). The latter assumption is analogous to one in [33]. We note that computing the defined objects and verifying the assumptions may not be easy for general nonlinear systems, which is rather expected.

The exposition in the paper is self-contained, without an explicit review of our previous work [18,19]. We first introduce the key control-theoretic concepts underlying dissipativity-based control in Section 2. Statistical aspects of dissipativity learning, including the sampling of trajectories and inference of the effective dual dissipativity set, are discussed in Section 3. A standardized DLC algorithm with a formal guarantee of control performance is formulated in Section 4. We examine such a standardized DLC framework and compare it with linear system identification-based optimal control through a case study on a two-phase chemical reactor in Section 5. Conclusions are given in Section 6.

2. Control-theoretic foundations

2.1. Dissipativity

We consider nonlinear systems in the form of

$$\dot{x}(t) = f(x(t), u(t), d(t)),\ \ y(t) = h(x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the vector of states, $y(t) \in \mathbb{R}^{n_y}$ is the vector of outputs, $u(t) \in \mathbb{R}^{n_u}$ is the vector of control inputs (manipulated variables), and $d(t) \in \mathbb{R}^{n_d}$ is the exogenous disturbances. The vector of inputs $v(t) = (u(t), d(t)) \in \mathbb{R}^{n_v}$ is stacked from manipulated variables and disturbances. The functions f and h are assumed to be Lipschitz continuous, satisfying $f(0, 0, 0) = 0$ and $h(0) = 0$, i.e., the origin is an equilibrium point of (1) giving zero outputs under zero inputs. An output feedback control law is considered as a Lipschitz continuous function $\kappa : \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$, leading to a closed-loop system:

$$\dot{x}(t) = f(x(t), \kappa(h(x(t))), d(t)). \tag{2}$$

For the design of κ, it is desirable that the closed-loop system (2) is asymptotically attracted to the origin from some neighborhood of the origin in the absence of disturbances, or is subject to limited impact of the disturbances. The concept of dissipativity, originally introduced in [35,36] and later developed in [37,38], provides a global description of a fundamental constraint on the input–output behavior of dynamical systems. We note that since a globally dissipative property is difficult to obtain, here we consider dissipativity in a more restricted but practical context.

We first define the set of admissible input signals.

Definition 1 (*Admissible Inputs*)**.** The set of admissible input signals on $[0, t]$, $t > 0$ is the collection of real n_v-dimensional vector-valued continuous functions on $[0, t]$ that has a squared L^2-norm not exceeding $t/2$, and for each of its components, if expressed as cosine series, the contribution to the L^2-norm from all the high-frequency terms with wave number over $N_\mathrm{f} \in \mathbb{N}$ cannot exceed a proportion of a small positive number ϵ_f, i.e.,

$$\begin{aligned}\mathcal{V}_{N_\mathrm{f},\epsilon_\mathrm{f}}(t) = \Bigg\{ v : [0, t] \to \mathbb{R}^{n_v} \Bigg| v^j(\tau) = \frac{a_0^j}{\sqrt{2}} + \sum_{i=1}^{\infty} a_i^j \cos\frac{i\pi\tau}{t} \\ \sum_{i=0}^{\infty}(a_i^j)^2 \le 1, \sum_{i=N_\mathrm{f}}^{\infty}(a_i^j)^2 \le \epsilon_\mathrm{f}\sum_{i=0}^{N_\mathrm{f}-1}(a_i^j)^2, j = 1, \dots, n_v \Bigg\}\end{aligned} \tag{3}$$

Such an admissible input signal set excludes very large or highly oscillatory signals. By using this definition, we are implicitly assuming that for the controller design of the system (1), its behavior under excessively large or oscillatory input signals does not contain any information that is of interest, nor does the controller necessarily result in such signals. This is an assumption on both manipulated inputs and exogenous disturbances. We also define the domain of states that are reachable from the origin within time t.

Definition 2 (*Reachable Domain*)**.** The reachable domain at time t (under admissible input signals) is the endpoint of all trajectories of system (1) on $[0, t]$ under input signals whose restriction on $[0, \tau]$ is admissible for all $\tau \in (0, t]$:

$$\begin{aligned}\mathcal{D}_{N_\mathrm{f},\epsilon_\mathrm{f}}(t) = \Bigg\{ x(t) \Bigg| \dot{x}(\tau) = f(x(\tau), v(\tau)), \tau \in [0, t] \\ v|_{[0,\tau]} \in \mathcal{V}_{N_\mathrm{f},\epsilon_\mathrm{f}}(\tau), \tau \in (0, t], x(0) = 0 \Bigg\}.\end{aligned} \tag{4}$$

For brevity we will usually omit the $N_\mathrm{f}, \epsilon_\mathrm{f}$ in the subscript. Clearly, if $x \in \mathcal{D}(t_1)$ for some $t_1 \ge 0$, then there exists v on $[0, t_1]$ driving the states from the origin to x; then for any $t_2 > t_1$, the signal v with a time delay of $t_2 - t_1$ drives the states from 0 to x in $[0, t_2]$, and hence $x \in \mathcal{D}(t_2)$. This implies $\mathcal{D}(t_1) \subseteq \mathcal{D}(t_2)$ whenever $0 \le t_1 \le t_2$, and hence the reachable domain at time t is equivalent to the reachable domain *within* time t. With these definitions, we reconstruct the dissipative control theory through an adaptation of the approach of [37].

Definition 3 (*Hill-Moylan Inequality*)**.** The system (1) is said to satisfy the Hill-Moylan inequality on $\mathcal{D}(T)$ for some $T > 0$, if for any $t \in [0, T]$ and $v : [0, t] \to \mathbb{R}^{n_v}$ such that $v|_{[0,\tau]} \in \mathcal{V}(\tau)$ for all $\tau \in (0, t]$, the resulting trajectory starting from $x(0) = 0$ on $[0, t]$ satisfies

$$\int_0^t s(y(\tau), v(\tau))d\tau \ge 0 \tag{5}$$

for some continuous function $s : \mathbb{R}^{n_y} \times \mathbb{R}^{n_v} \to \mathbb{R}$. Such a function s is called the supply rate function.

Now we establish the dissipativity property of the system. For this we define the storage function and the concept of early reachable domain.

Definition 4 (*Storage Function*). If the system (1) satisfies the Hill-Moylan inequality on $\mathcal{D}(T)$ for some $T > 0$, then the following function $V(x)$ defined on $\mathcal{D}(T)$, which is positive semidefinite with $V(0) = 0$, is called the storage function:

$$V(x) = \min_{\substack{v|_{[0,\tau]}\in\mathcal{V}(\tau),\ \tau\in(0,t],\ t\le T\\ x(0)=0,\ x(t)=x}} \int_0^t s(y(\tau), v(\tau))d\tau. \tag{6}$$

We refer to any such minimizing input signal from the origin to x as an effective reaching signal.

The minimum in the above definition is well defined due to the Lipschitz continuity of the dynamics, continuity of the storage function, and completeness of the admissible input signal set. Obviously, the minimum can always be found for an input signal with $t = T$ (by using the time-delay argument). However, for studying the dissipativity property, we should consider only the states for which an effective reaching signal exists for some $t < T$, so that this signal may be extended after t.

