LOCAL EXISTENCE AND UNIQUENESS IN SOBOLEV SPACES FOR FIRST-ORDER CONFORMAL CAUSAL RELATIVISTIC VISCOUS HYDRODYNAMICS

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Abstract. In this manuscript, we study the theory of conformal relativistic viscous hydrodynamics introduced in [4], which provided a causal and stable first-order theory of relativistic fluids with viscosity. Local existence and uniqueness of solutions to its equations of motion have been previously established in Gevrey spaces. Here, we improve this result by proving local existence and uniqueness of solutions in Sobolev spaces.

1. **Introduction.** Relativistic hydrodynamics is an essential tool in several branches of physics, including high-energy nuclear physics [3], astrophysics [33], and cosmology [38], and it is also a fertile source of mathematical problems (see, e.g., the monographs [2, 10, 11, 12, 33] and references therein). This paper is concerned with the local Cauchy problem to the equations of motion of relativistic viscous fluids.

More precisely, we consider the energy-momentum tensor for a relativistic conformal fluid given by

$$\mathcal{T}_{\alpha\beta} = (\varepsilon + A)(u_\alpha u_\beta + \frac{1}{3}\Pi_{\alpha\beta}) - \eta\sigma_{\alpha\beta} + u_\alpha Q_\beta + u_\beta Q_\alpha, \tag{1.1}$$

where

$$A = 3\chi\left(\frac{1}{\theta}u^\mu\nabla_\mu\theta + \frac{1}{3}\nabla_\mu u^\mu\right),$$

$$Q_\alpha = \lambda\left(\frac{1}{\theta}\Pi_\alpha^\mu\nabla_\mu\theta + u^\mu\nabla_\mu u_\alpha\right),$$

$$\sigma_{\alpha\beta} = \Pi_\alpha^\mu\nabla_\mu u_\beta + \Pi_\beta^\mu\nabla_\mu u_\alpha - \frac{2}{3}\Pi_{\alpha\beta}\nabla_\mu u^\mu.$$

Above, ε is the fluid's energy density; u is the fluid's four-velocity, which satisfies the constraint

$$g_{\alpha\beta}u^\alpha u^\beta = -1, \tag{1.2}$$

where g is the spacetime metric[1]; Π is the projection onto the space orthogonal to u, given by $\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$; θ is the temperature that satisfies $\varepsilon = \varepsilon_0\theta^4$, where $\varepsilon_0 > 0$ a constant; η, χ, and λ are transport coefficients, which are known functions of ε and model the viscous effects in the fluid; and ∇ is the covariant derivative associated with the metric g. Indices are raised and lowered using the spacetime metric, lowercase Greek indices vary from 0 to 3, Latin indices vary from 1 to 3, repeated indices are summed over their range, and expressions such as z_α, $w_{\alpha\beta}$, etc. represent the components of a vector or tensor with respect to a system of coordinates $\{x^\alpha\}_{\alpha=0}^3$ in spacetime, where the coordinates are always chosen so that $x^0 = t$ represents a time coordinate. We will consider the fluid dynamics in a fixed background, so that the metric g is given.

The equations of motion are given by

$$\nabla_\alpha\mathcal{T}_\beta^\alpha = 0 \tag{1.3}$$

supplemented by the constraint (1.2).

We now state our result. After the statement, we discuss our assumptions and provide some further context. We note that in view of (1.2), it suffices to provide the components of u tangent to $\{t = 0\}$ as initial data; this explains the statement involving the projector $\mathcal{P}$ in the Theorem, where we recall that u is the fluid's four-velcity. We also observe that in view of the relation $\varepsilon = \varepsilon_0\theta^4$ between the energy density and the temperature, it suffices to provide initial data for θ (which is what we in fact use in Section 3). The conclusion of the Theorem, however, is more appropriately stated in terms of ε, since fluid dynamic equations are more commonly written in terms of the energy density (this is particularly the case for perfect fluids, which provide a basis for comparisons with viscous ones).

Theorem 1.1. *Let g be the Minkowski metric on $\mathbb{R}\times\mathbb{T}^3$, where $\mathbb{T}^3$ is the three-dimensional torus. Let $\eta : (0,\infty)\to(0,\infty)$ be an analytic function, $\chi = a_1\eta$, and $\lambda = a_2\eta$, where a_1 and a_2 are positive constants satisfying $a_1 > 4$ and $a_2 \geq 3a_1/(a_1 - 1)$. Let $\theta_{(0)} \in H^r(\mathbb{T}^3,\mathbb{R})$, $\theta_{(1)} \in H^{r-1}(\mathbb{T}^3,\mathbb{R})$, $u_{(0)} \in H^r(\mathbb{T}^3,\mathbb{R}^3)$, and $u_{(1)} \in H^{r-1}(\mathbb{T}^3,\mathbb{R}^3)$ be given, where H^r is the Sobolev space and $r > 7/2$. Assume that $\theta_{(0)} \geq C > 0$ for some constant C. Let $\varepsilon_{(0)} = \varepsilon_0\theta_{(0)}^4$ and $\varepsilon_{(1)} = 4\varepsilon_0\theta_{(0)}^3\theta_{(1)}$, where $\varepsilon_0 > 0$ is a constant.*

