Systems & Control Letters 139 (2020) 104673

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Sampled-data containment control for double-integrator agents with dynamic leaders with nonzero inputs☆

Yong Ding, Wei Ren *

Department of Electrical and Computer Engineering, University of California, Riverside, CA 92521, USA

ARTICLE INFO

Article history:
Received 25 October 2019
Received in revised form 7 March 2020
Accepted 15 March 2020
Available online 11 April 2020

Keywords:
Containment control
Dynamic leaders
Sampled-data setting
Double-integrator dynamics

ABSTRACT

The objective of containment control in multi-agent systems is to design control algorithms for the followers to converge to the convex hull spanned by the leaders. Sampled-data based containment control algorithms are suitable for the cases where the power supply and sensing capacity are limited, due to their low-cost and energy-saving features resulting from discrete sensing and interactions. In addition, sampled-data control has advantages in performance, price and generality. On the other hand, when the agents have double-integrator dynamics and the leaders are dynamic with nonzero inputs, the existing algorithms are not directly applicable in a sampled-data setting. To this end, this paper proposes a sampled-data based containment control algorithm for a group of double-integrator agents with dynamic leaders with nonzero inputs under directed communication networks. By applying the proposed control algorithm, the followers converge to the convex hull spanned by the dynamic leaders with bounded position and velocity containment control errors, and the ultimate bound of the overall containment error is proportional to the sampling period. A numerical simulation is presented to illustrate the proposed algorithm.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

Due to the advantages in achieving group performance with low operation cost and flexible scalability, and potential practical applications in vehicle formation, sensor networks, cooperative surveillance, and so on [1,2], distributed cooperative control of a group of robots/agents have drawn massive attention from various scientific communities. Consensus is an important research subject in distributed cooperative control of multi-agent systems, where all the agents reach an agreement on a state of interest. A number of distributed consensus algorithms have been proposed to solve the consensus problems for a group of agents with no leader [3,4] and one leader [5,6]. When there are multiple leaders in the group, the objective is to solve the containment control problem [7], where the followers are to converge to the convex hull spanned by the leaders. Several natural phenomena exhibit the relationship between leaders and followers in the containment control problem. For instance, several sheepdogs gather a flock of sheep and guide them safely to a desired location [8]. Another biological example is provided in [9,10], where female silkworm moths are capable of releasing a certain kind of pheromone to attract male moths to swarm in tight geometrical configurations. On the other hand, the containment control problem has practical applications. For instance, several robots capable of self-navigation are able to guide a group of agents to cross a partially known area [10]. Also, the containment control problem has applications in coordination of a group of robots [11].

A number of algorithms have been reported in the literature to deal with the containment control problem under various scenarios. For instance, containment control algorithms are proposed for a group of single-integrator agents [12,13], double-integrator agents [11,13,14], and agents with general linear dynamics [15] and Euler–Lagrange dynamics [16]. The aforementioned results are derived for continuous-time cases, which require continuous sensing and interaction among agents. However, when each agent has limited power supply and sensing capacities, energy saving becomes one of the main factors that the designers have to take into account. Because of the advantages in cost reduction, the event-triggered and discrete-time containment control algorithms are studied in the literature.

Several different event-triggered containment control algorithms are proposed in the literature. See [17–23] for instance. These event-triggered containment control algorithms require that each agent continuously monitor the communication channels and certain states, and continuously compute and check the event-triggering functions to see whether they exceed some threshold. These actions will cost additional energy and resources.

☆ This work was supported by National Science Foundation under Grant ECCS-1920798.

* Corresponding author.
E-mail addresses: dingyong1014@hotmail.com (Y. Ding), ren@ece.ucr.edu (W. Ren).

https://doi.org/10.1016/j.sysconle.2020.104673
0167-6911/© 2020 Elsevier B.V. All rights reserved.

It is also worth noting that in [17–23], the leaders' inputs are either zero or designed to drive the leaders to some stationary locations, which are simpler than the case where the dynamic leaders' inputs can be arbitrary as long as they are bounded.

Discrete-time containment control algorithms are proposed for multi-agent systems with single-integrator dynamics [10,24], double-integrator dynamics [13,25–27], higher-order integrator dynamics [28,29], and general discrete-time linear dynamics [15]. The containment control problem for heterogeneous multi-agent systems is addressed in [30], where the followers have single- and double-integrator dynamics and the leaders are single-integrator agents.

Among these discrete-time containment control algorithms, the sampled-data based ones stand out because of their advantages in performance/accuracy, price and generality. Also it is more coincident with practical applications in real life. For instance, sampled-data based algorithms are proposed in [27] and [13] to solve the containment control problem for multiple agents with fractional-order double-integrator dynamics and ordinary double-integrator dynamics, respectively.

In the above mentioned discrete-time containment control algorithms, however, the leaders' inputs remain zero, which greatly simplifies analysis and design. One natural question arises is how to solve the containment control problem for the case where leaders' inputs are nonzero. In this case, discontinuous algorithms are usually used to achieve containment control for continuous-time single- and double-integrator agents [11,12]. However, the discontinuous algorithms proposed in [11,12] require each agent to continuously interact with its neighbors, and it is not clear whether it is applicable for double-integrator agents in a sampled-data setting. A solution to the question is provided in [10] for discrete-time higher-order-integrator agents if the leaders' trajectories are described by polynomial functions. Such trajectories can be generated by integrator agents with polynomial inputs. However, it is not directly applicable when the followers' dynamics become complicated and the leaders' inputs are non-polynomial as considered in this paper. Also, note that to implement the discrete-time containment control algorithm in [10], each double-integrator follower needs to store a great amount of historical state information to update its controller.

In the sampled-data setting, there exist new challenges for the containment control of double-integrator agents with dynamic leaders with nonzero inputs. The coexistence of the sampled-data setting, double-integrator dynamics and dynamic leaders with nonzero inputs, makes the containment control problem more difficult and complicated, and renders the existing related results in the literature inapplicable. Therefore, the development of new sampled-data containment control algorithm is needed for double-integrator agents with dynamic leaders with nonzero inputs. Inspired by the above observations, in this paper, we address the containment control problem in a sampled-data setting for double-integrator agents with multiple dynamic leaders with nonzero inputs under directed communication networks. The contributions of this paper are two-fold. First, a sampled-data based containment control algorithm is proposed for double-integrator agents, which eliminates the requirement of continuous sensing and interactions. It is more suitable for practical applications, since continuous sensing and interaction are not energy-efficient, and demand a larger portion of energy on board compared with periodic ones. Second, the proposed algorithm is proposed for the case where there are multiple dynamic leaders with nonzero inputs, which is one of the main differences distinguishing our work from the existing discrete-time distributed containment control algorithms in the literature. By the proposed algorithm, we show that all the followers converge to the convex hull spanned by the leaders with bounded errors. Both the collective position and velocity containment control errors are bounded, and the ultimate bound of the overall containment control error is proportional to the sampling period.

The remainder of this paper is arranged as follows. In Section 2, some preliminaries are presented, and the containment control problem is introduced. Section 3 shows that by the proposed algorithm, the containment control problem is solved with bounded errors. A numerical example is provided in Section 4 to explain the main results and a few concluding remarks are made in Section 5.

2. Preliminaries

For a given vector $x \in \mathbb{R}^p$, $\|x\|_2$ denotes the two-norm of x. For a scalar r, $|r|$ denotes the absolute value of r. For a complex number c, $\mathrm{Re}\{c\}$ and $\mathrm{Im}\{c\}$ denote the real and imaginary parts of c, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. $\otimes$ denotes the Kronecker product. $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ denotes the $m \times n$ dimensional zero matrix, and for simplicity, let $\mathbf{0}_m = \mathbf{0}_{m\times 1}$. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix.

2.1. Graph theory

For a multi-agent system consisting of n agents, the interaction topology can be modeled by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. Each edge, denote by $(j, i) \in \mathcal{E}$, means that node j is a neighbor of node i, and that node i can obtain information from node j. Self edges (i, i) are not considered here. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_k \in \mathcal{V}$.