Definition 5 (*Early Reachable Domain*). The early reachable domain $\mathring{\mathcal{D}}(T)$ is the subset of $\mathcal{D}(T)$ with states x such that an effective reaching signal for x defined in (6) exists on some time t strictly less than T. We call any such effective reaching signal an early reaching signal.

Now we may follow a similar approach to [37] to establish the dissipativity property on the reachable domain.

Lemma 1 (*Hill-Moylan Lemma*). *Suppose that the Hill-Moylan inequality (5) holds. Then for any $x_1 \in \mathring{\mathcal{D}}(T)$, suppose that it has an early reaching signal that is defined on $[0, t_1]$ for some $0 < t_1 < T$, can be extended to some $t_2 \in [t_1, T]$ without violating the admissibility, and drives the states to x_2. Then*

$$V(x_2) - V(x_1) \le \int_{t_1}^{t_2} s(y(t), v(t))dt. \tag{7}$$

We refer to such a property as the dissipativity of storage function V with respect to supply rate s on $\mathcal{D}(T)$, and (7) as the dissipative inequality.

Proof. According to the definition of the storage function (6),

$$V(x_2) - V(x_1) = \min_{\substack{v|_{[0,\tau]}\in\mathcal{V}(\tau),\ \tau\in(0,t],\ t\le T\\ x(0)=0,\ x(t)=x_2}} \int_0^{t_2} s(\tau)d\tau - \int_0^{t_1} s(\tau)d\tau. \tag{8}$$

The lemma is proved by relaxing the minimum term to this hypothetical trajectory on $[0, t_1]$ from the origin to x_1 continued on $[t_1, t_2]$ from x_1 to x_2. □

Therefore, to acquire the knowledge about the dissipativity property of the system (1), one needs to solve the following problem, which we will further discuss later.

Problem 1 (*Determination of the Supply Rate Function*). Given hyperparameters N_f, ϵ_f and T, find (a set of) continuous functions s satisfying the Hill-Moylan inequality.

2.2. Dissipative controller synthesis

Suppose that we now know a supply rate function s (or a set of supply rate functions) of the system. According to the Hill-Moylan Lemma, the system is dissipative with respect to supply rate s for some storage function V. We now consider how the dissipativity property leads to results in controller synthesis to guarantee closed-loop stability. Key to the desirable stability property is to shape the closed-loop supply rate function with bounded nonconcavity:

$$s(y, \kappa(y), d) \le \beta\|d\|^2 - \|\kappa(y)\|^2 - \|y\|^2, \tag{9}$$

for some $\beta > 0$, so that the dissipative inequality (7) constrains the L^2-gain from d to the performance outputs $z = (y, u)$ not to exceed $\beta^{1/2}$.

To guarantee the dissipative inequality throughout the time, we need to prevent the concepts defined in the previous subsection on a finite time horizon from becoming ill-defined when extended to infinite time. Hence, we define for each controller a closed-loop forward invariant set in which the admissibility and effective reaching signal concepts are recursively preserved.

Definition 6 (*Effective Reachable Domain*). The effective reachable domain $\check{\mathcal{D}}^\kappa(T) = \check{\mathcal{D}}^\kappa_{N_f,\epsilon_f}(T)$ of the output feedback control law κ is such a subset of $\mathring{\mathcal{D}}$ satisfying the following condition. For any point $x_0 \in \mathring{\mathcal{D}}(T)$ and for any exogenous disturbance signal that is admissible on $[0, t]$ for all $t > 0$, the closed-loop system starting from x_0 at time 0 under control $u = \kappa(y) = \kappa(h(x))$ remains in $\check{\mathcal{D}}$, and the input signal retains the admissibility of the effective reaching signal for all $t > 0$.

It appears to be difficult to characterize the effective reachable domain and prove whether it is a connected open set. For simplicity, we make the following assumption.

Assumption 1 (*Closed-loop Invariance in the Effective Reachable Domain*). Assume that when the output-feedback law κ is chosen within a predefined range of interest $\mathcal{K}$, there is a neighborhood of the origin $\check{\mathcal{D}}$ that is in the intersection of all the effective reachable domains, i.e., $\check{\mathcal{D}}(T) \subseteq \cap_{\kappa\in\mathcal{K}}\check{\mathcal{D}}^\kappa(T)$.

With a slight abuse of terminology, we still call this $\check{\mathcal{D}}(T)$ the effective reachable domain. Clearly, it is the domain on which the dissipative inequality (7) holds recursively. Moreover, since we only discuss the dissipativity on the effective reachable domain $\check{\mathcal{D}}(T)$ under controller κ, it suffices to have a relaxed definition of supply rate, called effective supply rate function. Of course, since $\check{\mathcal{D}}(T)$ is difficult to know, so is the effective supply rate.

Definition 7 (*Effective Supply Rate*). An effective supply rate function is a continuous function s such that for any state in $\check{\mathcal{D}}(T)$, the Hill-Moylan inequality holds for some of its early reaching signals. We refer to the set of supply rate functions as $\mathcal{S}$ and the set of effective supply rate functions as $\check{\mathcal{S}}$. Then $\check{\mathcal{S}} \supseteq \mathcal{S}$.

Now we establish the guarantee of L^2-stability according to the following proposition, for which the proof is obtained by combining (7) and (9).

Lemma 2 (*Stability of Dissipative Control*). *Suppose that there exist an output-feedback control law $\kappa \in \mathcal{K}$ and an effective supply rate function $s \in \check{\mathcal{S}}$ satisfying (9) for some $\beta > 0$. Then under the control law κ, for any $t > 0$, if the disturbances are admissible on $[0, \tau]$ for all $\tau \in (0, t]$, then we have a nonnegative constant C such that*

$$\int_0^t \left(\|y(\tau)\|^2 + \|u(\tau)\|^2\right) d\tau \le \beta \int_0^t \|d(\tau)\|^2 d\tau + C \tag{10}$$

holds for any initial condition $x(0) \in \check{\mathcal{D}}(T)$, and hence the exogenous disturbance has a L^2-gain to $z = (y, u)$ not exceeding $\beta^{1/2}$. In other words, the closed-loop system is L^2-stable.

Hence the controller synthesis problem based on dissipativity is formally stated as the following problem.

Problem 2 (*Determination of the Control Law*)**.** Given a (or a set of) supply rate function s with respect to which the system is dissipative on $\mathcal{D}_{N_f,\epsilon_f}(T)$, find an output feedback control law κ such that $s(y,\kappa(y),d) \le -\|y\|^2 - \|\kappa(y)\|^2 + \beta\|d\|^2$ for some (or the smallest) $\beta > 0$.