Then, there exists a $T > 0$, a function

$$\varepsilon \in C^0([0,T), H^r(\mathbb{T}^3,\mathbb{R})) \cap C^1([0,T), H^{r-1}(\mathbb{T}^3,\mathbb{R})) \cap C^2([0,T), H^{r-2}(\mathbb{T}^3,\mathbb{R})),$$

and a vector field

$$u \in C^0([0,T), H^r(\mathbb{T}^3,\mathbb{R}^4)) \cap C^1([0,T), H^{r-1}(\mathbb{T}^3,\mathbb{R}^4)) \cap C^2([0,T), H^{r-2}(\mathbb{T}^3,\mathbb{R}^4))$$

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[1]By "metric" we always mean a "Lorentzian metric."

such that equations (1.2) *and* (1.3) *hold on* $[0, T] \times \mathbb{T}^3$, *and satisfy* $\varepsilon(0, \cdot) = \varepsilon_{(0)}$, $\partial_t \varepsilon(0, \cdot) = \varepsilon_{(1)}$, $\mathcal{P}u(0, \cdot) = u_{(0)}$, *and* $\mathcal{P}\partial_t u(0, \cdot) = u_{(1)}$, *where* ∂_t *is the derivative with respect to the first coordinate in* $[0, T] \times \mathbb{T}^3$ *and* $\mathcal{P}$ *is the canonical projection from the tangent bundle of* $[0, T] \times \mathbb{T}^3$ *onto the tangent bundle of* $\mathbb{T}^3$. *Moreover,* (ε, u) *is the unique solution with the stated properties.*

Relativistic viscous hydrodynamics is important in many areas of physics, such as in the study of the quark-gluon-plasma that forms in heavy ion-collisions [3, 24] or in neutron star mergers [1]. In fact, in the heavy-ions community, for example, all current studies within the scope of fluid theories are done using viscous models [3, 24].

One of the main challenges in the theory of relativistic viscous hydrodynamics is to construct physically meaningful theories that respect causality, (linear) stability, and local existence and uniqueness of solutions. Despite the importance of relativistic viscous hydrodynamics, very few models have been showed to be causal, stable, and to admit local existence and uniqueness of solutions, and typical results of this nature have been only partial [5, 13, 14, 21, 28]. The literature on this topic is vast and we refer the reader to [3, 17, 18, 24, 25, 26, 33, 35, 36] and references therein for discussion and background.

The energy-momentum (1.1) was introduced in [4], where a new approach to the formulation of relativistic viscous hydrodynamics was proposed for the case of a conformal fluid. Conformal fluids satisfy the property that the ratio between any two transport coefficients is constant [3, 7], which explains our assumption that χ and λ are a multiple of η. Under the assumptions on a_1 and a_2 stated in the Theorem, the equations of motion derived from (1.1) (i.e., (1.2) and (1.3)) were showed to be causal and linearly stable in [4, 16]. The physical significance of our assumptions on η, χ, and λ is highlighted by the fact that causality and linear stability are basic physical properties of relativistic viscous fluids. Further physical properties of equations (1.2) and (1.3), such as the non-relativistic limit, entropy production, and connections with kinetic theory, are also discussed in [4].

In the works [4, 16], it is was also showed that equations (1.2) and (1.3) admit local existence and uniqueness of solutions in Gevrey spaces (see Section 3.1 for the definition of Gevrey functions). The goal of this manuscript is to extend these result by establishing local existence and uniqueness of solutions in Sobolev spaces[2].

We work on $\mathbb{T}^3$ for simplicity, since using the domain of dependence property (proved in [16]) one can adapt the proof to $\mathbb{R}^3$. The assumption $\theta_{(0)} \geq C > 0$, on the other hand, is crucial. Without it the equations can degenerate. This can be seen more clearly from the matrix $\mathcal{A}^\alpha$ in Section 3: some entries of $\mathcal{A}^\alpha$ involve θ^{-1}, but multiplying the system by θ would cause the matrix $\mathcal{A}^\alpha$ to become singular. It is possible that the proper context to understand the situation when θ (or ε) can vanish is that of a free-boundary dynamics. However, free-boundary problems remain largely open even in the case of a relativistic perfect fluid [8, 9, 15, 20, 22, 23, 27, 30, 31, 32].

We remark that it is likely that our results can be extended to non-conformal fluids, but we focus here on the conformal case as this seems to provide the simplest setting where the equations simplify considerably, while being at the same time of direct physical relevance (since conformal viscous fluids are used in the study of

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[2]However, only the case of a fixed background Minkowski metric is treated here, whereas in [4, 16] the coupling of (1.1) with Einstein's equations has also been studied.

the quark-gluon plasma). In fact, historically conformal fluids have served as an efficient test-case of ideas in the study of viscous relativistic hydrodynamics. In this regard, we point out that for an energy-momentum tensor of the form (1.1) but with a general barotropic equation of state, causality and local existence and uniqueness of solutions in Gevrey spaces have been established in [6].

We also stress that our results can be generalized to curved fixed backgrounds, but we focused on Minkowski space for simplicity and because the most physically relevant system for conformal relativistic viscous fluids is the quark-gluon-plasma, which evolves on a Minkowski background. We stress, however, that while our proof could be adapted to a curved background, considering coupling to Einstein's equations is significantly more complicated. This is because in our derivation of the new system of equations of Section 2, we commute derivatives, which would produce terms proportional to the the Riemann tensor. Such terms will be lower order (in fact, zero in Minkowski) in terms of the fluid variables whenever the metric is given (i.e., in a fixed background), but they will be top-order when coupled to Einstein's equations.

2. **A new system of equations.** In this section we derive a new system of equations that will allow us to establish Theorem 1.1. In order to do so, throughout this section, we assume to be given a sufficiently regular solution to (1.2)-(1.3).

