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Brief paper

Continuous-time distributed Nash equilibrium seeking algorithms for non-cooperative constrained games ☆

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ABSTRACT

This paper studies the Nash equilibrium seeking problem for non-cooperative games subject to set and nonlinear inequality constraints. The cost function for each player and the constrained function are determined by all the players' decision variables. Each player is assigned a constrained set while all the players are subject to a coupling nonlinear inequality constraint. A continuous-time distributed seeking algorithm using local information interaction is proposed, where the players deliver/receive information unidirectionally over a directed network. In particular, a distributed observer is first introduced for each player to estimate all the others' decision variables. Then, by using these estimates, a seeking algorithm is synthesized with a projection operator. Based on the time-scale separation approach, it is shown that the proposed continuous-time distributed seeking algorithm guarantees the convergence of the strategy profile to an arbitrarily small neighborhood of the generalized Nash equilibrium satisfying a KKT condition. An illustrative example is finally presented to validate the theoretical results.

1. Introduction

Significant attention has been paid to the research on game theory due to its various applications in traffic, communication networks, energy and other fields in recent years (Koshal, Nedić, & Shanbhag, 2016; Lei & Shanbhag, 2018; Liu, Cheng, Yu, Zhong, & Lei, 2018; Pisarski & de Wit, 2016; Shakarami, Persis, & Monshizadeh, 2019; Zheng, Cai, Chen, Li, & Zhang, 2015). With the rapid development of game theory, the Nash equilibrium seeking in non-cooperative games is of considerable interest from both theoretical and application perspectives (Cominetti, Facchinei, & Lasserre, 2012). In the general Nash equilibrium seeking problem, each player attempts to minimize its cost function by responding to its own and other players' actions. This requires a fully connected network such that a full observation over all the players' actions in the network can be performed (Cominetti et al., 2012; Contreras, Klusch, & Krawczyk, 2004). To relax this connectivity requirement, especially for a large-scale network, distributed algorithms borrowed from the consensus idea for multi-agent systems are proposed. In such distributed algorithms, the players minimize their cost functions via local information interaction with only adjacent players.

Nowadays, a variety of discrete-time distributed Nash equilibrium seeking algorithms have been proposed. In particular, a gossip-based methodology was developed for seeking the Nash equilibrium of non-cooperative games in Salehisadaghiani and Pavel (2016). Moreover, in Salehisadaghiani, Shi, and Pavel (2019), an altering direction method of multipliers was introduced into the distributed Nash equilibrium seeking under partial-decision information. In Poveda, Teel, and Nesic (2015), a discrete-time stochastic algorithm was proposed such that the players took actions in both simultaneous and asynchronous manners. Besides, by considering a shared affine constraint into the games, multiple discrete-time distributed algorithms based on operator splitting methods were proposed in Pavel (2020) and Yi and Pavel (2019a, 2019b). In addition, by imposing coupling constraints of the

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average of the decision variables, a discrete-time distributed Nash equilibrium seeking algorithms was developed for aggregative games in Grammatico (2017). Furthermore, instead of the linear or set constraint considered in the previous references, a nonlinear constraint on the decision variable was taken into account in Paccagnan, Gentile, Parise, Kamgarpour, and Lygeros (2019); and correspondingly, seeking algorithms were designed for the aggregative games.

On the other hand, without necessity of determining the step size as done in the discrete-time seeking algorithms, various continuous-time distributed seeking algorithms have also been proposed recently. In particular, a continuous-time semi-decentralized seeking algorithm was proposed for monotone aggregative games with linear and set constraints in Persis and Grammatico (2019a). By introducing consensus-based distributed observers, continuous-time distributed algorithms were developed such that the Nash equilibrium was locally reached in Ratliff, Burden, and Sastry (2016) and Ye and Hu (2017). Moreover, by considering decoupling nonlinear and set constraints, continuous-time distributed gradient-based Nash equilibrium seeking algorithms were designed for noncooperative games in Gadjov and Pavel (2019), Lu, Jing, and Wang (2019) and Persis and Grammatico (2019b). Furthermore, rather than the multi-player game, the multi-cluster game was studied in Zeng, Chen, Liang, and Hong (2019), and a continuous-time seeking strategy was proposed therein. Nevertheless, the aforementioned continuous-time distributed seeking algorithms are in terms of an undirected topology, which means a synchronous bilateral interaction between adjacent players. This might not be practical in some real applications.

In this paper, a continuous-time distributed Nash equilibrium seeking algorithm is proposed for non-cooperative constrained games over a directed communication topology. Each player is assigned a cost function and a constrained set, while a nonlinear inequality constraint is imposed on all the players' actions represented by decision variables. The objective is to seek a generalized Nash equilibrium which satisfies the set and inequality constraints such that each cost function coupled by all the decision variables is unilaterally minimized. In particular, by developing a distributed observer to estimate other players' decision variables, a distributed seeking algorithm with a projection operator is proposed. It is shown in terms of the time-scale separation approach that, given a strong connectivity topology condition, all the decision variables uniformly ultimately converge to an arbitrarily small neighborhood of the generalized Nash equilibrium satisfying a KKT condition. Compared with the previous continuous-time distributed seeking algorithms (Gadjov & Pavel, 2019; Lu et al., 2019; Persis & Grammatico, 2019a, 2019b; Ratliff et al., 2016; Ye & Hu, 2017), the main contributions herein lie in three aspects. First, this paper simultaneously considers the set constraint and coupling nonlinear inequality constraint existing in the general noncooperative game. To overcome the constraint impact, a continuous-time distributed Nash equilibrium seeking algorithm resorting to a distributed observer is proposed. Second, instead of the impractical bidirectional interaction over the undirected topology, the players deliver/receive information to/from their neighbors unidirectionally in this paper. Such an information interaction manner is characterized by a directed topology. It is shown that the strong connectivity topology condition is sufficient for seeking the generalized Nash equilibrium of interest successfully. Last but not the least, the time-scale separation approach is used for the convergence analysis. Specifically, it is first shown that the distributed observer, considered as the fast dynamics, guarantees the estimation errors converging to an arbitrarily small neighborhood of the origin within finite time. By using this result, it is next shown that the seeking algorithm, considered as the slow dynamics, guarantees the decision variables uniformly ultimately converging to an arbitrarily small neighborhood of the Nash equilibrium of interest.