Define a row-stochastic matrix $D = [d_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$, and assume that $d_{ii} > 0$, $d_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $d_{ij} = 0$ otherwise.

2.2. Problem statement

Consider a network of n agents whose interactions are represented by the directed graph $\mathcal{G}$. Each agent i has double-integrator dynamics given by

$$\dot{r}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i = 1, \ldots, n,$$

where $r_i(t) \in \mathbb{R}^p$ and $v_i(t) \in \mathbb{R}^p$ denote the position and velocity of agent i at time t, respectively, and $u_i(t)$ is the corresponding control input. In this paper, we consider a sampled-data setting where the agents have continuous-time dynamics while the control inputs are based on zero-order hold and the interactions with neighbors are made at discrete sampling times. Then the system can be discretized as

$$\begin{aligned} r_i[k+1] &= r_i[k] + Tv_i[k] + \frac{T^2}{2}u_i[k] \\ v_i[k+1] &= v_i[k] + Tu_i[k], \end{aligned} \tag{1}$$

where T is the sampling period, k is the discrete-time index, and $r_i[k] \in \mathbb{R}^p$, $v_i[k] \in \mathbb{R}^p$ and $u_i[k] \in \mathbb{R}^p$ represent the position, velocity, and control input of the ith agent at $t = kT$, respectively.

We adopt the definitions of the leaders and the followers used in [31]. That is, an agent is called a leader if and only if it has no neighbor, and otherwise it is called a follower. Without loss of generality, let $\mathscr{F} = \{1, \ldots, m\}$ and $\mathscr{L} = \{m + 1, \ldots, n\}$ denote the follower set and the leader set, respectively. Therefore, the row-stochastic matrix D associated with the directed graph $\mathcal{G}$ can be written as

$$D = \begin{bmatrix} D_1 & D_2 \\ \mathbf{0}_{(n-m)\times m} & I_{(n-m)\times(n-m)} \end{bmatrix}$$

where $D_1 \in \mathbb{R}^{m\times m}$ and $D_2 \in \mathbb{R}^{m\times(n-m)}$. We assume that $\mathcal{G}$ satisfies the following assumption.

Assumption 1. For each of the followers, there is at least one leader that has a directed path to the follower.

Lemma 1 (*[15]*)**.** *Under Assumption 1, the matrix D_1 has all the eigenvalues within the unit circle, and each entry of $(I_m - D_1)^{-1}D_2$ is nonnegative, and each row of $(I_m - D_1)^{-1}D_2$ has a sum equal to* 1. □

Definition 1. Let $\mathcal{C}$ be a set in a real vector space $\mathcal{S} \subseteq \mathbb{R}^p$. The set is convex if, for any x and y in $\mathcal{C}$, the point $(1-\alpha)x+\alpha y \in \mathcal{C}$ for any $\alpha \in [0, 1]$. The convex hull for a set of points $\mathcal{X} := \{x_1, \dots, x_m\}$ in $\mathcal{S}$, denoted by $\mathrm{Co}(\mathcal{X})$, is the minimal convex set containing all points in $\mathcal{X}$, that is, $\mathrm{Co}(\mathcal{X}) := \{\sum_{i=1}^m \beta_i x_i \mid x_i \in \mathcal{X}, \beta_i \geq 0, \sum_{i=1}^m \beta_i = 1\}$.

In this paper, the objective is to solve the containment control problem, that is to design $u_i[k]$ for follower i, $i \in \mathscr{F}$, by using its own and neighbors' states, $\{r_j\}_{j\in\mathcal{N}_i\cup\{i\}}$ and $\{v_j\}_{j\in\mathcal{N}_i\cup\{i\}}$, such that all followers' positions and velocities converge to the convex hull spanned by the dynamic leaders' positions and velocities, respectively, which are given by $\mathrm{Co}(\{r_l\}_{l\in\mathscr{L}})$ and $\mathrm{Co}(\{v_l\}_{l\in\mathscr{L}})$, respectively.

Note that by properly designing $u_i[k]$ for leader $i \in \mathscr{L}$, the leader will be able to follow a certain desired trajectory. Then, the leaders are capable of guiding the followers through a certain region safely and reach the desired location. We assume that the leaders' inputs are pre-designed and satisfy the following condition.

Assumption 2. The inputs of the leaders are bounded, i.e., for any $j \in \mathscr{L}$, $\left\|u_j[k]\right\|_2 \leq c_1$, where c_1 is a positive constant.

3. Sampled-data containment control

In order to solve the multi-agent containment control problem, we consider the following controller for follower i as

$$u_i[k] = \sum_{j\in\mathscr{L}\cup\mathscr{F}} d_{ij}\left(\frac{v_j[k]-v_j[k-1]}{T} - \gamma_1\{r_i[k]-r_j[k]\} - \gamma_2\{v_i[k]-v_j[k]\}\right), \quad i \in \mathscr{F} \tag{2}$$

where d_{ij} is the (i, j) the entry of the matrix D, and $\gamma_1, \gamma_2 > 0$ are constant. Essentially, the term $\frac{v_j[k]-v_j[k-1]}{T}$ make use of past data to approximate the acceleration of agent j. Therefore, each follower only uses its own and its neighbors' current and previous velocities as well as the current positions to update its control input, which means the algorithm (2) can be implemented in reality.

Define the position and velocity containment control errors for follower i as $x_i[k] = \sum_{j=1}^n d_{ij}(r_i[k] - r_j[k])$ and $y_i[k] = \sum_{j=1}^n d_{ij}(v_i[k] - v_j[k])$, respectively. Define the collective position and velocity containment control errors as $X[k] = \left(x_1^\top[k], \dots, x_m^\top[k]\right)^\top$ and $Y[k] = \left(y_1^\top[k], \dots, y_m^\top[k]\right)^\top$, respectively. Using (2) for (1), we have

$$\begin{aligned} X[k+1] &= \left(A_{11}\otimes I_p\right)X[k] + \left(A_{12}\otimes I_p\right)Y[k] \\ &\quad - \left(\frac{T}{2}D_1\otimes I_p\right)Y[k-1] + \left(\frac{T}{2}D_2\otimes I_p\right)\Delta[k] \\ Y[k+1] &= \left(A_{21}\otimes I_p\right)X[k] + \left(A_{22}\otimes I_p\right)Y[k] \\ &\quad - \left(D_1\otimes I_p\right)Y[k-1] + \left(D_2\otimes I_p\right)\Delta[k], \end{aligned}$$

where

$$\begin{aligned} A_{11} &= \left(1 - \frac{T^2}{2}\gamma_1\right)I_m + \frac{T^2}{2}\gamma_1 D_1, \\ A_{12} &= \left(T - \frac{T^2}{2}\gamma_2\right)I_m + \left(\frac{T}{2} + \frac{T^2}{2}\gamma_2\right)D_1, \\ A_{21} &= -T\gamma_1\left(I_m - D_1\right), \\ A_{22} &= \left(1 - T\gamma_2\right)I_m + \left(1 + T\gamma_2\right)D_1. \end{aligned}$$

and $\Delta[k] = 2\boldsymbol{v}_L[k] - \boldsymbol{v}_L[k+1] - \boldsymbol{v}_L[k-1]$ with $\boldsymbol{v}_L[k] = \left(v_{m+1}^\top[k], \dots, v_n^\top[k]\right)^\top$. Define $Z[k+1] = \left(X^\top[k+1], Y^\top[k+1], X^\top[k], Y^\top[k]\right)^\top$. It then follows that

$$Z[k+1] = \widetilde{A}Z[k] + \widetilde{B}\Delta[k], \tag{3}$$

where $\widetilde{A} = A \otimes I_p$ with

$$A = \begin{bmatrix} A_{11} & A_{12} & \mathbf{0}_{m\times m} & -\frac{T}{2}D_1 \\ A_{21} & A_{22} & \mathbf{0}_{m\times m} & -D_1 \\ I_m & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & I_m & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \end{bmatrix}, \tag{4}$$

and $\widetilde{B} = \left[\frac{T}{2}, 1, 0, 0\right]^\top \otimes \left(D_2 \otimes I_p\right)$.