Using (1.2) to decompose $\nabla_\alpha \mathcal{T}_\beta^\alpha$ in the directions parallel and orthogonal to u, we can rewrite (1.3) as

$$u^\alpha \nabla_\alpha A + \frac{4}{3} A \nabla_\alpha u^\alpha + \nabla_\alpha Q^\alpha + Q_\alpha u^\lambda \nabla_\lambda u^\alpha - \frac{1}{2}\eta \sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{4}{3\chi}\varepsilon_0 \theta^4 A = 0, \quad (2.1a)$$

$$\frac{1}{3}\Pi_\mu^\alpha \nabla_\alpha A + \frac{4}{3} A u^\alpha \nabla_\alpha u_\mu - \eta \nabla_\alpha \sigma_\mu^\alpha + \frac{\eta}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} u_\mu + 3\eta \sigma_{\mu\lambda} u^\alpha \nabla_\alpha u^\lambda + u^\alpha \nabla_\alpha Q_\mu$$

$$- u_\mu Q^\lambda u^\alpha \nabla_\alpha u_\lambda + \nabla_\alpha u^\alpha Q_\mu + Q^\alpha \nabla_\alpha u_\mu + \frac{4\varepsilon}{3\lambda} Q_\mu - \frac{3\eta}{\lambda}\sigma_{\mu\nu} Q^\nu = 0. \quad (2.1b)$$

Introducing

$$S_\alpha{}^\beta = \Pi_\alpha^\mu \nabla_\mu u^\beta, \quad S^\alpha = u^\mu \nabla_\mu u^\alpha,$$

we find

$$u^\mu \nabla_\mu A + \nabla_\mu Q^\mu + r_1 = 0, \quad (2.2a)$$

$$\Pi^{\alpha\mu}\nabla_\mu A + 3u^\mu \nabla_\mu Q^\alpha + B_\nu{}^{\alpha\mu\lambda}\nabla_\lambda S_\mu{}^\nu + r_2 = 0, \quad (2.2b)$$

$$-\frac{1}{\chi}\Pi^{\alpha\mu}\nabla_\mu A + \frac{3}{\lambda}u^\mu \nabla_\mu Q^\alpha - 3u^\mu \nabla_\mu S^\alpha + \Pi^{\alpha\mu}\nabla_\mu S_\nu{}^\nu + r_3 = 0, \quad (2.2c)$$

$$u^\mu \nabla_\mu S_\alpha{}^\beta - \Pi_\alpha^\nu \nabla_\nu S^\beta + r_4 = 0, \quad (2.2d)$$

$$\frac{1}{\theta}u^\mu \nabla_\mu \theta + \frac{1}{3}\nabla_\mu u^\mu + r_5 = 0, \quad (2.2e)$$

$$\frac{1}{\theta}\Pi^{\alpha\mu}\nabla_\mu \theta + u^\mu \nabla_\mu u^\alpha + r_6 = 0, \quad (2.2f)$$

where

$$B_\nu{}^{\alpha\mu\lambda} = -3\eta(\delta_\nu^\alpha \Pi^{\mu\lambda} + \delta_\nu^\lambda \Pi^{\alpha\mu} - \frac{2}{3}\delta_\nu^\mu \Pi^{\alpha\lambda}),$$

and r_i, $i = 1, \dots, 6$ are smooth functions of A, Q^α, S^α, $S_\alpha{}^\beta$, θ, and u^α; no derivative of such quantities appears in the r_i's. Above and throughout, δ is the Kronecker delta.

The derivation of (2.2) is as follows: equations (2.2a) and (2.2b) are equations (2.1a) and (2.1b), respectively; equations (2.2e) and (2.2f) are simply the definition of A and Q^α; equations (2.2c) and (2.2d) follow from contracting the identities

$$\nabla_\mu \nabla_\nu \theta - \nabla_\nu \nabla_\mu \theta = 0, \quad \nabla_\mu \nabla_\nu u^\alpha - \nabla_\nu \nabla_\mu u^\alpha = R_{\mu\nu}{}^\alpha{}_\lambda u^\lambda = 0,$$

with u^μ and then with Π^ν_λ. We also used the identities

$$\frac{1}{\theta} \nabla_\alpha \theta = -\frac{1}{3\chi} u_\alpha A + \frac{1}{\lambda} Q_\alpha + \frac{1}{3} u_\alpha S_\mu{}^\mu - \Pi_{\alpha\mu} S^\mu, \quad \nabla_\alpha u^\beta = -u_\alpha S^\beta + S_\alpha{}^\beta.$$

We write equations (2.2) as a quasilinear first order system for the the variable $\boldsymbol{\Psi} = (A, Q^\alpha, S^\alpha, S_0{}^\alpha, S_1{}^\alpha, S_2{}^\alpha, S_3{}^\alpha, \theta, u^\alpha)^T$, with ^{T} being the transpose, as

$$\mathcal{A}^\alpha \nabla_\alpha \boldsymbol{\Psi} + \mathcal{R} = 0, \tag{2.3}$$

