Nonlinear Constraints on Relativistic Fluids Far from Equilibrium

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New constraints are found that must necessarily hold for Israel-Stewart-like theories of fluid dynamics to be causal far away from equilibrium. Conditions that are sufficient to ensure causality, local existence, and uniqueness of solutions in these theories are also presented. Our results hold in the full nonlinear regime, taking into account bulk and shear viscosities (at zero chemical potential), without any simplifying symmetry or near-equilibrium assumptions. Our findings provide fundamental constraints on the magnitude of viscous corrections in fluid dynamics far from equilibrium.

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Introduction.—Relativistic fluid dynamics is essential to the state-of-the-art modeling of the quark-gluon plasma (QGP) formed in ultrarelativistic heavy-ion collisions (see [1–3]). However, despite its wide use and significant success, it remains unclear why such a fluid dynamical description is applicable given that local deviations from equilibrium in nucleus-nucleus collisions can be very large, especially at early times [4–6]. In fact, typical fluidlike signatures involving anisotropic flow [7] persist even in small systems formed in proton-nucleus and proton-proton collisions at sufficiently high multiplicity [8–14]. Such findings have motivated a series of new investigations on the foundations of relativistic viscous fluid dynamics [15–18] and their subsequent extension toward the far-from-equilibrium regime relevant for heavy-ion collisions [19–45].

The viscous fluid description of the QGP is currently based on ideas from Israel and Stewart (IS) [46,47] (see, also, Mueller [48]), who proposed a way to fix the long-standing acausality [49] and instability [50] problems of the relativistic generalization of Navier-Stokes (NS) equations derived by Eckart [51] and Landau and Lifshitz [52]. The general mechanism introduced by IS to try to avoid such issues assumes that dissipative currents such as the shear stress tensor, $\pi_{\mu\nu}$, and the bulk scalar, Π, are new degrees of freedom [53,54] which obey nonlinear relaxation equations describing how such quantities relax to their relativistic NS limits within relaxation time scales τ_π and τ_Π. The same principle is also at play in modern formulations of fluid dynamics put forward by Ref. [55] and Ref. [56], which are currently employed in numerical simulations (see, for instance, [57]).

It is well known that the IS-like theories are linearly stable around equilibrium [58–61]. But physically sensible relativistic theories of fluid dynamics must also be causal, i.e., the equations of motion must be hyperbolic, and the propagation of information must be, at most, the speed of light [62]. Also, the Cauchy problem must be locally well posed [63], i.e., given initial conditions, one must show that the equations admit a unique solution. A common misconception in the field is that IS-like theories have already been proven to be causal a long time ago in Refs. [46,58,59,64–66]. This is not the case. Those early works only considered linearized disturbances around equilibrium, where the background fields $\pi_{\mu\nu}$ and Π vanish and the corresponding linear disturbances are small. Such a linearized analysis says nothing about the nonlinear regime, even for small $\pi_{\mu\nu}$ and Π. The far-from-equilibrium regime, in particular, is necessarily nonlinear as $\pi_{\mu\nu}$ and Π can be as large as the local equilibrium pressure P.

Hence, it is not known if IS theories are, indeed, sensible in the regime probed by high energy hadronic collisions. Understanding the far-from-equilibrium properties of such theories is also crucial to reliably assess the role of viscous effects in early universe cosmology [67]. Here, we make essential steps toward solving this critical problem by finding conditions (in the form of simple algebraic inequalities that can be checked at every step of the evolution) that must necessarily hold for IS-like theories to be causal in the nonlinear regime. We also present conditions that are sufficient to ensure causality, local existence, and uniqueness of solutions of IS-like theories. Our results are the first in the literature that hold in the full nonlinear regime, with

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bulk and shear viscosities (at zero chemical potential), in three spatial dimensions, without recurring to any symmetry or near-equilibrium assumptions.

The equations of motion.—Using the Landau frame definition of the hydrodynamic variables [52], the energy-momentum tensor of the fluid can be written as [We use units $c = \hbar = k_B = 1$. The spacetime metric signature is $(-+++)$. Greek indices run from 0 to 3, Latin indices from 1 to 3.] $T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$, where u^μ is the fluid's four-velocity (with $u_\mu u^\mu = -1$), ε is the energy density, $P = P(\varepsilon)$ is the equilibrium pressure defined by an equation of state, $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projector orthogonal to the flow, $g_{\mu\nu}$ is the spacetime metric, $\pi_{\mu\nu} = \pi_{\nu\mu}$, $\pi^{\mu\nu}u_\mu = 0$, and $\Delta_{\mu\nu}\pi^{\mu\nu} = 0$. We focus on high energy collisions, and thus, we only investigate, here, the case of zero chemical potentials. Conservation of energy and momentum implies that $\nabla_\mu T^{\mu\nu} = 0$, which can be written as ($c_s^2 = dP/d\varepsilon$ is the equilibrium speed of sound squared)

$$u^\alpha \nabla_\alpha \varepsilon + (\varepsilon + P + \Pi)\nabla_\alpha u^\alpha + \pi^\alpha_\mu \nabla_\alpha u^\mu = 0,$$
$$(\varepsilon + P + \Pi)u^\beta \nabla_\beta u_\alpha + c_s^2 \Delta^\beta_\alpha \nabla_\beta \varepsilon + \Delta^\beta_\alpha \nabla_\beta \Pi + \Delta^\beta_\alpha \nabla_\mu \pi^\mu_\beta = 0. \tag{1}$$

Here, we consider the case where the dissipative currents $\{\pi^{\mu\nu}, \Pi\}$ satisfy the following equations (Note that our metric signature is different than in [56].), derived using the Denicol-Niemi-Molnar-Rischke formalism [56], and commonly used in heavy-ion collision applications

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi = -\zeta \nabla_\mu u^\mu - \delta_{\Pi\Pi}\Pi \nabla_\mu u^\mu - \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{2a}$$

$$\tau_\pi \Delta^{\mu\nu}_{\alpha\beta} u^\lambda \nabla_\lambda \pi^{\alpha\beta} + \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} - \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \tag{2b}$$

where $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$ is the shear tensor, $\Delta^{\mu\nu}_{\alpha\beta} = (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha)/2 - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$, $A_\lambda^{\langle\mu}B^{\nu\rangle\lambda} = \Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\lambda}B^\beta_\lambda$, and η, ζ are the shear and bulk viscosities, respectively. All the transport coefficients, $\{\eta, \zeta, \tau_\Pi, \tau_\pi, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \tau_{\pi\pi}, \lambda_{\pi\Pi}\}$, can depend on the ten dynamical variables $\{\varepsilon, u_\mu, \pi_{\mu\nu}, \Pi\}$ (so, in principle, they may even depend on the dissipative tensors) but not on their derivatives. Explicit expressions for transport coefficients in models can be found, for instance, in [56,68,69].

We note that $\{\eta, \zeta, \tau_\pi, \tau_\Pi\}$ are the only coefficients that remain after linearization around equilibrium where $\pi^{\mu\nu} = 0$ and $\Pi = 0$. This shows why linearized analyses [58,59] necessarily miss the effects from the other coefficients, $\{\delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \tau_{\pi\pi}, \lambda_{\pi\Pi}\}$, which contribute to the nonlinear evolution. However, other nonlinear terms such as $\pi_{\mu\nu}\pi^{\mu\nu}$, Π^2, $\pi^{\mu\nu}\Pi$, $\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha}$, which appear in [56], could have been trivially added to the equations as they do not contribute to a causality analysis since they do not involve derivatives of the fields. Nevertheless, there are still some other nonlinear terms that can be considered such as $\pi_\alpha^{\langle\mu}\Omega^{\nu\rangle\alpha}$, where $\Omega_{\mu\nu} = (\Delta^\alpha_\mu \nabla_\alpha u_\nu - \Delta^\alpha_\nu \nabla_\alpha u_\mu)/2$ is the vorticity and, also, $\Omega_\alpha^{\langle\mu}\Omega^{\nu\rangle\alpha}$ [3]. The former will be investigated in a separate publication. The latter contributes with derivatives of the fields to the principal part of the system of equations and, thus, a different analysis than presented here would be required.