For computational tractability of the controller synthesis, we consider supply rate functions in quadratic forms:

$$s(y,u,d) = \begin{bmatrix} y^\top & u^\top & d^\top \end{bmatrix} \Pi \begin{bmatrix} y \\ u \\ d \end{bmatrix} \tag{11}$$

and hence the supply rate function is represented by a symmetric matrix $\Pi \in \mathbb{R}^{(n_y+n_v)\times(n_y+n_v)}$. We also consider controllers in linear form $\kappa(y) = Ky$. The condition in Lemma 2 thus results in a semidefinite programming problem of finding a feasible or optimal solution of (K,β) satisfying the following matrix inequality

$$\begin{bmatrix} I & K^\top & 0 \\ 0 & 0 & I \end{bmatrix} \left(\Pi + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\beta I \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ K & 0 \\ 0 & I \end{bmatrix} \preceq 0. \tag{12}$$

Remark 1 (*PID Control*)**.** Although we have represented the controller in the form of proportional feedback, the semidefinite programming formulation can be extended to more general linear controller forms such as PID controllers, by simply augmenting the output variables with its derivatives and the state variables with an integrator. If the set of supply rate functions is accurately known and the optimal controller gain is optimized over all supply rate functions, then we obtain theoretically the optimal dissipative P/PI/PID controller in the L^2-sense within the controller range $\mathcal{K}$.

Remark 2 (*Tracking Control*)**.** For tracking control tasks where the goal is to drive the system towards a reference trajectory rather than to the predefined steady state (origin), it suffices to split the process variables (v,x,y) into the corresponding reference variables $(\bar{v},\bar{x},\bar{y})$ (whose trajectory is specified a priori) and deviation variables $(\tilde{v},\tilde{x},\tilde{y})$ (so that the goal is to drive the deviations to zero), and correspondingly redefine the reachable domain, storage function, and supply rate functions based on the origin of deviation variables. This was discussed in our previous work [19].

2.3. Dissipativity and dual dissipativity sets

Suppose that the model of the system (1) is unknown. In order to find a (or a set of) supply rate function parameterized by a quadratic form (11) in a data-driven setting, we first note that the Hill-Moylan inequality (5) can be rewritten as

$$\langle \Pi, \int_0^t \begin{bmatrix} y(\tau) \\ u(\tau) \\ d(\tau) \end{bmatrix} \begin{bmatrix} y(\tau)^\top & u(\tau)^\top & d(\tau)^\top \end{bmatrix} d\tau \rangle \ge 0, \tag{13}$$

where the inner product $\langle M_1, M_2\rangle$ for any two symmetric matrices M_1 and M_2 is defined as trace(M_1M_2), with $\langle M,M\rangle = \|M\|^2$ being the squared Frobenius norm. We note that Π as a representation of the supply function is a property of the system, and the integral above is a property of the excitation input signal. Hence we introduce the following definitions in dual.

Definition 8 (*Dissipativity Parameters and Sets*)**.** The matrix Π is called the (matrix of) dissipativity parameters. The set of dissipativity parameters such that the corresponding supply rate function (11) is in $\mathcal{S}$ is called the dissipativity set (still denoted as $\mathcal{S}$). The dissipativity parameter set Π is said to be effective if the corresponding supply rate is effective. The set of effective dissipativity parameters is called the effective dissipativity set (still denoted as $\check{\mathcal{S}}$).

Definition 9 (*Dual Dissipativity Parameters and Sets*)**.** The dual dissipativity parameter of an excitation input signal on $[0,t]$ that is admissible on all $[0,\tau]$ for $\tau \in (0,t]$ is defined based on the resulting excited trajectory as

$$\Gamma = \int_0^t \begin{bmatrix} y(\tau) \\ u(\tau) \\ d(\tau) \end{bmatrix} \begin{bmatrix} y(\tau)^\top & u(\tau)^\top & d(\tau)^\top \end{bmatrix} d\tau. \tag{14}$$

The set of dual dissipativity parameters is called the dual dissipativity set (denoted as $\mathcal{G}$), and the set of dual dissipativity parameters of effective input signals is called the effective dual dissipativity set (denoted as $\check{\mathcal{G}}$).

It directly follows from these definitions that $\mathcal{S} \subseteq \check{\mathcal{S}}$, $\mathcal{G} \supseteq \check{\mathcal{G}}$, and $\mathcal{S}$ and $\check{\mathcal{S}}$ are the dual cones of $\mathcal{G}$ and $\check{\mathcal{G}}$, respectively, i.e.,

$$\begin{aligned} \mathcal{S} &= \mathcal{G}^* = \{\Pi | \langle \Pi, \Gamma\rangle \ge 0, \forall \Gamma \in \mathcal{G}\} \\ \check{\mathcal{S}} &= \check{\mathcal{G}}^* = \{\Pi | \langle \Pi, \Gamma\rangle \ge 0, \forall \Gamma \in \check{\mathcal{G}}\}. \end{aligned} \tag{15}$$

The dissipativity sets and dual dissipativity sets defined above are also hyper-parameterized by N_f, ϵ_f and T. These symbols are omitted for brevity. For a quadratic supply rate function and linear feedback laws, we can restate Lemma 2 as

Lemma 3 (*Stability of Dissipative Control*)**.** *Suppose that there exist $\Pi \in \check{\mathcal{G}}^*$, $K \in \mathcal{K}$, and $\beta > 0$ satisfying (12). Then the closed-loop system is L^2-stable in $\check{\mathcal{D}}$.*

Therefore, the task of obtaining a dissipative controller for system (1), involving the determination of supply rate function (Problem 1) and the output-feedback control (Problem 2), is stated as follows.

Problem 3 (*Dissipativityb-based Control*)**.** Given hyperparameters N_f, ϵ_f and T, obtain $\check{\mathcal{G}}$ and its dual cone $\check{\mathcal{S}} = \check{\mathcal{G}}^*$, and solve for a control law K satisfying (12) with some $\Pi \in \check{\mathcal{S}}$ and $\beta > 0$.

3. Statistical foundations

In a data-driven setting, the determination of the effective dual dissipativity set $\check{\mathcal{G}}$ and its dual cone for the system (1) is done on the basis of statistical inference. That is, an estimation of $\check{\mathcal{G}}$, denoted as $\hat{\mathcal{G}}$ and its dual cone $\hat{\mathcal{S}}$, as an estimation of $\check{\mathcal{S}}$, should be obtained from some data samples of the system. In this section, we discuss the effect of the sampling and inference steps, and provide guidelines on how to collect samples and conduct inference.