where $\mathcal{R} = (r_1, \ldots, r_6)^T$ and $\mathcal{A}^\alpha$ is given by

$$\mathcal{A}^\alpha = \begin{bmatrix} u^\alpha & \delta^\alpha_\nu & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0 & 0_{1\times4} \\ \Pi^{\mu\alpha} & 3u^\alpha I_4 & 0_{4\times4} & B_\nu{}^{\mu0\alpha} & B_\nu{}^{\mu1\alpha} & B_\nu{}^{\mu2\alpha} & B_\nu{}^{\mu3\alpha} & 0_{4\times1} & 0_{4\times4} \\ -\frac{\Pi^{\mu\alpha}}{\chi} & \frac{3u^\alpha}{\lambda} I_4 & -3u^\alpha I_4 & \delta^0_\nu \Pi^{\mu\alpha} & \delta^1_\nu \Pi^{\mu\alpha} & \delta^2_\nu \Pi^{\mu\alpha} & \delta^3_\nu \Pi^{\mu\alpha} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -\Pi^\alpha_0 I_4 & u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -\Pi^\alpha_1 I_4 & 0_{4\times4} & u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -\Pi^\alpha_2 I_4 & 0_{4\times4} & 0_{4\times4} & u^\alpha I_4 & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -\Pi^\alpha_3 I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha I_4 & 0_{4\times1} & 0_{4\times4} \\ 0 & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & \frac{u^\alpha}{\theta} & \frac{\delta^\alpha_\nu}{3} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & \frac{\Pi^{\mu\alpha}}{\theta} & I_4\, u^\alpha \end{bmatrix}.$$

Equation (2.3) is the main equation we will use to derive estimates. Observe that $\boldsymbol{\Psi}$ has 30 components and the matrices $\mathcal{A}^\alpha$ are 30×30 matrices.

3. **Diagonalization.** Here, we show that under assumptions consistent with those of Theorem 1.1, we can diagonalize the principal part of (2.3).

Proposition 1. *Let ξ be a timelike vector and assume that $\chi > 4\eta > 0$ and that $\lambda \geq \frac{3\chi\eta}{\chi - \eta}$. Then:*
   (i) $\det(\mathcal{A}^\alpha \xi_\alpha) \neq 0$;
   (ii) *For any spacelike vector ζ, the eigenvalue problem $\mathcal{A}^\alpha(\zeta_\alpha + \Lambda \xi_\alpha)V = 0$ has only real eigenvalues Λ and a complete set of eigenvectors V.*

Remark 1. In practice we will take $\xi = (1, 0, 0, 0)$ and $\zeta = (0, \zeta_1, \zeta_2, \zeta_3)$. We note that the assumptions on χ, λ, and η on Theorem 1.1 imply the assumptions on these coefficients in the Proposition.

Proof. Let a and b be the projection of $\zeta + \Lambda\xi$ on the direction orthogonal and parallel to u, i.e., $a^\alpha = \Pi^{\alpha\mu}(\zeta_\mu + \Lambda\xi_\mu)$ and $b = (\zeta_\alpha + \Lambda\xi_\alpha)u^\alpha$. Then

$$a^\mu a_\mu$$
$$= \Pi^{\alpha\mu}\Pi_{\alpha\nu}(\zeta_\mu + \Lambda\xi_\mu)(\zeta^\nu + \Lambda\xi^\nu) = (g^{\alpha\mu} + u^\alpha u^\mu)(\zeta_\mu + \Lambda\xi_\mu)(g_{\alpha\nu} + u_\alpha u_\nu)(\zeta^\nu + \Lambda\xi^\nu)$$
$$= (\zeta_\alpha + \Lambda\xi_\alpha)(\zeta^\alpha + \Lambda\xi^\alpha) + b^2.$$

To simplify the notation, set $\Xi_\alpha = \zeta_\alpha + \Lambda\xi_\alpha$. Then

$$\det(\Xi_\alpha \mathcal{A}^\alpha)$$

$$= \det \begin{bmatrix} b & \Xi^T & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0 & 0_{1\times4} \\ a & 3bI_4 & 0_{4\times4} & D_\nu{}^{\mu0} & D_\nu{}^{\mu1} & D_\nu{}^{\mu2} & D_\nu{}^{\mu3} & 0_{4\times1} & 0_{4\times4} \\ -\frac{a}{\chi} & \frac{3b}{\lambda}I_4 & -3bI_4 & \delta_\nu^0 a & \delta_\nu^1 a & \delta_\nu^2 a & \delta_\nu^3 a & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_0 I_4 & u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_1 I_4 & 0_{4\times4} & bI_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_2 I_4 & 0_{4\times4} & 0_{4\times4} & bI_4 & 0_{4\times4} & 0_{4\times1} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_3 I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & bI_4 & 0_{4\times1} & 0_{4\times4} \\ 0 & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & \frac{b}{\theta} & \frac{\Xi^T}{3} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & \frac{a}{\theta} & bI_4 \end{bmatrix}$$

$$= m_1 m_2,$$

where we write Ξ^T to emphasize that Ξ^T represents a 1×4 piece, and $D_\nu{}^{\alpha\mu} = B_\nu{}^{\alpha\mu\lambda}\Xi_\lambda$. m_2 is given by

$$m_2 = \det \begin{bmatrix} \frac{b}{\theta} & \frac{\Xi^T}{3} \\ \frac{a}{\theta} & bI_4 \end{bmatrix} = \frac{b^3}{3\theta}(3b^2 - \Pi_{\alpha\beta}\Xi^\alpha\Xi^\beta),$$

whereas

$$m_1 = \det \begin{bmatrix} b & \Xi^T & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} \\ a & 3bI_4 & 0_{4\times4} & D_\nu{}^{\mu0} & D_\nu{}^{\mu1} & D_\nu{}^{\mu2} & D_\nu{}^{\mu3} \\ -\frac{a}{\chi} & \frac{3b}{\lambda}I_4 & -3bI_4 & a\delta_\nu^0 & a\delta_\nu^1 & a\delta_\nu^2 & a\delta_\nu^3 \\ 0_{4\times1} & 0_{4\times4} & -a_0 I_4 & bI_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_1 I_4 & 0_{4\times4} & bI_4 & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_2 I_4 & 0_{4\times4} & 0_{4\times4} & bI_4 & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & -a_3 I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & bI_4 \end{bmatrix} \tag{3.1}$$