Causality.—Causality is the concept in relativity theory asserting that no information propagates faster than the speed of light and no closed timelike curves exist (so the future cannot influence the past). See Refs. [70–74] for a mathematically precise definition of causality. Causality can be investigated by determining the characteristic manifolds associated with a system of partial differential equations [75,76]. Let us write equations (1)–(2) as $A^\alpha \nabla_\alpha \Psi = F(\Psi)$, where we defined the vector $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})$, the 22×22 matrix

$$A^\alpha = \begin{bmatrix} u^\alpha & \rho\delta^\alpha_\nu + \pi^\alpha_\nu & 0_{1\times1} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} \\ c_s^2\Delta^{\mu\alpha} & \rho u^\alpha \delta^\mu_\nu - \pi^\alpha_\nu u^\mu & \Delta^{\mu\alpha} & \delta^\alpha_0 I_4 & \delta^\alpha_1 I_4 & \delta^\alpha_2 I_4 & \delta^\alpha_3 I_4 \\ 0_{4\times1} & E^\alpha_\nu & \tau_\Pi u^\alpha & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & C^{0\delta\alpha}_\nu & 0_{4\times1} & \tau_\pi u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & C^{1\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & \tau_\pi u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & C^{2\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & \tau_\pi u^\alpha I_4 & 0_{4\times4} \\ 0_{4\times1} & C^{3\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & \tau_\pi u^\alpha I_4 \end{bmatrix}, \tag{3}$$

and $F(\Psi)$ is a vector that does not contain derivatives of the variables. Above, we also defined $\rho = \varepsilon + P + \Pi$, $E^{\alpha}_{\nu} = (\zeta + \delta_{\Pi\Pi}\Pi)\delta^{\alpha}_{\nu} + \lambda_{\Pi\pi}\pi^{\alpha}_{\nu}$, $B^{\mu\lambda\alpha}_{\nu} = \frac{1}{2}(\Delta^{\mu\alpha}\delta^{\lambda}_{\nu} + \Delta^{\lambda\alpha}\delta^{\mu}_{\nu} - \frac{2}{3}\Delta^{\mu\lambda}\delta^{\alpha}_{\nu})$, and

$$C^{\sigma\delta\alpha}_{\nu} = \left[(2\eta + \lambda_{\pi\Pi}\Pi)\delta^{\sigma}_{\mu}\delta^{\delta}_{\lambda} + \frac{\tau_{\pi\pi}}{2}\pi^{\sigma}_{\lambda}\delta^{\delta}_{\mu} + \frac{\tau_{\pi\pi}}{2}\pi^{\delta}_{\lambda}\delta^{\sigma}_{\mu}\right]B^{\mu\lambda\alpha}_{\nu} - \frac{\tau_{\pi\pi}}{3}\Delta^{\sigma\delta}\pi^{\alpha}_{\nu} + \delta_{\pi\pi}\pi^{\sigma\delta}\delta^{\alpha}_{\nu} - \tau_{\pi}(\pi^{\sigma}_{\nu}u^{\delta} + \pi^{\delta}_{\nu}u^{\sigma})u^{\alpha}.$$

The characteristic surfaces $\{\Phi(x) = 0\}$ are determined by the principal part of the equations by solving the characteristic equation $\det(A^{\alpha}\xi_{\alpha}) = 0$, with $\xi_{\alpha} = \nabla_{\alpha}\Phi$ [77]. The system is causal if, for any ξ_i, it holds that (C1) the roots $\xi_0 = \xi_0(\xi_i)$ of the characteristic equation are real and (C2) $\xi_{\alpha} = (\xi_0, \xi_i)$ is spacelike or lightlike. Condition (C2) implies that the characteristic surfaces $\{\Phi(x) = 0\}$ are timelike or lightlike, indicating that no information is superluminal. For instance, for an ideal fluid (where $\Pi = 0$ and $\pi_{\mu\nu} = 0$), the characteristic velocities are determined by the speed of sound and causality implies that $c_s^2 \leq 1$ [63].

From (3), it is clear that the characteristics associated with the evolution depend on the dissipative tensors $\{\pi^{\mu\nu}, \Pi\}$. Therefore, the true causal behavior of IS theories is necessarily a far-from-equilibrium property of the fluid, and linear analyses around equilibrium cannot be used to establish causality and well posedness in IS theories. The computation of the characteristics defined by (3), which is needed for a causality analysis, is extremely involved and is presented in the Supplemental Material [78]. Below, we present the main consequences of such calculations.

Let Λ_{α}, $\alpha = 0, 1, 2, 3$, be the eigenvalues of the π^{μ}_{ν}. The eigenvalues are such that $\Lambda_0 = 0$, since u_{μ} is in the kernel of π^{μ}_{ν} ($u_{\mu}\pi^{\mu}_{\nu} = 0$), and $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$, so that the trace is kept zero. Without loss of generality, let us take $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3$ with $\Lambda_1 \leq 0 \leq \Lambda_3$. Now, we state our assumptions, which are the following: (A1) for the transport coefficients and relaxation times, suppose that $\tau_{\Pi}, \tau_{\pi} > 0$ and $\eta, \zeta, \tau_{\pi\pi}, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \lambda_{\pi\Pi}, c_s^2 \geq 0$; (A2) for the fluid variables, suppose that $\varepsilon > 0$, $P \geq 0$, and $\varepsilon + P + \Pi > 0$; finally, we also assume that (A3) $\varepsilon + P + \Pi + \Lambda_a > 0$, $a = 1, 2, 3$. Then, the following conditions are necessary for causality, i.e., if any of the inequalities below is not satisfied, then the system is not causal:

$$(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0, \tag{4a}$$

$$\varepsilon + P + \Pi - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}\Lambda_3 \geq 0, \tag{4b}$$

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_a + \Lambda_d) \geq 0, \quad a \neq d, \tag{4c}$$

$$\varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_d + \Lambda_a) \geq 0, \quad a \neq d, \tag{4d}$$

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d + \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_{\Pi}} + (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0, \tag{4e}$$

$$\varepsilon + P + \Pi + \Lambda_d - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d - \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] - \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_{\Pi}} - (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0, \tag{4f}$$

where (4c)–(4f) must hold for $a, d = 1, 2, 3$. The proof that (4) are necessary conditions for causality under assumptions (A1)–(A3) is given in the Supplemental Material [78]. Here, we discuss the significance of this result.

We stress that assumptions (A1) and (A2) are standard in heavy-ion collision applications [57], and (A3) is a very natural assumption since $P + \Pi + \Lambda_a$ for $a = 1, 2, 3$ may be interpreted as the pressure in each spatial axis in the local rest frame. Thus, if (A3) is violated, the theory would have a pathology in the sense that fluid elements would have negative inertia, i.e., the acceleration is opposite to the force given by the negative of the gradient of pressure. Furthermore, it is natural to make assumptions that hold close to equilibrium, and since (A2) guarantees $\varepsilon + P + \Pi > 0$, for small deviations from equilibrium, Λ_a will be small, giving $\varepsilon + P + \Pi + \Lambda_a > 0$. That said, we stress that, although (A3) is expected to hold near equilibrium, it is, itself, not a near-equilibrium assumption.

Conditions (4) could never have been found using a linearized analysis, as they depend on Π and Λ_a, both of which vanish in equilibrium. Consequently, if, in any fluid dynamic simulation in heavy-ion collisions that employs (1)–(2), the necessary conditions above are not fulfilled, causality is necessarily violated. It is important to point out that this causality violation has nothing to do with the ability of numerical schemes to produce a solution, a point we shall return to at the end of the Letter.

While the above conditions must hold for the system to be causal, they are not sufficient conditions, i.e., by themselves, conditions (A1)–(A3) and (4) do not assure the system to be causal (see the Supplemental Material

[78]). Therefore, it is important to have conditions that are sufficient for causality. In this regard, assume, again, that (A1)–(A3) hold. Then, the following conditions are sufficient to ensure that causality holds, i.e., if they are satisfied, then the system is causal:

$$(\varepsilon+P+\Pi-|\Lambda_1|)-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\geq 0, \tag{5a}$$

$$(2\eta+\lambda_{\pi\Pi}\Pi)-\tau_{\pi\pi}|\Lambda_1|>0, \tag{5b}$$

$$\tau_{\pi\pi}\leq 6\delta_{\pi\pi}, \tag{5c}$$

$$\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\geq 0, \tag{5d}$$

$$\begin{aligned}&\frac{1}{3\tau_\pi}[4\eta+2\lambda_{\pi\Pi}\Pi+(3\delta_{\pi\pi}+\tau_{\pi\pi})\Lambda_3]\\&\quad+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi}+|\Lambda_1|+\Lambda_3c_s^2\\&\quad+\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi})(\Lambda_3+|\Lambda_1|)^2}{\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3}\\&\quad\leq(\varepsilon+P+\Pi)(1-c_s^2),\end{aligned} \tag{5e}$$

$$\begin{aligned}&\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(\tau_{\pi\pi}-6\delta_{\pi\pi})|\Lambda_1|]\\&\quad+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2\geq 0,\end{aligned} \tag{5f}$$

$$1\geq\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi})(\Lambda_3+|\Lambda_1|)^2}{[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|]^2}, \tag{5g}$$

$$\begin{aligned}&\frac{1}{3\tau_\pi}[4\eta+2\lambda_{\pi\Pi}\Pi-(3\delta_{\pi\pi}+\tau_{\pi\pi})|\Lambda_1|]\\&\quad+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2\\&\quad\geq\frac{(\varepsilon+P+\Pi+\Lambda_2)(\varepsilon+P+\Pi+\Lambda_3)}{3(\varepsilon+P+\Pi-|\Lambda_1|)}\\&\quad\times\left\{1+\frac{2[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)+\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3]}{\varepsilon+P+\Pi-|\Lambda_1|}\right\},\end{aligned} \tag{5h}$$

where condition (5h) can be dropped if $\delta_{\pi\pi}=\tau_{\pi\pi}=0$. The detailed proof can be found in the Supplemental Material [78]. Since (4) must hold for causality, they must be satisfied for any set of conditions that imply causality, and it is possible to verify that (5) imply (4) under assumptions (A1)–(A3). When shear viscous effects are neglected, (5) reduces to the conditions for the bulk viscosity case found in [79].