The following sections are organized as follows. Section 2 presents some useful preliminaries. Section 3 describes the game problem to be addressed. Section 4 introduces the main results including the seeking algorithm synthesis and the convergence analysis. Section 5 performs a simulation to validate the proposed theoretical results. Section 6 draws final conclusions.

Notations. In what follows, $\mathbb{R}^n$ denotes real vectors of dimension n, $\mathbb{R}^+$ denotes non-negative real numbers, I_n is an $n \times n$ unit vector, $\mathbf{1_n}$ is an n-dimensional vector with all its entries being 1, $\|\cdot\|$ denotes the Euclidean norm of a real vector or the Frobenius norm of a real matrix, $\mathrm{col}(x_1, x_2, \ldots, x_n) = [x_1^T, x_2^T, \ldots, x_n^T]^T$, $\mathrm{diag}\{y_1, y_2, \ldots, y_n\}$ is a diagonal matrix with diagonal entries being scalars y_1 to y_n, $\mathrm{sgn}(y) = [\mathrm{sgn}(y_1), \mathrm{sgn}(y_2), \ldots, \mathrm{sgn}(y_n)]^T$ for $y = [y_1, y_2, \ldots, y_n]^T$ with $\mathrm{sgn}(\cdot)$ being the sign function, and $\vartheta_{\max}(\cdot)$ and $\vartheta_{\min}(\cdot)$ denote the maximum and minimum eigenvalues of a square matrix. Given a positive constant μ, the μ-neighborhood around a fixed point $\bar{x} \in \mathbb{R}^n$ is defined by a compact set $\{x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq \mu\}$. Moreover, let ∇f be the gradient of a function f and $\mathcal{J}F$ be the Jacobian matrix of a map F.

2. Preliminaries

2.1. Function properties

Given a closed set $\Omega \subseteq \mathbb{R}^n$, a differentiable function $f : \Omega \to \mathbb{R}$ is said to be convex if $(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0$, $\forall x, y \in \Omega$. Moreover, such f is locally θ-Lipschitz if $\|f(x) - f(y)\| \leq \theta\|x - y\|$, $\forall x, y \in \Omega$. In addition, a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone over set Ω if $(x - y)^T(F(x) - F(y)) \geq 0$, $\forall x, y \in \Omega$, and F is said to be ω-strongly monotone if there exists a positive constant ω such that $(x - y)^T(F(x) - F(y)) \geq \omega\|x - y\|^2$, $\forall x, y \in \Omega$ (Bauschke & Combettes, 2011; Facchinei & Kanzow, 2007). According to Liang, Yi, and Hong (2017), a differentiable map F is ω-strongly monotone over set Ω if and only if the corresponding Jacobian matrix $\mathcal{J}F(x)$ is positive definite for each $x \in \Omega$.

2.2. Projection operator

A set $\Omega \subseteq \mathbb{R}^n$ is called convex if $\mu x_1 + (1 - \mu)x_2 \in \Omega$ for any $x_1, x_2 \in \Omega$ and any $\mu \in [0, 1]$. Given a closed convex set $\Omega \subseteq \mathbb{R}^n$, we define the projection operator $P_\Omega(x) : \mathbb{R}^n \to \Omega$ as $P_\Omega(x) = \arg\min_{\omega \in \Omega} \|x - \omega\|$. Next, two useful results concerned with the projection operator are presented as follows.

Lemma 2.1 (*Aubin & Cellina, 1984*)**.** *Given a closed convex set $\Omega \subseteq \mathbb{R}^n$, the following inequalities hold:*

$$(x - P_\Omega(x))^T(P_\Omega(x) - y) \geq 0, \quad x \in \mathbb{R}^n, \quad y \in \Omega, \tag{1}$$

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \quad x, y \in \mathbb{R}^n. \tag{2}$$

Lemma 2.2. *Consider the following system:*

$$\dot{x} = -x + P_\Xi(h(x, u)), \tag{3}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the system state and input, $\Xi = \{\xi_1 \leq x \leq \xi_2\}$ is a closed convex set, and $h(x, u)$ is a differentiable function. The following results hold:
(1) if $x(0) \in \Xi$, then $x(t) \in \Xi$, $\forall t \geq 0$, and
(2) if $x(0) \notin \Xi$, there exists a positive constant γ such that $|x(t)| \leq \gamma$, $\forall t \geq 0$.

Proof. See Appendix A. ■

Remark 2.1. In terms of Lemma 2.2, it also follows that $\dot{x}$ in (3) is bounded given that x is bounded and $P_{\Xi}(h(x,u)) \in [\xi_1, \xi_2]$. Moreover, following the argument in Lemma 2.2, we can also obtain that $x(t) \in \Xi_1 = \{x \geq \xi_1\}$ or $\Xi_2 = \{x \leq \xi_2\}$, $\forall t \geq 0$ given $x(0) \in \Xi_1$ or Ξ_2.

2.3. Graph theory

The information interaction among agents is built by a topology graph $\mathcal{G} \triangleq \{\mathcal{V}, \mathcal{E}\}$, where the node set is defined by $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$ and the edge set is defined by $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In a directed graph, $(j, i) \in \mathcal{E}$ means that node i has access to node j's information but not vice versa; and in such a case, node j is called a neighbor of node i. All the neighbors of node i are included in set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. A path from node i to node j is a sequence of directed edges. A directed graph is called strongly connected if each node has a path to every other node. Moreover, the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with a directed graph is defined such that $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise; and its Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

3. Problem statement

Consider an n-player non-cooperative game subject to coupled constraints. The set of players is denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$. Each player $i \in \mathcal{V}$ is assigned a decision variable $x_i \in \mathbb{R}$ and a cost function $f_i(x) : \mathbb{R}^n \to \mathbb{R}$, where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ is the strategy profile of this game. Suppose that player j's decision variable is available to player i only if player j is a neighbor of player i. In addition, consider each decision variable x_i being subject to a set constraint:

$$x_i \in \Omega_i, \tag{4}$$

where $\Omega_i \subset \mathbb{R}$ is a closed convex set; and the strategy profile x is constrained by a nonlinear inequality satisfying

$$\chi = \{x \in \mathbb{R}^n | g(x) \leq 0\}, \tag{5}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is the constrained function. Define $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. Then the feasible strategy set of this game is $\mathcal{Q} = \Omega \cap \chi$. For notational simplicity, we just consider the non-cooperative game with one-dimensional decision variables. The corresponding results can be straightforwardly extended to the higher-dimensional case with some augmented operations.