$$= b^9 \det \begin{bmatrix} 1 & \Xi^T & 0_{1\times4} \\ a & 3b^2 I_4 & D_\nu{}^{\mu\alpha}a_\alpha \\ -\frac{a}{\chi} & \frac{3b^2 I_4}{\lambda} & -3b^2 I_4 + a^\mu a_\nu \end{bmatrix} \tag{3.2}$$

$$= b^9 \det \begin{bmatrix} 3b^2 I_4 & a & E_\nu^\mu \\ \Xi^T & 1 & 0_{1\times4} \\ \frac{3b^2}{\lambda}I_4 & -\frac{a}{\chi} & -3b^2 I_4 + a^\mu a_\nu \end{bmatrix} \tag{3.3}$$

$$= 27b^{15} \det \begin{bmatrix} 3b^2 - a^\mu a_\mu & -\Xi_\mu E_\nu^\mu \\ \frac{\lambda+\chi}{\lambda\chi}a & 3b^2 I_4 - a^\mu a_\nu + \frac{E_\nu^\mu}{\lambda} \end{bmatrix} \tag{3.4}$$

$$= 27b^{15} \det \begin{bmatrix} F & d_\nu \\ c^\mu & h_\nu^\mu \end{bmatrix} = \frac{27b^{15}}{F^3} \det(Fh_\nu^\mu - c^\mu d_\nu) \tag{3.5}$$

$$= \frac{27b^{15}}{F^3} \det(FG\delta_\nu^\mu - H_\nu a^\mu) = 27b^{15}G^3(FG - H_\mu a^\mu) \tag{3.6}$$

$$= 27b^{15}G^3\left(FG - \frac{\lambda+\eta}{\lambda}F(a^\mu a_\mu) - \kappa\right). \tag{3.7}$$

We now detail how the computations (3.1)-(3.7) were carried out. These computations made successive use of the formula

$$\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2) \tag{3.8}$$

$$= \det(M_4)\det(M_1 - M_2 M_4^{-1} M_3) \tag{3.9}$$

when M_1^{-1} or M_4^{-1} exist, and we defined

$$E_\nu^\mu = -3\eta(a^\alpha a_\alpha \delta_\nu^\mu + a^\mu \Xi_\nu - \frac{2}{3}a^\mu a_\nu),$$

$$F = 3b^2 - a^\mu a_\mu,$$

$$d_\nu = -2\eta a^\alpha a_\alpha(a_\nu - 3\Xi_\nu),$$

$$c^\mu = \frac{\lambda + \chi}{\lambda\chi}a^\mu,$$

$$h_\nu^\mu = 3b^2\delta_\nu^\mu - a^\mu a_\nu + \frac{E_\nu^\mu}{\lambda} = 3(b^2 - \frac{a^\alpha a_\alpha \eta}{\lambda})\delta_\nu^\mu - \frac{\lambda - 2\eta}{\lambda}a^\mu a_\nu - \frac{3\eta}{\lambda}a^\mu \Xi_\nu,$$

$$G = 3(b^2 - \frac{a^\alpha a_\alpha \eta}{\lambda}),$$

$$H_\nu = F(\frac{\lambda - 2\eta}{\lambda}a_\nu + \frac{3\eta}{\lambda}\Xi_\nu) + \frac{\lambda + \chi}{\lambda\chi}d_\nu,$$

$$H_\mu a^\mu = \frac{\lambda + \eta}{\lambda}F(a^\mu a_\mu) + \kappa,$$

$$\kappa = c^\mu d_\mu = \frac{4\eta(\lambda + \chi)}{\lambda\chi}(a^\alpha a_\alpha)^2. \tag{3.10}$$

From (3.1) to (3.2) we used (3.8) by setting

$$M_1 = \begin{bmatrix} b & \Xi^T & 0_{1\times 4} \\ a & 3bI_4 & 0_{4\times 4} \\ -\frac{a}{\chi} & \frac{3b}{\lambda}I_4 & -3bI_4 \end{bmatrix}$$

with M_2, M_3, and M_4 following accordingly. Although $\det(M_4) = b^{16}$, we multiplied lines 2 to 9 by b and divided column 1 by b. Then, the overall multiplicative factor was modified by $b^{16}b^{-8}b^1 = b^9$, resulting in (3.2). After that, we performed the following permutations in (3.2): the fifth line was brought to the first line after 4 line permutations and the fifth column became the first column after 4 column permutations, obtaining (3.3), where E^μ was defined in (3.10). From (3.3) to (3.4) we made again use of (3.8) by setting $M_1 = 3b^2 I_4$, where M_2, M_3, and M_4 are chosen accordingly. The resulting matrix has the overall factor multiplied by $\det M_1 = 81b^8$, but since we multiplied the first line of the resulting matrix by $3b^2$, it reduces to $27b^6$ and, then, by changing the sign of the last 4 lines, Eq. (3.4) is obtained. The first equality in (3.5) corresponds to (3.4) with the definitions that appear in (3.10). Setting $F = M_1$, where M_2, M_3, and M_4 are chosen accordingly, we have applied equation (3.9) to the second equality. The first equality of (3.6) corresponds to the second equality in (3.5) by using the definitions in (3.10). From the first to the second equality in (3.6), we used the formula

$$\det(A\delta_\nu^\mu + \alpha^\mu \beta_\nu) = A^4 + A^3\alpha^\mu \beta_\mu$$

with $A = FG$, $\alpha^\mu = a^\mu$, and $\beta_\nu = -H_\nu$. Finally,

$$\det(\Xi_\alpha \mathcal{A}^\alpha) = m_1 m_2 = \frac{9b^{18}}{\theta}G^3(3b^2 - a^\mu a_\mu)(FG - F\frac{\lambda + \eta}{\lambda}a^\mu a_\mu - \kappa).$$