Conditions (A1)–(A3) and (5) also ensure the unique local solvability of the initial-value problem in the class of quasianalytic functions: given initial data of sufficient regularity satisfying (5), there exists a unique solution to the nonlinear equations taking the given initial data, defined for a certain time interval. Thus, if (A1)–(A3) and (5) hold, the evolution of the viscous fluid is guaranteed to be well defined and causal even far from equilibrium where the gradients (and, hence, $\pi^{\mu\nu}$ and Π) are large. This is relevant for determining the properties of hydrodynamic attractors [20] under general flow conditions [27,80] and, also, for an overall validation of fluid dynamics descriptions of extreme systems, such as proton-proton collisions. Moreover, while, here, we focus on applications to heavy-ion collisions, so $g_{\mu\nu}$ is the Minkowski metric, the methods of [79] can be adapted to show that our conclusions hold when (1)–(2) are coupled to Einstein's equations. Hence, our results are also crucial for determining the far-from-equilibrium behavior of viscous fluids in general relativity, which may be relevant to neutron star mergers [81]. The technical details of these statements are provided in the Supplemental Material [78].

When we linearize the equations around the equilibrium, terms involving $\tau_{\pi\pi}, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \lambda_{\pi\Pi}$ drop out and, thus, (A1) can be replaced by $\tau_\pi, \tau_\Pi > 0$, $\eta, \zeta, c_s^2 \geq 0$, and (A2) and (A3) can be replaced by $\varepsilon + P > 0$ and $P \geq 0$. Then, conditions (5) become necessary and reduce to $\varepsilon + P > 0$, $\varepsilon + P - (\eta/\tau_\pi) \geq 0$, and $1/(\varepsilon + P)(4\eta/3\tau_\pi + \zeta/\tau_\Pi) \leq 1 - c_s^2$. These conditions coincide with the corresponding well-known results previously found in [58,59] that ensure causality and stability in the linearized regime around equilibrium.

It is instructive to compare the causal propagation modes of the full nonlinear theory determined here with that of the dynamics linearized about equilibrium. Linearizing Eqs. (1)–(2) around equilibrium, we find four distinct modes of propagation which correspond to the flow lines, the sound waves, and shear waves at two distinct speeds. These are the same ones found in the previous works [46,58,59,64–66], where, there, the authors find, in addition, a second longitudinal mode (second sound) due to the fact that their equations also include a conserved current. In the nonlinear case, we found six distinct propagation modes corresponding to the flow lines, the sound waves, and shear waves at four distinct speeds. This, again, highlights how one misses an important part of the dynamics by looking only at linearizations around equilibrium: there are two additional speeds allowed for the shear waves that collapse onto the remaining two upon linearization, so that these additional velocities are not visible in the linearized analysis.

We presented two sets of conditions for causality, namely, conditions that are necessary and conditions that are sufficient. Further studies must be done to discover conditions that are necessary and sufficient, i.e., conditions

that ensure the system to be causal if and only if they hold. This is an extremely challenging task given the complexity of the characteristic equation in the nonlinear problem.

Conformal limit.—To obtain some physical understanding about our nonlinear constraints, consider a conformal fluid [55], i.e., $\Pi = 0$, $P = \varepsilon/3$, $\delta_{\pi\pi} = 4\tau_\pi/3$, with η/s and $\tau_\pi T$ being constants (here, $T \sim \varepsilon^{1/4}$ is the temperature and $s \sim T^3$ is the equilibrium entropy density). Assume, for simplicity, that all the other transport coefficients vanish (as in [82]). The necessary conditions in (4) then impose that $\Lambda_a/(\varepsilon + P) \geq -1 + (\eta/s)(1/\tau_\pi T)$, so none of the eigenvalues of π^μ_ν can be too negative. Also, when $\Lambda_a/(\varepsilon + P) > -1 + (\eta/s)(1/\tau_\pi T)$, the eigenvalues are also limited from above since (4e) gives $\Lambda_a/(\varepsilon + P) \leq 1 - (2/\tau_\pi T)(\eta/s)$. Using typical values motivated by heavy-ion collision applications, $\eta/s = 1/(4\pi)$ [83] and $\tau_\pi T = 5\eta/s$ [84], one then finds $-4/5 < \Lambda_a/(\varepsilon + P) \leq 3/5$. This implies that the relative magnitude of the shear stress tensor, $\sqrt{\pi_{\mu\nu}\pi^{\mu\nu}/(\varepsilon + P)^2}$, cannot be arbitrarily large. Using a NS initial condition where $\pi_{\mu\nu} \sim -2\eta\sigma_{\mu\nu}$ at the initial time τ_0, the corresponding normalized eigenvalues would be parametrically given by $\Lambda_a/(\varepsilon + P) \sim (\eta/s)/[\tau_0 T(\tau_0, \vec{x})]$ (assuming $\sigma_{\mu\nu} \sim 1/\tau_0$ in the initial state). Given that our conditions imply that, roughly, $|\Lambda_a/(\varepsilon + P)| \lesssim 1$, for $\tau_0 = 0.6$ fm and $\eta/s = 1/(4\pi)$, causality issues will be found where $T(\tau_0, \vec{x}) \lesssim 30$ MeV, which is below the typical values for the freeze-out temperature. However, in initial state models where the initialized $\pi_{\mu\nu}/(\varepsilon + P)$ is large and strongly deviates from NS, the estimate above does not apply, and causality violations may appear in hot regions of the plasma as well. A detailed numerical study is needed to assess the importance of our results to current simulations of heavy-ion collisions.

Conclusions.—In this Letter, we established, for the first time, that causality, in fact, holds for the full set of nonlinear equations in IS-like theories without the need for symmetry assumptions and in the presence of both shear and bulk viscosity. All our conditions are simple algebraic inequalities among the dynamical variables that can be easily checked in a given system or simulation. Previous attempts to go beyond the linear regime were restricted to $1 + 1$ dimensions [60] or assumed strong symmetry conditions [61,85], which, in practice, also corresponds to partial differential equations with only one spatial variable. Without such restrictions, the only other work where nonlinear causality has been shown for IS-like systems is [79], which only included bulk viscous effects. We have also studied the Cauchy problem for (1)–(2), establishing that it is well defined, so that it is meaningful to talk about solutions.

Prior to our work, unless a numerical code was specifically tailored to detect causality violations of the underlying equations, which typically is not a feature present in standard codes, one could only identify whether a numerical simulation of (1)–(2) violated causality if this caused (a) a breakdown of the simulation, (b) a manifestly spurious solution, or (c) clear nonphysical behavior. These constraints are all too weak, as we now explain. For illustration, consider $-\partial_t^2\psi + (1 + \psi)\Delta\psi = 0$, where Δ is the Laplacian. This is a nonlinear wave equation with (nonlinear) speed given by $\sqrt{1 + \psi}$ for (For $\psi < -1$, the equation is no longer a wave equation, becoming elliptic, and it is a degenerate wave equation when $\psi = -1$.) $\psi > -1$. Indeed, the characteristics are given by $\xi_0 = \pm\sqrt{1 + \psi}|\vec{\xi}|$. Therefore, the solutions are not causal when $\psi > 0$, but are causal for $-1 < \psi \leq 0$. Nevertheless, the equation remains hyperbolic as long as $\psi > -1$. Standard hyperbolic theory (see, e.g., [86]) ensures that, given smooth initial data $\psi|_{t=0}$ and $\partial_t\psi|_{t=0}$, there exists a unique smooth solution defined for some time. So any numerical scheme that is able to track the unique solution will produce results in both the acausal and causal cases $\psi > 0$ and $-1 < \psi \leq 0$, respectively. This makes it extremely difficult to infer violations of causality using (a) or (b) as criteria. Exactly the same situation can happen in simulations of (1)–(2). We also note that linearizing the equation about the "equilibrium" $\psi = 0$ gives $-\delta\psi_{tt} + \Delta\delta\psi = 0$, which is always causal, reinforcing, again, the idea that causality cannot always be obtained from linearizations.

Criteria (c) has also limited applicability. First, there are different mechanisms that can produce nonphysical solutions. Thus, it is still important to understand whether unphysical behavior is being caused by causality violation or some other mechanism, such as running beyond the limit where the effective description is valid. Second, relativistic fluids in the far-from-equilibrium regime, such as the QGP, may exhibit unexpected behavior, so one needs to be careful to differentiate genuine exotic features from those that are consequences of running a simulation in a superluminal regime. This may be particularly relevant to heavy-ion simulations where the values of the fields drop extremely rapidly at the edges of the QGP at early times and in the cold or dilute regions of plasma where a rescaling of dissipative tensors has been employed [87–90]. Third, numerical simulations of relativistic fluids must be based on equations of motion that respect causality, a fundamental physical principle in relativity.