Given the aforementioned constrained non-cooperative game, the objective herein is to develop a Nash equilibrium seeking algorithm such that the strategy profile reaches the generalized Nash equilibrium. For the development of the seeking algorithm, except that the constrained function g is known to all the players, each cost function f_i and constrained set Ω_i are only available to their corresponding players, while all the others, even their neighbors, have no access to these pieces of information. First, we formalize the definition of the generalized Nash equilibrium, which follows the one introduced in Cominetti et al. (2012) and Facchinei and Kanzow (2007).

Definition 3.1 (*Generalized Nash Equilibrium*)**.** A strategy profile $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is called a generalized Nash equilibrium of the constrained non-cooperative game if

$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*), \quad x_i \in \mathcal{Q}_i(x_{-i}^*), \tag{6}$$

where $x_{-i} = [x_1, \ldots, x_{i-1}, x_{i+1} \cdots, x_n]$ and $\mathcal{Q}_i(x_{-i}^*) = \{x_i \in \Omega_i : g(x_i, x_{-i}^*) \leq 0\}$, $i \in \mathcal{V}$.

Actually, the generalized Nash equilibrium is an optimal strategy profile incorporated in set $\mathcal{Q}$ in the sense that no player can further reduce its associated cost function by unilaterally changing its own decision variable. Moreover, to clarify the subsequent analysis, we define two vectors

$$F(x) = [\nabla_{x_1} f_1(x), \nabla_{x_2} f_2(x), \ldots, \nabla_{x_n} f_n(x)]^T, \tag{7}$$

$$G(x) = [\nabla_{x_1} g(x), \nabla_{x_2} g(x), \ldots, \nabla_{x_n} g(x)]^T. \tag{8}$$

Before moving on, several fundamental assumptions associated with the considered game are made as follows.

Assumption 3.1 (*Existence Condition*)**.** The feasible strategy set $\mathcal{Q}$ is non-empty, i.e., $\mathcal{Q} \neq \emptyset$.

Assumption 3.2 (*Differentiability and Convexity*)**.** Each cost function $f_i(x)$ and the constrained function $g(x)$ are at least twice continuously differentiable with respect to $x \in \mathbb{R}^n$, and they are convex with respect to $x_i \in \mathbb{R}$ for fixed $x_{-i} \in \mathbb{R}^{n-1}$.

Assumption 3.3 (*Monotonicity*)**.** The map $F(x)$ formulated in (7) is ω-strongly monotone over $\mathbb{R}^n$, and the map $G(x)$ in (8) is monotone over $\mathbb{R}^n$.

Remark 3.1. The existence condition claimed in Assumption 3.1 guarantees the existence of the generalized Nash equilibrium introduced in Definition 3.1 (Bazaraa, Sherali, & Shetty, 2006). Then, the differentiable condition claimed in Assumption 3.2, also adopted in Ye and Hu (2017), guarantees that the constrained function $g(x)$ and the gradient functions $\nabla_{x_i} f_i(x)$ and $\nabla_{x_i} g(x)$, $i \in \mathcal{V}$ are locally θ-Lipschitz over any given closed convex set (Khalil, 2002), while the convex condition therein is a sufficient condition for determining a KKT condition which characterizes the Nash equilibrium of interest into an equality (Cominetti et al., 2012). Under Assumption 3.3, it is trivial to show that $\mathcal{J}F(x)$ is positive definite and $\mathcal{J}G(x)$ is positive semi-definite. This property is useful in the subsequent convergence analysis. Furthermore, according to Liang et al. (2017), the strong monotonicity of $F(x)$ guarantees the uniqueness of the Nash equilibrium of interest.

In order to show the convergence of the strategy profile to the generalized Nash equilibrium, we first characterize the Nash equilibrium of interest. Intuitively, a KKT condition for the considered game is determined in the following theorem, which indicates that the strategy profile satisfying this KKT condition is exactly the generalized Nash equilibrium introduced in Definition 3.1.

Theorem 3.1 (*Cominetti et al., 2012*)**.** *Suppose that Assumptions 3.1–3.3 hold. The strategy profile x^* is a generalized Nash equilibrium in the sense of Definition 3.1 if and only if there exist optimal multipliers $\lambda_i^* \geq 0$, $i \in \mathcal{V}$ such that the following KKT condition holds:*

$$x_i^* - P_{\Omega_i}(x_i^* - \nabla_{x_i} f_i(x^*) - \lambda_i^* \nabla_{x_i} g(x^*)) = 0, \tag{9a}$$

$$g(x^*) \leq 0, \quad \lambda_i^* g(x^*) = 0, \quad i \in \mathcal{V}. \tag{9b}$$

4. Main results

In this section, a distributed seeking algorithm is developed such that the strategy profile x converges to the generalized Nash equilibrium introduced in Definition 3.1. A directed graph is applied to describe the information exchange relationship among players. Introducing a distributed observer, we first propose a continuous-time distributed seeking algorithm for the studied non-cooperative game subject to both the set and inequality constraints. Then, the system stability is analyzed by using the time-scale separation approach.

4.1. Seeking algorithm synthesis

To guarantee that the strategy profile x converges to the generalized Nash equilibrium, a seeking algorithm using the projection operator is proposed as follows:

$$\dot{x}_i = -x_i + P_{\Omega_i}(x_i - \nabla_{x_i} f_i(x) - \lambda_i \nabla_{x_i} g(x)), \tag{10a}$$

$$\dot{\lambda}_i = -\lambda_i + P_{\mathbb{R}^+}(\lambda_i + g(x)), \quad i \in \mathcal{V}, \tag{10b}$$

where $x_i(0) \in \mathbb{R}$ and $\lambda_i(0) \geq 0$. In terms of Lemma 2.2 and Remark 2.1, it is not hard to show that $|x_i(t)| \leq \gamma$ and $\lambda_i(t) \geq 0$, $\forall t \geq 0$ given $\lambda_i(0) \geq 0$ for $i \in \mathcal{V}$, where $\gamma > 0$ is a bound parameter. Next, a lemma characterizing the equilibrium $\bar{x} = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]^T$ and $\bar{\lambda} = [\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n]^T$ of algorithm (10) is presented as follows.