The eigenvalues of $\widetilde{A}$ play an important role in determining the solution of (3). Therefore, we investigate the eigenvalues of $\widetilde{A}$ in the following. We first present three useful lemmas before moving on.

Lemma 2 (*Generalized Schur's Formula [32]*)**.** *Let $M_{ij} \in \mathbb{R}^{n\times n}$, $i, j \in \mathcal{M}$, where $\mathcal{M} = \{1, \dots, m\}$, and*

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix}.$$

If M_{ij}, $i, j \in \mathcal{M}$ pairwise commute, i.e., $M_{ij}M_{ls} = M_{ls}M_{ij}$ for all possible pairs of indices i, j and l, s, then

$$\det(M) = \det\left(\sum_{\pi\in\mathcal{S}_m} \mathrm{sgn}(\pi)M_{1\pi(1)}M_{2\pi(2)}\dots M_{m\pi(m)}\right),$$

where $\det(\cdot)$ denotes the determinant of a matrix, π is a permutation, set $\mathcal{S}_m$ denotes the set of all possible permutations of the $\mathcal{M}$, and $\mathrm{sgn}(\pi)$ denotes the parity of the permutation π. □

Lemma 3. *Let $P(z)$ be a polynomial of order three with complex coefficients in the form of $P(z) = z^3 + \alpha_1 z^2 + \alpha_2 z + \alpha_3$, where $\alpha_i = p_i + \boldsymbol{j}q_i$, $i = 1, \dots, 3$ and $\boldsymbol{j}$ is the imaginary unit. The polynomial $P(z)$ has all its zeros in the open left half of the z-complex plane if and only if $p_1 > 0$, $p_1^2 p_2 + p_1 q_1 q_2 - p_1 p_3 - q_2^2 > 0$, and $\det(M_3) > 0$ where*

$$M_3 = \begin{bmatrix} p_1 & p_3 & p_5 & -q_2 & -q_4 \\ 1 & p_2 & p_4 & -q_1 & -q_3 \\ 0 & p_1 & p_3 & 0 & -q_2 \\ 0 & q_2 & q_4 & p_1 & p_3 \\ 0 & q_1 & q_3 & p1 & p_2 \end{bmatrix}.$$

Proof. This lemma is a special case of the Theorem 3.2 in [33], and the proof is thus omitted. □

Lemma 4 ([34]). *The matrix $M \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $g(x) = a_0 + a_1 x + \cdots + a_k x^k$ be a polynomial, and let $g(M) = a_0 I_n + a_1 M + \cdots + a_k M^k$. Then the eigenvalues of $g(M)$ are $g(\lambda_1), \ldots, g(\lambda_n)$.* □

With the above three lemmas, we can obtain the following results on the eigenvalues of the matrix $\widetilde{A}$.

Lemma 5. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalue of D_1. The matrix $\widetilde{A}$ has all eigenvalues within the unit circle if and only if there exist positive scalars T, γ_1 and γ_2 such that*

$$\frac{2\gamma_2}{T\gamma_1} > \frac{1-|\lambda_i|^2}{|1-\lambda_i|^2}, \quad i = 1, \ldots, m, \tag{5}$$

and

$$\left(\frac{2\gamma_2}{T} - \frac{1-|\lambda_i|^2}{|1-\lambda_i|^2}\gamma_1\right)^2 \left[\frac{2(1-|\lambda_i|^2)}{|1-\lambda_i|^2} - T\gamma_2\right] - \frac{16(\mathrm{Im}\{\lambda_i\})^2\gamma_2^3}{|1-\lambda_i|^4 T} > 0, \quad i = 1, \ldots, m, \tag{6}$$

hold. In addition, such positive scalars T, γ_1 and γ_2 always exist.

Proof. First, we prove that the matrix A defined in (4) has all eigenvalues within the unit circle if and only if there exist positive scalars T, γ_1 and γ_2 such that (5) and (6) hold. Note that the characteristic polynomial of A is given by

$$\begin{aligned}
&\det(sI_{4m} - A) \\
&= \det\left(\begin{bmatrix} sI_m - A_{11} & -A_{12} & \mathbf{0}_{m\times m} & \frac{T}{2}D_1 \\ -A_{21} & sI_m - A_{22} & \mathbf{0}_{m\times m} & D_1 \\ -I_m & \mathbf{0}_{m\times m} & sI_m & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & -I_m & \mathbf{0}_{m\times m} & sI_m \end{bmatrix}\right) \\
&= \det\left(s\Big[s(sI_m - A_{11})(sI_m - A_{22}) - sA_{12}A_{21} + (sI_m - A_{11})D_1 + \frac{T}{2}D_1 A_{21}\Big]\right) \\
&= \det\left(s\left\{\left[s^3 - \left(2 - \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s^2 + \left(1 + \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s\right]I_m + \left[\left(-1 - \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s^2 + \left(2 - \frac{T^2}{2}\gamma_1 + T\gamma_2\right)s - 1\right]D_1\right\}\right),
\end{aligned}$$

where we have used Lemma 2 to obtain the second to the last equality because $sI_m - A_{11}$, $-A_{12}$, $\mathbf{0}_{m\times m}$, $\frac{T}{2}D_1$, $-A_{21}$, $sI_m - A_{22}$, D_1, $-I_m$ and sI_m commute pairwise. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of D_1. Then by Lemma 4 and the fact that the determinant of a matrix is the product of its eigenvalues, it holds that $\det[g_1(s)I_m + g_2(s)D_1] = \prod_{i=1}^m [g_1(s) + g_2(s)\lambda_i]$, where g_1 and g_2 are two polynomial functions of s. Thus, it follows that

$$\begin{aligned}
&\det(sI_{4m} - A) \\
&= \prod_{i=1}^m \left(s\left\{s^3 - \left(2 - \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s^2 + \left(1 + \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s + \left[\left(-1 - \frac{T^2}{2}\gamma_1 - T\gamma_2\right)s^2 + \left(2 - \frac{T^2}{2}\gamma_1 + T\gamma_2\right)s - 1\right]\lambda_i\right\}\right).
\end{aligned}$$

Thus, the roots of $\det(sI_{4m} - A) = 0$ either equal to zero or satisfy

$$\begin{aligned} s^3 &+ \left[-2 - \lambda_i + (1-\lambda_i)\left(\frac{T^2}{2}\gamma_1 + T\gamma_2\right)\right]s^2 \\ &+ \left[1 + 2\lambda_i + (1-\lambda_i)\left(\frac{T^2}{2}\gamma_1 - T\gamma_2\right)\right]s - \lambda_i = 0. \end{aligned} \tag{7}$$

It is trivial when the roots of $\det(sI_{4m} - A) = 0$ are zero. Note that the matrix A has all eigenvalues within the unit circle if and only if, for any eigenvalue of D_1, the roots of (7) all lie inside the unit circle. Instead of computing the roots of (7) directly, we apply the bilinear transformation $s = \frac{z+1}{z-1}$ to (7), which yields

$$\begin{aligned} (1-\lambda_i)T^2\gamma_1 z^3 + 2(1-\lambda_i)T\gamma_2 z^2 + (1-\lambda_i)(4 - T^2\gamma_1)z \\ + 4(1+\lambda_i) - 2(1-\lambda_i)T\gamma_2 = 0. \end{aligned} \tag{8}$$