The results we presented here are an important step in addressing all these difficulties, as one can check if (A1)–(A3), (4), or (5) hold at any moment in numerical simulations [Comparing with the example of the equation for ψ above, this would be similar to monitoring the value of $\sqrt{1 + \psi}$: if $\psi > 1$, then the system is not causal, which is the analog of (5), whereas causality is guaranteed if $-1 < \psi \leq 0$, which is the analog of (5).] since all the quantities involved in our inequalities can be readily extracted in numerical simulations [3]. We also note that

our results apply, in particular, to the initial conditions, so (4) and (5) can be used to rule out initial conditions that violate causality or to select initial conditions for which causality holds. This can be particularly relevant to further constrain the physical assumptions behind the modeling of initial conditions in QGP simulations. There are many subtleties involved in numerically solving the IS equations, including possible violations of causality caused entirely as an artifact of the numerical simulation. Thus, it is important to distinguish between such numerically caused unphysical phenomena from true violation of causality of the underlying equations. Our new causality criteria can be instrumental in such analyses.

In sum, in this Letter, we established, for the first time in the literature, conditions to settle the longstanding questions concerning causality in Israel-Stewart theories in the nonlinear, far-from-equilibrium regime. As such, our general results provide the most stringent tests to date for determining the validity of relativistic fluid dynamic approaches in heavy-ion collisions, astrophysics, and cosmology.

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Nonlinear Constraints on Relativistic Fluids Far From Equilibrium
SUPPLEMENTAL MATERIAL

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In this Supplemental Material, in Section II we provide the proof that conditions (4) are necessary for causality, in Section III we provide the proof that conditions (5) are sufficient for causality, and in Section IV we establish local existence and uniqueness of solutions to the initial-value problem for equations (1)-(2). All these results depend on a careful analysis of the roots of the characteristic equation $\det(A^\alpha\xi_\alpha) = 0$. Thus, we first present in Section I a suitable factorization of $\det(A^\alpha\xi_\alpha)$. In Section V we show that conditions (4), albeit necessary, are not sufficient for causality. In Section VI we provide the formal definition of causality and comment on why, in our case, it can be reduced to conditions (C1) and (C2). Since causality is intrinsically tied to concepts of relativity theory, we refer to the standard literature (e.g., [1]) for further background. Throughout this Supplemental Material, we continue to use the notation and definitions of the paper.

I. THE CHARACTERISTIC EQUATION

Letting $b = u^\alpha\xi_\alpha$, $v^\mu = \Delta^{\mu\nu}\xi_\nu$, and $w^\mu = \pi^{\mu\nu}\xi_\nu$, the characteristic determinant can be written as

$$\det(A^\alpha\xi_\alpha) = b^{13}\tau_\pi^{16}\tau_\Pi \det\begin{bmatrix} b & \rho\xi_\nu + w_\nu \\ bc_s^2 v^\mu & \rho b^2\delta^\mu_\nu - bw_\nu u^\mu - \frac{\bar{C}^\mu_\nu}{\tau_\pi} - \frac{v^\mu\tilde{E}_\nu}{\tau_\Pi} \end{bmatrix} = b^{14}\tau_\pi^{16}\tau_\Pi \det[M], \tag{S1}$$

where $M = [M^\mu_\nu]_{4\times4}$ with $M^\mu_\nu = \rho b^2\delta^\mu_\nu - bw_\nu u^\mu - \frac{\bar{C}^\mu_\nu}{\tau_\pi} - \frac{v^\mu\tilde{E}_\nu}{\tau_\Pi} - c_s^2 v^\mu(\rho\xi_\nu + w_\nu)$, $\tilde{E}_\nu = E^\alpha_\nu\xi_\alpha = (\zeta + \delta_{\Pi\Pi}\Pi)\,\xi_\nu + \lambda_{\Pi\pi}w_\nu$, and

$$\begin{aligned}\bar{C}^\delta_\nu \;=\; & C^{\sigma\delta\alpha}_\nu\xi_\alpha\xi_\sigma = \frac{1}{2}\left[(2\eta + \lambda_{\pi\Pi}\Pi)\xi_\mu\delta^\delta_\lambda + \frac{\tau_{\pi\pi}}{2}w_\lambda\delta^\delta_\mu + \frac{\tau_{\pi\pi}}{2}\pi^\delta_\lambda\xi_\mu\right]\left(v^\mu\delta^\lambda_\nu + v^\lambda\delta^\mu_\nu - \frac{2}{3}\Delta^{\mu\lambda}\xi_\nu\right) \\ & -\frac{\tau_{\pi\pi}}{3}v^\delta w_\nu + \delta_{\pi\pi}w^\delta\xi_\nu - b\tau_\pi(w_\nu u^\delta + b\pi^\delta_\nu).\end{aligned} \tag{S2}$$

Since $\pi^{\mu\nu}$ is symmetric and traceless, it can be diagonalized, so the eigenvalue problem $\pi^\mu_\nu e^\nu_A = \Lambda_A e^\mu_A$, with $A = 0, 1, 2, 3$, defines an orthonormal set of eigenvectors $e^\mu_{A=0} = u^\mu$, $e^\mu_{A=a} = e^\mu_a$ with real eigenvalues Λ_a for $a = 1, 2, 3$ in the sense that $g_{\mu\nu}e^\mu_A e^\nu_B = \eta_{AB}$ where $\eta_{AB} = diag(-1, 1, 1, 1)$. The eigenvalues are such that $\Lambda_0 = 0$ and $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$. Without any loss of generality, let us take $\Lambda_1 \le \Lambda_2 \le \Lambda_3$ with $\Lambda_1 \le 0 \le \Lambda_3$ so that the trace is kept zero (note that if $\pi^\mu_\nu \neq 0$, this allows degeneracies to occur with multiplicity up to two). Since $\{e^\mu_A\}$ is a complete set in $\mathbb{R}^4$, we may define a tetrad of dual vectors $\{e^A_\nu\}$ by setting $e^A_\nu \equiv \eta^{AB}(e_B)_\nu$ so that[1] $\delta^B_A = e^\nu_A e^B_\nu$. Also, the following completeness relation holds: $\delta^\mu_\nu = \sum_A e^\mu_A e^A_\nu = -u^\mu u_\nu + \sum_a e^\mu_a (e_a)_\nu$. Therefore, the components of any four-vector z^μ relative to the tetrad $\{e^\mu_A\}$ are defined by $z^A \equiv z^\nu e^A_\nu$. We can then use this to define $v_A \equiv e^\mu_A v_\mu$ and $\xi_A \equiv e^\mu_A\xi_\mu$. Given that $\xi^\mu = -bu^\mu + \sum_a v^a e^\mu_a$ $(a = 1, 2, 3)$ one finds that $\xi_{A=0} = -\xi^{A=0} = b$ while $\xi_a = v_a$. Furthermore, $w_A \equiv e^\mu_A w_\mu = e^\mu_A\pi_{\mu\nu}\xi^\nu = \Lambda_A\xi_A = \Lambda_A v_A$, where

[1] From now on, repeated Latin indexes are not summed unless explicitly stated.

we used that $\Lambda_0 = 0$ and again $\xi_a = v_a$ (note also that $v^a = v_a$ since $\eta_{ab} = \delta_{ab}$). Using these observations, we can show that the determinant $\det(M)$ needed for the characteristics in (S1) is given by

$$\begin{aligned}
&\det(M) = \det(E^{-1}ME) = m_0 m_1 m_2 m_3 \\
\times &\left[1 - \sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a}\right. \\
&\left. - \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{\substack{a,b \\ a<b}} \frac{(\Lambda_a - \Lambda_b)^2 \hat{v}_a^2 \hat{v}_b^2}{\bar{m}_a \bar{m}_b}\right],
\end{aligned} \tag{S3}$$

where $E = [e^\mu_A]_{4\times4}$, $E^{-1} = [e^B_\nu]_{4\times4}$, and $E^{-1}ME = [e^A_\mu M^\mu_\nu e^\nu_B]_{4\times4}$. Also, we defined above $m_0 = \rho\left(b^2 - \sum_a \mathfrak{g}_a v_a^2\right)$, $\mathfrak{g}_a = \frac{2(2\eta + \lambda_{\pi\Pi}\Pi) + \tau_{\pi\pi}\Lambda_a}{4\rho\tau_\pi}$, $m_a = (\rho + \Lambda_a)b^2 - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi)(v\cdot v) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a v\cdot v + \sum_c \Lambda_c v_c^2\right)$, $\hat{v}_a = v_a/\sqrt{v\cdot v}$ (assuming $v \neq 0$), and $\bar{m}_0 = m_0/(v\cdot v), \bar{m}_a = m_a/(v\cdot v)$. Note that $\sum_a \hat{v}_a^2 = \sum_a v_a^2/(v\cdot v) = 1$ since $v\cdot v = v^\mu v_\mu = \sum_a v_a^2$. Assuming $v \neq 0$ is allowed because $v = 0$ does not lead to nontrivial roots $b \neq 0$ of the characteristic equation if assumptions (A1)–(A3) hold.