3.1. Sampling of input excitations

In order to obtain estimations of the dissipativity and dual dissipativity sets, one needs to create samples of admissible input signals to excite the system (1) from the origin, collect the resulting output trajectories, and calculate the corresponding dual dissipativity parameters Γ. We note that each admissible input signal in $\mathcal{V}_{N_f,\epsilon_f}(T)$ is identical to an infinite-dimensional point (series) a in $\mathcal{A}_{N_f,\epsilon_f}^{n_v}(T)$, where

$$\mathcal{A}_{N_f,\epsilon_f}(T) = \left\{ (a_0, a_1, \dots) \middle| \sum_{i=0}^{\infty} a_i^2 \le 1, \sum_{i=N_f}^{\infty} a_i^2 \le \epsilon_f \sum_{i=0}^{N_f-1} a_i^2 \right\}. \tag{16}$$

Sampling from $\mathcal{A}_{N_f,\epsilon_f}^{n_v}(T)$ as an infinite-dimensional set is apparently not tractable. Hence we consider sampling on its approximation as a finite-dimensional unit ball centered at the origin:

$$\hat{\mathcal{A}}_{N_f,\epsilon_f}(T) = \left\{ (a_0, a_1, \dots, a_{N_f}) \middle| \sum_{i=0}^{N_f-1} a_i^2 \le 1 \right\} = \mathbb{B}_{N_f}(0,1). \tag{17}$$

Every element of $\hat{\mathcal{A}}(T)$ is also in $\mathcal{A}(T)$ if suffixed with infinite number of zeros, and in this sense $\hat{\mathcal{A}}(T)$ is a subset of $\mathcal{A}(T)$. For each $a \in \mathcal{A}(T)$, one can obviously find a corresponding $\hat{a} \in \hat{\mathcal{A}}(T)$ such that $\|a - \hat{a}\| \leq \epsilon_{\mathrm{f}}\|\hat{a}\|$ by grounding the numbers after the N_{f}th to zero.

Therefore, if one samples in $\hat{\mathcal{A}}(T)$ in a sufficiently "proportionally dense" manner, such that for any $\hat{a} \in \hat{\mathcal{A}}(T)$, there exists a sample a' that is close enough to $\hat{a}$ with $\|a' - \hat{a}\| \leq \epsilon_{\hat{a}}\|a'\|$ for some small $\epsilon_{\hat{a}} > 0$, then for any $a \in \mathcal{A}(T)$, there exists a sample a' with $\|a' - a\| \leq \epsilon_a\|a'\|$ for some $\epsilon_a > 0$. We note that the ℓ^2-distance on the space of Fourier coefficients equals the L^2-distance of the input signals. Hence a proportionally dense sampling on $\hat{\mathcal{A}}(T)$ implies a proportionally dense sampling of admissible input signals, which also implies a proportionally dense sampling on the dual dissipativity set $\mathcal{G}$.

Lemma 4 (*Denseness of Sampling on the Dual Dissipativity Set*)**.** *If for any input signal $v \in \mathcal{A}(T)$, there exists a sample input signal $v^{(p)} \in \hat{\mathcal{A}}(T)$ (where p is the sample index) such that $\|v - v^{(p)}\|_{L^2([0,T])} \leq \epsilon_v\|v^{(p)}\|_{L^2([0,T])}$ for some small $\epsilon_v > 0$ (assuming $\epsilon_v < 1$), then there exist constants $C_0, C_1, C_2 > 0$ such that the corresponding dual dissipativity parameters Γ and $\Gamma^{(p)}$ satisfy*

$$\|\Gamma - \Gamma^{(p)}\| \leq C_0(1 + C_1 e^{C_2 T})\epsilon_v \mathrm{trace}(\Gamma^{(p)}). \tag{18}$$

Proof. Consider the incremental dynamics

$$\Delta\dot{x} = f(x + \Delta x, v + \Delta v) - f(x, v), \quad \Delta y = h(x + \Delta x) - h(x). \tag{19}$$

Let the f and h have Lipschitz constants $L_{f,x}$, $L_{f,v}$ and $L_{h,x} > 0$, respectively. Then we obtain

$$\|\Delta\dot{x}\| \leq L_{f,x}\|\Delta x\| + L_{f,v}\|\Delta v\|, \quad \|\Delta y\| \leq L_{h,x}\|\Delta x\|. \tag{20}$$

Applying Grönwall inequality and Cauchy–Schwartz inequality, one can verify that

$$\|\Delta y\|_{L^2([0,T])} \leq \frac{L_{h,x}L_{f,v}}{2L_{f,x}} e^{L_{f,x}T}\|\Delta v\|_{L^2([0,T])}. \tag{21}$$

The resulting difference of the dual dissipativity parameters can be shown to be upper bounded by

$$\|\Delta\Gamma\|^2 \leq \frac{1}{3}\left(\frac{L_{h,x}^2 L_{f,v}^2}{4L_{f,x}^2} e^{2L_{f,x}T} + 1\right)^2 (2\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2)\|\Delta v\|_{L^2}^2, \tag{22}$$

and hence

$$\|\Delta\Gamma\| \leq \frac{2}{\sqrt{3}}\left(1 + \frac{L_{h,x}^2 L_{f,v}^2}{4L_{f,x}^2} e^{2L_{f,x}T}\right)\epsilon_v\|v\|_{L^2}. \tag{23}$$

This leads to the conclusion of the lemma with $C_0 = 2/\sqrt{3}$, $C_1 = L_{h,x}^2 L_{f,v}^2 / 4L_{f,x}^2$ and $C_2 = 2L_{f,x}$, since $\mathrm{trace}(\Gamma^{(p)}) = \|\Delta v\|_{L^2}^2 + \|\Delta y\|_{L^2}^2$. □

Remark 3 (*Dense Sampling and Curse of Dimensionality*)**.** The idea of dense sampling on the space of trajectories to accurately learn input–output properties was earlier proposed by Montenbruck & Allgöwer [33] in the context of operator norm over-estimation. In our approach, through the admissibility concept in Definition 1, we handle this in a Euclidean space of Fourier coefficients. As the dimension of the input signal increases, the dense sampling requires a drastic increase of the number of trajectories to be collected. This curse of dimensionality poses a limitation of dissipativity learning approaches for systems with high-dimensional inputs.

If the sampling is dense on $\mathcal{G}$, then the remaining task is to select the samples that are useful to estimate the effective subset $\check{\mathcal{G}}$, and perform the inference of $\check{\mathcal{G}}$. Such a data cleaning step is a common practice in data-driven techniques. From a theoretical point of view, we can only assume that the user is given a priori a proper strategy to choose samples that are sufficiently informative to approximately cover the effective dual dissipativity set.

Assumption 2 (*Effective Sample Selection Oracle*)**.** Assume that for any dual dissipativity parameter Γ in the effective dual dissipativity set $\check{\mathcal{G}}$, there is a selected sample $\Gamma^{(p)}$ among the set of samples $\Gamma^{(1)}, \dots, \Gamma^{(P)}$ such that

$$\|\Gamma - \Gamma^{(p)}\| \leq \epsilon_\Gamma \mathrm{trace}(\Gamma^{(p)}) \tag{24}$$

for some $\epsilon_\Gamma > 0$. Alternatively we may assume that the samples are directly generated so that the above inequality is satisfied.

Since such an oracle is unavailable in reality, it must be approximated by a heuristic rule. Here we propose a simple heuristic based on the intuition that the effective reaching signal should tend to result in a smaller extent of oscillations in the inputs and outputs. If expressed as Fourier series, these input signals should have larger (in absolute values) low-frequency coefficients than the high-frequency coefficients. Hence we sort the components of the sample points in $\hat{\mathcal{A}}_{N_{\mathrm{f}},\epsilon_{\mathrm{f}}}(T)$ in the descending order of absolute values and use the sorted $a \in \hat{\mathcal{A}}(T)$ to replace the original one. We shall refer to this rule as the *sorting heuristics* later.