Such bilinear transformation maps the left half of the complex z-plane to the interior of the unit circle in the s-plane, it then follows that (7) has all the roots within the unit circle if and only if (8) has all the roots in the open left half of the complex plane. Since Assumption 1 holds, it follows that $|1-\lambda_i| > 0$ by Lemma 1. Note that both γ_1 and the sampling period T are positive. Then (8) is equivalent to

$$z^3 + \alpha_1 z^2 + \alpha_2 z + p_3 + \boldsymbol{j} q_3 = 0. \tag{9}$$

where $\alpha_1 = \frac{2T\gamma_2}{T^2\gamma_1}$, $\alpha_2 = \frac{4-T^2\gamma_1}{T^2\gamma_1}$, $p_3 = \frac{4(1-|\lambda_i|^2)}{|1-\lambda_i|^2 T^2\gamma_1} - \frac{2T\gamma_2}{T^2\gamma_1}$, and $q_3 = \frac{8\mathrm{Im}\{\lambda_i\}}{|1-\lambda_i|^2 T^2\gamma_1}$. Denote by $P(z, \lambda_i)$ the left hand side of (9) for some given λ_i. Note that for a given λ_i, $P(z, \lambda_i)$ is a polynomial in the indeterminate z of degree 3. Then for a given λ_i, by Lemma 3, $P(z, \lambda_i)$ has all the zeros in the open left half of the complex plane if and only if the positive scalars T, γ_1 and γ_2 satisfy

$$f_j^i > 0, \quad j = 1, \ldots, 3, \quad i = 1, \ldots, m, \tag{10}$$

with $f_1^i = \alpha_1$, $f_2^i = \alpha_1\alpha_2 - p_3$, and $f_3^i = (\alpha_1\alpha_2 - p_3)^2 p_3 - q_3^2\alpha_1^3$. It is easy to see that (5) follows from $f_1^i > 0$ and $f_2^i > 0$, $i = 1, \ldots, m$. Note that $f_3^i > 0$ can be written as (6). Thus, the matrix A has all eigenvalues within the unit circle if and only if there exist positive scalars T, γ_1 and γ_2 such that (5) and (6) hold.

By the fact that μ is an eigenvalue of $\widetilde{A}$ if and only if μ is also an eigenvalue of A, we conclude that $\widetilde{A}$ has all eigenvalues within the unit circle if and only if there exist positive scalars T, γ_1 and γ_2 such that (5) and (6) hold.

In the following, we show that such positive scalars T, γ_1 and γ_2 always exist. Obviously, $f_1^i > 0\ \forall i = 1, \ldots, m$. We rewrite f_2^i and f_3^i, $i = 1, \ldots, m$ as

$$\begin{aligned}
f_2^i &= \left(\frac{8\gamma_2}{\gamma_1^2}\right)\beta^3 - \left[\frac{4(1-|\lambda_i|^2)}{|1-\lambda_i|^2\gamma_1}\right]\beta^2, \\
f_3^i &= 32\left\{\left[\frac{8(1-|\lambda_i|^2)\gamma_2^2}{|1-\lambda_i|^2\gamma_1^5}\right]\beta^8 - \left[\frac{8(1-|\lambda_i|^2)^2\gamma_2}{|1-\lambda_i|^4\gamma_1^4} + \frac{4\gamma_2^3}{\gamma_1^5} + \frac{16\big(\mathrm{Im}\{\lambda_i\}\big)^2\gamma_2^3}{|1-\lambda_i|^4\gamma_1^5}\right]\beta^7 + \left[\frac{4(1-|\lambda_i|^2)\gamma_2^2}{|1-\lambda_i|^2\gamma_1^4} + \frac{2(1-|\lambda_i|^2)^3}{|1-\lambda_i|^6\gamma_1^3}\right]\beta^6 - \left[\frac{(1-|\lambda_i|^2)^2\gamma_2}{|1-\lambda_i|^4\gamma_1^3}\right]\beta^5\right\},
\end{aligned}$$

where $\beta = \frac{1}{T}$. Note that f_2^i and f_3^i are two polynomials in the indeterminate β of degree 3 and 8, respectively. The leading coefficients (the coefficient of the term with the highest degree) of f_2^i and f_3^i are $e_2^i := \frac{8\gamma_2}{\gamma_1^2}$, and $e_3^i := \frac{256(1-|\lambda_i|^2)\gamma_2^2}{|1-\lambda_i|^2\gamma_1^5}$, respectively. We can see that $e_2^i > 0\ \forall i = 1, \ldots, m$, since $\gamma_1, \gamma_2 > 0$. When Assumption 1 holds, the eigenvalues of D_1 are located inside the unit circle by Lemma 1, which implies that $1-|\lambda_i|^2>0$ $\forall i=1,\ldots,m$. It then holds that $e_3^i>0$ $\forall i=1,\ldots,m$. Then it follows that, for any eigenvalue of D_1, and any given positive constants γ_1 and γ_2,

$$\lim_{T\to 0^+} f_j^i=+\infty,\quad j=2,3.$$

Hence, given any eigenvalue of D_1, and for any positive finite constants γ_1 and γ_2, there always exists a positive constant $\overline{T}_{\lambda_i}$ such that for any $T<\overline{T}_{\lambda_i}$, $f_2^i>0$ and $f_3^i>0$ hold. Let $\overline{T}=\min_{i=1,\ldots,m}\overline{T}_{\lambda_i}$. When $T<\overline{T}$, f_j^i, $j=1,\ldots,3$, hold for any eigenvalues of D_1, which implies the existence of these three positive scalars T, γ_1 and γ_2 such that (10) holds for any eigenvalue of D_1. $\square$

Theorem 1. *Let Assumptions 1 and 2 hold. If the positive scalars T, γ_1 and γ_2 satisfy (10) for any eigenvalue of D_1, using the algorithms (2) for (1), the followers converge to the convex hull spanned by the leaders with bounded position and velocity containment control error, and the overall containment control error, $\|X[k]\|_2+\|Y[k]\|_2$, is ultimately bounded by $2c_1c_2T\sqrt{n-m}\,\|\widetilde{B}\|_2/(1-\rho)$, where c_1 is given in Assumption 2, and positive constant c_2 and $\rho\in[0,1)$ satisfy $\|\widetilde{A}^j\|_2\le c_2\rho^j$, $j\ge 0$.*

Proof. It follows that the solution of (3) is

$$Z[k]=\widetilde{A}^kZ[0]+\sum_{i=0}^{k-1}\widetilde{A}^{k-i-1}\widetilde{B}\Delta[i].$$

Then, it holds that

$$\begin{aligned}\|Z[k]\|_2&\le\left\|\widetilde{A}^kZ[0]\right\|_2+\left\|\sum_{i=0}^{k-1}\widetilde{A}^{k-i-1}\widetilde{B}\Delta[i]\right\|_2\\&\le\left\|\widetilde{A}^k\right\|_2\|Z[0]\|_2+2\sqrt{n-m}Tc_1\left\|\sum_{i=0}^{k-1}\widetilde{A}^{k-i-1}\right\|_2\left\|\widetilde{B}\right\|_2,\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\|\Delta[i]\|_2&=\|2\boldsymbol{v}_L[i]-\boldsymbol{v}_L[i+1]-\boldsymbol{v}_L[i-1]\|_2\\&\le\|\boldsymbol{v}_L[i+1]-\boldsymbol{v}_L[i]\|_2+\|\boldsymbol{v}_L[i]-\boldsymbol{v}_L[i-1]\|_2\\&\le 2\sqrt{n-m}Tc_1\end{aligned}$$

holds for all i if Assumption 2 holds. Since Assumption 1 holds, and by Lemma 5, if the positive scalars T, γ_1, and γ_2 satisfy (10) for any eigenvalue of D_1, the matrix $\widetilde{A}$ has all the eigenvalues within the unit circle. Then by [35,36], there exist two finite positive constants c_2 and $\rho\in[0,1)$ such that $\|\widetilde{A}^j\|_2\le c_2\rho^j$. Then, we have $\|Z[k]\|_2\le c_2\rho^k\|Z[0]\|_2+2Tc_1c_2\sqrt{n-m}(1-\rho^k)\|\widetilde{B}\|_2/(1-\rho)<\infty$, which implies that both the position and velocity containment errors are bounded. It also follows that $\lim_{k\to\infty}\|Z[k]\|_2\le 2Tc_1c_2\sqrt{n-m}\,\|\widetilde{B}\|_2/(1-\rho)$, since $\lim_{k\to\infty}\rho^k=0$. Therefore, it holds that $\lim_{k\to\infty}(\|X[k]\|_2+\|Y[k]\|_2)\le\lim_{k\to\infty}\sqrt{2\|X[k]\|_2^2+2\|Y[k]\|_2^2}=\lim_{k\to\infty}\|Z[k]\|_2\le 2Tc_1c_2\sqrt{n-m}\,\|\widetilde{B}\|_2/(1-\rho)$. This completes the proof. $\square$

Remark 1. The ultimate overall containment control error is proportional to the sampling period T. As $T\to 0$, $\|X[k]\|_2+\|Y[k]\|_2\to 0$, which implies that the position and velocity containment errors for each follower approach zero eventually.