The roots ξ of $\det(A^\alpha \xi_\alpha) = 0$ in Eq. (S1) are the 14 roots from $b = u^\alpha \xi_\alpha = 0$ and the 8 roots from $\det(M) = 0$ in Eq. (S3) which consist of the 2 roots from $m_0 = 0$ and the 6 roots coming from the zeros of

$$f(k) = \bar{m}_1 \bar{m}_2 \bar{m}_3 G(k), \tag{S4}$$

where we defined $k \equiv b^2/v\cdot v$ and

$$\begin{aligned}
G(k) = 1 &- \sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a} \\
&- \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{\substack{a,b \\ a<b}} \frac{(\Lambda_a - \Lambda_b)^2 \hat{v}_a^2 \hat{v}_b^2}{\bar{m}_a \bar{m}_b}.
\end{aligned} \tag{S5}$$

In this notation $\det(M) = m_0(v\cdot v)^3 f(k)$ because we used the definition $\bar{m}_a = m_a/v\cdot v$. Note that although $G(k)$ has $\bar{m}_a$ appearing in denominators, these are canceled by the multiplication of $G(k)$ by $\bar{m}_1\bar{m}_2\bar{m}_3$ in the definition of $f(k)$. Thus, $f(k)$ is a polynomial of degree 3 in k (of degree 6 in b) and is defined for all values of $k \in \mathbb{R}$. Then, it is possible to factorize $f(k)$ as

$$f(k) = \left[\prod_{a=1}^{3}(\varepsilon + P + \Pi + \Lambda_a)\right](k - k_1)(k - k_2)(k - k_3), \tag{S6}$$

where k_1, k_2, k_3 as the three roots of $f(k)$. Note that for the sake of brevity, we have suppressed the dependence on $\hat{v}$ in writing $G(k)$ and $f(k)$ (to be more precise, these should have been written as $G(k, \hat{v}), f(k, \hat{v})$).

Conditions (C1) and (C2) for causality demand that all the 22 roots $\xi_0 = \xi_0(\xi_i)$ of $\det(A^\alpha \xi_\alpha) = 0$ are real and satisfy $\xi_\alpha \xi^\alpha = -b^2 + v\cdot v \geq 0$, i.e., $0 \leq k \leq 1$. The 14 roots $b = 0$ are causal. Thus, the rest the analysis of necessary conditions in Section II will focus on the remaining roots defined by $f(k) = 0$. We summarize this in the following important statement:

The system is causal if and only if for all for all $\hat{v}$ on the unit sphere, the roots of $\bar{m}_0(k, \hat{v}) = 0$ and $f(k, \hat{v}) = 0$ are real and $0 \leq k \leq 1$. (C3)

II. DERIVATION OF NECESSARY CONDITIONS FOR CAUSALITY

Here we establish that conditions (4) are necessary (but not sufficient, see Section V) for causality. More precisely, we establish the following Theorem.

Theorem 1. *Let* $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\dots,3}$ *be a smooth solution to equations (1)-(2) in Minkowski space, with* $u_\mu u^\mu = -1$ *and* $\pi_{\mu\nu}$ *satisfying* $\pi^\mu_\mu = 0$ *and* $u^\mu \pi_{\mu\nu} = 0$. *Suppose that (A1)-(A3) hold. If any of conditions (4) is not satisfied, then* Ψ *is not causal in the sense of Definition 4 (see Section VI).*

Proof of Theorem 1: Our derivation of necessary conditions for causality is via the following reasoning. Causality requires that conditions (C1) and (C2) hold for all ξ_i. Thus, in order to violate causality, it suffices to show that for some ξ_i, (C1) or (C2) fails. Suppose now that we find a condition, say Z, for which we can exhibit one ξ_i such that (C1) or (C2) fail, i.e., we obtain the statement "Z implies non-causality." This statement is logically equivalent to "Causality implies non-Z." This means that non-Z is a necessary condition for causality: if it is violated, the system is not causal. In our case, conditions like Z will be inequalities among the scalars of the problem (e.g., the relaxation times, eigenvalues Λ_a, etc.) of the form $A > B$, whose negation is then $A \leq B$. The latter is then the necessary condition we are looking for: if $A \leq B$ does not hold, the system is not causal.

Recall that (C1) and (C2) is equivalent to (C3), so in view of the foregoing discussion, we aim to violate (C3). With the choice $\hat{v}_a = \delta_{ad}$, one can write $m_0 = \rho(v \cdot v)(k - \mathfrak{g}_d) = 0$. Under our assumptions, the only root is $k = \mathfrak{g}_d$. Since we need $0 \leq k \leq 1$, as discussed, and since $\mathfrak{g}_1 \leq \mathfrak{g}_2 \leq \mathfrak{g}_3$, causality if violated if $\mathfrak{g}_1 < 0$, leading to condition (4a), or if $\mathfrak{g}_3 > 1$, leading to condition (4b).

As for the roots of $f(k)$, we may note that now in $f(k) = \bar{m}_1 \bar{m}_2 \bar{m}_3 G(k)$ we have

$$\bar{m}_a = (\varepsilon + P + \Pi + \Lambda_a)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d) \tag{S7}$$

and

$$G(k) = 1 - \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}}{\bar{m}_d} \tag{S8}$$

because we have set $\hat{v}_a = \delta_{ad}$. We may therefore rewrite

$$f(k) = \bar{m}_a \bar{m}_b \left[\bar{m}_d - \left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}\right], \tag{S9}$$

where $a \neq b$ and $a, b \neq d$. Setting each of the factors m_a, m_b equal to zero, we obtain the roots

$$k = \frac{\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d)}{\varepsilon + P + \Pi + \Lambda_a}, \quad a \neq d. \tag{S10}$$

Causality is violated if $k < 0$, leading to condition (4c), of if $k > 1$, leading to condition (4d). The remaining root in (S9) is obtained when the term in brackets vanishes, giving

$$\begin{aligned} k = &\frac{\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d}{\varepsilon + P + \Pi + \Lambda_d} \\ &+ \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}}{\varepsilon + P + \Pi + \Lambda_d}. \end{aligned} \tag{S11}$$

Causality is violated if $k < 0$, leading to (4e), or if $k > 1$, leading to (4f). This finishes the proof. □

III. DERIVATION OF SUFFICIENT CONDITIONS FOR CAUSALITY

Here we establish that conditions (5) are sufficient for causality. More precisely, we have:

Theorem 2. *Let $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\dots,3}$ be a smooth solution to equations (1)-(2) in Minkowski space, with $u_\mu u^\mu = -1$ and $\pi_{\mu\nu}$ satisfying $\pi^\mu_\mu = 0$ and $u^\mu \pi_{\mu\nu} = 0$. Suppose that (A1)-(A3) and (5) hold. Then Ψ is causal in the sense of Definition 4 (see Section VI).*

Proof of Theorem 2: As discussed in Section I, the 14 roots $b = 0$ are causal and do not need any further treatment. The remaining 8 roots that come from $\det(M) = 0$ are, again, the two roots of m_0 and the six roots of $f(k)$ defined in (S4). We begin by analyzing the two roots of m_0. Recalling that $v = 0$ does not lead a nontrivial root of $\det(A^\alpha \xi_\alpha) = 0$, we see that the roots of m_0 are given by $b^2 = k = \sum_a \mathfrak{g}_a \hat{v}_a^2$. For these roots we need to check (according to (C3)) that

$$0 \leq \sum_a \mathfrak{g}_a \hat{v}_a^2 \leq 1. \tag{S12}$$

(A3) together with conditions (5a) and (5b) give $0 \leq \mathfrak{g}_1 \leq \mathfrak{g}_2 \leq \mathfrak{g}_3 \leq 1$. From $\mathfrak{g}_1 \leq \sum_a \mathfrak{g}_a \hat{v}_a^2 \leq \mathfrak{g}_3$, we see that (S12) is satisfied.

Now we analyze the remaining 6 roots of $\det(M) = 0$ coming from $f(k)$ defined in Eq. (S4) and written explicitly as a polynomial in (S6). We will show further below that the three roots k_i in (S6) are real. But let us first show that any real root of f must lie within $[0, 1]$. Since f is a cubic polynomial, it either has only one real root, say s_1, or three real roots, in which case we can order them as $k_1 \leq k_2 \leq k_3$ in (S6). Invoking (5a), we see that in the first case f is negative to the left of s_1 and positive to its right, and in the second case that f is a growing cubic polynomial except in the interval between the roots k_1 and k_3. In either situation, any real root will be between 0 and 1 if

$$f(k < 0) < 0, \tag{S13}$$

and

$$f(k > 1) > 0. \tag{S14}$$

Let us first verify the inequality (S14). For $k > 1$

$$\bar{m}_a(k > 1) \geq k(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3, \tag{S15}$$

where we have used $-2|\Lambda_1| \leq \Lambda_a + \sum_c \Lambda_c \hat{v}_c^2 \leq 2\Lambda_3$. Now, observe that

$$k(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 > (\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3$$

for $k > 1$, hence the condition (5a) lead us to $\bar{m}_a(k \geq 1) > 0$. This guarantees that

$$\bar{m}_1(k > 1)\bar{m}_2(k > 1)\bar{m}_3(k > 1) > 0.$$

To obtain $f(k > 1) > 0$ in (S14), we therefore need $G(k > 1) > 0$. By means of (5c) and (5d),

$$\begin{aligned} &-\sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a(k > 1)} \\ &> -\frac{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_3] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi} + (\varepsilon + P + \Pi + \Lambda_3)c_s^2}{\varepsilon + P + \Pi - |\Lambda_1| - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \end{aligned} \tag{S16}$$

as well as

$$-\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{a<b}\frac{(\Lambda_a - \Lambda_b)^2 \hat{v}_a^2 \hat{v}_b^2}{\bar{m}_a(k > 1)\bar{m}_b(k > 1)}$$

$$> -\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3-\Lambda_1)^2}{\left[\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]^2}, \tag{S17}$$

and thus,

$$G(k>1) > 1-\frac{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(6\delta_{\pi\pi}-\tau_{\pi\pi})\Lambda_3]+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi}+(\varepsilon+P+\Pi+\Lambda_3)c_s^2}{\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3}$$
$$-\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]^2}. \tag{S18}$$

Note that we have used $\max_{a,b}(\Lambda_a-\Lambda_b)^2=(\Lambda_3-\Lambda_1)^2=(\Lambda_3+|\Lambda_1|)^2$, which follows from the ordering of the eigenvalues Λ_a. Hence (5e) implies $G(k)>0$ for $k>1$.