Lemma 4.1. *For an equilibrium $(\bar{x}, \bar{\lambda})$ of algorithm (10), $\bar{x}$ is the generalized Nash equilibrium introduced in Definition 3.1.*

Proof. In terms of (10b), $\dot{\bar{\lambda}}_i = 0$ implies that

$$\bar{\lambda}_i = P_{\mathbb{R}^+}(\bar{\lambda}_i + g(\bar{x})), \quad i \in \mathcal{V}.$$

If $\bar{\lambda}_i = 0$, then $P_{\mathbb{R}^+}(g(\bar{x})) = 0$. This implies that $g(\bar{x}) \leq 0$. If $\bar{\lambda}_i > 0$, then $\bar{\lambda}_i + g(\bar{x}) > 0$. This implies that $g(\bar{x}) = 0$. It thus follows that

$$\bar{\lambda}_i g(\bar{x}) = 0, \;\; g(\bar{x}) \leq 0, \quad i \in \mathcal{V}. \tag{11}$$

Next, by considering the fact that $\bar{\lambda}_i \geq 0$, it follows from (10a) that each $\bar{x}_i$ satisfies

$$\bar{x}_i - P_{\Omega_i}(\bar{x}_i - \nabla_{x_i} f_i(\bar{x}) - \bar{\lambda}_i \nabla_{x_i} g(\bar{x})) = 0, \quad i \in \mathcal{V}. \tag{12}$$

By comparing (11) and (12) with (9), it is evident that equilibrium $(\bar{x}, \bar{\lambda})$ satisfies the KKT condition (9). Hence, according to Theorem 3.1, $\bar{x}$ is exactly the generalized Nash equilibrium in the sense of Definition 3.1. ■

It is worthwhile to point out that, the seeking algorithm (10) is feasible under a fundamental condition that each player's decision variable is accessible to every other player. In other words, the seeking algorithm (10) is in a centralized manner. To address the game problem introduced in Section 3 in a distributed fashion, motivated by Ratliff et al. (2016) and Ye and Hu (2017), the following distributed observer is proposed for $i \in \mathcal{V}$:

$$\begin{cases} \dot{\hat{x}}_{ij} = -\eta\left[\sum_{k\in\mathcal{N}_i} a_{ik}(\hat{x}_{ij} - \hat{x}_{kj}) + a_{ij}(\hat{x}_{ij} - x_j)\right], \; j \in \mathcal{V}\backslash\{i\}, \\ \hat{x}_{ii} = x_i, \end{cases} \tag{13}$$

where $\hat{x}_{ij}$ denotes the player i's estimate on the jth decision variable for $j \in \mathcal{V}$, η is a positive tunable parameter, and a_{ij} is the (i, j)-entry of the adjacency matrix A associated with the underlying graph $\mathcal{G}$. Note that the true x_j is just available to the neighbors of agent j (i.e., $a_{ij} \neq 0$) in the distributed observer (13). However, unlike Ye and Hu (2017); Ratliff et al. (2016) in terms of an undirected topology, the distributed observer (13) is based on a directed topology. This means that the information delivery therein is unidirectional.

Next, define the estimate vector $\hat{x}_i = [\hat{x}_{i1}, \hat{x}_{i2}, \ldots, \hat{x}_{in}]^T$ for $i \in \mathcal{V}$ and introduce a closed convex set $\Lambda = \{\lambda \in \mathbb{R} \mid 0 \leq \lambda \leq \lambda_{\max}\}$, where $\lambda_{\max}$ is a given positive constant. By implementing $\hat{x}_i$ instead of the unavailable strategy profile x, the seeking algorithm (10) is revised as

$$\dot{x}_i = -x_i + P_{\Omega_i}(x_i - \nabla_{x_i} f_i(\hat{x}_i) - \lambda_i \nabla_{x_i} g(\hat{x}_i)), \tag{14a}$$

$$\dot{\lambda}_i = -\lambda_i + P_{\Lambda}(\lambda_i + g(\hat{x}_i)), \quad i \in \mathcal{V}, \tag{14b}$$

where $x_i(0) \in \mathbb{R}$ and $\lambda_i(0) \in \Lambda$. In terms of Lemma 2.2, the seeking algorithm (14) ensures that each decision variable is bounded by $|x_i(t)| < \gamma$, $\forall t \geq 0$ with a bound parameter $\gamma > 0$, and that each multiplier $\lambda_i(t) \in \Lambda$, $\forall t \geq 0$ given $\lambda_i(0) \in \Lambda$. Note from the seeking algorithm (14) that, player i updates its individual variables x_i and λ_i using its private information including the cost function f_i and the constrained set Ω_i, as well as $\hat{x}_i$ from the distributed observer (13), whereas the others' private information is unnecessary.

4.2. Convergence analysis

In this subsection, we show that the seeking algorithm (14) drives the strategy profile to an arbitrarily small neighborhood of the generalized Nash equilibrium introduced in Definition 3.1 under a strong connectivity topology condition. The proof is in terms of the time-scale separation approach. The underlying idea behind this approach lies in that the distributed observer (13) and the seeking algorithm (14) are considered as a composite system with fast dynamics (13) and slow dynamics (14). By following this idea, the analysis can be divided into two parts. The first one focuses on the initialized time interval $[0, T_1]$. In particular, Theorem 4.1 shows that the distributed observer (13) guarantees the estimated error converging to a μ-neighborhood of the origin in finite time T_1 and remaining within it afterwards. Such a μ-neighborhood can be made arbitrarily small by sufficiently increasing the parameter η. Using this result, Theorem 4.2 in the second part for $[T_1, \infty)$ shows that the strategy profile x driven by the seeking algorithm (14) converges to a small neighborhood of the generalized Nash equilibrium of interest.