Remark 2. The discrete-time controller (2) is robust to bounded state disturbance. Consider that $r_i[k+1]=r_i[k]+Tv_i[k]+\frac{T^2}{2}u_i[k]+d_i^r[k]$, $v_i[k+1]=v_i[k]+Tu_i[k]+d_i^v[k]$, where $d_i^r[k]$ and $d_i^v[k]$ are the position and velocity disturbances, respectively. Since (3) is a linear time-invariant system, then under the state disturbances, the followers are still capable of converging to the convex hull with bounded errors, the values of which depend on the bounded disturbances, in addition to the sampling period and the choices of positive scalars γ_1 and γ_2.

The real-world communication environment may be corrupted by noise, that is, each agent has access to noisy state information received from its neighbors. In the present case, each agent i receives $\widehat{r}_j[k]=r_j[k]+\boldsymbol{n}_j^r[k]$ and $\widehat{v}_j[k]=v_j[k]+\boldsymbol{n}_j^v[k]$ from its neighbor i, where $\boldsymbol{n}_j^r$ and $\boldsymbol{n}_j^v\in\mathbb{R}^p$ are noise vectors. The entries of each noise vector are drawn independently from some identical zero-mean distribution. Then by using noisy transmitted position and velocity information in the control law (2), that is, replacing $r_j[k]$ and $v_j[k]$ with $\hat{r}_j[k]$ and $\hat{v}_j[k]$, respectively, the followers are to converge to the convex hull spanned by the leaders with bounded error in expectation. The variance of the resulting overall containment control error is also bounded.

Remark 3. The containment control problem for double-integrator agents in a sampled-data setting is also investigated in [13]. However, in [13], the leaders are with zero inputs, which can be included as a special case of the present paper. Moreover, when the leaders has nonzero inputs, the sampled-data containment algorithm proposed in [13] does not work anymore. It is also worth noting that the analysis and controller design have been greatly simplified under the assumption of zero inputs for the leaders.

3.1. Selection of the sampling period

The sampling period T plays an essential role in ensuring that the matrix $\widetilde{A}$ has all its eigenvalues inside the unit circle and thus the convergence. Although it has been proven from Lemma 5, that given a communication network $\mathcal{G}$ satisfying Assumption 1 and some positive scalars γ_1 and γ_2, one can always find small enough sampling period T such that (5) and (6) hold, it is still not clear about how to choose appropriate sampling period T. We address this problem in the following.

Given any positive scalars γ_1 and γ_2, we can obtain the solution to (5) as

$$\mathcal{S}_1=\left\{T\;\middle|\;0<T<\min_{i=1,\ldots,m}\left\{\frac{2|1-\lambda_i|^2\gamma_2}{(1-|\lambda_i|^2)\gamma_1}\right\}\right\}.\tag{11}$$

The inequality (6) can be equivalently expressed as

$$a_3^iT^3-a_2^iT^2+a_1^iT-a_0^i<0,\tag{12}$$

where $a_3^i=(1-|\lambda_i|^2)^2|1-\lambda_i|^2\gamma_1^2\gamma_2$, $a_2^i=2(1-|\lambda_i|^2)\gamma_1\left[(1-|\lambda_i|^2)^2\gamma_1+2|1-\lambda_i|^4\gamma_2^2\right]$, $a_1^i=4|1-\lambda_i|^2\gamma_2\left\{\left[|1-\lambda_i|^4+4\left(\mathrm{Im}\{\lambda_i\}\right)^2\right]\gamma_2^2+2(1-|\lambda_i|^2)^2\gamma_1\right\}$ and $a_0^i=8(1-|\lambda_i|^2)|1-\lambda_i|^4\gamma_2^2$. The left-hand side of the inequality in (12), denoted by $g_i(T)$, is a polynomial of T with order 3. There are at most three roots for $g_i(T)=0$, and then set $\mathcal{S}_2^i$ can be obtained. Let $\mathcal{S}_2=\bigcap_{i=1}^m\mathcal{S}_2^i$. Therefore, we have the following corollary.

Proposition 1. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalues of D_1. The matrix $\widetilde{A}$ has all eigenvalues within the unit circle if and only if the sampling period $T\in\mathcal{S}_1\cap\mathcal{S}_2$.* $\square$

Note that from Lemma 5, a small enough sampling period T always exists such that the eigenvalues of $\widetilde{A}$ are located inside the unit circle. The following corollary gives a rough idea on how to choose a small enough sampling period given the underlying communication networks and positive scalars γ_1 and γ_2.

Corollary 1. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalue of D_1. Given positive scalars γ_1 and γ_2, the matrix $\widetilde{A}$ has all eigenvalues within the unit circle if the sampling period $T \in (0, T_a)$ where*

$$T_a = \min_{i=1,\ldots,m}\left\{\frac{2(1-|\lambda_i|^2)|1-\lambda_i|^2}{\left[|1-\lambda_i|^4+4\left(\mathrm{Im}\{\lambda_i\}\right)^2\right]\gamma_2^2+2(1-|\lambda_i|^2)^2\gamma_1}\right\}.$$

Proof. The sets of $\mathcal{S}_1$ has been derived in (11). We focus on solving the inequality (12). Note that $a_3^iT^3 - a_2^iT^2 < 0$ if $0 < T < \frac{a_2^i}{a_3^i}$, and $a_1^iT - a_0^i < 0$ if $0 < T < \frac{a_0^i}{a_1^i}$. Then, the inequality (12) holds if $0 < T < \min\left\{\frac{a_2^i}{a_3^i}, \frac{a_0^i}{a_1^i}\right\}$. In addition, it can be verified that $\frac{a_0^i}{a_1^i} < \frac{2|1-\lambda_i|^2\gamma_2}{(1-|\lambda_i|^2)\gamma_1} < \frac{a_2^i}{a_3^i}$. Therefore, if $0 < T < \min_{i=1,\ldots,m}\left\{\frac{a_0^i}{a_1^i}\right\} = T_a$, (5) and (6) hold. □

Note that Lemma 5 and Proposition 1 give necessary and sufficient conditions such that $\widetilde{A}$ has all its eigenvalues inside the unit circle, and Corollary 1 only provides a conservative interval for the sampling period T, which is a sufficient condition.

It can be seen that the sampling period T should be small enough if the design parameter γ_1 and γ_2 are chosen to be large numbers. This observation coincides with the proof of Lemma 5. Also, note that as $T \to 0$, the controller (2) turns into a continuous-time controller, and it is well-known that the followers are to converge to the convex hull as long as γ_1 and γ_2 are positive. However, the resulting continuous-time controller as $T \to 0$ cannot be implemented in practice since each agent's input depends on its neighbors' inputs while the neighbors' inputs depend on their neighbors' inputs, which creates algebraic loops. In contrast, the introduced control algorithm (2) uses data from neighboring agents and can be implemented in a distributed manner in reality.

Given a graph satisfying Assumption 1, and any γ_1 and γ_2, to implement control algorithm (2) in practice, one can calculate the value of T_a given in Proposition 1, and select a valid sampling period T in the range $(0, T_a)$. If the interaction topology among the followers is undirected, all the eigenvalues of D_1 are real and inside the unit circle, then a simplified result can be obtained.