We set $\det(\Xi_\alpha \mathcal{A}^\alpha)$ equal to zero to find the eigenvalues and eigenvectors. Thus, we need to find the roots Λ of $b = 0$ with multiplicity 18, $G = 0$ (which gives a total of two roots with multiplicity 3), $3b^2 - a^\mu a_\mu$ (which gives a total of 2 roots with multiplicity 1), and $FG - F\frac{\lambda+\eta}{\lambda}a^\mu a_\mu - \kappa = 0$ (which gives a total of 4 roots with multiplicity 1), and the corresponding eigenvectors in all cases.

$\underline{b^{18} = 0}$ gives

$$\Lambda_1 = -\frac{u^\alpha \zeta_\alpha}{u^\beta \xi_\beta}.$$

There are 18 corresponding linearly independent eigenvectors given by

$$\begin{bmatrix} 0 \\ w_a^\nu \\ 0_{25\times1} \end{bmatrix}, \begin{bmatrix} 0_{26\times1} \\ w_a^\nu \end{bmatrix}, \begin{bmatrix} \chi f_\lambda^\lambda \\ 0_{8\times1} \\ f_0^\nu \\ f_1^\nu \\ f_2^\nu \\ f_3^\nu \\ 0_{5\times1} \end{bmatrix},$$

where $w_a^\nu = \{w_1^\nu = u^\nu, w_2^\nu, w_3^\nu\}$ are 3 linearly independent vectors orthogonal to $\zeta_\lambda + \Lambda_1 \xi_\lambda$, and f_λ^ν totalizes 16 components that define the entries in the last vector. However, since these 16 components are constrained by the 4 equations $\chi f_\lambda^\lambda a^\mu + D_\nu{}^{\mu\lambda} f_\lambda^\nu = 0$ (where a^α is as above but with $\Lambda = \Lambda_1$), we end up with 12 independent entries. Then, $3 + 3 + 12 = 18$, which equals the multiplicity of the root Λ_1.

$\underline{3b^2 - a^\mu a_\mu = 0}$ can be written as $b^2 - \beta a^\mu a_\mu = 0$, where $\beta = \frac{1}{3}$. The roots are then $\Lambda_{2,\pm} = (-u^\mu \zeta_\mu u^\nu \xi_\nu + \beta \Pi^{\mu\nu} \xi_\mu \zeta_\nu \pm \sqrt{\Delta})/((u^\mu \xi_\mu)^2 (1 - \beta) - \beta \xi^\mu \xi_\mu)$, where

$$\begin{aligned} \Delta =& \beta(((u^\mu \xi_\mu)^2 - \Pi^{\mu\nu} \xi_\mu \xi_\nu)(\Pi^{\alpha\beta} \zeta_\alpha \zeta_\beta - (u^\alpha \zeta_\alpha)^2) + (u^\mu \xi_\mu u^\nu \zeta_\nu + \Pi^{\mu\nu} \xi_\mu \zeta_\nu)^2 \\ &+ (1 - \beta)(\Pi^{\mu\nu} \xi_\mu \xi_\nu \Pi^{\alpha\beta} \zeta_\alpha \zeta_\beta - (\Pi^{\mu\nu} \xi_\mu \zeta_\nu)^2)). \end{aligned}$$

We note that these roots are always real when $0 < \beta < 1$ because $\Pi_{\alpha\beta} \xi^\alpha \xi^\beta < (\xi_\alpha u^\alpha)^2$, $\Pi_{\alpha\beta} \zeta^\alpha \zeta^\beta > (\zeta_\alpha u^\alpha)^2$, and $(\Pi^{\mu\nu} \xi_\mu \zeta_\nu)^2 \leq \Pi^{\mu\nu} \xi_\mu \xi_\nu \Pi^{\alpha\beta} \zeta_\alpha \zeta_\beta$. Thus, $\Lambda_{2,\pm}$ has two distinct roots giving two linearly independent eigenvectors.

$\underline{G^3 = 0}$ can also be written as $b^2 - \beta a^\mu a_\mu = 0$, where $\beta = \frac{\eta}{\lambda}$. The roots are written the same way as $\Lambda_{2,\pm}$ with the particularity that now each one has multiplicity 3. We note that these roots are real because $0 < \beta < 1$. The corresponding eigenvectors are

$$\begin{bmatrix} C_\pm \\ D_\pm^\nu \\ e_\pm^\nu \\ \frac{a_0^\pm e_\pm^\nu}{b_\pm} \\ \frac{a_1^\pm e_\pm^\nu}{b_\pm} \\ \frac{a_2^\pm e_\pm^\nu}{b_\pm} \\ \frac{a_3^\pm e_\pm^\nu}{b_\pm} \\ 0_{5\times1} \end{bmatrix},$$

where $a_\pm$ is as a above but with $\Lambda = \Lambda_{3,\pm}$, $b_\pm$ is as b above but with $\Lambda = \Lambda_{3,\pm}$ (so that $b_\pm^2 = \beta(a_\pm)^\mu (a_\pm)_\mu$),