First, define the estimated error $\tilde{x}_i = \hat{x}_i - x$ for $i \in \mathcal{V}$ and the column stack vector $\tilde{x} = \mathrm{col}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$. Also, define $\tilde{x}_{*j} = \mathrm{col}(\tilde{x}_{1j}, \ldots, \tilde{x}_{(j-1)j}, \tilde{x}_{(j+1)j}, \ldots, \tilde{x}_{nj})$ for $j \in \mathcal{V}$. It follows from (13) that it satisfies the following dynamics:

$$\begin{aligned} \dot{\tilde{x}}_{*j} &= -\eta(L_j \tilde{x}_{*j} + \bar{L}_j \tilde{x}_{*j}) - \dot{x}_j \mathbf{1_n} \\ &= -\eta M_j \tilde{x}_{*j} - \dot{x}_j \mathbf{1_n}, \quad j \in \mathcal{V}, \end{aligned} \tag{15}$$

where L_j is the Laplacian matrix L of graph $\mathcal{G}$ without the jth row and jth column, $\bar{L}_j = \mathrm{diag}\{a_{1j}, \ldots, a_{(j-1)j}, a_{(j+1)j}, \ldots, a_{nj}\}$, and $M_j = L_j + \bar{L}_j$. Given $j \in \mathcal{V}$, if graph $\mathcal{G}$ is strongly connected, it is obvious that player j has a directed path to any other player. In such a case, according to Qu (2009, Theorem 4.25), matrix M_j, $j \in \mathcal{V}$ is non-singular, and there exists a positive definite diagonal matrix $\Xi_j = \mathrm{diag}\{\xi_{1j}, \ldots, \xi_{(j-1)j}, \xi_{(j+1)j}, \ldots, \xi_{nj}\}$ such that matrix $\Upsilon_j = \Xi_j M_j + M_j^T \Xi_j$ is symmetric and positive definite. Moreover, it follows from Lemma 2.2 that, by using the seeking algorithm (14), there exists a positive constant $\bar{\gamma}$ such that $\max_{j\in\mathcal{V}} |\dot{x}_j(t)| < \bar{\gamma}$, $\forall t \geq 0$.

The following theorem shows that the estimated error $\tilde{x}$ can be driven to a neighborhood of the origin in finite time and such a neighborhood can be made arbitrarily small by choosing a sufficiently large parameter η.

Theorem 4.1. *Consider the distributed observer (13). Suppose that Assumption 3.1–3.3 hold, and the underlying graph $\mathcal{G}$ is strongly connected. Given any finite time $T_1 > 0$, the estimated error $\tilde{x}$ converges to a μ-neighborhood of the origin within T_1, where*

$$\mu = \sqrt{\frac{1}{\beta_3}\left[e^{-\frac{\eta\beta_1 T_1}{2}}\Big(\sum_{i=1}^{n} \tilde{x}_{*j}(0)^T \Xi_j \tilde{x}_{*j}(0)\Big)^{\frac{1}{2}} + \frac{\beta_2}{\eta\beta_1}\right]}, \tag{16}$$

with $\beta_1 = \min_{j\in\mathcal{V}}\{\vartheta_{\min}(\Upsilon_j)\}/\max_{j\in\mathcal{V}}\{\vartheta_{\max}(\Xi_j)\}$, $\beta_2 = 2\sqrt{2}\bar{\gamma}\max_{j\in\mathcal{V}}\{\sum_{i=1,i\neq j}^{n} \xi_{ij}\}/\min_{j\in\mathcal{V}}\{\vartheta_{\min}(\Xi_j)\}$ *and* $\beta_3 = \sqrt{\min_{j\in\mathcal{V}}\{\vartheta_{\min}(\Xi_j)\}}$.

Proof. Assign a Lyapunov function $W=\sum_{j=1}^{n}\tilde{x}_{*j}^T\varXi_j\tilde{x}_{*j}$. Its derivative along (15) satisfies

$$\begin{aligned}\dot{W}&=-\eta\sum_{j=1}^{n}\tilde{x}_{*j}^T(\varXi_jM_j+M_j^T\varXi_j)\tilde{x}_{*j}+2\sum_{j=1}^{n}\tilde{x}_{*j}^T\varXi_j\dot{x}_j\mathbf{1_{n-1}}\\&\le-\eta\sum_{j=1}^{n}\tilde{x}_{*j}^T\Upsilon_j\tilde{x}_{*j}+2\sum_{j=1}^{n}\Big(\sum_{i=1,i\neq j}^{n}\xi_{ij}\Big)\|\tilde{x}_{*j}\||\dot{x}_j|\\&\le-\eta\sum_{j=1}^{n}\vartheta_{\min}(\Upsilon_j)\|\tilde{x}_{*j}\|^2+2\bar{\gamma}\sum_{j=1}^{n}\Big(\sum_{i=1,j\neq j}^{n}\xi_{ij}\Big)\|\tilde{x}_{*j}\|\\&\le-\eta\beta_1W+\beta_2\sqrt{W}.\end{aligned}\tag{17}$$

Then, take $V=\sqrt{W}$. When $W\neq 0$, it follows from (17) that

$$\dot{V}=-\frac{\eta\beta_1}{2}V+\frac{\beta_2}{2}.\tag{18}$$

When $W=0$, the Dini derivative of V satisfies $D^+V\le\beta_2/2$. Hence, D^+V satisfies (18) all the time. In terms of Comparison Principle (Khalil, 2002), it follows that

$$V(t)\le e^{-\frac{\eta\beta_1}{2}t}V(0)+(1-e^{-\frac{\eta\beta_1}{2}t})\frac{\beta_2}{\eta\beta_1}.\tag{19}$$

This implies that $V(t)\le e^{-\eta\beta_1T_1/2}V(0)+\beta_2/(\eta\beta_1)$, $t\ge T_1$ for any $T_1>0$. By considering the definition of V, this further implies that $\|\tilde{x}(t)\|\le\mu$, $t\in[T_1,\infty)$. ■

Remark 4.1. In fact, β_1, β_2 and β_3 given in (16) are related to the edge weights and initial decision variables $\hat{x}_i(0)$. Hence, given a time-invariant network topology and the initial decision variables, β_1, β_2 and β_3 are determined accordingly as constants. It then follows from (16) that the μ-neighborhood can be made sufficiently small if a sufficiently large parameter η is chosen beforehand. In addition, if we extend the finite time T_1 to infinity, it further follows from the analysis in Theorem 4.1 that the estimated error $\tilde{x}$ is bounded and ultimately converges to set $\mathcal{Z}=\{\tilde{x}\in\mathbb{R}^{n\times n}\mid\|\tilde{x}\|\le\beta_2/(\eta\beta_1\beta_3)\}$. Note that a sufficiently large parameter η also guarantees a sufficiently small ultimate convergence bound. This effectively improves the estimation accuracy.

In addition, based on Lemma 2.2, each decision variable x_i, $i\in\mathcal{V}$ driven by the seeking algorithm (14) cannot escape to infinity. Hence, we just need to study the convergence of the strategy profile for the time interval $[T_1,\infty)$.