Corollary 2. *Suppose that Assumption 1 holds, and the interaction topology among the followers is undirected. Given positive scalars γ_1 and γ_2, the matrix $\widetilde{A}$ has all eigenvalues within the unit circle if the sampling period $T \in \left(0, \min_{i=1,\ldots,m}\left\{\frac{2(1-\lambda_i^2)}{(1-\lambda_i)^2\gamma_2^2+2(1-\lambda_i^2)^2\gamma_1}\right\}\right)$.*

3.2. Design of scalars γ_1 and γ_2

Though for any given scalars, γ_1 and γ_2, Corollary 1 provides a way to select a valid sampling period T, it is also important to design the scalars, γ_1 and γ_2, under given T. It is because sometimes, small enough sampling period cannot be guaranteed due to economic constraint and energy consumption issues, and each sampling device has limits on its sampling frequency. We provide the following results on the design of γ_1 and γ_2 given any sampling period T.

Proposition 2. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalue of D_1. Given the sampling period T, the matrix $\widetilde{A}$ has all eigenvalues within the unit circle if the positive scalars γ_1 and γ_2 are chosen such as*

$$0 < \gamma_1 < \min_{i=1,\ldots,m}\left\{\frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}\left(\frac{2\gamma_2}{T}-\sqrt{\phi_i}\right)\right\}, \tag{13}$$

$$0 < \gamma_2 < \min_{i=1,\ldots,m}\left\{\frac{2(1-|\lambda_i|^2)|1-\lambda_i|^2}{\left[|1-\lambda_i|^4+4\left(\mathrm{Im}\{\lambda_i\}\right)^2\right]T}\right\}, \tag{14}$$

where $\phi_i = \frac{16(\mathrm{Im}\{\lambda_i\})^2\gamma_2^3}{|1-\lambda_i|^2T[2(1-|\lambda_i|^2)-|1-\lambda_i|^2T\gamma_2]}$, $i = 1, \ldots, m$.

Proof. By (6), it is easy to see that

$$\gamma_2 < \frac{2(1-|\lambda_i|^2)}{|1-\lambda_i|^2T}, \quad i = 1, \ldots, m. \tag{15}$$

Note that (6) is equivalent to

$$\left(\frac{2\gamma_2}{T} - \frac{1-|\lambda_i|^2}{|1-\lambda_i|^2}\gamma_1\right)^2 - \phi_i < 0, \quad i = 1, \ldots, m, \tag{16}$$

where ϕ_i is given in the statement. The left hand side of (16) has two zeros, i.e.,

$$\gamma_{11,i}, \text{ and } \gamma_{12,i} = \frac{|1-\lambda_i|^2}{1-|\lambda_i|^2}\left(\frac{2\gamma_2}{T} \pm \sqrt{\phi_i}\right).$$

Then the solution to (16) is $\gamma_1 > \max_{i=1,\ldots,m}\{\gamma_{11,i}\}$ or $\gamma_1 < \min_{i=1,\ldots,m}\{\gamma_{12,i}\}$. Note that $\gamma_1 > \max_{i=1,\ldots,m}\{\gamma_{11,i}\}$ contradicts (5). Hence, we have (13). Since $\gamma_1 > 0$, and in order to ensure such choice of γ_1 exist, it requires that $\gamma_{12,i} > 0\ \forall i = 1, \ldots, m$, which yields that $\gamma_2 < \min_{i=1,\ldots,m}\left\{\frac{2(1-|\lambda_i|^2)|1-\lambda_i|^2}{[|1-\lambda_i|^4+4(\mathrm{Im}\{\lambda_i\})^2]T}\right\}$. Combining (15), we have (14). Therefore, if γ_1 and γ_2 are selected to respectively satisfy (13) and (14), (5) and (6) hold, which implies that all the eigenvalues of matrix $\widetilde{A}$ are inside the unit circle. □

It can be seen that the two design parameters γ_1 and γ_2 should be small if the sampling period T is chosen to be a large number.

In practice, given the graph $\mathcal{G}$ satisfying Assumption 1 and the sampling period T, one can first choose γ_2 satisfying (14), and then choose γ_1 satisfying (13) given the selected γ_2. Such choices of γ_1 and γ_2 ensure that (5) and (6) hold. If the interaction topology among followers is undirected, a simplified result can be obtained.

Corollary 3. *Suppose that Assumption 1 holds, and the interaction topology among the followers is undirected. Given the sampling period T, the matrix $\widetilde{A}$ has all eigenvalues within the unit circle if the positive scalars γ_1 and γ_2 are respectively chosen such as $0 < \gamma_1 < \min_{i=1,\ldots,m}\left\{\frac{2(1-\lambda_i)^2\gamma_2}{(1-\lambda_i^2)T}\right\}$ and $0 < \gamma_2 < \min_{i=1,\ldots,m}\left\{\frac{2(1-\lambda_i^2)}{(1-\lambda_i)^2T}\right\}$.* □

Remark 4. The design of the sampling period T and parameters γ_1 and γ_2 depends on the eigenvalues of the matrix D_1, which is related to the underlying interaction graph. In real applications, one can always let the weights d_{ij}, $j \in \mathcal{N}_i \cup \{i\}$ for agent i to be $\frac{1}{|\mathcal{N}_i|+1}$, which is valid since it ensures that D is a row-stochastic matrix. The number of possible values of D such that the underlying interaction graph satisfies Assumption 1, is finite since there are a finite number of agents. Note that the choices of the sampling period T and the design parameters γ_1 and γ_2 depend on each other. If the parameters γ_1 and γ_2 are fixed, for each possible D, one can select T by Corollary 1. Among these values of T, the smallest one can be selected for implementation in practice. If the sampling period T is fixed, valid scalars γ_1 and γ_2 can be selected in a similar manner.

3.3. Two special cases

3.3.1. Discrete-time single-integrator agents

When the agents have single-integrator dynamics given by

$$r_i[k+1] = r_i[k] + Tu_i[k], \tag{17}$$

we implement the following control law for follower $i \in \mathscr{F}$

$$u_i[k] = \sum_{j\in\mathscr{L}\cup\mathscr{F}} d_{ij}\left(\frac{r_j[k]-r_j[k-1]}{T} - \gamma\{r_i[k]-r_j[k]\}\right), \tag{18}$$

where γ is positive constant to be determined. Define the containment control error for follower i as $x_i[k] = \sum_{j=1}^{n} d_{ij}(r_i[k] - r_j[k])$. Define the collective containment control error vector as $X[k] = \left(x_1^\top[k], \ldots, x_m^\top[k]\right)^\top$. Let $W[k] = \left(X^\top[k+1], X^\top[k]\right)^\top$. Then we have

$$W[k+1] = \widetilde{A}_1 W[k] + \widetilde{B}_1 \Delta_L[k], \tag{19}$$

where $\widetilde{A}_1 = A_1 \otimes I_p$ and $\widetilde{B}_1 = \begin{bmatrix} D_2 \otimes I_p \\ \mathbf{0}_{mp\times(n-m)p} \end{bmatrix}$ with

$$A_1 = \begin{bmatrix} (1-T\gamma)I_m + (1+T\gamma)D_1 & -D_1 \\ I_m & \mathbf{0}_{m\times m} \end{bmatrix}. \tag{20}$$

Lemma 6. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalue of D_1. Then $\theta_i > 0$ holds, where $\theta_i = \frac{2|1-\lambda_i|^2\left[2(1-\mathrm{Re}\{\lambda\}_i)-|1-\lambda_i|^2\right]}{|1-\lambda_i|^4+4[\mathrm{Im}\{\lambda_i\}]^2}$. If the positive scalars T and γ satisfy*

$$T\gamma < \min\left\{1, \min_{i=1,\ldots,m} \theta_i\right\}, \tag{21}$$

then the matrix $\widetilde{A}_1$ has all eigenvalues within the unit circle.