It now remains to verify the inequality (S13). In this case, when $k<0$

$$\bar{m}_a(k<0) = -|k|(\varepsilon+P+\Pi+\Lambda_a)-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a+\sum_c\Lambda_c\hat{v}_c^2\right)$$
$$< -\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)+\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|. \tag{S19}$$

From condition (5b), one has that $\bar{m}_a(k\le 0)<0$. Then,

$$f(k<0)=\bar{m}_1(k<0)\bar{m}_2(k<0)\bar{m}_3(k<0)G(k<0)<0$$

if, and only if, $G(k<0)>0$. Due to $\bar{m}_a(k\le 0)<0$ together with (5c) and (5d), we obtain that

$$\sum_a\frac{\left\{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(6\delta_{\pi\pi}-\tau_{\pi\pi})\Lambda_a]+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi}+(\rho+\Lambda_a)c_s^2\right\}\hat{v}_a^2}{-\bar{m}_a(k<0)}$$
$$> \frac{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi-(6\delta_{\pi\pi}-\tau_{\pi\pi})|\Lambda_1|]+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2}{-m_a(k<0)}. \tag{S20}$$

Condition (5f) guarantees that $\sum_a\ldots>0$ in the above inequality. Moreover,

$$-\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{a<b}\frac{(\Lambda_a-\Lambda_b)^2\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a(k<0)\bar{m}_b(k<0)}$$
$$> -\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}. \tag{S21}$$

where we used (S19) and $(\Lambda_3+|\Lambda_1|)^2=\max_{a,b}(\Lambda_a-\Lambda_b)^2$ again. Now, since

$$G(k<0) > 1-\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}, \tag{S22}$$

we have $G(k<0)>0$ from condition (5g), finally implying $f(k<0)<0$.

It remains to establish the reality of the roots k_i in (S6). To do that, let us write $G(k)$ as

$$G(k) \;=\; 1-\sum_a\frac{R_a\hat{v}_a^2}{\bar{m}_a}-\sum_{\substack{a,b\\a<b}}\frac{S_{ab}\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a\bar{m}_b} \tag{S23}$$

and

$$\bar{m}_a = \rho_a k - r_a, \tag{S24}$$

where

$$\begin{aligned}
R_a &= \frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2 && \text{(S25)}\\
S_{ab} &= \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_a - \Lambda_b)^2, && \text{(S26)}\\
\rho_a &= \rho + \Lambda_a = \varepsilon + P + \Pi + \Lambda_a, && \text{(S27)}\\
r_a &= \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a + \sum_c \Lambda_c \hat{v}_c^2\right). && \text{(S28)}
\end{aligned}$$

Note, in particular, that $\bar{r}_1 \leq r_a \leq \bar{r}_3$, where $\bar{r}_{1,3} \equiv \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}\Lambda_{1,3}}{2\tau_\pi} > 0$ from (5b). By applying conditions (5) one has that $R_a, S_{ab}, \rho_a, r_a \geq 0$. Then, $f(k)$ can be written as

$$\begin{aligned}
&f(k) = \bar{m}_1\bar{m}_2\bar{m}_3 - \bar{m}_1\bar{m}_2 R_3\hat{v}_3^2 - \bar{m}_2\bar{m}_3 R_1\hat{v}_1^2 - \bar{m}_3\bar{m}_1 R_2\hat{v}_2^2 - \bar{m}_1 S_{23}\hat{v}_2^2\hat{v}_3^2 - \bar{m}_2 S_{13}\hat{v}_1^2\hat{v}_3^2\\
&-\bar{m}_3 S_{12}\hat{v}_1^2\hat{v}_2^2\\
&= a_3k^3 + a_2k^2 + a_1k + a_0, && \text{(S29)}
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= -\big(r_1r_2r_3 + r_1r_2R_3\hat{v}_3^2 + r_2r_3R_1\hat{v}_1^2 + r_1r_3R_2\hat{v}_2^2 - r_1S_{23}\hat{v}_2^2\hat{v}_3^2 - r_2S_{13}\hat{v}_1^2\hat{v}_3^2\\
&\quad -r_3S_{12}\hat{v}_1^2\hat{v}_2^2\big), && \text{(S30)}\\
a_1 &= \rho_1r_2r_3 + \rho_2r_1r_3 + \rho_3r_1r_2 + (\rho_1r_2 + \rho_2r_1)R_3\hat{v}_3^2 + (\rho_2r_3 + \rho_3r_2)R_1\hat{v}_1^2\\
&\quad +(\rho_3r_1 + \rho_1r_3)R_2\hat{v}_2^2 - \rho_1S_{23}\hat{v}_2^2\hat{v}_3^2 - \rho_2S_{13}\hat{v}_1^2\hat{v}_3^2 - \rho_3S_{12}\hat{v}_1^2\hat{v}_2^2, && \text{(S31)}\\
a_2 &= -(\rho_1\rho_2r_3 + \rho_1\rho_3r_2 + \rho_2\rho_3r_1 + \rho_1\rho_2R_3\hat{v}_3^2 + \rho_2\rho_3R_1\hat{v}_1^2 + \rho_1\rho_3R_2\hat{v}_2^2), && \text{(S32)}\\
a_3 &= \rho_1\rho_2\rho_3. && \text{(S33)}
\end{aligned}$$

In view of (5), we have $a_3 > 0$ and $a_2 < 0$. Since all coefficients of $f(k)$ are real, then at least one of the roots must be real, say $k = s_1 \in \mathbb{R}$ is the real root. Then, we know that the other two roots s_2 and s_3 are real or complex conjugate, i.e., $s_3^* = s_2$. Let us assume that s_2 and s_3 can be imaginary and set $s_{2,3} = k_R \pm ik_I$, $k_I \neq 0$. By using Vieta's formula $s_1 + s_2 + s_3 = -\frac{a_2}{a_3} = \frac{|a_2|}{a_3} > 0$ we obtain that

$$\frac{|a_2|}{a_3} - 1 \leq 2k_R = \frac{|a_2|}{a_3} - s_1 \leq \frac{|a_2|}{a_3}. \tag{S34}$$

Thus, the following condition holds,

$$\frac{3\rho_1(\bar{r}_1 + R_1)}{\rho_2\rho_3} - 1 < 2k_R < \frac{3\rho_3(\bar{r}_3 + R_3)}{\rho_1\rho_2} \tag{S35}$$

because the real root $s_1 \in [0, 1]$ when (5a)–(5g) apply, as we have already showed. Since we are assuming $s_{2,3} = k_R \pm ik_I$, where $k_I \neq 0$, we have that $\bar{m}_a(s_{2,3}) = \rho_a k_R - r_a \pm ik_I$ cannot be zero (unless $k_I = 0$ and the roots are real). Consequently, from (S23) we obtain that $f(s_{2,3}) = 0$ lead us to $G(s_{2,3}) = 0$, where $s_{2,3}$ must obey the above conditions implied by f being a cubic polynomial, in particular the condition on k_R in (S35). Thus, let us split $G(s_{2,3})$ in (S19) into $G_R(s_{2,3}) + iG_I(s_{2,3})$, where $G_R(s_{2,3}) = \Re[G(s_{2,3})]$ and $G_I(s_{2,3}) = \Im[G(s_{2,3})]$. In particular,

$$G_I(s_{2,3}) = \pm k_I \sum_a \frac{\hat{v}_a^2}{|\bar{m}_a|^2}\left[\rho_a R_a + \sum_{\substack{b\\ b>a}} \frac{[\rho_a(\rho_b k_R - \bar{r}_b) + \rho_b(\rho_a k_R - \bar{r}_a)]S_{ab}\hat{v}_b^2}{|\bar{m}_b|^2}\right]. \tag{S36}$$

To show that the roots are real, if suffices to have $G_I(s_{2,3}) \neq 0$. We distinguish two cases. If $S_{ab} = 0$ then $G_I(s_{2,3}) \neq 0$ because we assumed $k_I \neq 0$. This means that in this case the roots must all be real. On the other hand, if $S_{ab} \neq 0$ and $\rho_1 R_1 - \bar{r}_3 > 0$, then Eq. (S36) also gives $G_I(s_{2,3}) \neq 0$, because then the sum over b in (S36) is > 0. To check that $\rho_1 R_1 - \bar{r}_3 > 0$, note first that (5a) guarantees that $\rho_a > r_a$. Then, by means of (S35), we obtain that

$$\rho_1 k_R - \bar{r}_3 \;>\; \frac{\rho_1}{2}\left(\frac{3\rho_1(R_1+\bar{r}_1)}{\rho_2\rho_3} - 1 - \frac{2\bar{r}_3}{\rho_1}\right) \geq 0 \tag{S37}$$

because of condition (5h), and this implies $\rho_1 k_R - \bar{r}_3 > 0$. Since we have already showed that any real root of $f(k)$ must lie within $[0,1]$, this finishes our proof. □

IV. LOCAL EXISTENCE AND UNIQUENESS

In this Section, we establish the local existence and uniqueness of solutions to the Cauchy problem. Below, $\mathcal{G}$ is the space of Gevrey functions or quasi-analytic functions.