Define $\hat{x}=\mathrm{col}(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)$ and $\varGamma=\mathrm{diag}\{\lambda_1,\lambda_2,\dots,\lambda_n\}$, and introduce

$$\phi=\begin{bmatrix}x\\\lambda\end{bmatrix},\quad\bar{F}(\phi)=\begin{bmatrix}F(x)+\varGamma G(x)\\-g(x)\mathbf{1_n}\end{bmatrix},\tag{20a}$$

$$\varPhi=\varOmega\times\varLambda^n,\quad H(\phi)=P_\varPhi(\phi-\bar{F}(\phi)),\tag{20b}$$

where $\varLambda^n=\varLambda\times\varLambda\times\cdots\times\varLambda$, and $F(x)$ and $G(x)$ have been defined in (7) and (8). Thus, the seeking algorithm (14) can be rewritten in the following compact form:

$$\dot{\phi}=-\phi+P_\varPhi(\phi-\bar{F}'(\hat{x},\lambda)),\tag{21}$$

where $\bar{F}'(\hat{x},\lambda)$ is a column stack vector composed in sequence by $\nabla_{x_i}f_i(\hat{x}_i)+\lambda_i\nabla_{x_i}g(\hat{x}_i)$ and $-g(\hat{x}_i)$, $i\in\mathcal{V}$. We further introduce a function:

$$S=(\bar{F}(\phi)-\bar{F}(\bar{\phi}))^T(\phi-H(\phi))+\frac{1}{2}\|\phi-\bar{\phi}\|^2,\tag{22}$$

where $\bar{\phi}=\mathrm{col}(\bar{x},\bar{\lambda})$, and $\bar{x}$ and $\bar{\lambda}$ have been introduced in Lemma 4.1. Before presenting the main result, we propose a proposition to indicate the positive definiteness of function S.

Proposition 4.1. *Under Assumptions 3.1–3.3, function S in (22) is positive definite with respect to $\phi-\bar{\phi}$.*

Proof. See Appendix B. ■

In terms of Theorem 4.1 and Proposition 4.1, we next present the convergence result of the seeking algorithm (14) in the following theorem.

Theorem 4.2. *Consider the distributed observer (13) and the seeking algorithm (14). Suppose that Assumptions 3.1–3.3 hold, and the underlying graph $\mathcal{G}$ is strongly connected. Then the strategy profile x ultimately converges to a $\mu'(\mu)$-neighborhood of the generalized Nash equilibrium in the sense of Definition 3.1.*

Proof. Note from the analysis of Lemma 4.1 that $\bar{\lambda}$ conforms to (11). It is trivial to show that there always exists a $\bar{\lambda}$ such that each $\bar{\lambda}_i\in\varLambda$ given $\lambda_{\max}>0$. This implies that $\bar{\phi}=H(\bar{\phi})$. Assign a Lyapunov function in the form of (22). According to Proposition 4.1, S is positive definite with respect to $\phi-\bar{\phi}$. Its derivative along (21) satisfies

$$\begin{aligned}\dot{S}=&(\bar{F}(\phi)-\bar{F}(\bar{\phi})+\mathcal{J}F(\phi)(\phi-H(\phi))+\phi-\bar{\phi})^T(-\phi)\\&+P_\varPhi(\phi-\bar{F}'(\hat{x},\lambda))\\=&-(\bar{F}(\phi)-\bar{F}(\bar{\phi})+\mathcal{J}F(\phi)(\phi-H(\phi))+\phi-\bar{\phi})^T(\phi\\&-H(\phi))+(\bar{F}(\phi)-\bar{F}(\bar{\phi})+\mathcal{J}F(\phi)(\phi-H(\phi))+\phi-\bar{\phi})^T\\&\cdot H(\phi)-P_\varPhi(\phi-\bar{F}'(\hat{x},\lambda)).\end{aligned}\tag{23}$$

In view of Lemma 2.2, the seeking algorithm (14) guarantees that there exists a positive constant $\bar{\gamma}_1$ such that $|x(t)|\le\bar{\gamma}_1$, $\forall t\ge 0$. This, according to Assumption 3.2, guarantees that the constrained function $g(x)$, and the gradient functions $\nabla_{x_i}f_i(x)$ and $\nabla_{x_i}g_i(x)$ are locally θ-Lipschitz over this domain. It then follows from Theorem 4.1 that $\|\tilde{x}(t)\|\le\mu$, $\forall t\ge T_1$. Thus, in terms of Lemma 2.1, it follows that for $t\ge T_1$,

$$\begin{aligned}&\|-H(\phi)+P_\varPhi(\phi-\bar{F}'(\hat{x},\lambda))\|\le\|\bar{F}(\phi)-\bar{F}'(\hat{x},\lambda)\|\\\le&\sum_{i=1}^{n}\Big(|\nabla_{x_i}f_i(x)-\nabla_{x_i}f_i(\hat{x}_i)|+|\lambda_i(\nabla_{x_i}g(x)-\nabla_{x_i}g(\hat{x}_i))|\\&+|g(x)-g(\hat{x}_i)|\Big)\\\le&\sqrt{2}(2+\lambda_{\max})\theta\|\tilde{x}\|\le\kappa\mu,\end{aligned}$$

where $\kappa=\sqrt{2}(2+\lambda_{\max})\theta$. Moreover, by considering Assumption 3.2 and the facts that $|x|\le\bar{\gamma}_1$ and $\lambda_i\in\varLambda$, $i\in\mathcal{V}$, there exists a positive constant ϖ such that

$$\|(\bar{F}(\phi)-\bar{F}(\bar{\phi})+(\phi-H(\phi))\mathcal{J}F(\phi))^T(\phi-\bar{\phi})\|\le\varpi.$$