Remark 4 (*Negligibly Small Excitations*)**.** The condition of proportionally dense sampling is in fact not realizable with a finite sample set, since the excitation input signal may be arbitrarily small. We need to compromise such excessively small signals, which drive the states from the origin to a small neighborhood. In that case the guaranteed L^2-stability degrades into a practical stability on the effective reaching domain excluding such a small neighborhood.

3.2. Inference of effective dissipativity set

Now suppose that a dense sampling is performed on $\mathcal{G}$ and effective samples $\Gamma^{(p)}$, $p = 1, \dots, P$ are selected such that the condition in Assumption 2 is satisfied. A statistical inference procedure is subsequently used to give an approximate description of the effective dual dissipativity set $\check{\mathcal{G}}$. It is easily verifiable that, if such a set $\hat{\mathcal{G}}$ estimated from samples may leave a sample out for a small enough distance, i.e., for any $\Gamma^{(p)}$ there exists a $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that for a small $\epsilon_{\hat{\Gamma}} > 0$, $\|\Gamma^{(p)} - \hat{\Gamma}\| \leq \epsilon_{\hat{\Gamma}}\|\hat{\Gamma}\|$, then there exists a $0 < \epsilon_\Gamma^+ \leq \epsilon_\Gamma + \epsilon_{\hat{\Gamma}} + \epsilon_\Gamma\epsilon_{\hat{\Gamma}}\sqrt{n_y + n_v} > 0$ such that $\|\Gamma^{(p)} - \hat{\Gamma}\| \leq \epsilon_\Gamma^+\|\hat{\Gamma}\|$. Hence for any $\Gamma \in \check{\mathcal{G}}$ there exists $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that $\|\Gamma - \hat{\Gamma}\| \leq \epsilon\|\hat{\Gamma}\|$ for some $0 < \epsilon < \epsilon_\Gamma^+ + \epsilon_{\hat{\Gamma}}$. Simply stated, if the sampling is effective enough and the inference of $\check{\mathcal{G}}$ is close to the samples, then the estimated $\hat{\mathcal{G}}$ is close to the actual $\check{\mathcal{G}}$.

The following proposition establishes the impact of the accuracy of $\hat{\mathcal{G}}$ on the inference of its dual cone $\hat{\mathcal{S}}$.

Lemma 5 (*Accuracy of Effective Dual Dissipativity Set*)**.** *Suppose that for any $\Gamma \in \check{\mathcal{G}}$ there exists a $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that $\|\Gamma - \hat{\Gamma}\| \leq \epsilon\|\hat{\Gamma}\|$, then*

$$\check{\mathcal{S}} \supseteq \bigcap_{\hat{\Gamma}\in\hat{\mathcal{G}}} \left\{\hat{\Pi} \,|\, \langle\hat{\Pi}, \hat{\Gamma}\rangle \geq \epsilon\|\hat{\Pi}\| \cdot \|\hat{\Gamma}\|\right\}. \tag{25}$$

The right-hand side can be viewed as a "modified dual cone" of $\hat{\mathcal{G}}$ (the angle between any its element and any element of $\hat{\mathcal{G}}$ is an acute angle not exceeding $\arccos\epsilon$). Therefore for any $\hat{\Pi} \in \hat{\mathcal{S}}$, there exists a $\Pi \in \check{\mathcal{S}}$ with $\|\Pi - \hat{\Pi}\| \leq \delta\|\hat{\Pi}\|$ with $\delta \in [0, 2\sin\arcsin(\epsilon/2)]$.

Proof. According to the condition of the lemma, we have

$$\check{\mathcal{G}} \subseteq \{\hat{\Gamma} + \Delta\|\hat{\Gamma}\| | \hat{\Gamma} \in \hat{\mathcal{G}}, \|\Delta\| \leq \epsilon\}. \tag{26}$$

Hence $\check{\mathcal{S}}$ is a superset of the dual cone of the right-hand side of the "$\subseteq$" symbol. Since for each $\hat{\Gamma}$,

$$\{\hat{\Gamma} + \Delta\|\hat{\Gamma}\||, \|\Delta\| \leq \epsilon\}^* = \{\hat{\Pi} | \langle \hat{\Pi}, \hat{\Gamma}\rangle + \|\hat{\Gamma}\| \min_{\|\Delta\|\leq\epsilon} \langle \hat{\Pi}, \Delta\rangle \geq 0\}, \tag{27}$$

where the minimization leads to a $-\epsilon\|\hat{\Pi}\| \cdot \|\hat{\Gamma}\|$ term, by taking the intersection over all $\hat{\Gamma} \in \hat{\mathcal{G}}$ we see the conclusion. □

For the estimation of the efficient reaching domain $\hat{\mathcal{G}}$, any anomaly detection, one-class classification, probability density estimation, or hull algorithm is in principle suitable. However, for computational tractability of the controller synthesis, we need the estimate $\hat{\mathcal{G}}$ to have a simple convex form such as polyhedron, polyhedral cone, ellipsoid, or second-order cone. In our previous works [18,19] we have used support vector machine and probability estimation with independent bi-exponential distributions for polyhedral sets. Here we provide the procedure for inferring $\hat{\mathcal{G}}$ as a second-order cone.

Note that any $\Gamma^{(p)}$ must be a semidefinite matrix, whose trace is positive except for the singular case of the zero-input trajectory, which we can assume does not exist in the sample. We first normalize all the samples of dual dissipativity parameters to trace 1:

$$\Gamma_+^{(p)} = \Gamma^{(p)}/\|\Gamma^{(p)}\|, \; p = 1, \dots, P. \tag{28}$$

Then a principal component analysis (PCA) algorithm is applied to the trace-1 samples. For this purpose, a basis $\{E_k\}_{k=1}^K$ for (n_y+n_v)-order symmetric matrices is chosen, which vectorizes the samples into

$$\gamma^{(p)} = [\langle E_k, \Gamma_+^{(p)}\rangle]_{k=1}^K. \tag{29}$$

By finding the sample average $\bar{\gamma}$ and the diagonal matrix of component-wise standard deviations $D = \mathrm{diag}(D_k, k = 1, \dots, K)$, the vectorized samples are whitened by translation and scaling:

$$\begin{aligned} &\bar{\gamma} = \frac{1}{P}\sum_{p=1}^{P}\gamma^{(p)}, \; D_k = \frac{1}{P-1}\sum_{p=1}^{P}(\gamma_k^{(p)} - \bar{\gamma}_k)^2, \\ &\gamma_+^{(p)} = D^{-1}(\gamma^{(p)} - \bar{\gamma}). \end{aligned} \tag{30}$$