Theorem 3. *Consider the Cauchy problem for equations (1)-(2) in Minkowski space, with initial data* $\mathring{\Psi} = (\mathring{\varepsilon}, \mathring{u}^\nu, \mathring{\Pi}, \mathring{\pi}^{0\nu}, \mathring{\pi}^{1\nu}, \mathring{\pi}^{2\nu}, \mathring{\pi}^{3\nu})_{\nu=0,\dots,3}$ *given on* $\{t=0\}$. *Assume that the data satisfies the constraints*[2] $\mathring{u}^\nu \mathring{u}_\nu = -1$, $\mathring{u}^\nu$ *is future-pointing,* $\mathring{\pi}^\nu_\nu = 0$, *and* $\mathring{\pi}^\nu_\mu \mathring{u}^\mu = 0$. *Suppose that (A1)-(A3) and (5) hold for* $\mathring{\Psi}$ *in a strict form (i.e.* $<$ *instead of* $\leq$, $>$ *instead of* $\geq$*). Finally, assume that* $\mathring{\Psi} \in \mathcal{G}^\delta(\{t=0\})$, *where* $1 \leq \delta < 20/19$. *Then, there exist a* $T > 0$ *and a unique* $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\dots,3}$ *defined on* $[0,T) \times \mathbb{R}^3$ *such that* Ψ *is a solution to (1)-(2) in* $[0,T) \times \mathbb{R}^3$ *and* $\Psi = \mathring{\Psi}$ *on* $\{t=0\}$. *Moreover, the solution* Ψ *is causal in the sense of Definition 4 (see Section VI).*

Proof of Theorem 3: The calculations provided in Section I and in the proof of Theorem 2 imply that, under the assumptions, the characteristic polynomial of the system evaluated at the initial data is a product of strictly hyperbolic polynomials. One also sees that intersection of the interior of the characteristic cones defined by these strictly hyperbolic polynomials has non-empty interior and lies outside the light-cone defined by the metric. Under these circumstances we can apply theorems A.18, A.19, and A.23 of [2] to conclude the result (the remaining assumptions of these theorems are easily verified in our case). □

For the sake of brevity, we refer readers to [3] for a definition of $\mathcal{G}^\delta$, making only the following remarks. The case of $\delta = 1$ corresponds to the space of analytic functions, of which $\mathcal{G}^\delta$ with $\delta > 1$ is a generalization. This is why $\mathcal{G}$ is sometimes referred to as the space of quasi-analytic functions. The usefulness of Gevrey functions to the study of hyperbolic problems is at least two-fold. On the one hand, one can prove very general existence and uniqueness theorems for Gevrey data given on a non-characteristic surface that are akin to the Cauchy-Kovalewskaya theorem for analytic data. On the other hand, an advantage of Gevrey maps over analytic ones is that one can construct Gevrey functions that are compactly supported; hence one can appeal to the type of localization arguments that are so useful in the study of hyperbolic equations. This is particularly important when one is considering coupling to Einstein's equations.

While typical evolution problems consider solutions in more general function spaces than $\mathcal{G}^\delta$, we stress that ours is the very first existence and uniqueness result for equations (1)-(2). In other words, while it is desirable to extend our result to more general function spaces, Theorem 3 is important because it shows, for the very first time in the literature, that the initial value problem for equation (1)-(2) is well-defined, so that it is meaningful to talk about solutions.

We remark that the diagonalization of $\pi_{\mu\nu}$ was carried out in terms of orthonormal frames which can be defined for any Lorentzian metric. Also, our computations are manifestly covariant. Thus, the results of

[2] Alternatively, we could have only unconstrained data be prescribed and obtain the full set of data from the stated constraints. For example, we could have $\mathring{u}^i$ prescribed and define u^0 so that $\mathring{u}^\nu$ is unit time-like and future pointing.

Theorems 1, 2, and 3, remain true in a general globally hyperbolic space-time, as mentioned in the main text. Moreover, as also mentioned in the main text, the result extends to the case when (1)-(2) are coupled to Einstein's equations. This follows by computing the characteristic determinant of the coupled system and observing that it factors into the product of the characteristic determinant of (1)-(2), which we analyzed here, and the characteristic determinant of Einstein's equations. The argument is the same as given in [4].

V. INSUFFICIENCY OF CONDITIONS FOR CAUSALITY

In this Section, we show that conditions (4), albeit necessary, are not sufficient for causality. We do this by showing that causality can be violated if we only assume (A1)-(A3) and (4).

Thus, suppose that (A1)-(A3) and (4) hold. Consider the case where (Ins1) $\delta_{\pi\pi} = \tau_{\pi\pi}/4$, $\delta_{\Pi\Pi} = 0$, $\zeta + \lambda_{\Pi\pi}\Lambda_a \geq 0$, $\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi} > 0$, and $1 - c_s^2 - \frac{\tau_{\pi\pi}}{3\tau_\pi} - \frac{\lambda_{\Pi\pi}}{\tau_\Pi} < 0$. Also, the parameters as well as c_s^2 obey the necessary conditions (4). Assume also that (Ins2) $\Lambda_3 = \Lambda_2 > 0$, i.e., Λ_3 is a degenerated eigenvalue. Then, we may write

$$G(k) = 1 - \sum_a \frac{R_a \hat{v}_a^2}{\bar{m}_a} - \sum_{\substack{a,b \\ a<b}} \frac{S_{ab}\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a\bar{m}_b}, \tag{S38}$$

where

$$R_a = \frac{1}{6\tau_\pi}\left[2\eta + \lambda_{\pi\Pi}\Pi + \frac{\tau_{\pi\pi}}{2}\Lambda_a\right] + \frac{\zeta + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\varepsilon + P + \Pi + \Lambda_a)c_s^2 \tag{S39}$$

and

$$S_{ab} = \frac{\tau_{\pi\pi}}{6\tau_\pi}\left(\frac{\Lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_a - \Lambda_b)^2. \tag{S40}$$

From (4a) together with the above choices we have that $R_a, S_{ab} > 0$. Now, let us define

$$\tilde{m}_a \equiv \varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_a. \tag{S41}$$

Then, (4f) can be written as

$$\tilde{m}_d - R_d \geq 0, \tag{S42}$$

culminating into $\tilde{m}_d > 0$. Note that this must hold for any $d = 1, 2, 3$. Let us consider the case where a_1 is such that $\tilde{m}_{a_1} - R_{a_1} = \min_d(\tilde{m}_d - R_d)$. Thus, if (S42) is verified for $d = a_1$, it is automatically true for for all $d = 1, 2, 3$. Now, we may choose the constraint in the parameters such that (Ins3) $\tilde{m}_{a_1} - R_{a_1} = 0$, what is in accord with (S42). The remainder of this proof relies on the choice $\hat{v}_{a_1} = \sqrt{1-\epsilon^2}$, $\hat{v}_{a_2} = \epsilon$, and $\hat{v}_{a_3} = 0$ for $\epsilon \in (0, 1)$. Moreover, we make the assumption (Ins4) that if $a_1 = 3, 2$, then $a_2 = 2, 3$ while if $a_1 = 1$, then a_2 can be either 2 or 3. Thus, one can clearly see that

$$f(k) = \bar{m}_{a_3}\left(\bar{m}_{a_1}\bar{m}_{a_2} - \bar{m}_{a_1}R_{a_2}\epsilon^2 - \bar{m}_{a_2}R_{a_1}(1-\epsilon^2) - S_{a_1a_2}\epsilon^2(1-\epsilon^2)\right), \tag{S43}$$

$$\begin{aligned}\bar{m}_d &= (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\left[\Lambda_d + \Lambda_{a_1}(1-\epsilon^2) + \Lambda_{a_2}\epsilon^2\right] \\ &= \bar{m}_d^0 - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_{a_2} - \Lambda_{a_1})\epsilon^2,\end{aligned} \tag{S44}$$

where we defined

$$\bar{m}_d^0 \equiv (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_d + \Lambda_{a_1}\right).$$

From (4d) one may easily verify that $\bar{m}_d^0(k \geq 1) \geq 0$. In particular,

$$\bar{m}_{a_1}^0(k=1) = \tilde{m}_{a_1} > 0 \tag{S45}$$

from (S42), while $\bar{m}_{a_2,a_3}^0(k=1) > \tilde{m}_{a_2,a_3} > 0$. (Ins2) enables us to write (note that $a_2 \neq 1$ according to (Ins4))

$$\Lambda_{a_1}(1-\epsilon^2) + \Lambda_{a_2}\epsilon^2 \begin{cases} = \Lambda_3 = \Lambda_2, \text{ if } a_1 = 2, a_2 = 3 \text{ or } a_1 = 3, a_2 = 2, \\ < \Lambda_3, \text{ if } a_1 = 1 \;\forall\; \epsilon \in (0,1) \end{cases}, \tag{S46}$$

which results into

$$\bar{m}_d \geq (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_d + \Lambda_3), \tag{S47}$$

and gives $\bar{m}_d(k \geq 1) \geq 0$ due to (4d) and $\bar{m}_{2,3}(k \geq 1) > 0$ because $\tilde{m}_d > 0$ from (S42).