Accordingly, for $t\ge T_1$, $\dot{S}(t)$ satisfies

$$\begin{aligned}\dot{S}\le&-(\bar{F}(\phi)-\bar{F}(\bar{\phi})+\mathcal{J}\bar{F}(\phi)(\phi-H(\phi))+\phi-\bar{\phi})^T\\&\cdot(\phi-H(\phi))+\kappa\varpi\mu\\=&-(\bar{F}(\phi)-\bar{F}(\bar{\phi})+\mathcal{J}\bar{F}(\phi)(\phi-H(\phi))+\phi-\bar{\phi})^T\\&\cdot(\phi-\bar{\phi}-H(\phi)+\bar{\phi})+\kappa\varpi\mu\\=&-(\phi-H(\phi))^T\mathcal{J}\bar{F}(\phi)(\phi-H(\phi))-\|\phi-\bar{\phi}\|^2\\&-\bar{F}(\phi)-\bar{F}(\bar{\phi})^T(\phi-\bar{\phi})-(H(\phi)-\bar{\phi})^T\bar{F}(\bar{\phi})\\&-(\phi-\bar{F}(\phi)-H(\phi))^T(H(\phi)-\bar{\phi})\\&+(\phi-H(\phi)+\phi-\bar{\phi})^T(H(\phi)-\phi+\phi-\bar{\phi})+\kappa\varpi\mu.\end{aligned}\tag{24}$$

Based on Assumption 3.3, it is trivial to show that the Jacobian matrix $\mathcal{J}\bar{F}(\phi)$ is positive semi-definite. This implies that $-(\phi-H(\phi))^T\mathcal{J}\bar{F}(\phi)(\phi-H(\phi))\le 0$. It then follows from the analysis in Proposition 4.1 that $-(\bar{F}(\phi)-\bar{F}(\bar{\phi}))^T(\phi-\bar{\phi})\le-\omega\|x-\bar{x}\|^2$.

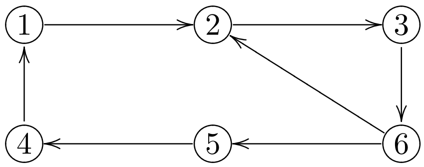

Fig. 1. Fixed topology among players.

Moreover, constrained by the variational inequality (Facchinei & Kanzow, 2007), we have that $-(H(\phi)-\bar{\phi})^T\bar{F}(\bar{\phi}) \le 0$. Furthermore, in terms of Lemma 2.1, we have that $-(\phi-\bar{F}(\phi)-H(\phi))(H(\phi)-\bar{\phi}) \le 0$. It then follows that, for $t \ge T_1$, $\dot{S}(t)$ satisfies

$$\begin{aligned}\dot{S} &\le -\omega\|x-\bar{x}\|^2 - \|\phi - H(\phi)\|^2 + \kappa\varpi\mu \\ &\le -\omega\|x-\bar{x}\|^2 + \kappa\varpi\mu. \end{aligned} \tag{25}$$

This implies that $\dot{S}$ is negative definite provided that $\|x-\bar{x}\| > \sqrt{\kappa\varpi\mu/\omega}$. Consequently, according to Khalil (2002, Theorem 4.18) it can be concluded that $x-\bar{x}$ is bounded and ultimately converges to set $\bar{\varXi} = \{x \in \mathbb{R}^n \mid \|x\| \le \sqrt{\kappa\varpi\mu/\omega}\}$. Since it has been shown in Lemma 4.1 that $\bar{x} = x^*$, it finally follows that x ultimately converges to the μ'-neighborhood of x^* with $\mu' = \sqrt{\kappa\varpi\mu/\omega}$. ■

Remark 4.2. According to the expression of parameter μ' in the analysis of Theorem 4.2, the convergence accuracy of the strategy profile x to the generalized Nash equilibrium is determined by parameter μ. As shown in Theorem 4.1, parameter μ can be driven arbitrarily small by specifying a sufficiently large parameter η. Towards this end, increasing parameter η enables the strategy profile x to converge to an arbitrarily small neighborhood of the generalized Nash equilibrium.

5. Simulations

In this section, an example is given to verify the effectiveness of the proposed distributed seeking algorithm.

Suppose that there is a non-cooperative game consisting of six players. The cost function $f_i(x)$ assigned to each player is $f_i(x) = m_i f(x)$, where $m_1 = 1$, $m_2 = 5$, $m_3 = 2$, $m_4 = 3$, $m_5 = 2$, $m_6 = 4$ and

$$\begin{aligned} f(x) =& 5x_1^2 + 2x_1x_2 + 5x_2^2 + x_2x_3 + x_2x_5 + \frac{5}{2}x_3^2 + x_3x_4 \\ &+ x_4^2 + 2x_4x_5 + 3x_5^2 + 3x_5x_6 + x_6^2 - 2x_1 + 3x_2 \\ &- 8x_3 - 6x_4 + 10x_5 - x_6. \end{aligned}$$

The network topology among these six players is illustrated in Fig. 1, which can be examined to be strongly connected. The parameters in the seeking algorithms are chosen as $\eta = 100$ and $\lambda_{\max} = 10$. Consider that the strategy profile is subject to a set constraint: $\Omega_i = \{|x_i| \le 1\}$, $i \in \mathcal{V}$ and an inequality constraint: $g(x) = \sum_{i=1}^{6} x_i^2 + \sum_{i=1}^{5} x_i x_{i+1} - 4 \le 0$. It can be calculated that the generalized Nash equilibrium is $(x^*)_1 = [0.2708, -0.3542, 1, 0.7, -1, 1]^T$. Fig. 2 shows that each decision variable driven by the seeking algorithm (14) converges to the small neighborhood of the generalized Nash equilibrium.

6. Conclusion

This paper investigates the Nash equilibrium seeking problem for non-cooperative games subject to set and nonlinear inequality constraints. Each player is assigned a cost function and a constrained set, while all the cost functions, as well as the constrained function, are coupled by all the decision variables. A distributed observer is first developed such that each player obtains all others' decision variables. Using these estimates, we then propose a continuous-time seeking algorithm with a projection operator. By using the time-scale separation approach, it is proven that the proposed seeking algorithm achieves the convergence of the strategy profile to an arbitrarily small neighborhood of the generalized Nash equilibrium satisfying a KKT condition.

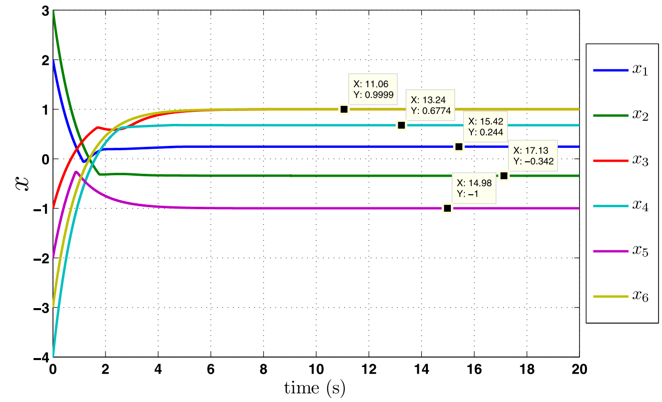

Fig. 2. Trajectories of decision variables with set and inequality constraints.