$$C_\pm = -\frac{\lambda}{\lambda + \chi}((2\lambda + \chi)(e_\pm)^\mu (\Xi_\pm)_\mu - \frac{\lambda}{3\eta}(2\eta + \chi)(a_\pm)^\mu (e_\pm)_\mu),$$

$$D_\pm^\mu = \frac{\lambda + \chi}{3b_\pm^2 \lambda}((a_\pm)^\nu (e_\pm)_\nu (a_\pm)^\mu - 3b_\pm^2 \chi(e_\pm)^\mu - (e_\pm)^\nu D_\nu{}^{\mu\lambda}(a^\pm)_\lambda),$$

where $\Xi_\pm$ is as Ξ above but with $\Lambda = \Lambda_{3,\pm}$, and $e_\pm$ obeys the following constraint

$$\frac{\lambda + \chi}{\lambda\chi}c_\pm b_\pm - \frac{3\eta}{\lambda}(\Xi_\pm)^\mu (e_\pm)_\mu + \frac{2\eta - \lambda}{\lambda}(a_\pm)^\mu (e_\pm)_\mu = 0.$$

Thus, the eigenvectors are written in terms of 3 independent components of e^μ for each root, giving a total of 6 eigenvectors.

$\underline{FG - F\frac{\lambda+\eta}{\lambda}a^\mu a_\mu - \kappa = 0}$ can be written as

$$9\lambda\chi b^4 - 6(\lambda + 2\eta)\chi a^\mu a_\mu b^2 + \lambda(\chi - 4\eta)(a^\mu a_\mu)^2 = 0.$$

This is a quadratic equation for b^2 that has positive discriminant, i.e.,

$$(a^\mu a_\mu)^2 \eta\chi(\lambda^2 + \eta\chi + \lambda\chi) > 0.$$

In order to obtain real roots Λ, we need

$$0 < \frac{b^2}{a^\mu a_\mu} = \frac{2\chi(\lambda + \eta) \pm \sqrt{\eta\chi(\lambda^2 + \eta\chi + \lambda\chi)}}{3\lambda\chi} \leq 1.$$

This gives the condition

$$2\chi(\lambda + \eta) - \sqrt{\eta\chi(\lambda^2 + \eta\chi + \lambda\chi)} > 0,$$

which is satisfied in view of $\chi > 4\lambda$, and

$$\frac{2\chi(\lambda + \eta) + \sqrt{\eta\chi(\lambda^2 + \eta\chi + \lambda\chi)}}{3\lambda\chi} \leq 1,$$

which is satisfied in view of $\lambda \geq \frac{3\chi\lambda}{\chi - \lambda}$. We also observe that these four roots are distinct, so that we obtain four linearly independent eigenvectors.

Finally, we notice that condition (i) can be verified upon setting $\zeta = 0$ in the above computations. $\qquad\square$

From the above Proposition, we immediately obtain:

Corollary 1. *Assume that $\chi > 4\eta > 0$ and that $\lambda \geq \frac{3\chi\eta}{\chi - \eta}$. Then, the system* (2.3) *can be written as*

$$\nabla_0 \boldsymbol{\Psi} + \tilde{\mathcal{A}}^i \nabla_i \boldsymbol{\Psi} = \tilde{\mathcal{R}}, \tag{3.11}$$

where $\tilde{\mathcal{A}}^i = (\mathcal{A}^0)^{-1}\mathcal{A}^i$ and $\tilde{\mathcal{R}} = -(\mathcal{A}^0)^{-1}\mathcal{R}$, and the eigenvalue problem $(\tilde{\mathcal{A}}^i \zeta_i - \Lambda I)V = 0$ possesses only real eigenvalues Λ and a set of complete eigenvectors V.

Existence and uniqueness of solutions to (3.11) is proved in the next Proposition.

Proposition 2. *Given $\boldsymbol{\Psi}_{(0)} \in H^r(\mathbb{T}^3, \mathbb{R}^{30})$, $r > \frac{5}{2}$, under the assumptions of Corollary 1, there exist a $T > 0$ and a unique*

$$\boldsymbol{\Psi} \in C^0([0, T), H^r(\mathbb{T}^3, \mathbb{R}^{30})) \cap C^1([0, T), H^{r-1}(\mathbb{T}^3, \mathbb{R}^{30}))$$

that is a solution to (3.11) *with initial data $\boldsymbol{\Psi}_{(0)}$.*

Proof. The proof of the Proposition is essentially contained in the proof of [37, Proposition 2.2, Section 16.2] and [37, Proposition 2.1, Section 16.2]. There, a proof is given for strictly hyperbolic systems. Inspecting the proof, one sees that the strict hyperbolicity assumption is used to diagonalize the system. In our case, although we do not have a strictly hyperbolic system, we were able to show that the system can still be diagonalized. Once one has a system in diagonal form, the proof given for the diagonalized system in [37, Proposition 2.2, Section 16.2] can be applied here. $\qquad\square$

Remark 2. We make two important observations regarding the proof of Proposition 2. First, it implies a proof of existence and uniqueness of solutions to (2.3). Second, the proof is based on energy estimates, which in particular ensure that a H^r solution $\boldsymbol{\Psi}$ to (2.3), $r > 5/2$, defined on a time interval $[0, T]$, satisfies

$$\|\boldsymbol{\Psi}\|_{C^0([0,T],H^r(\mathbb{T}^3,\mathbb{R}^{30}))} \leq C_T \|\boldsymbol{\Psi}_{(0)}\|_{H^r(\mathbb{T}^3,\mathbb{R}^{30})},$$

where $C_T > 0$ is a continuous increasing function of T. Using the equations of motion, we can then further derive a bound for $\|\boldsymbol{\Psi}\|_{C^1([0,T],H^{r-1}(\mathbb{T}^3,\mathbb{R}^{30}))}$ in term of C_T and the initial data. Combining these norm-bounds with the estimates for the difference of solutions (which are used to establish [37, Proposition 2.1, Section 16.2]), we also obtain an estimate in H^{r^-} for the difference of two solutions in terms of the difference of their data.