Proof. The proof can be derived by following a similar analysis of Lemma 3.3 in [6] and the properties of the Kronecker product, thus is omitted here. □

Corollary 4. *Suppose that Assumptions 1 hold and the leaders' inputs are bounded, i.e., $\|u_i[k]\|_2 \le c_3$, $i \in \mathscr{L}$, where c_3 is a positive constant. If the positive scalars T and γ satisfy (21), using the algorithms (18) for (17), the followers converge to the convex hull spanned by the leaders with bounded position containment error, and the ultimate bound of overall containment control error, $\|X[k]\|_2$, is $c_3c_4T\sqrt{2(n-m)}\left\|\widetilde{B}_1\right\|_2/(1-\rho_1)$, where positive constant c_4 and $\rho_1 \in [0, 1)$ satisfy $\left\|\widetilde{A}_1^j\right\|_2 \le c_4\rho_1^j$, $j \ge 0$.*

Proof. By following a similar analysis in the proof of Theorem 1, it can be obtained that $\lim_{k\to\infty}\|W[k]\|_2 \le 2c_3c_4T\sqrt{n-m}\left\|\widetilde{B}_1\right\|_2/(1-\rho_1)$. Therefore, it holds that $\lim_{k\to\infty}\|X[k]\|_2 = \frac{1}{2}\lim_{k\to\infty}\sqrt{(\|X[k]\|_2+\|X[k-1]\|_2)^2} \le \frac{\sqrt{2}}{2}\lim_{k\to\infty}\|W[k]\|_2 \le c_3c_4T\sqrt{2(n-m)}\left\|\widetilde{B}_1\right\|_2/(1-\rho_1)$. □

3.3.2. Discrete-time double-integrator agents

When the agent i's model is discretized as

$$\begin{aligned} r_i[k+1] &= r_i[k] + Tv_i[k] \\ v_i[k+1] &= v_i[k] + Tu_i[k] \end{aligned} \tag{22}$$

where $r_i[k] \in \mathbb{R}^p$, $v_i[k] \in \mathbb{R}^p$ and $u_i[k] \in \mathbb{R}^p$ represent the position, velocity, and control input of the ith agent at $t = kT$, respectively.

Use the same definitions of $x_i[k]$ and $y_i[k]$, respectively, for the position and velocity containment errors of the follower i, and $X[k]$ and $Y[k]$, respectively, for the collective position and velocity containment control errors. Using (2) for (22) for each follower, and defining $Z[k+1] = \left(X^\top[k+1], Y^\top[k+1], X^\top[k], Y^\top[k]\right)^\top$, we then have the same form of system as

$$Z[k+1] = \widetilde{A}_2 Z[k] + \widetilde{B}_2 \Delta[k],$$

with $\widetilde{A}_2 = A_2 \otimes I_p$ and $\widetilde{B}_2 = [1, 1, 0, 0]^\top \otimes \left(D_2 \otimes I_p\right)$, where

$$A_2 = \begin{bmatrix} I_m & TI_m & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \\ \bar{A}_{21} & \bar{A}_{22} & \mathbf{0}_{m\times m} & -D_1 \\ I_m & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & I_m & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \end{bmatrix}$$

with $\bar{A}_{21} = -T\gamma_1(I_m - D_1)$ and $\bar{A}_{22} = (1-T\gamma_2)I_m + (1+T\gamma_2)D_1$.

Lemma 7. *Suppose that Assumption 1 holds. Let λ_i be the ith eigenvalue of D_1. The matrix $\widetilde{A}_2$ has all eigenvalues within the unit circle if and only if there exist positive scalars T, γ_1 and γ_2 such that*

$$\frac{2\gamma_2}{\gamma_1} > \left(\frac{1-|\lambda_i|^2}{|1-\lambda_i|^2}+1\right)T, \quad i = 1,\ldots,m, \tag{23}$$

and

$$\left[\frac{1}{T\gamma_1} - \frac{1-|\lambda_i|^2}{|1-\lambda_i|^2(2\gamma_2-T\gamma_1)}\right]^2\left[\frac{4(1-|\lambda_i|^2)}{|1-\lambda_i|^2(2T\gamma_2-T^2\gamma_1)} - 1\right] - \frac{16(\mathrm{Im}\{\lambda_i\})^2}{|1-\lambda_i|^4T^2\gamma_1^2} > 0, \quad i = 1,\ldots,m. \tag{24}$$

In addition, such scalars T, γ_1 and γ_2 always exist.

Proof. The proof can be obtained by following a similar analysis procedure in the proof of Lemma 5, thus is omitted here. □

Corollary 5. *Suppose that Assumptions 1 and 2 hold. If the positive scalars T, γ_1 and γ_2 satisfy (23) and (24), the followers converge to the convex hull spanned by the leaders with bounded position and velocity containment error, and the ultimate bound of overall containment control error, $\|X[k]\|_2+\|Y[k]\|_2$, is $2c_1c_5T\sqrt{n-m}\left\|\widetilde{B}_2\right\|_2/(1-\rho_2)$, where c_1 is given in Assumption 2, and positive constants c_5 and $\rho_2 \in [0, 1)$ satisfy $\left\|\widetilde{A}_2^j\right\|_2 \le c_5\rho_2^j$, $j \ge 0$.* □

Remark 5. The containment control problem for agents with the same model as (22) has been addressed in [25]. However, in [25], the leaders' dynamics are assumed to be the same as the followers with zero control inputs. The proposed algorithm (2) can deal with the case where the leaders have bounded nonzero inputs, which is more general, and the corresponding result takes into account the more realistic sampled-data setting.

4. Simulation

We provide a simulation to illustrate the results obtained in previous section.

Consider a group of ten agents, which are labeled as $1, \ldots, 10$. Denote by $\mathscr{F} = \{1, \ldots, 6\}$ and $\mathscr{L} = \{7, \ldots, 10\}$ the sets of the followers and the leaders, respectively. The directed communication network is shown in Fig. 1. Let $(r_{x_i}[k], r_{y_i}[k])$ and $(v_{x_i}[k], v_{y_i}[k])$ be the coordinates of agent i's position and velocity at time k, respectively. The input of the ith leaders is chosen to be $u_i[k] = -\frac{1}{(i-6)^2}\sin(\frac{1}{i-6}k) + 0.01(i-6)^2e^{-0.1(i-6)k}$, $i \in \mathscr{L}$. Set the sampling period T to 0.1, and choose $\gamma_1 = 0.9$ and $\gamma_2 = 1.25$. The control law (2) is implemented for all the followers with dynamics (1). The resulting trajectories of positions and velocities are shown in Fig. 2. It can be seen that both positions and velocities of all the followers converge to the convex hull spanned by those of the four leaders.

5. Conclusion

In this paper, we have proposed a sampled-data based containment control algorithm for a group of double-integrator

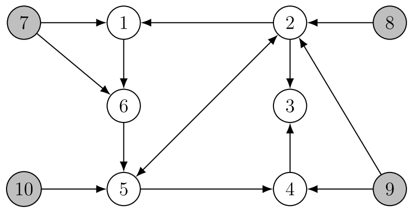
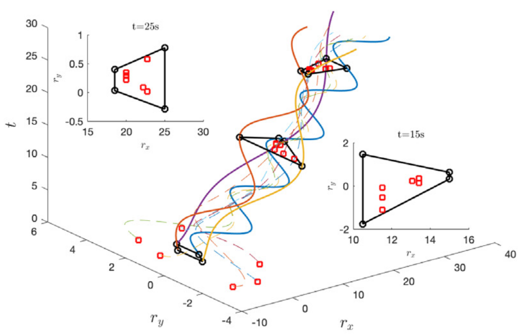
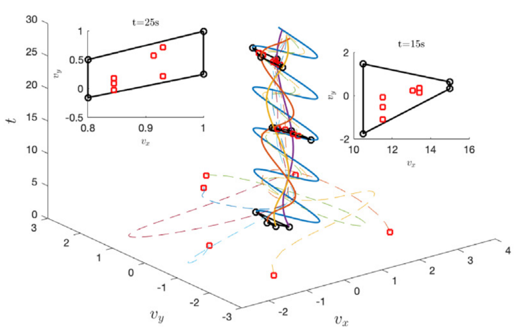


Fig. 1. Directed network topology for a group of ten agents, which are labeled from 1 to 10. There are four leaders, which are denoted by gray-filled circles. The rest are the followers.