Appendix A. Proof of Lemma 2.2

First, we prove the invariant set of $\varXi$ by contradiction. Suppose that this is not true. In such a case, based on the continuity of x, there exists a time t' such that $x(t') = \xi_1$ with $\dot{x}(t') < 0$ or $x(t') = \xi_2$ with $\dot{x}(t') > 0$. Considering the projection operator, we have that $\xi_1 \le P_{\varXi}(h(x,u)) \le \xi_2$. Hence, it follows from (3) that there exists a constant $\xi \in [\xi_1, \xi_2]$ such that

$$\dot{x}(t') = -\xi_1 + \xi \ge 0 \text{ or } \dot{x}(t') = -\xi_2 + \xi \le 0. \tag{A.1}$$

This brings in contradiction. Therefore, given $x(0) \in \varXi$, it follows that $x(t) \in \varXi$, $\forall t \ge 0$.

Next, we focus on the boundedness of x when $x(0) \notin \varXi$. Assign a positive function $W = x^2/2$. Its derivative along (3) satisfies

$$\dot{W} \le -2W + \sqrt{2W}\bar{\xi}, \tag{A.2}$$

where $\bar{\xi} = \max\{|\xi_1|, |\xi_2|\}$. Take $V = \sqrt{W}$. It then follows that its Dini derivative satisfies $D^+V \le -V + \sqrt{2}\bar{\xi}/2$. According to Comparison Principle (Khalil, 2002), it follows that

$$V(t) \le e^{-t}V(0) + (1-e^{-t})\frac{\sqrt{2}}{2}\bar{\xi} \le \gamma, \tag{A.3}$$

where $\gamma = \max\{V(0), \sqrt{2}\bar{\xi}/2\}$. This completes the proof.

Appendix B. Proof of Proposition 4.1

Note from the analysis of Lemma 4.1 that $\bar{\lambda}$ conforms to (11). When $g(\bar{x}) = 0$, $\bar{\lambda}_i > 0$ can be specified arbitrarily. In such a case, there always exists a $\bar{\lambda}_i \in \varLambda$ no matter which $\lambda_{\max}$ is chosen as. When $g(\bar{x}) < 0$, it follows from (11) that $\bar{\lambda}_i = 0$, making no difference to the choice of $\lambda_{\max}$. Therefore, given $\lambda_{\max} > 0$, there must exist $\bar{\lambda}$ such that each $\bar{\lambda}_i \in \varLambda$. This further implies that $\bar{\phi} = H(\bar{\phi})$. In such a case, according to Lemma 2.1, S satisfies $S = S_1 + S_2$, where

$$S_1 = (\bar{F}(\phi) - \bar{F}(\bar{\phi}))^T(\phi - \bar{\phi}),$$

$$S_2 = -(\bar{F}(\phi) - \bar{F}(\bar{\phi}))^T(H(\phi) - \bar{\phi}) + \frac{1}{2}\|\phi - \bar{\phi}\|^2.$$

Under Assumption 3.3, S_1 satisfies

$$\begin{aligned} S_1 =& (F(x) - F(\bar{x}))^T(x - \bar{x}) \\ &+ \sum_{i=1}^{n}\big((\lambda_i \nabla_{x_i} g(x) - \bar{\lambda}_i \nabla_{x_i} g(\bar{x}))(x_i - \bar{x}_i) \\ &+ (g(x) - g(\bar{x}))(\lambda_i - \bar{\lambda}_i)\big) \\ \geq& \omega\|x - \bar{x}\|^2 + \sum_{i=1}^{n} \lambda_i (g(\bar{x}) - g(x) - \nabla_{x_i} g(x)(\bar{x}_i - x_i)) \\ &+ \sum_{i=1}^{n} \bar{\lambda}_i (g(x) - g(\bar{x}) - \nabla_{x_i} g(\bar{x})(x_i - \bar{x}_i)) \\ \geq& \omega\|x - \bar{x}\|^2 \geq 0. \end{aligned} \tag{B.1}$$

Next, by considering the variational inequality $\bar{F}(\bar{\phi})^T(H(\phi) - \bar{\phi}) = \bar{F}(\bar{\phi})^T(H(\phi) - H(\bar{\phi})) \geq 0$, S_2 satisfies

$$\begin{aligned} S_2 \geq& -\bar{F}(\phi)^T(H(\phi) - \bar{\phi}) + \frac{1}{2}\|\phi - \bar{\phi}\|^2 \\ =& (\phi - \bar{F}(\phi) - H(\phi))^T(H(\phi) - \bar{\phi}) \\ &- (\phi - H(\phi))^T(H(\phi) - \bar{\phi}) + \frac{1}{2}\|\phi - \bar{\phi}\|^2 \\ \geq& -(\phi - H(\phi))^T(H(\phi) - \bar{\phi}) + \frac{1}{2}\|\phi - \bar{\phi}\|^2 \\ =& \|\phi - H(\phi)\|^2 - (\phi - H(\phi))^T(\phi - \bar{\phi}) + \frac{1}{2}\|\phi - \bar{\phi}\|^2 \\ \geq& \frac{1}{4}\|\phi - \bar{\phi}\|^2, \end{aligned} \tag{B.2}$$

where the second inequality follows from Lemma 2.1, and the last inequality follows from the fact that $\pm a^T b \geq -\frac{1}{4}\|a\|^2 - \|b\|^2$ for $\forall a, b \in \mathbb{R}^n$. Substituting (B.1) and (B.2) into S yields that $S \geq \frac{1}{4}\|\phi - \bar{\phi}\|^2$. This implies that S is positive definite with respect to $\phi - \bar{\phi}$.

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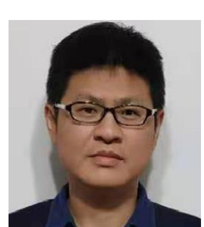

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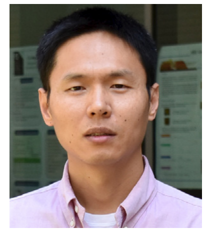

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