The roots of f are the roots of $\bar{m}_{a_3}$ and the roots in the term in brackets in (S43). Let us define it as

$$\begin{aligned} \tilde{f}(k) &\equiv \bar{m}_{a_1}\bar{m}_{a_2} - \bar{m}_{a_1}R_{a_2}\epsilon^2 - \bar{m}_{a_2}R_{a_1}(1-\epsilon^2) - S_{a_1a_2}\epsilon^2(1-\epsilon^2) \\ &= \bar{m}_{a_1}\bar{m}_{a_2}G(k), \end{aligned} \tag{S48}$$

where

$$G(k) = 1 - \frac{R_{a_2}\epsilon^2}{\bar{m}_{a_2}} - \frac{R_{a_1}(1-\epsilon^2)}{\bar{m}_{a_1}} - \frac{S_{a_1a_2}\epsilon^2(1-\epsilon^2)}{\bar{m}_{a_1}\bar{m}_{a_2}}. \tag{S49}$$

Note that since $\epsilon \in (0,1)$, the terms $\bar{m}_{a_1,a_2}(\bar{k})$ cannot be zero if $\bar{k}$ is a root of $\tilde{f}$ due to the term $S_{a_1a_2}$. Also, because $\tilde{f}(k) = (\rho + \Lambda_{a_1})(\rho + \Lambda_{a_2})k^2 + \mathcal{O}(k)$ is a positive function to the right of the larger real root due to (Ins3), then $\tilde{f}(k > 1) > 0$, or equivalently $G(k > 1) > 0$, guarantees that there is no real root for $k > 1$. Because (Ins1) leads to $R_a, S_{ab} > 0$, and since $\bar{m}_a(k > 1) > \bar{m}_a(k = 1)$, then condition $G(k > 1) > 0$ is equivalent to $G(k = 1) \geq 0$. In other words we must have that

$$1 - \frac{R_{a_2}\epsilon^2}{\bar{m}_{a_2}(k=1)} - \frac{R_{a_1}(1-\epsilon^2)}{\bar{m}_{a_1}(k=1)} - \frac{S_{a_1a_2}\epsilon^2(1-\epsilon^2)}{\bar{m}_{a_1}(k=1)\bar{m}_{a_2}(k=1)} \geq 0. \tag{S50}$$

Since $\epsilon < 1$ we can expand (S45) in powers of it and, after using (S45) and (Ins3), obtain the causality condition

$$\left\{1 - \frac{\tau_{\pi\pi}}{4\tau_\pi\tilde{m}_{a_1}}(\Lambda_{a_2} - \Lambda_{a_1}) - \frac{R_{a_2}}{\bar{m}_{a_2}^0(k=1)} - \frac{S_{a_1a_2}}{\tilde{m}_{a_1}\bar{m}_{a_2}^0(k=1)}\right\}\epsilon^2 + \mathcal{O}(\epsilon^4) \geq 0. \tag{S51}$$

Now, by writing

$$\bar{m}_{a_2}^0(k=1) = \tilde{m}_{a_1} + (\Lambda_{a_2} - \Lambda_{a_1})\left(1 - \frac{\tau_{\pi\pi}}{4\tau_\pi}\right)$$

and

$$R_{a_2} = R_{a_1} + (\Lambda_{a_2} - \Lambda_{a_1})\left(c_s^2 + \frac{\tau_{\pi\pi}}{12\tau_\pi} + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\right),$$

and by means of (Ins3) we may rewrite

$$1 - \frac{R_{a_2}}{\bar{m}_{a_2}^0(k=1)} = \frac{\Lambda_{a_2} - \Lambda_{a_1}}{\bar{m}_{a_2}^0(k=1)}\left(1 - c_s^2 - \frac{\tau_{\pi\pi}}{3\tau_\pi} - \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\right) \leq 0. \tag{S52}$$

Note that (S52) is negative or zero because of (Ins2), (Ins3), and (Ins4). From (Ins2) and (Ins4), if $a_1 = 2, 3$, then $a_2 = 3, 2$ and $\Lambda_{a_2} - \Lambda_{a_1} = 0$ while if $a_1 = 1$, then $a_2 = 2, 3$ and $\Lambda_{a_2} - \Lambda_1 > 0$, resulting in $\Lambda_{a_2} - \Lambda_{a_1} \geq 0$, while (Ins1) makes (S52) negative or zero. As a consequence of (S52), the term proportional to ϵ^2 in the LHS of (S51) is negative and, for some small value of $\epsilon \in (0,1)$ it must become the leading term, turning the LHS of (S51) strictly negative. Then, one concludes that the system is not causal and the necessary conditions (4) are not sufficient.

VI. FORMAL DEFINITION OF CAUSALITY AND CONDITIONS (C1) AND (C2)

Here we present the precise mathematical definition of causality and how it relates to conditions (C1) and (C2). Causality can be defined as follows (see [5, page 620] or [6, Theorem 10.1.3] for more details).

Definition 4. Let $(\mathcal{M}, g)$ be the Minkowski space. Consider in $\mathcal{M}$ a system of partial differential equations for an unknown ψ, which we write as $\mathcal{P}\psi = 0$, where $\mathcal{P}$ is a differential operator (which is allowed to depend on ψ)[3]. Let φ be a solution to the system. We say that φ is causal if the following holds true: given a Cauchy surface $\Sigma \subset \mathcal{M}$, for any point x in the future of Σ, $\varphi(x)$ depends only on $\varphi|_{J\ (x)\cap\Sigma}$, where $J^-(x)$ is the causal past of x.

The case of most interest is when the Cauchy surface is the hypersurface $\{t = 0\}$ where initial data is prescribed. We also note that since we are working in Minkowski space, $J^-(x)$ is simply the past light-cone with vertex at x. The situation in Definition 4 is illustrated in Fig. 1. In particular, causality implies that $\varphi(x)$ remains unchanged if the the values of φ along Σ are altered only outside $J^-(x) \cap \Sigma$. Observe that this definition says that $\varphi(x)$ can only be influenced by points in the past of x that are causally connected to x, so no information is allowed to propagate faster than the speed of light.

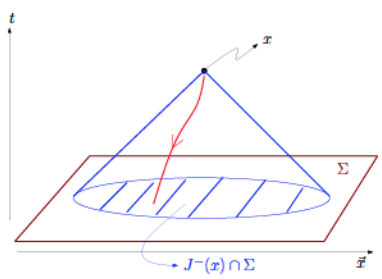

FIG. 1: (color online) Illustration of causality. $J^-(x)$ is the past light-cone with vertex at x. Points inside $J^-(x)$ can be joined to a point x in space-time by a causal past directed curve (e.g. the red line). The value of $\varphi(x)$ depends only on $\varphi|_{J\ (x)\cap\Sigma}$.

The connection between Definition 4 and conditions (C1) and (C2) is via the characteristics of the system $\mathcal{P}\psi = 0$. It is beyond the scope of this Supplemental Material to provide a detailed description of the connections between Definition 4 and the system's characteristics. We refer readers to Appendix A of [2], [7, Chapter VI], and [8]. Here, we restrict ourselves to the following comments. Finite speed of propagation is a property of hyperbolic equations. For such equations, there exist domains of dependence that show precisely how the value of a solution at a point x is determined solely by values within a domain of dependence in the past with "vertex" at x (this is exactly the generalization of the past light-cone). The domain of dependence, in turn, is determined by the system's characteristics. While it is mathematically possible for hyperbolic equations to exhibit domains of dependence where information propagates faster than the speed of light (see, again, our discussion in the Conclusion), for solutions to be causal (i.e., to not have faster-than-light signals), the domains of dependence must always lie inside the light-cones. This is equivalent to the statement (C1) and (C2) that we have used. Definition 4 can be generalized to arbitrary globally hyperbolic spaces, which is needed for the aforementioned generalization of our Theorems to this setting. Again, we refer to Appendix A of [2], [7, Chapter VI], and [8].

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[3] In coordinates, this system of PDEs would be represented by $P^I_K \psi^K = 0$, $I, K = 1, \ldots, N$, where $\{\psi^K\}_{K=1}^N$ are local representations of ψ, and $\mathcal{P}^I_K$ are differential operators (possibly depending on ψ^K).

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