(a) Position.



(b) Velocity.

Fig. 2. Position and velocity trajectories of the agents with dynamics (1) under a directed network topology presented in Fig. 1. The followers' input is implemented with (2). The solid lines denote the trajectories of the leaders' positions and velocities, and the dashed lines denote the followers' positions and velocities. The black circles and red squares denote the positions (velocities) of the leaders and the followers, respectively. The areas formed by four connecting black lines are convex hull spanned by the leaders. Two snapshots at $t = 15$ s and $t = 25$ s show that all the followers' positions and velocities are within the convex hull spanned by the leaders.

agents under directed communication networks. This algorithm contributes the solution to the discrete-time containment control problem with dynamic leaders whose inputs are nonzero. It has been shown that, by applying the proposed control algorithm, the containment control problem is solved with bounded position and velocity containment control errors, and the ultimate bound of the overall containment control error is proportional to the sampling period. A numerical simulation is presented to illustrate the proposed algorithm.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Yong Ding: Writing - original draft, Software, Formal analysis, Investigation. **Wei Ren:** Conceptualization, Writing - review & editing.

References

[1] P. Ogren, E. Fiorelli, N.E. Leonard, Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment, IEEE Trans. Automat. Control 49 (8) (2004) 1292–1302.

[2] W. Ren, R.W. Beard, E.M. Atkins, Information consensus in multivehicle cooperative control, IEEE Control Syst. Mag. 27 (2) (2007) 71–82.

[3] R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. Automat. Control 49 (9) (2004) 1520–1533.

[4] W. Ren, R.W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, IEEE Trans. Automat. Control 50 (5) (2005) 655–661.

[5] Y. Hong, J. Hu, L. Gao, Tracking control for multi-agent consensus with an active leader and variable topology, Automatica 42 (7) (2006) 1177–1182.

[6] Y. Cao, W. Ren, Y. Li, Distributed discrete-time coordinated tracking with a time-varying reference state and limited communication, Automatica 45 (5) (2009) 1299–1305.

[7] M. Ji, G. Ferrari-Trecate, M. Egerstedt, A. Buffa, Containment control in mobile networks, IEEE Trans. Automat. Control 53 (8) (2008) 1972–1975.

[8] R. Vaughan, N. Sumpter, J. Henderson, A. Frost, S. Cameron, Experiments in automatic flock control, Robot. Auton. Syst. 31 (1–2) (2000) 109–117.

[9] M.A. Haque, M. Egerstedt, C.F. Martin, First-order, networked control models of swarming silkworm moths, in: Proceedings of the American Control Conference, 2008, pp. 3798–3803.

[10] L. Cheng, Y. Wang, W. Ren, Z.-G. Hou, M. Tan, Containment control of multiagent systems with dynamic leaders based on a PI^n-type approach, IEEE Trans. Cybern. 46 (12) (2016) 3004–3017.

[11] Y. Cao, D. Stuart, W. Ren, Z. Meng, Distributed containment control for multiple autonomous vehicles with double-integrator dynamics: algorithms and experiments, IEEE Trans. Control Syst. Technol. 19 (4) (2011) 929–938.

[12] Y. Cao, W. Ren, M. Egerstedt, Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks, Automatica 48 (8) (2012) 1586–1597.

[13] H. Liu, G. Xie, L. Wang, Necessary and sufficient conditions for containment control of networked multi-agent systems, Automatica 48 (7) (2012) 1415–1422.

[14] J. Li, W. Ren, S. Xu, Distributed containment control with multiple dynamic leaders for double-integrator dynamics using only position measurements, IEEE Trans. Automat. Control 57 (6) (2011) 1553–1559.

[15] Z. Li, W. Ren, X. Liu, M. Fu, Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders, Internat. J. Robust Nonlinear Control 23 (5) (2013) 534–547.

[16] J. Mei, W. Ren, G. Ma, Distributed containment control for Lagrangian networks with parametric uncertainties under a directed graph, Automatica 48 (4) (2012) 653–659.

[17] W. Zou, Z. Xiang, Event-triggered containment control of second-order nonlinear multi-agent systems, J. Franklin Inst. B 356 (17) (2019) 10421–10438.

[18] G. Miao, J. Cao, A. Alsaedi, F.E. Alsaadi, Event-triggered containment control for multi-agent systems with constant time delays, J. Franklin Inst. B 354 (15) (2017) 6956–6977.

[19] T.-H. Cheng, Z. Kan, J.R. Klotz, J.M. Shea, W.E. Dixon, Decentralized event-triggered control of networked systems-part 2: Containment control, in: Proceedings of the American Control Conference, 2015, pp. 5444–5448.

[20] W. Liu, C. Yang, Y. Sun, J. Qin, Observer-based event-triggered containment control of multi-agent systems with time delay, Internat. J. Systems Sci. 48 (6) (2017) 1217–1225.

[21] W. Chen, D. Ding, X. Ge, Q.-L. Han, G. Wei, $\mathcal{H}_\infty$ Containment control of multiagent systems under event-triggered communication scheduling: The finite-horizon case, IEEE Trans. Cybern. (2018).

[22] W. Zhang, Y. Tang, Y. Liu, J. Kurths, Event-triggering containment control for a class of multi-agent networks with fixed and switching topologies, IEEE Trans. Circuits Syst. I. Regul. Pap. 64 (3) (2017) 619–629.

[23] K. Liu, Z. Ji, G. Xie, R. Xu, Event-based broadcasting containment control for multi-agent systems under directed topology, Internat. J. Control 89 (11) (2016) 2360–2370.

[24] D. Wang, W. Wang, Necessary and sufficient conditions for containment control of multi-agent systems with time delay, Automatica 103 (2019) 418–423.

[25] M. Asgari, H. Atrianfar, Necessary and sufficient conditions for containment control of heterogeneous linear multi-agent systems with fixed time delay, IET Control Theory Appl. 13 (13) (2019) 2065–2074.

[26] H. Xia, W.X. Zheng, J. Shao, Event-triggered containment control for second-order multi-agent systems with sampled position data, ISA Trans. 73 (2018) 91–99.

[27] H. Liu, G. Xie, M. Yu, Necessary and sufficient conditions for containment control of fractional-order multi-agent systems, Neurocomputing 323 (2019) 86–95.

[28] X. Wang, L. Shi, J. Shao, Containment control for high-order multi-agent systems with heterogeneous time delays, IET Control Theory Appl. 12 (9) (2018) 1340–1348.

[29] J. Shao, L. Shi, W.X. Zheng, Y. Cheng, Cooperative containment control in time-delayed multi-agent systems with discrete-time high-order dynamics under dynamically changing topologies, J. Franklin Inst. B 356 (5) (2019) 2441–2462.

[30] J. Shao, L. Shi, W.X. Zheng, T.-Z. Huang, Containment control for heterogeneous multi-agent systems with asynchronous updates, Inform. Sci. 436 (2018) 74–88.

[31] Y. Cao, W. Ren, Containment control with multiple stationary or dynamic leaders under a directed interaction graph, in: Proceedings of the IEEE Conference on Decision and Control, and the Chinese Control Conference, 2009, pp. 3014–3019.

[32] I. Kovacs, D.S. Silver, S.G. Williams, Determinants of block matrices and schurs formula, Amer. Math. Monthly 106 (5) (1999) 950–952.

[33] E. Frank, On the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc. 52 (2) (1946) 144–157.

[34] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge university press, 2012.

[35] J.P. LaSalle, The Stability and Control of Discrete Processes, Vol. 62, Springer Science & Business Media, 2012.

[36] Z.-P. Jiang, Y. Wang, Input-to-state stability for discrete-time nonlinear systems, Automatica 37 (6) (2001) 857–869.