3.1. **Solution to the original system.** Our proof of Theorem 1.1 is based on an approximation argument combined with the existence, uniqueness, and energy estimates valid for (2.3), as explained in Remark 2. The idea is similar to the one used to establish existence and uniqueness for the relativistic Euler equations by coupling it with the vorticity evolution [19], wherein (i) one derives a separate evolution equation for the vorticity (which depends on first derivatives of the velocity); (ii) proves existence for the system coupled to the vorticity as if the latter were an independent variable; (iii) uses the fact that the analytic Cauchy problem for the original equations can be solved, and (iv) finally obtains solutions to the original system by approximating the data by analytic data. In our case, we will use an approximation by Gevrey functions, defined as follows (see, e.g., [34]). The Gevrey space $G^s(\mathbb{T}^3, \mathbf{R}^m)$, $s \geq 1$, consist of C^∞ maps $f \colon \mathbb{T}^3 \to \mathbb{R}^m$ such that, for every compact set $K \subseteq \mathbb{T}^3$, there exists a constant $C > 0$ such that, for all multiindices α and all $x \in K$, it holds that

$$|D^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s.$$

Using Taylor's estimates, one sees in particular that the case $s = 1$ corresponds to analytic functions. The usefulness of Gevrey functions to the study of hyperbolic problems is two-fold. On the one hand, one can prove very general existence and uniqueness theorems for Gevrey data given on a non-characteristic surface that are akin to the Cauchy-Kovalewskaya theorem for analytic data. On the other hand, an advantage of Gevrey functions over analytic ones is that one can construct Gevrey functions that are compactly supported; hence one can appeal to the type of localization arguments that are so useful in the study of hyperbolic equations (this is particularly important when one is considering coupling to Einstein's equations, as in [16]). One can show that Gevrey functions are dense in C^∞ and then establish the density of Gevrey functions in Sobolev spaces [29].

For Gevrey-regular data G^s (for suitable s, but the precise value of s is not important here), equations (1.2) and (1.3) admit a unique Gevrey-regular solution [4, 16]. For this, we observe that, in the spirit of obtaining Gevrey regular solutions as a generalization of the Cauchy-Kovalewskaya theorem, we use in an essential manner the fact that the surface $\{t = 0\}$ is non-characteristic for the system (see [16]). A solution to (1.2) and (1.3) for Sobolev regular data, as in Theorem 1.1, thus follows by a standard approximation argument similar to [19], which we now outline.

Consider the initial data $\mathcal{I} = (\varepsilon_{(0)}, \varepsilon_{(1)}, u_{(0)}, u_{(1)}) \in H^r$ for (1.2)-(1.3) as in the assumptions of Theorem 1.1, and let $\mathcal{I}_k$ be a sequence of Gevrey regular data

converging to $\mathcal{I}$ in H^r. For each k, let $V_k = (\varepsilon_k, u_k)$ be the Gevrey regular solution to (1.2)-(1.3) with data $\mathcal{I}_k$, whose existence is ensured by [4, 16]. In view of the way (2.3) was derived from (1.2)-(1.3), for each k, we obtain a Gevrey regular solution $\boldsymbol{\Psi}_k$ to (2.3), with $\boldsymbol{\Psi}_k$ defined in terms of V_k according to the definitions of Section 2.

Let $\boldsymbol{\Psi}_0$ be initial data for (2.3) constructed out of $\mathcal{I}$, i.e., we define $\boldsymbol{\Psi}_0$ in terms of $\mathcal{I}$ using the definitions of Section 2. This is possible since the entries of $\boldsymbol{\Psi}_0$ will be simple algebraic expressions in terms of $\mathcal{I}$.

Let $\boldsymbol{\Psi}$ be the solution to (2.3) with data $\boldsymbol{\Psi}_0$. Note that we do not assume that $\boldsymbol{\Psi}$ is given in terms of the original fluid variables via the relations of Section 2 since at this point we do not yet have a solution to (1.2)-(1.3) with data $\mathcal{I}$. In other words, the entries of $\boldsymbol{\Psi}$ are treated as independent variables; at this point the only relation between $\boldsymbol{\Psi}$ and the original system (1.2)-(1.3) is that $\boldsymbol{\Psi}_0$ is constructed out of $\mathcal{I}$.

The estimates for solutions to (2.3) (see Remark 2) imply that as $\mathcal{I}_k \to \mathcal{I}$ in H^r, $\boldsymbol{\Psi}_k(t)$ converges to $\boldsymbol{\Psi}(t)$ in H^{r^-}, and thus the solutions $V_k(t)$ to (1.2)-(1.3) converge to a limit $V(t)$ in H^{r^-}, and this limit is in fact in H^r. Since $r > 7/2$, we can pass to the limit in the equations (1.2)-(1.3) satisfied by V_k to conclude that V solves (1.2)-(1.3) as well (and that $\boldsymbol{\Psi}$ is in fact given in terms of V by the same expressions that define $\boldsymbol{\Psi}_k$ in terms of V_k). By construction, V takes the data $\mathcal{I}$.

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