Then the ellipsoidal range of $\gamma_+^{(p)}$, $p = 1, \dots, P$ is found by a singular value decomposition (SVD) of the horizontally stacked matrix of samples,

$$\frac{1}{\sqrt{P-1}}\begin{bmatrix}\gamma_+^{(1)} & \dots & \gamma_+^{(P)}\end{bmatrix} = USV^\top, \tag{31}$$

where U and V are orthogonal matrices, and $S \in \mathbb{R}^{K\times P}$ is a diagonal matrix of singular values (SVs) $s_i \geq 0$ (in descending order), i.e., if subscribing the column vectors of U and V and the SVs, then the right-hand side above is $\sum_{i=1}^{\min(K,P)} s_i u_i v_i^\top$. Typically sufficiently small SVs are neglected by choosing the number of principal values J according to a certain rule, then

$$\frac{1}{\sqrt{P-1}}\begin{bmatrix}\gamma_+^{(1)} & \dots & \gamma_+^{(P)}\end{bmatrix} \approx U_{1:J}S_{1:J}V_{1:J}^\top, \tag{32}$$

which implies that the sample covariance matrix is approximately $U_{1:J}S_{1:J}^2U_{1:J}^\top$ (the subscript $1:J$ stands for columns 1 to J for U and V, and also rows 1 to J for S). Suppose that the samples are normally distributed. Then the vectors of principle components (PCs)

$$\eta^{(p)} = S_{1:J}^{-1}U_{1:J}^\top\gamma_+^{(p)} \in \mathbb{R}^J, \; p = 1, \dots, P \tag{33}$$

should be samples from a standard normal distribution, for which the ℓ^2-norm is the Hotelling statistics, and a threshold hyperparameter $\Theta > 0$ can be chosen.

Through the PCA, an estimation $\hat{\mathcal{G}}$ is obtained as a confidence region in the form of

$$\begin{aligned} \hat{\mathcal{G}} &= \{\eta | \|\eta\| \leq \Theta\} = \{\gamma_+ | \gamma_+ = U_{1:J}S_{1:J}\eta, \|\eta\| \leq \Theta\} \\ &= \{\gamma = \bar{\gamma} + DU_{1:J}S_{1:J}\eta, \|\eta\| \leq \Theta\} \end{aligned} \tag{34}$$

Call the following matrix as the mixing matrix:

$$G = DU_{1:J}S_{1:J}. \tag{35}$$

Then

$$\hat{\mathcal{G}} = \left\{\Gamma \middle| \Gamma = r\sum_{k=1}^{K}\gamma_k E_k, \gamma = \bar{\gamma} + G\eta, r \geq 0, \|\eta\| \leq \Theta\right\}, \tag{36}$$

whose dual cone is expressed as

$$\hat{\mathcal{S}} = \left\{\Pi \middle| \Pi = \sum_{k=1}^{K}\pi_k E_k, \langle \pi, \bar{\gamma}\rangle - \Theta\|G^\top\pi\| \geq 0\right\}. \tag{37}$$

3.3. The effect of learning on control

Finally, we consider the impact of the statistical errors on the resulting control performance in terms of the upper bound on the L^2-gain. Call the third matrix on the left-hand side of the matrix inequality (12) as the loop-closing matrix, and denote it as H. The perturbation on the resulting L^2-gain is then considered by the modification of the matrix inequality.

Lemma 6 (*Resulting Control Performance*)**.** *Let*

$$H = \begin{bmatrix} I & 0 \\ K & 0 \\ 0 & I \end{bmatrix}. \tag{38}$$

Suppose that the matrix inequality

$$H^\top\left(\hat{\Pi} + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\beta I\end{bmatrix}\right)H \preceq 0, \tag{39}$$

holds for some $\hat{\Pi} \in \hat{\mathcal{S}}$, and that there exists a $\Pi \in \check{\mathcal{S}}$ such that $\|\Pi - \hat{\Pi}\| \leq \delta\|\hat{\Pi}\|$. Then the square L^2-gain from exogenous d to $z = (y, u)$ is guaranteed to be upper bounded by $\beta + \delta_\beta$ on $\check{\mathcal{D}}$ with $\delta_\beta = (1+\beta)\delta\|\hat{\Pi}\|/(1 - \delta\|\hat{\Pi}\|)$, if $\beta\|\hat{\Pi}\| \leq 1$.

Proof. Since for any symmetric matrix the spectral radius cannot exceed its Frobenius norm, $\|\Pi - \hat{\Pi}\| \leq \delta\|\hat{\Pi}\|$ implies that

$$\Pi - \hat{\Pi} - \begin{bmatrix} \delta\|\hat{\Pi}\|I & 0 & 0 \\ 0 & \delta\|\hat{\Pi}\|I & 0 \\ 0 & 0 & \delta\|\hat{\Pi}\|I \end{bmatrix} \preceq 0. \tag{40}$$

By multiplying the left-hand side by $H^\top$ on the left and H on the right, adding it to (39), and multiplying by $1/(1 - \delta\|\hat{\Pi}\|)$, we obtain:

$$H^\top\left(\frac{1}{1 - \delta\|\hat{\Pi}\|}\Pi + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (\beta + \delta_\beta)I\end{bmatrix}\right)H \preceq 0. \tag{41}$$

Since $\check{\mathcal{S}}$ is a cone, when $\Pi \in \check{\mathcal{S}}$, $\frac{1}{1-\delta\|\hat{\Pi}\|}\Pi \in \check{\mathcal{S}}$. According to Lemma 2, the conclusion is proved. □

Due to the possibility that the adopted schemes of sample generation, selection and statistical inference may not be effective enough to cover a significant part of $\check{\mathcal{G}}$ (satisfying the condition of Lemma 5), the above-mentioned perturbed control performance

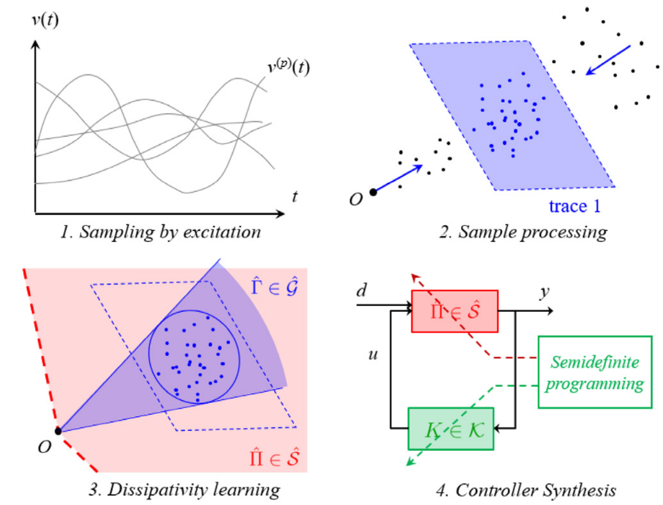

Fig. 1. Procedures involved in the DLC framework.

is guaranteed only on a subset of $\check{\mathcal{D}}$ that is positive-invariant under control laws in $\mathcal{K}$ without violating admissibility of input signals. Finally we assume that this true domain of attraction Ω is still a neighborhood of the origin with a satisfactory size.

4. The DLC framework

4.1. General DLC algorithm and its performance

At this point we summarize the results in the previous two sections. Generally, the DLC approach to input–output data-driven model-free control contains the following procedures, for which an illustration is given in Fig. 1.

1. *Sampling by excitation*: Densely choose admissible input signals by the vector of Fourier coefficients from (17), and heuristically select effective samples (e.g., using the sorting heuristics). Alternatively, samples may be generated directly by a heuristic rule.
2. *Sample processing*: Calculate the dual dissipativity parameters of the sample trajectories by (14). Then choose matrix basis and vectorize the dual dissipativity parameters normalized to trace 1 as in (28) and (29).
3. *Dissipativity learning*: Perform a PCA procedure on the vector samples as in (30) and (32) to find the mixing matrix by (35). Characterize the dissipativity set by (37).
4. *Controller synthesis*: Find a solution (or the optimal solution) of K satisfying the matrix inequality (39) with some $\hat{\Pi} \in \hat{\mathcal{S}}$ and some (or the smallest) $\beta \geq 0$.

The performance guarantee of DLC is formalized as the following theorem, which follows from the lemmas in the previous two sections.

Theorem 1 (*Performance of DLC*)**.** *Suppose that*

- *Assumption 1 (on the existence of an effective reaching domain in the vicinity of the origin) and Assumption 2 (on the selection of dense samples around the effective reaching domain) hold;*
- *For any $\Gamma \in \hat{\mathcal{G}}$, there exists $\hat{\Gamma} \in \hat{\mathcal{G}}$ such that $\|\Gamma - \hat{\Gamma}\| \leq \epsilon \|\hat{\Gamma}\|$ for some $\epsilon > 0$;*
- *There exist $K \in \mathcal{K}$, $\hat{\Pi} \in \hat{\mathcal{S}}$, and $\beta \geq 0$ satisfying (39), with H specified in (38);*
- $2 \sin \arcsin(\epsilon/2) \cdot \|\hat{\Pi}\| < 1$.

Then on $\check{\mathcal{D}}$ the closed-loop system under $u = Ky$ is L^2-stable, with the L^2-gain from d to $z = (y, u)$ upper bounded by $\beta + \delta_\beta$, where $\delta_\beta = (1+\beta)\delta\|\hat{\Pi}\|/(1-\delta\|\hat{\Pi}\|)$.

4.2. Solution of the DLC algorithm

For controller synthesis where we consider to seek the optimal solution $(K, \hat{\Pi}, \beta) \in \mathcal{K} \times \hat{\mathcal{S}} \times [0, +\infty)$, it often helps to accelerate the semidefinite programming problem solution by tightening the feasible region. We first impose constraints $\hat{\Pi}_{vv} \succeq 0$, which indicates that an excitation at the origin cannot lead to any further decrease of the storage function. The two-letter subscript stands for the matrix block whose rows and columns correspond to the indicated process variables, i.e.,

$$\Pi = \begin{bmatrix} \Pi_{yy} & \Pi_{yu} & \Pi_{yd} \\ \Pi_{uy} & \Pi_{uu} & \Pi_{ud} \\ \Pi_{dy} & \Pi_{du} & \Pi_{dd} \end{bmatrix} = \begin{bmatrix} \Pi_{yy} & \Pi_{yv} \\ \Pi_{vy} & \Pi_{vv} \end{bmatrix}. \tag{42}$$

We then note that the matrix inequality (39) is non-convex but multi-convex in $(\hat{\Pi}, \beta)$ and K as two groups of variables. An iterative algorithm can hence be adopted [19], i.e., in each iteration, K is first fixed to solve the remaining variables, and then the remaining variables are fixed to update K.

To update K, we let $\Pi' = \hat{\Pi} + \mathrm{diag}(I, I, -\beta I)$ and rewrite (39) as

$$\begin{bmatrix} \Pi'_{yy} + K^\top \Pi'^\top_{yu} + \Pi'_{yu} K + K^\top \Pi'_{uu} K & \Pi'_{yd} + K^\top \Pi'_{ud} \\ \Pi'^\top_{yd} + \Pi'^\top_{ud} K & \Pi'_{dd} \end{bmatrix} \preceq 0. \tag{43}$$

When (39) is satisfied by some Π and β, the bottom-right block Π'_{dd} is negative semidefinite. The Schur complement of the above matrix over the Π'_{dd} block is therefore

$$\begin{aligned} &K^\top \left(\Pi'_{uu} - \Pi'_{ud}\Pi'^{-1}_{dd}\Pi'^\top_{ud}\right) K + K^\top \left(\Pi'^\top_{yu} - \Pi'_{ud}\Pi'^{-1}_{dd}\Pi'^\top_{yd}\right) \\ &+ \left(\Pi'_{yu} - \Pi'_{yd}\Pi'^{-1}_{dd}\Pi'^\top_{ud}\right) K + \left(\Pi'_{yy} - \Pi'_{yd}\Pi'^{-1}_{dd}\Pi'^\top_{yd}\right). \end{aligned} \tag{44}$$

Since $\Pi'_{uu} \succeq 0$, $\Pi'_{dd} \preceq 0$, and hence $\Pi'_{uu} - \Pi'_{ud}\Pi'^{-1}_{dd}\Pi'^\top_{ud} \succeq 0$, the negative semidefiniteness of the Schur complement is best achieved by a controller gain matrix of

$$K = \mathrm{proj}_{\mathcal{K}} \left(-\left(\Pi'_{uu} - \Pi'_{ud}\Pi'^{-1}_{dd}\Pi'^\top_{ud}\right)^{-1} \left(\Pi'^\top_{yu} - \Pi'_{ud}\Pi'^{-1}_{dd}\Pi'^\top_{yd}\right)\right) \tag{45}$$

where the projection operator $\mathrm{proj}_{\mathcal{K}}(\cdot)$ means the element of $\mathcal{K}$ with the smallest distance to the object in the parentheses, which is computationally tractable if $\mathcal{K}$ is convex.

Hence each iteration executes the following three steps.

1. Obtain $(\hat{\Pi}, \beta)$ from

$$\begin{aligned} \min\ \beta \quad \text{s.t. } & H^\top \left(\hat{\Pi} + \mathrm{diag}(I, I, -\beta I)\right) H \preceq 0 \\ & \hat{\Pi} \in \hat{\mathcal{S}},\ \hat{\Pi}_{vv} \succeq 0,\ \beta \geq 0; \end{aligned} \tag{46}$$

2. Let $\Pi' = \hat{\Pi} + \mathrm{diag}(I, I, -\beta I)$ and update K by (45);
3. Reset the loop-closing matrix H according to (38).

4.3. Tuning of the DLC algorithm

The DLC algorithm contains multiple hyperparameters in the sampling and dissipativity learning steps, which may affect the performance of the final controller. Specifically, in the sampling stage, the hyperparameters for the set of admissible input signals, namely N_f (number of significant terms in Fourier series) and ϵ_f (allowable truncation error), affect the range of excitation signals generated, and T (time span of excitations) affects the reachable domain. Generally, if N_f is too high or ϵ_f is too low, there may be an abuse of signals with high-frequency oscillations. On the contrary, the allowed bandwidth of the signals may be too shallow. Both extremes can restrict the effectiveness of the excitation signals. T should also be well chosen so that the reachable domain is of a proper size, in which the control performance is of interest.

For the statistical inference of effective reaching domain, two hyperparameters are involved if using PCA for a second-order cone estimation, namely the number of PCs J and the threshold of the Hotelling statistics Θ. These two hyperparameters represent the reduced dimension and the confidence level, respectively. Two analogous hyperparameters appeared in a polyhedral cone estimation scheme used in our previous work [19]. If J and Θ are large, the estimated $\hat{\mathcal{G}}$ is large and hence the estimated dissipativity set $\hat{\mathcal{S}}$ is small, which may result in the conservativeness (namely sub-optimality) of controller performance. On the other hand, if J and Θ are too small, $\hat{\mathcal{S}}$ is large, which may bring the risk of obtaining unrealistically optimistic L^2-performance. Therefore, J and Θ need to be tuned suitably. Since J affects the dimension of the feasible region of the semidefinite programming problem, we expect that the impact of J on the control performance is more significant than that of Θ, and needs to be tuned in priority.

5. Case study: two-phase chemical reactor

In this section we examine DLC with the regulating control of a two-phase reactor, whose first-principles model, considered as the true dynamics, was described in [39]. We consider the outlet flow rates F_V and F_L simultaneously (scaled by 1 mol/s) and the heating rate Q (scaled by 25 kW) as two manipulated inputs, and the vapor phase composition y_A and temperature T as two outputs (scaled by 0.001 and 0.5 K, respectively). Two disturbances in F_{B0} and T_{A0} (scaled by 1 mol/s and 2.5 K, respectively) are considered. We generate 1000 trajectory samples excited by random 5-term Fourier series in 120 s. For PID controller synthesis, the outputs are augmented with their integrals and derivatives, and hence the dual dissipativity parameters are 10×10 matrices, vectorized into 100-dimensional vectors by column-major order. From PCA we find that for retaining 99% (99.9%, 99.99%) of data variations (sum of squares of SVs), only the 4 (10, 19) largest SVs are needed.

Setting the number of PCs as 4 and confidence level as 90% (i.e., $\hat{\mathcal{G}}$ covers 90% of the samples) and following the procedure in Section 4.2, a DLC-PID controller is obtained:

$$u(t) = \begin{bmatrix} -1.163 & 0.493 \\ 1.711 & -2.924 \end{bmatrix} y(t) + \begin{bmatrix} 1.463 & 7.351 \\ -0.617 & 0 \end{bmatrix} \text{h}^{-1} \cdot \int_0^t y(\tau)d\tau + \begin{bmatrix} -0.813 & 5.037 \\ 11.178 & -18.996 \end{bmatrix} \text{s} \cdot \frac{dy}{dt}(t). \tag{47}$$

DLC-PI and DLC-P controllers are also synthesized following the same procedures. To test the control performance, we generate the disturbances as randomized piecewise constant signals and simulate the closed-loop trajectories in 3600 s, as shown in Fig. 2. The control performance is measured by the sum of integrated squares of output errors and control inputs (ISE+ISC $= \frac{1}{3600}\int_0^{3600}(\|y(t)\|^2 + \|u(t)\|^2)dt$). The ISE+ISC indices are 2.5846, 2.4316, and 2.5345 for the PID, PI and P controllers, respectively, among which the PI controller slightly outperforms the other two, indicating that it suffices to use a PI or even a P controller for disturbance rejection around the steady state. The DLC controllers are also compared to the open-loop trajectories (with fixed $u = 0$) which result in significantly larger deviations in the outputs, with ISE + ISC = 35.0907. By further perturbing the 4 gains in the DLC-P controller, we observe that the lowest ISE+ISC value achievable by P controllers under the specific disturbance scenario in this simulation is approximately 2.17 (by letting $K_{11} = 0$ and increasing K_{21}, K_{22} by 50%), with which the performance of the DLC-P controller has a fairly small gap of about 14%. This gap is due to the intrinsic conservativeness of the L^2 synthesis formulation used in DLC, which aims at mitigating the worst-case disturbance.

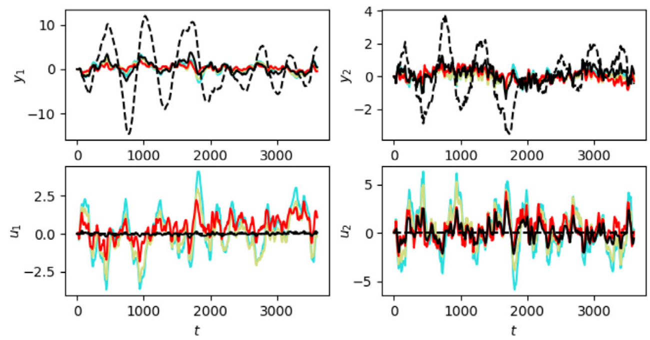

Fig. 2. Simulated closed-loop trajectories under the DLC-P (blue), DLC-PI (green), and DLC-PID (red) controllers compared to LQG (black solid) and open-loop trajectories (black dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We compare the performance of DLC to an LQG controller based on linear system identification using the Kalman–Ho algorithm [40], which finds the minimal realization of linear time-invariant (LTI) systems. Given a desirable reduced order of states, the algorithm can be used for LTI system identification with an approximate SVD. For the two-phase reactor, we choose a sampling time of 10 s and specify the order of states as 6. Under state disturbance and output noise covariance matrices $W = I$, $V = 0.1I$ and weighting matrices $Q = I$, $R = I$, a linear quadratic Gaussian (LQG) controller is obtained. The simulation result shows that the LQG controller also well rejects the disturbances (with ISE + ISC = 2.6766). By varying the number of states in the Kalman–Ho algorithm, we found that the 6-state LQG controller has the best achievable performance among such linear system identification-based optimal control (ISE + ISC = 2.6766), which is out-competed by the DLC-PI controller (ISE + ISC = 2.4316) by approximately 9.2%. Moreover, by comparing the trajectories in Fig. 2, we observe that DLC and LQG choose different ways to reject exogenous disturbances. For DLC, the magnitudes of inputs and outputs turn out to be more balanced, while the LQG prefers to tradeoff larger output deviations (especially y_1) for smaller inputs (especially u_1), which does not change significantly when V are tuned to different values such as $0.01I$, $0.001I$ and $0.0001I$. This may be related to the different ways that these two types of controllers are synthesized. The dissipativity learning relies on the overall (integrated) input–output responses over an excitation time period, while linear system identification is based on the incremental responses during each short sampling time.

6. Conclusions

In this paper we have studied fundamental properties of the DLC framework for input–output data-driven control. The nominal control performance of DLC and the impact of the dissipativity learning procedures, including the sampling of excitation signals and statistical inference of the effective dual dissipativity set, were formalized. Concisely stated, *DLC achieves nearly L^2-optimal control within an effective reachable domain, assuming that the sampling and statistical inference steps are sufficiently accurate with respect to the effective dual dissipativity set*. DLC was implemented on a two-phase chemical reactor, and its performance was compared to linear system identification-based LQG control. The extension of DLC to more general plant dynamics, controllers and large-scale systems, as well as its in-depth comparison with other data-driven control frameworks such as reinforcement learning, will be addressed in our upcoming works.

CRediT authorship contribution statement

Wentao Tang: Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing - original draft. **Prodromos Daoutidis:** Conceptualization, Funding acquisition, Project administration, Resources, Supervision, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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